Using LJF as a Framework for Proof Systems

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Abstract

In this work we show how to use the focused intuitionistic logic system LJF as a framework for encoding several different intuitionistic and classical proof systems. The proof systems are encoded in a strong level of adequacy, namely the level of (open) derivations. Furthermore we show how to prove *relative completeness* between the different systems. By relative completeness we mean that the systems prove the same formulas. The proofs of relative completeness exploit the encodings to give, in most cases, fairly simple proofs. This work is heavily based on the recent work by Nigam and Miller, which uses the focused linear logic system LLF to encode the same proof systems as we do. Our work shows that the features of linear logic are not needed for the full adequacy result, and furthermore we show that even though encoding in LLF is more generic and streamlined, the encoding in LJF sometimes gives simpler, more natural encodings and easier proofs.

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Contents

1 Introduction

In recent work by Nigam and Miller [11] they propose to use a *focused* proof system for classical linear logic as a logic framework for representing different object-level proof systems.

In a focused proof system the connectives are divided into two groups. The *negative* or *asynchronous* connectives are usually associated with invertible rules, whereas the *positive* or *synchronous* connective in general not need to have invertible rules. For a system with two-sided sequents like intuitionistic logic the negative connectives would be the ones with invertible right-introduction rules and the positive the ones with invertible left-introduction rules. Formulas are assigned to the groups according to their top most connective. Atoms are arbitrarily assigned to either the negative or positive group. We say that a formula or connective assigned to the negative (resp. positive) group has negative (resp. positive) *polarity*.

Derivations in a focused proof system are divided into two *phases* corresponding to the groups above. The negative or asynchronous phase applies all invertible rules to the sequent. The positive or synchronous phase *focuses* on a particular formula and then keeps applying the remaining rules until the formula becomes negative or an atom. As all rules applied in the negative phase are invertible we can exploit the "dont-care" non-determinism of the invertible rules and see the entire negative phase as one single *macro-rule*, which applies all negative rules in one go. The macro-rule concept is critical for the strong encodings shown later. The positive phase may contain "dont-know" non-determinism (e.g. whihc branch to prove in a disjunction), but because it is fixed by the focused formula we can also see the synchronous phase as consisting of one macro-rule, although there may be several applicable macro-rules.

Focusing was first developed for classical linear logic by Andreoli [1], and later adopted into various proof systems (e.g. [3, 15]). An important result, is that regardless of how the polarity is assigned to atoms, the system is complete with respect to classical linear logic.

In the work by Nigam and Miller they show how to encode a long range of different intuitionistic and classical proof systems using Andreoli's focused system LLF. The encoded systems are strongly related to the object systems, as there is a provable bijection between the *open derivations* in LLF and open derivations in the object system. To establish the bijection the assignment of polarity to atoms play a big role, as the polarity can be used to enforce either forward or backward reasoning.

In this work we show that the approach to encoding object-level systems using a focused proof system is not dependent on the linear aspects of LLF, and we show how to encode the same systems, and get the full completeness of open derivations, using the focused intuitionistic system LJF [7]. Furthermore we compare the LJF encodings to the LLF encodings and comment on the differences.

In addition to encoding different systems Nigam and Miller uses the meta-logic encodings to prove properties of the object-logic systems. In the paper they focus on *relative completeness* which is completeness with respect to provability. They show how to relate the different proof systems back to a generic set of rules with the inclusion/exclusion of specific structural rules.

In this work we also look at relative completeness, but instead of relating back to a generic set of rules we relate the different systems to each other. The reason is that the encodings of intuitionistic and classical systems are rather different as we have to encode them differently using an intuitionistic meta-logic. But in the end we get the same results and in several places the proofs seems easier to do.

A disclaimer: the work in this report is based on the article by Nigam and Miller [11] therefore there will be a lot of similarities, for instance, all the systems we have encoded, are found in their article. We have made some changes to the systems on the syntactic level and a few other changes, these changes will be described in the later sections. Furthermore most of the references here are from the article as well, they are included here as convenience for the reader.

2 Focused intuitionistic logic

As framework for hosting different object-level systems we use the focused intuitionistic system LJF [7]. The choice of LJF is somewhat arbitrary and we think that a similar intuitionistic focused system like for instance the system by Pfenning [15] could be used as well. An important criterion of the system is the possibility of assigning arbitrary polarities to atoms and still retaining completeness with regards to unfocused intuitionistic logic.

The rest of this section presents LJF. For a more complete description the reader is referred to the original paper on LJF.

The formulas in LJF are given by the following grammar:

$$\begin{aligned} A ::= \quad & M \mid \mathbf{false} \mid \mathbf{true} \mid \\ & A_1 \supset A_2 \mid A_1 \wedge^- A_2 \mid A_1 \wedge^+ A_2 \mid A_1 \vee A_2 \mid \\ & \forall x A \mid \exists x A \end{aligned}$$

where M is an atom. We note that there are no negation ($\neg A$ can be defined as $A \supset \mathbf{false}$), and that there are two conjunctions. The two conjunctions are equivalent in terms of intuitionistic provability, but have different focused proofs as one is positive and the other is negative. We refer to the original paper for more information.

Atoms in LJF are arbitrary assigned either positive or negative polarity. Positive formulas are given by the following:

$$P ::= M_P \mid \mathbf{false} \mid \mathbf{true} \mid A_1 \wedge^+ A_2 \mid A_1 \vee A_2 \mid \exists x A$$

where M_P is a positive atom. Negative formulas are given by the following:

$$N ::= M_N \mid A_1 \supset A_2 \mid A_1 \wedge^- A_2 \mid \forall x A$$

where M_N is a negative atom.

The sequents in LJF have one of the following forms:

1. $[\Gamma], \Theta \longrightarrow \mathcal{R}$ is an unfocused sequent. Γ and Θ are multisets and Γ only contains negative formulas and atoms. $\mathcal{R}$ is either a formula R or a bracketed formula $[R]$. The formulas inside brackets are saved for the synchronous phase.
2. $[\Gamma] \xrightarrow{A} [R]$ is a left-focused sequent with focus on A.
3. $[\Gamma] -_A\!\!\rightarrow$ is a right-focused sequent with focus on A.

The rules for LJF are given in Figure 1. The initial rules ends the proof if the atom has the right polarity. The decision rules pick a formula from either the left or the right side and continues with focus on that formula. If a formula is positive in the left-focused phase or negative in the right-focused phase, focus is lost by one of the reaction rules. Furthermore in the asynchronous phase, synchronous formulas are inserted into the bracketed context by one of the remaining reaction rules. The introduction rules decompose a specific formula. For the synchronous phase the focused formula is decomposed and for the asynchronous phase one of the asynchronous formulas are decomposed.

As mentioned in the introduction we can see the two phases as applying macro-rules which takes an entire phase in one go. Consider for example the possible derivations of $[\Gamma] -_{A_1 \vee (A_2 \wedge^+ A_3)}\!\!\rightarrow$ where A_1, A_2, A_3 are negative:

$$\dfrac{\dfrac{[\Gamma] \longrightarrow A_1}{[\Gamma] -_{A_1}\!\!\rightarrow}}{[\Gamma] -_{A_1 \vee (A_2 \wedge^+ A_3)}\!\!\rightarrow} \qquad \dfrac{\dfrac{\dfrac{[\Gamma] \longrightarrow A_2}{[\Gamma] -_{A_2}\!\!\rightarrow} \quad \dfrac{[\Gamma] \longrightarrow A_3}{[\Gamma] -_{A_3}\!\!\rightarrow}}{[\Gamma] -_{A_2 \wedge^+ A_3}\!\!\rightarrow}}{[\Gamma] -_{A_1 \vee (A_2 \wedge^+ A_3)}\!\!\rightarrow}$$

These can be seen as applications of one of two different macro-rules:

Initial Rules

$$\frac{}{[\Gamma] \xrightarrow{M_N} [M_N]}\ \mathsf{I_L} \qquad \frac{}{[M_P, \Gamma] -_{M_P}\!\!\rightarrow}\ \mathsf{I_R}$$

Decision Rules

$$\frac{[N, \Gamma] \xrightarrow{N} [R]}{[N, \Gamma] \longrightarrow [R]}\ \mathsf{L_F} \qquad \frac{[\Gamma] -_{P}\!\!\rightarrow}{[\Gamma] \longrightarrow [P]}\ \mathsf{R_F}$$

Reaction Rules

$$\frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]}\ \mathsf{R_L} \qquad \frac{[\Gamma] \longrightarrow N}{[\Gamma] -_{N}\!\!\rightarrow}\ \mathsf{R_R}$$

$$\frac{[C, \Gamma], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, C \longrightarrow \mathcal{R}}\ []_L \qquad \frac{[\Gamma], \Theta \longrightarrow [D]}{[\Gamma], \Theta \longrightarrow D}\ []_R$$

Introduction Rules

$$\frac{}{[\Gamma], \Theta, \mathbf{false} \longrightarrow \mathcal{R}}\ \mathbf{false}_{\mathsf{L}}$$

$$\frac{[\Gamma], \Theta \longrightarrow \mathcal{R}}{[\Gamma], \Theta, \mathbf{true} \longrightarrow \mathcal{R}}\ \mathbf{true}_{\mathsf{L}} \qquad \frac{}{[\Gamma] -_{\mathbf{true}}\!\!\rightarrow}\ \mathbf{true}_{\mathsf{R}}$$

$$\frac{[\Gamma] \xrightarrow{A_i} [R]}{[\Gamma] \xrightarrow{A_1 \wedge^- A_2} [R]}\ \wedge^-_{\mathsf{L}} \qquad \frac{[\Gamma], \Theta \longrightarrow A_1 \qquad [\Gamma], \Theta \longrightarrow A_2}{[\Gamma], \Theta \longrightarrow A_1 \wedge^- A_2}\ \wedge^-_{\mathsf{R}}$$

$$\frac{[\Gamma], \Theta, A_1, A_2 \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A_1 \wedge^+ A_2 \longrightarrow \mathcal{R}}\ \wedge^+_{\mathsf{L}} \qquad \frac{[\Gamma] -_{A_1}\!\!\rightarrow \qquad [\Gamma] -_{A_2}\!\!\rightarrow}{[\Gamma] -_{A_1 \wedge^+ A_2}\!\!\rightarrow}\ \wedge^+_{\mathsf{R}}$$

$$\frac{[\Gamma], \Theta, A_1 \longrightarrow \mathcal{R} \qquad [\Gamma], \Theta, A_2 \longrightarrow \mathcal{R}}{[\Gamma], \Theta, A_1 \vee A_2 \longrightarrow \mathcal{R}}\ \vee_{\mathsf{L}} \qquad \frac{[\Gamma] -_{A_i}\!\!\rightarrow}{[\Gamma] -_{A_1 \vee A_2}\!\!\rightarrow}\ \vee_{\mathsf{R}}$$

$$\frac{[\Gamma] -_{A_1}\!\!\rightarrow \qquad [\Gamma] \xrightarrow{A_2} [R]}{[\Gamma] \xrightarrow{A_1 \supset A_2} [R]}\ \supset_{\mathsf{L}} \qquad \frac{[\Gamma], \Theta, A_1 \longrightarrow A_2}{[\Gamma], \Theta \longrightarrow A_1 \supset A_2}\ \supset_{\mathsf{R}}$$

$$\frac{[\Gamma], \Theta, A[c/x] \longrightarrow \mathcal{R}}{[\Gamma], \Theta, \exists x A \longrightarrow \mathcal{R}}\ \exists_{\mathsf{L}}{}^c \qquad \frac{[\Gamma] -_{A[t/x]}\!\!\rightarrow}{[\Gamma] -_{\exists x A}\!\!\rightarrow}\ \exists_{\mathsf{R}}$$

$$\frac{[\Gamma] \xrightarrow{A[t/x]} [R]}{[\Gamma] \xrightarrow{\forall x A} [R]}\ \forall_{\mathsf{L}} \qquad \frac{[\Gamma], \Theta \longrightarrow A[c/x]}{[\Gamma], \Theta \longrightarrow \forall x A}\ \forall_{\mathsf{R}}{}^c$$

Figure 1: The proof system LJF [7]. M_P is a positive atom, M_N is a negative atom. P is positive, N is negative. C is negative or an atom, D is positive or an atom. c is not free in Γ, Θ or $\mathcal{R}$. $i \in \{1, 2\}$.

$$\frac{[\Gamma] \longrightarrow A_1}{[\Gamma] \xrightarrow{}_{A_1 \vee (A_2 \wedge^+ A_3)}} \qquad \frac{[\Gamma] \longrightarrow A_2 \qquad [\Gamma] \longrightarrow A_3}{[\Gamma] \xrightarrow{}_{A_1 \vee (A_2 \wedge^+ A_3)}} \quad (*)$$

Note that we use double lines when we contract several rule applications into one.

As an example for the asynchronous phase, consider the possible derivations of $[\Gamma], (A_1 \vee A_2) \wedge^+ (A_3 \wedge^+ A_4) \longrightarrow [R]$ where the A_i's are negative:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Gamma, A_1, A_3, A_4] \longrightarrow [R]}{\vdots}}{[\Gamma], A_1, A_3, A_4 \longrightarrow [R]}}{[\Gamma], A_1, A_3 \wedge^+ A_4 \longrightarrow [R]} \quad \dfrac{\dfrac{\dfrac{[\Gamma, A_2, A_3, A_4] \longrightarrow [R]}{\vdots}}{[\Gamma], A_2, A_3, A_4 \longrightarrow [R]}}{[\Gamma], A_2, A_3 \wedge^+ A_4 \longrightarrow [R]}}{\dfrac{[\Gamma], A_1 \vee A_2, A_3 \wedge^+ A_4 \longrightarrow [R]}{[\Gamma], (A_1 \vee A_2) \wedge^+ (A_3 \wedge^+ A_4) \longrightarrow [R]}}$$

$$\dfrac{\dfrac{\dfrac{[\Gamma, A_1, A_3, A_4] \longrightarrow [R]}{\vdots}}{[\Gamma], A_1, A_3, A_4 \longrightarrow [R]} \quad \dfrac{\dfrac{[\Gamma, A_2, A_3, A_4] \longrightarrow [R]}{\vdots}}{[\Gamma], A_2, A_3, A_4 \longrightarrow [R]}}{\dfrac{[\Gamma], A_1 \vee A_2, A_3, A_4 \longrightarrow [R]}{\dfrac{[\Gamma], A_1 \vee A_2, A_3 \wedge^+ A_4 \longrightarrow [R]}{[\Gamma], (A_1 \vee A_2) \wedge^+ (A_3 \wedge^+ A_4) \longrightarrow [R]}}}$$

These combine into a single macro-rule for the asynchronous phase:

$$\frac{[\Gamma, A_1, A_3, A_4] \longrightarrow [R] \qquad [\Gamma, A_2, A_3, A_4] \longrightarrow [R]}{[\Gamma], (A_1 \vee A_2) \wedge^+ (A_3 \wedge^+ A_4) \longrightarrow [R]}$$

In the rest of this report we consider two LJF derivations with the same macro-rule derivation as the same derivations. We view formulas like $A_1 \vee (A_2 \wedge^+ A_3)$ as a *synthetic connective* with the introduction rules given by (*). This identification of derivations is needed to get full completeness of open derivations for the different systems.

Like the original focused proof system for linear logic, LLF [1], assignment of polarity to atoms does not affect provability. Therefore LJF is sound and complete with respect to intuitionistic logic. In the following and in the rest of this work $\vdash_I$ will stand for provability in an unspecified intuitionistic system. (For instance the sequent calculus of Gentzen [5].)

Theorem 2.1. *For all formulas A and for any assignment of polarities to atoms:*

$$\vdash_{\mathrm{I}} A \quad \textit{if and only if} \quad [\,] \longrightarrow A$$

Proof. The proof is available in the original article [7]. $\square$

In the encodings of the proof systems we will use the bracketed context to hold the encoded rules. The following corollary allows us to use the connection to intuitionistic logic for sequents with non-empty contexts.

Corollary 2.2. *If A is an atom or a positive formula, and Γ consists of atoms and negative formulas then:*

$$\Gamma \vdash_{\mathrm{I}} A \quad \textit{if and only if} \quad [\Gamma] \longrightarrow [A]$$

Proof. If $\Gamma = \{A_1, \ldots, A_n\}$ then by invertability of implication in intuitionistic logic we have that:

$$\Gamma \vdash_{\mathrm{I}} A \quad \text{if and only if} \quad \vdash_{\mathrm{I}} A_1 \supset (A_2 \supset \cdots (A_n \supset A) \cdots)$$

which by Theorem 2.1 means that:

$$\Gamma \vdash_{\mathrm{I}} A \quad \text{if and only if} \quad [\,] \longrightarrow A_1 \supset (A_2 \supset \cdots (A_n \supset A) \cdots)$$

which means that we have:

$$\Gamma \vdash_{\mathrm{I}} A \quad \text{if and only if} \quad [\,], \Gamma \longrightarrow A$$

and because A is an atom or a positive formula, and Γ consists of atoms and negative formulas then:

$$\Gamma \vdash_{\mathrm{I}} A \quad \text{if and only if} \quad [\Gamma] \longrightarrow [A]$$

$\square$

3 Encoding in LJF

The encodings of object-level formulas inside LJF atoms are the same as in Nigam and Miller's paper and we shortly give an overview here.

We assume a type of object-level formulas called *form* and a type of object-level terms called i. The object-level quantifiers have type $(i \rightarrow \mathit{form}) \rightarrow \mathit{form}$. The meta-level quantifiers have type $(\mathit{term} \rightarrow o) \rightarrow o$ where term is either i or $i \rightarrow \cdots i \rightarrow \mathit{form}$ (0 or more i's).

$$\frac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LJ}} A_1 \qquad \Gamma, A_1 \Rightarrow A_2, A_2 \vdash_{\mathrm{LJ}} C}{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LJ}} C} \Rightarrow_{\mathsf{L}} \quad \frac{\Gamma, A_1 \vdash_{\mathrm{LJ}} A_2}{\Gamma \vdash_{\mathrm{LJ}} A_1 \Rightarrow A_2} \Rightarrow_{\mathsf{R}}$$

$$\frac{\Gamma, A_1 \wedge A_2, A_i \vdash_{\mathrm{LJ}} C}{\Gamma, A_1 \wedge A_2 \vdash_{\mathrm{LJ}} C} \wedge_{\mathsf{L}} \quad \frac{\Gamma \vdash_{\mathrm{LJ}} A_1 \qquad \Gamma \vdash_{\mathrm{LJ}} A_2}{\Gamma \vdash_{\mathrm{LJ}} A_1 \wedge A_2} \wedge_{\mathsf{R}}$$

$$\frac{\Gamma, A_1 \vee A_2, A_1 \vdash_{\mathrm{LJ}} C \qquad \Gamma, A_1 \vee A_2, A_2 \vdash_{\mathrm{LJ}} C}{\Gamma, A_1 \vee A_2 \vdash_{\mathrm{LJ}} C} \vee_{\mathsf{L}} \quad \frac{\Gamma \vdash_{\mathrm{LJ}} A_i}{\Gamma \vdash_{\mathrm{LJ}} A_1 \vee A_2} \vee_{\mathsf{R}}$$

$$\frac{\Gamma, \forall x A, A[t/x] \vdash_{\mathrm{LJ}} C}{\Gamma, \forall x A \vdash_{\mathrm{LJ}} C} \forall_{\mathsf{L}} \quad \frac{\Gamma \vdash_{\mathrm{LJ}} A[c/x]}{\Gamma \vdash_{\mathrm{LJ}} \forall x A} \forall_{\mathsf{R}}{}^{c}$$

$$\frac{\Gamma, \exists x A, A[c/x] \vdash_{\mathrm{LJ}} C}{\Gamma, \exists x A \vdash_{\mathrm{LJ}} C} \exists_{\mathsf{L}}{}^{c} \quad \frac{\Gamma \vdash_{\mathrm{LJ}} A[t/x]}{\Gamma \vdash_{\mathrm{LJ}} \exists x A} \exists_{\mathsf{R}}$$

$$\frac{}{\Gamma, A \vdash_{\mathrm{LJ}} A} \mathsf{I} \quad \frac{\Gamma \vdash_{\mathrm{LJ}} A \qquad \Gamma, A \vdash_{\mathrm{LJ}} C}{\Gamma \vdash_{\mathrm{LJ}} C} \mathsf{Cut}$$

$$\frac{}{\Gamma, \bot \vdash_{\mathrm{LJ}} C} \bot_{\mathsf{L}} \quad \frac{}{\Gamma \vdash_{\mathrm{LJ}} \top} \top_{\mathsf{R}}$$

Figure 2: The proof system LJ. c is not free in Γ or C. $i \in \{1, 2\}$.

$$\frac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LK}} A_1, \Delta \qquad \Gamma, A_1 \Rightarrow A_2, A_2 \vdash_{\mathrm{LK}} \Delta}{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LK}} \Delta} \Rightarrow_{\mathsf{L}} \quad \frac{\Gamma, A_1 \vdash_{\mathrm{LK}} A_1 \Rightarrow A_2, A_2, \Delta}{\Gamma \vdash_{\mathrm{LK}} A_1 \Rightarrow A_2, \Delta} \Rightarrow_{\mathsf{R}}$$

$$\frac{\Gamma, A_1 \wedge A_2, A_i \vdash_{\mathrm{LK}} \Delta}{\Gamma, A_1 \wedge A_2 \vdash_{\mathrm{LK}} \Delta} \wedge_{\mathsf{L}} \quad \frac{\Gamma \vdash_{\mathrm{LK}} A_1 \wedge A_2, A_1, \Delta \qquad \Gamma \vdash_{\mathrm{LK}} A_1 \wedge A_2, A_2, \Delta}{\Gamma \vdash_{\mathrm{LK}} A_1 \wedge A_2, \Delta} \wedge_{\mathsf{R}}$$

$$\frac{\Gamma, A_1 \vee A_2, A_1 \vdash_{\mathrm{LK}} \Delta \qquad \Gamma, A_1 \vee A_2, A_2 \vdash_{\mathrm{LK}} \Delta}{\Gamma, A_1 \vee A_2 \vdash_{\mathrm{LK}} \Delta} \vee_{\mathsf{L}} \quad \frac{\Gamma \vdash_{\mathrm{LK}} A_i, A_1 \vee A_2, \Delta}{\Gamma \vdash_{\mathrm{LK}} A_1 \vee A_2, \Delta} \vee_{\mathsf{R}}$$

$$\frac{\Gamma, \forall x A, A[t/x] \vdash_{\mathrm{LK}} \Delta}{\Gamma, \forall x A \vdash_{\mathrm{LK}} \Delta} \forall_{\mathsf{L}} \quad \frac{\Gamma \vdash_{\mathrm{LK}} A[c/x], \forall x A, \Delta}{\Gamma \vdash_{\mathrm{LK}} \forall x A, \Delta} \forall_{\mathsf{R}}{}^{c}$$

$$\frac{\Gamma, \exists x A, A[c/x] \vdash_{\mathrm{LK}} \Delta}{\Gamma, \exists x A \vdash_{\mathrm{LK}} \Delta} \exists_{\mathsf{L}}{}^{c} \quad \frac{\Gamma \vdash_{\mathrm{LK}} A[t/x], \exists x A, \Delta}{\Gamma \vdash_{\mathrm{LK}} \exists x A, \Delta} \exists_{\mathsf{R}}$$

$$\frac{}{\Gamma, A \vdash_{\mathrm{LK}} A, \Delta} \mathsf{I} \quad \frac{\Gamma \vdash_{\mathrm{LK}} A, \Delta \qquad \Gamma, A \vdash_{\mathrm{LK}} \Delta}{\Gamma \vdash_{\mathrm{LK}} \Delta} \mathsf{Cut} \quad \frac{}{\Gamma, \bot \vdash_{\mathrm{LK}} \Delta} \bot_{\mathsf{L}} \quad \frac{}{\Gamma \vdash_{\mathrm{LK}} \top, \Delta} \top_{\mathsf{R}}$$

Figure 3: The proof system LK. c is not free in Γ or Δ. $i \in \{1, 2\}$.

To encode an object-level sequent we introduce two meta-level predicates $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ with type $form \to o$. The first one is used to encode hypotheses in the object-level sequent and the second one is used to encode conclusions. Both predicates can be applied to sets of object-level formulas generating sets of meta-level formulas in a straightforward manner.

In general an intuitionistic sequent $\Gamma \vdash C$ will be encoded as the sequent $\lfloor\Gamma\rfloor \vdash \lceil C\rceil$, and the classical sequent $\Gamma \vdash \Delta$ as $\lfloor\Gamma\rfloor, \lceil\Delta\rceil \vdash \mathbf{false}$. We will give more details on the encodings for each system in the next sections.

3.1 Sequent calculus

In this section we consider a proof system for intuitionistic sequent calculus (LJ) and classical sequent calculus (LK) [5]. These systems are given by the rules in Figure 2 and 3. The system LJ is a little different from the system LJ in the Nigam and Miller paper. We use a version of LJ where there are no empty right sides of the sequents. We use this version to give a simpler encoding, but we will return to the version with empty right sides in Section 3.4.

For the intuitionistic sequent calculus LJ we use a straightforward encoding. All propositions on the left of the LJ sequent are encoded using the $\lfloor\cdot\rfloor$ predicate to the left side of the LJF sequent. The conclusion is encoded on the right side of the LJF sequent using the $\lceil\cdot\rceil$ predicate.

For the left rules we see that they can only be applied if there exists a formula with the given connective. Which means that we must not release focus from atoms in the right focused sequent, and therefore $\lfloor\cdot\rfloor$ atoms has to be positive. The opposite is true for the right rules and therefore $\lceil\cdot\rceil$ atoms has to be negative.

The encoding of the rules for LJ is given in Figure 4. For simplicity we will use a shorthand when presenting rules and omit the leading universals. Therefore we will write:

$$\lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \supset \lfloor B \rfloor)$$

instead of:

$$\forall A \forall B \lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \supset \lfloor B \rfloor)$$

We use $\mathcal{L}_{\text{LJ}}$ to stand for the set of all encoded rules from Figure 4, similarly we will use $\mathcal{L}_{\text{X}}$ to stand for the set of all encoded rules for some system, X. Putting all this together we get that the sequent $\Gamma \vdash_{\text{LJ}} C$ is encoded as $[\mathcal{L}_{\text{LJ}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil C \rceil]$.

The encodings of the different rules are very straightforward as the object-level connectives are encoded by the same meta-level connectives. For the left rules, the principal formula imply the rest of the formulas, and for the right rules, the principal formula is implied by the rest of the formulas.

The encoding encodes LJ on the full level of completeness between (open) derivations, as shown by the following proposition:

Proposition 3.1. *Let $\Gamma \cup \{C\}$ be a set of LJ formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$\Gamma \vdash_{\text{LJ}} C \quad \textit{and} \quad [\mathcal{L}_{\text{LJ}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil C \rceil]$$

$(\Rightarrow_{\mathsf{L}})$	$\lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \supset \lfloor B \rfloor)$	$(\Rightarrow_{\mathsf{R}})$	$\lceil A \Rightarrow B \rceil \subset (\lfloor A \rfloor \supset \lceil B \rceil)$
$(\wedge_{\mathsf{L}})$	$\lfloor A \wedge B \rfloor \supset (\lfloor A \rfloor \wedge^- \lfloor B \rfloor)$	$(\wedge_{\mathsf{R}})$	$\lceil A \wedge B \rceil \subset (\lceil A \rceil \wedge^- \lceil B \rceil)$
$(\vee_{\mathsf{L}})$	$\lfloor A \vee B \rfloor \supset (\lfloor A \rfloor \vee \lfloor B \rfloor)$	$(\vee_{\mathsf{R}})$	$\lceil A \vee B \rceil \subset (\lceil A \rceil \vee \lceil B \rceil)$
$(\forall_{\mathsf{L}})$	$\lfloor \forall x A \rfloor \supset \forall x \lfloor A \rfloor$	$(\forall_{\mathsf{R}})$	$\lceil \forall x A \rceil \subset \forall x \lceil A \rceil$
$(\exists_{\mathsf{L}})$	$\lfloor \exists x A \rfloor \supset \exists x \lfloor A \rfloor$	$(\exists_{\mathsf{R}})$	$\lceil \exists x A \rceil \subset \exists x \lceil A \rceil$
(I)	$\lfloor A \rfloor \supset \lceil A \rceil$	(Cut)	$\lfloor A \rfloor \subset \lceil A \rceil$
$(\bot_{\mathsf{L}})$	$\lfloor \bot \rfloor \supset \mathbf{false}$	$(\top_{\mathsf{R}})$	$\lceil \top \rceil \subset \mathbf{true}$

Figure 4: Intuitionistic sequent calculus, $\mathcal{L}_{\mathrm{LJ}}$.

$(\Rightarrow_{\mathsf{L}})$	$\lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \vee \lfloor B \rfloor)$	$(\Rightarrow_{\mathsf{R}})$	$\lceil A \Rightarrow B \rceil \supset (\lfloor A \rfloor \wedge^+ \lceil B \rceil)$
$(\wedge_{\mathsf{L}})$	$\lfloor A \wedge B \rfloor \supset (\lfloor A \rfloor \wedge^- \lfloor B \rfloor)$	$(\wedge_{\mathsf{R}})$	$\lceil A \wedge B \rceil \supset (\lceil A \rceil \vee \lceil B \rceil)$
$(\vee_{\mathsf{L}})$	$\lfloor A \vee B \rfloor \supset (\lfloor A \rfloor \vee \lfloor B \rfloor)$	$(\vee_{\mathsf{R}})$	$\lceil A \vee B \rceil \supset (\lceil A \rceil \wedge^- \lceil B \rceil)$
$(\forall_{\mathsf{L}})$	$\lfloor \forall x A \rfloor \supset \forall x \lfloor A \rfloor$	$(\forall_{\mathsf{R}})$	$\lceil \forall x A \rceil \supset \exists x \lceil A \rceil$
$(\exists_{\mathsf{L}})$	$\lfloor \exists x A \rfloor \supset \exists x \lfloor A \rfloor$	$(\exists_{\mathsf{R}})$	$\lceil \exists x A \rceil \supset \forall x \lceil A \rceil$
(I)	$(\lfloor A \rfloor \wedge^+ \lceil A \rceil) \supset \mathbf{false}$	(Cut)	$\lfloor A \rfloor \vee \lceil A \rceil$
$(\bot_{\mathsf{L}})$	$\lfloor \bot \rfloor \supset \mathbf{false}$	$(\top_{\mathsf{R}})$	$\lceil \top \rceil \supset \mathbf{false}$

Figure 5: Classical sequent calculus, $\mathcal{L}_{\mathrm{LK}}$.

Proof. This is the first proof of a series of rather similar proof of full completeness. Formally the proof goes by induction on the derivation in both directions but we consider both directions in one go. An observation is that the macro-rules in LJF corresponds exactly to focusing on one of the different formulas from $\mathcal{L}_{\mathrm{LJ}}$. What we then have to show is that each rule in LJ corresponds exactly to focusing on the corresponding formula in $\mathcal{L}_{\mathrm{LJ}}$.

So to prove completeness one goes through all the rules and check correspondence. In each of these proofs we will show a couple of cases and leave the rest for the reader.

$(\Rightarrow_{\mathsf{L}})$:

$$\dfrac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LJ}} A_1 \qquad \Gamma, A_1 \Rightarrow A_2, A_2 \vdash_{\mathrm{LJ}} C}{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LJ}} C} \Rightarrow_{\mathsf{L}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor \Gamma \rfloor \cup \{\lfloor A_1 \Rightarrow A_2 \rfloor\}$)

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor A_1 \Rightarrow A_2 \rfloor}\!\!\rightarrow}\,\mathsf{I}_{\mathsf{R}} \qquad \dfrac{[\mathcal{K}] \longrightarrow [\lceil A_1 \rceil]}{[\mathcal{K}] -_{\lceil A_1 \rceil}\!\!\rightarrow}\,\mathsf{R}_{\mathsf{R}}, []_{\mathsf{R}} \qquad \dfrac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [\lceil C \rceil]}{[\mathcal{K}] \xrightarrow{\lfloor A_2 \rfloor} [\lceil C \rceil]}\,\mathsf{R}_{\mathsf{L}}, []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \supset (\lceil A_1 \rceil \supset \lfloor A_2 \rfloor)} [\lceil C \rceil]}\,2 \times \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil C \rceil]}\,\mathsf{L}_{\mathsf{F}}, 2 \times \forall_{\mathsf{L}}$$

$(\Rightarrow_{\mathsf{R}})$:

$$\frac{\Gamma, A_1 \vdash_{\mathrm{LJ}} A_2}{\Gamma \vdash_{\mathrm{LJ}} A_1 \Rightarrow A_2}\ \Rightarrow_{\mathsf{R}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor \Gamma \rfloor$)

$$\cfrac{\cfrac{\cfrac{}{[\mathcal{K}] \xrightarrow{\lceil A_1 \Rightarrow A_2 \rceil} [\lceil A_1 \Rightarrow A_2 \rceil]}\ \mathsf{I}_{\mathsf{L}} \qquad \cfrac{\cfrac{[\mathcal{K}, \lfloor A_1 \rfloor] \longrightarrow [\lceil A_2 \rceil]}{[\mathcal{K}], \lfloor A_1 \rfloor \longrightarrow \lceil A_2 \rceil}\ []_{\mathsf{L}}, []_{\mathsf{R}}}{[\mathcal{K}] -_{\lfloor A_1 \rfloor \supset \lceil A_2 \rceil}\!\rightarrow}\ \mathsf{R}_{\mathsf{R}}, \supset_{\mathsf{R}}}{[\mathcal{K}] \xrightarrow{\lceil A_1 \Rightarrow A_2 \rceil \subset (\lfloor A_1 \rfloor \supset \lceil A_2 \rceil)} [\lceil A_1 \Rightarrow A_2 \rceil]}\ \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil A_1 \Rightarrow A_2 \rceil]}\ \mathsf{L}_{\mathsf{F}}, 2 \times \forall_{\mathsf{L}}$$

$(\forall_{\mathsf{L}})$:

$$\frac{\Gamma, \forall x A, A[t/x] \vdash_{\mathrm{LJ}} C}{\Gamma, \forall x A \vdash_{\mathrm{LJ}} C}\ \forall_{\mathsf{L}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor \Gamma \rfloor \cup \{\lfloor \forall x A \rfloor\}$)

$$\cfrac{\cfrac{\cfrac{}{[\mathcal{K}] -_{\lfloor \forall x A \rfloor}\!\rightarrow}\ \mathsf{I}_{\mathsf{R}} \qquad \cfrac{\cfrac{[\mathcal{K}, \lfloor A[t/x] \rfloor] \longrightarrow [\lceil C \rceil]}{[\mathcal{K}] \xrightarrow{\lfloor A[t/x] \rfloor} [\lceil C \rceil]}\ \mathsf{R}_{\mathsf{L}}, []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\forall x \lfloor A \rfloor} [\lceil C \rceil]}\ \forall_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor \forall x A \rfloor \supset \forall x \lfloor A \rfloor} [\lceil C \rceil]}\ \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil C \rceil]}\ \mathsf{L}_{\mathsf{F}}, \forall_{\mathsf{L}}$$

(I):

$$\frac{}{\Gamma, A \vdash_{\mathrm{LJ}} A}\ \mathsf{I}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor \Gamma \rfloor \cup \{\lfloor A \rfloor\}$)

$$\cfrac{\cfrac{\cfrac{}{[\mathcal{K}] -_{\lfloor A \rfloor}\!\rightarrow}\ \mathsf{I}_{\mathsf{R}} \qquad \cfrac{}{[\mathcal{K}] \xrightarrow{\lceil A \rceil} [\lceil A \rceil]}\ \mathsf{I}_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A \rfloor \supset \lceil A \rceil} [\lceil A \rceil]}\ \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil A \rceil]}\ \mathsf{L}_{\mathsf{F}}, \forall_{\mathsf{L}}$$

$(\bot_{\mathsf{L}})$:

$$\dfrac{}{\Gamma, \bot \vdash_{\mathrm{LJ}} C}\ \bot_{\mathsf{L}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor \Gamma \rfloor \cup \{\lfloor \bot \rfloor\}$)

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor \bot \rfloor}\rightarrow}\ \mathsf{I}_{\mathsf{R}} \qquad \dfrac{}{[\mathcal{K}] \xrightarrow{\mathbf{false}} [\lceil C \rceil]}\ \mathsf{R}_{\mathsf{L}}, \mathbf{false}_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor \bot \rfloor \supset \mathbf{false}} [\lceil C \rceil]}\ \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil C \rceil]}\ \mathsf{L}_{\mathsf{F}}$$

□

To encode LK we use a well-known encoding [14] where both Γ and Δ are on the left side of the LJF sequent and **false** is on the right side. The hypotheses from Γ will be encoded using the $\lfloor \cdot \rfloor$ predicate and the conclusions will be encoded using the $\lceil \cdot \rceil$ predicate. As for LJ the rules need to force a specific connective therefore we must not release focus on either $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil$ atoms on the right. Which means that both sets of atoms must be positive.

The encoding of the rules for LK is given in Figure 5. The sequent $\Gamma \vdash_{\mathrm{LK}} \Delta$ is encoded as $[\mathcal{L}_{\mathrm{LK}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil] \longrightarrow [\mathbf{false}]$.

The encoding of the rules for LK have several differences from the encoding of the rules in LJ. First the principal formula always implies the rest of the formulas, because both the hypotheses and the conclusions are placed on the left of the LJF sequent. That also means the the right rules are negated which is why $\vee$ is encoded using $\wedge$ and vice-versa. The classical implication is encoded using $\vee$ same as the Cut rule. The I rule is encoded as an implication of false because of the **false** in the encoding of the sequent.

This encoding encode LK on the full level of completeness between (open) derivations, as shown by the following proposition:

Proposition 3.2. *Let Γ and Δ be sets of LK formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$\Gamma \vdash_{\mathrm{LK}} \Delta \quad \textit{and} \quad [\mathcal{L}_{\mathrm{LK}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil] \longrightarrow [\mathbf{false}]$$

Proof. This proof follows in the same way as the proof for LJ, we show some cases here:

$(\Rightarrow_{\mathsf{L}})$:

$$\dfrac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LK}} A_1, \Delta \qquad \Gamma, A_1 \Rightarrow A_2, A_2 \vdash_{\mathrm{LK}} \Delta}{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LK}} \Delta} \Rightarrow_{\mathsf{L}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LK}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil \cup \{\lfloor A_1 \Rightarrow A_2 \rfloor\}$)

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor A_1 \Rightarrow A_2 \rfloor}\!\!\longrightarrow} \mathsf{I_R} \qquad \dfrac{\dfrac{[\mathcal{K}, \lceil A_1 \rceil] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lceil A_1 \rceil \longrightarrow [\mathbf{false}]} []_{\mathsf{L}} \qquad \dfrac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lfloor A_2 \rfloor \longrightarrow [\mathbf{false}]} []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lceil A_1 \rceil \vee \lfloor A_2 \rfloor} [\mathbf{false}]} \mathsf{R_L}, \vee_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \supset (\lceil A_1 \rceil \vee \lfloor A_2 \rfloor)} [\mathbf{false}]} \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]} \mathsf{L_F}, 2 \times \forall_{\mathsf{L}}$$

($\Rightarrow_{\mathsf{R}}$):

$$\dfrac{\Gamma, A_1 \vdash_{\mathrm{LK}} A_1 \Rightarrow A_2, A_2, \Delta}{\Gamma \vdash_{\mathrm{LK}} A_1 \Rightarrow A_2, \Delta} \Rightarrow_{\mathsf{R}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LK}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil \cup \{\lceil A_1 \Rightarrow A_2 \rceil\}$)

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lceil A_1 \Rightarrow A_2 \rceil}\!\!\longrightarrow} \mathsf{I_R} \qquad \dfrac{\dfrac{[\mathcal{K}, \lfloor A_1 \rfloor, \lceil A_2 \rceil] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lfloor A_1 \rfloor, \lceil A_2 \rceil \longrightarrow [\mathbf{false}]} 2 \times []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \rfloor \wedge^{+} \lceil A_2 \rceil} [\mathbf{false}]} \mathsf{R_L}, \wedge^{+}_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lceil A_1 \Rightarrow A_2 \rceil \supset (\lfloor A_1 \rfloor \wedge^{+} \lceil A_2 \rceil)} [\mathbf{false}]} \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]} \mathsf{L_F}, 2 \times \forall_{\mathsf{L}}$$

(I):

$$\dfrac{}{\Gamma, A \vdash_{\mathrm{LK}} A, \Delta} \mathsf{I}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LK}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil \cup \{\lfloor A \rfloor, \lceil A \rceil\}$)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor A \rfloor}\!\!\longrightarrow} \mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}] -_{\lceil A \rceil}\!\!\longrightarrow} \mathsf{I_R}}{[\mathcal{K}] -_{\lfloor A \rfloor \wedge^{+} \lceil A \rceil}\!\!\longrightarrow} \wedge^{+}_{\mathsf{R}} \qquad \dfrac{}{[\mathcal{K}] \xrightarrow{\mathbf{false}} [\mathbf{false}]} \mathsf{R_L}, \mathbf{false}_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{(\lfloor A \rfloor \wedge^{+} \lceil A \rceil) \supset \mathbf{false}} [\mathbf{false}]} \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]} \mathsf{L_F}, \forall_{\mathsf{L}}$$

(Cut):

$$\frac{\Gamma \vdash_{\mathrm{LK}} A, \Delta \qquad \Gamma, A \vdash_{\mathrm{LK}} \Delta}{\Gamma \vdash_{\mathrm{LK}} \Delta}\ \mathsf{Cut}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LK}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil$)

$$\dfrac{\dfrac{\dfrac{[\mathcal{K}, \lfloor A \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lfloor A \rfloor \longrightarrow [\mathbf{false}]}\ []_{\mathsf{L}} \qquad \dfrac{[\mathcal{K}, \lceil A \rceil] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lceil A \rceil \longrightarrow [\mathbf{false}]}\ []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A \rfloor \vee \lceil A \rceil} [\mathbf{false}]}\ \mathsf{R_L}, \vee_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]}\ \mathsf{L_F}, \forall_{\mathsf{L}}$$

□

3.2 Natural deduction

In this section we consider an intuitionistic fragment of the system of natural deduction NJ based on the system by Sieg and Byrnes [19]. We use a formulation of the rules given by Pfenning [15], the rules are presented in Figure 6. We use sequents on the form $\Gamma \vdash_{\mathrm{NJ}} C \uparrow$ when C is obtained in a bottom-up way (reasoning on the conclusion), and sequents on the form $\Gamma \vdash_{\mathrm{NJ}} C \downarrow$ when C is obtained in a top-down way (reasoning from the hypotheses). When a rule has a $\uparrow(\downarrow)$ it means that the arrow can be either $\uparrow$ or $\downarrow$ but that all arrows in the same rule instance must be the same. Our formulation of NJ is slightly different from NJ in the Nigam and Miller paper. In our version of the ($\perp_{\mathsf{E}}$) rule the bottom sequent can be both a down and an up arrow sequent. The arrows can be used to differentiate *normal proofs*, but we do not consider those here, except for mentioning that the different ($\perp_{\mathsf{E}}$) in Nigam and Miller's paper is due to the fact that only their version is allowed in normal proofs.

We wish to encode NJ using $\lfloor\cdot\rfloor$ for the hypotheses and for the $\downarrow$ conclusions. We use $\lceil\cdot\rceil$ for the $\uparrow$ conclusions. Furthermore we encode the hypotheses on the left side of the LJF sequent and the conclusions on the right side, because NJ admits weakening on the left but not on the right.

The introduction rules dictate, like for the sequent calculus, that $\lceil\cdot\rceil$ atoms has to be negative. And because we want to loose focus for the elimination rules $\lfloor\cdot\rfloor$ atoms has to be negative as well. We could possible use a delay construction like $\mathbf{true} \supset \lfloor A \rfloor$ to loose focus but that would clutter the rules unnecessarily.

The encodings of the rules are given in Figure 7. Interestingly the rules are the same as for LJ, the only difference is the change of polarity. The sequent $\Gamma \vdash_{\mathrm{NJ}} A \downarrow$ is encoded as $[\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\lfloor A \rfloor]$. The sequent $\Gamma \vdash_{\mathrm{NJ}} A \uparrow$ is encoded as $[\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil A \rceil]$.

This encoding encodes NJ on the full level of completeness:

$$\frac{\Gamma \vdash_{\mathrm{NJ}} A_1 \Rightarrow A_2 \downarrow \qquad \Gamma \vdash_{\mathrm{NJ}} A_1 \uparrow}{\Gamma \vdash_{\mathrm{NJ}} A_2 \downarrow} \Rightarrow_{\mathsf{E}} \qquad \frac{\Gamma, A_1 \vdash_{\mathrm{NJ}} A_2 \uparrow}{\Gamma \vdash_{\mathrm{NJ}} A_1 \Rightarrow A_2 \uparrow} \Rightarrow_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{NJ}} A_1 \wedge A_2 \downarrow}{\Gamma \vdash_{\mathrm{NJ}} A_i \downarrow} \wedge_{\mathsf{E}} \qquad \frac{\Gamma \vdash_{\mathrm{NJ}} A_1 \uparrow \qquad \Gamma \vdash_{\mathrm{NJ}} A_2 \uparrow}{\Gamma \vdash_{\mathrm{NJ}} A_1 \wedge A_2 \uparrow} \wedge_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{NJ}} A_1 \vee A_2 \downarrow \qquad \Gamma, A_1 \vdash_{\mathrm{NJ}} C \uparrow(\downarrow) \qquad \Gamma, A_2 \vdash_{\mathrm{NJ}} C \uparrow(\downarrow)}{\Gamma \vdash_{\mathrm{NJ}} C \uparrow(\downarrow)} \vee_{\mathsf{E}} \qquad \frac{\Gamma \vdash_{\mathrm{NJ}} A_i \uparrow}{\Gamma \vdash_{\mathrm{NJ}} A_1 \vee A_2 \uparrow} \vee_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{NJ}} \forall x A \downarrow}{\Gamma \vdash_{\mathrm{NJ}} A[t/x] \downarrow} \forall_{\mathsf{E}} \qquad \frac{\Gamma \vdash_{\mathrm{NJ}} A[c/x] \uparrow}{\Gamma \vdash_{\mathrm{NJ}} \forall x A \uparrow} {\forall_{\mathsf{I}}}^{c}$$

$$\frac{\Gamma \vdash_{\mathrm{NJ}} \exists x A \downarrow \qquad \Gamma, A[c/x] \vdash_{\mathrm{NJ}} C \uparrow(\downarrow)}{\Gamma \vdash_{\mathrm{NJ}} C \uparrow(\downarrow)} {\exists_{\mathsf{E}}}^{c} \qquad \frac{\Gamma \vdash_{\mathrm{NJ}} A[t/x] \uparrow}{\Gamma \vdash_{\mathrm{NJ}} \exists x A \uparrow} \exists_{\mathsf{I}}$$

$$\frac{}{\Gamma, A \vdash_{\mathrm{NJ}} A \downarrow} \mathsf{I} \quad \frac{\Gamma \vdash_{\mathrm{NJ}} A \downarrow}{\Gamma \vdash_{\mathrm{NJ}} A \uparrow} \mathsf{M} \quad \frac{\Gamma \vdash_{\mathrm{NJ}} A \uparrow}{\Gamma \vdash_{\mathrm{NJ}} A \downarrow} \mathsf{S} \quad \frac{}{\Gamma \vdash_{\mathrm{NJ}} \top \uparrow} \top_{\mathsf{I}} \quad \frac{\Gamma \vdash_{\mathrm{NJ}} \bot \downarrow}{\Gamma \vdash_{\mathrm{NJ}} C \uparrow(\downarrow)} \bot_{\mathsf{E}}$$

Figure 6: The proof system NJ. c is not free in Γ or C. $i \in \{1, 2\}$.

$(\Rightarrow_{\mathsf{E}})$	$\lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \supset \lfloor B \rfloor)$	$(\Rightarrow_{\mathsf{I}})$	$\lceil A \Rightarrow B \rceil \subset (\lfloor A \rfloor \supset \lceil B \rceil)$
$(\wedge_{\mathsf{E}})$	$\lfloor A \wedge B \rfloor \supset (\lfloor A \rfloor \wedge^- \lfloor B \rfloor)$	$(\wedge_{\mathsf{I}})$	$\lceil A \wedge B \rceil \subset (\lceil A \rceil \wedge^- \lceil B \rceil)$
$(\vee_{\mathsf{E}})$	$\lfloor A \vee B \rfloor \supset (\lfloor A \rfloor \vee \lfloor B \rfloor)$	$(\vee_{\mathsf{I}})$	$\lceil A \vee B \rceil \subset (\lceil A \rceil \vee \lceil B \rceil)$
$(\forall_{\mathsf{E}})$	$\lfloor \forall x A \rfloor \supset \forall x \lfloor A \rfloor$	$(\forall_{\mathsf{I}})$	$\lceil \forall x A \rceil \subset \forall x \lceil A \rceil$
$(\exists_{\mathsf{E}})$	$\lfloor \exists x A \rfloor \supset \exists x \lfloor A \rfloor$	$(\exists_{\mathsf{I}})$	$\lceil \exists x A \rceil \subset \exists x \lfloor A \rfloor$
(M)	$\lfloor A \rfloor \supset \lceil A \rceil$	(S)	$\lfloor A \rfloor \subset \lceil A \rceil$
$(\bot_{\mathsf{E}})$	$\lfloor \bot \rfloor \supset \mathbf{false}$	$(\top_{\mathsf{I}})$	$\lceil \top \rceil \subset \mathbf{true}$

Figure 7: Natural deduction, $\mathcal{L}_{\mathrm{NJ}}$.

Proposition 3.3. *Let $\Gamma \cup \{C\}$ be a set of NJ formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$\Gamma \vdash_{\mathrm{NJ}} C \downarrow \quad \textit{and} \quad [\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\lfloor C \rfloor]$$

$$\Gamma \vdash_{\mathrm{NJ}} C \uparrow \quad \textit{and} \quad [\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil C \rceil]$$

Proof. The proof is similar to the proofs for LJ and LK, one remark is that the initial rule for NJ does not have a formula in $\mathcal{L}_{\mathrm{NJ}}$ but the proof goes through because focusing on an atom succeeds exactly when the conclusion is the same atom, corresponding to the initial rule in NJ. We show a couple of the remaining cases here:

$(\Rightarrow_{\mathsf{E}})$:

$$\frac{\Gamma \vdash_{\mathrm{NJ}} A_1 \Rightarrow A_2 \downarrow \qquad \Gamma \vdash_{\mathrm{NJ}} A_1 \uparrow}{\Gamma \vdash_{\mathrm{NJ}} A_2 \downarrow} \Rightarrow_{\mathsf{E}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{NJ}} \cup \lfloor \Gamma \rfloor$)

$$
\dfrac{
\dfrac{[\mathcal{K}] \longrightarrow [\lfloor A_1 \Rightarrow A_2 \rfloor]}{[\mathcal{K}] -_{\lfloor A_1 \Rightarrow A_2 \rfloor}\!\rightarrow}\,\mathsf{R_R}, []_{\mathsf{R}}
\qquad
\dfrac{[\mathcal{K}] \longrightarrow [\lceil A_1 \rceil]}{[\mathcal{K}] -_{\lceil A_1 \rceil}\!\rightarrow}\,\mathsf{R_R}, []_{\mathsf{R}}
\qquad
\dfrac{}{[\mathcal{K}] \xrightarrow{\lfloor A_2 \rfloor} \lfloor A_2 \rfloor}\,\mathsf{I_L}
}{
\dfrac{[\mathcal{K}] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \supset (\lceil A_1 \rceil \supset \lfloor A_2 \rfloor)} [\lfloor A_2 \rfloor]}{[\mathcal{K}] \longrightarrow [\lfloor A_2 \rfloor]}\,\mathsf{L_F}, 2 \times \forall_{\mathsf{L}}
}\,2 \times \supset_{\mathsf{L}}
$$

$(\vee_{\mathsf{E}})$:

$$
\dfrac{\Gamma \vdash_{\mathrm{NJ}} A_1 \vee A_2 \downarrow \qquad \Gamma, A_1 \vdash_{\mathrm{NJ}} C \uparrow(\downarrow) \qquad \Gamma, A_2 \vdash_{\mathrm{NJ}} C \uparrow(\downarrow)}{\Gamma \vdash_{\mathrm{NJ}} C \uparrow(\downarrow)}\,\vee_{\mathsf{E}}
$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{NJ}} \cup \lfloor \Gamma \rfloor$ and F either $\lfloor C \rfloor$ or $\lceil C \rceil$)

$$
\dfrac{
\dfrac{[\mathcal{K}] \longrightarrow [\lfloor A_1 \vee A_2 \rfloor]}{[\mathcal{K}] -_{\lfloor A_1 \vee A_2 \rfloor}\!\rightarrow}\,\mathsf{R_R}, []_{\mathsf{R}}
\qquad
\dfrac{
\dfrac{[\mathcal{K}, \lfloor A_1 \rfloor] \longrightarrow [F]}{[\mathcal{K}], \lfloor A_1 \rfloor \longrightarrow [F]}\,[]_{\mathsf{L}}
\qquad
\dfrac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [F]}{[\mathcal{K}], \lfloor A_2 \rfloor \longrightarrow [F]}\,[]_{\mathsf{L}}
}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \rfloor \vee \lfloor A_2 \rfloor} [F]}\,\mathsf{R_L}, \vee_{\mathsf{L}}
}{
\dfrac{[\mathcal{K}] \xrightarrow{\lfloor A_1 \vee A_2 \rfloor \supset (\lfloor A_1 \rfloor \vee \lfloor A_2 \rfloor)} [F]}{[\mathcal{K}] \longrightarrow [F]}\,\mathsf{L_F}, 2 \times \forall_{\mathsf{L}}
}\,\supset_{\mathsf{L}}
$$

$(\vee_{\mathsf{I}})$:

$$
\dfrac{\Gamma \vdash_{\mathrm{NJ}} A_i \uparrow}{\Gamma \vdash_{\mathrm{NJ}} A_1 \vee A_2 \uparrow}\,\vee_{\mathsf{I}}
$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{NJ}} \cup \lfloor \Gamma \rfloor$)

$$
\dfrac{
\dfrac{}{[\mathcal{K}] \xrightarrow{\lceil A_1 \vee A_2 \rceil} [\lceil A_1 \vee A_2 \rceil]}\,\mathsf{I_L}
\qquad
\dfrac{\dfrac{[\mathcal{K}] \longrightarrow [\lceil A_i \rceil]}{[\mathcal{K}] -_{\lceil A_i \rceil}\!\rightarrow}\,\mathsf{R_R}, []_{\mathsf{R}}}{[\mathcal{K}] -_{\lceil A_1 \rceil \vee \lceil A_2 \rceil}\!\rightarrow}\,\vee_{\mathsf{R}}
}{
\dfrac{[\mathcal{K}] \xrightarrow{\lceil A_1 \vee A_2 \rceil \subset (\lceil A_1 \rceil \vee \lceil A_2 \rceil)} [\lceil A_1 \vee A_2 \rceil]}{[\mathcal{K}] \longrightarrow [\lceil A_1 \vee A_2 \rceil]}\,\mathsf{L_F}, 2 \times \forall_{\mathsf{L}}
}\,\supset_{\mathsf{L}}
$$

$(\bot_{\mathsf{E}})$:

$$
\dfrac{\Gamma \vdash_{\mathrm{NJ}} \bot \downarrow}{\Gamma \vdash_{\mathrm{NJ}} C \uparrow(\downarrow)}\,\bot_{\mathsf{E}}
$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{NJ}} \cup \lfloor \Gamma \rfloor$ and F either $\lfloor C \rfloor$ or $\lceil C \rceil$)

$$
\dfrac{\dfrac{\dfrac{[\mathcal{K}] \longrightarrow [\lfloor\bot\rfloor]}{[\mathcal{K}] -_{\lfloor\bot\rfloor}\!\!\rightarrow}\ \mathsf{R_R}, []_\mathsf{R} \qquad \dfrac{}{[\mathcal{K}] \xrightarrow{\mathbf{false}} [F]}\ \mathsf{R_L}, \mathbf{false}_\mathsf{L}}{[\mathcal{K}] \xrightarrow{\lfloor\bot\rfloor\supset\mathbf{false}} [F]}\ \supset_\mathsf{L}}{[\mathcal{K}] \longrightarrow [F]}\ \mathsf{L_F}
$$

□

3.3 Generalized elimination rules

In this section we consider another system of natural deduction, where the form of elimination rules used for $\vee, \exists$ are used for all connectives. The system is similar to systems by Schroeder-Heister [18] and Von Plato [21]. We consider a form with sequents annotated with $\uparrow$ and $\downarrow$ (called GEA), like for NJ, and a form without (called GE). The rules are given in Figure 8 (not annotated) and Figure 9 (annotated). For GEA our formulation is slightly different from the formulation in the Nigam and Miller paper. The difference is the same as for the NJ system. (The ($\bot_\mathsf{E}$) rule.)

For GE we have that $\lfloor\cdot\rfloor$ atoms are positive and $\lceil\cdot\rceil$ atoms are negative, the choice is dictated by the I rule which must be able to focus on both A's. The encodings of the rules are given in Figure 10 and the sequent $\Gamma \vdash_{\mathrm{GE}} A$ is encoded as $[\mathcal{L}_{\mathrm{GE}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil A\rceil]$.

Besides from the missing cut rule the rules are different from the LJ rules in one way: the elimination rules uses $\lceil\cdot\rceil$ instead of $\lfloor\cdot\rfloor$. $\lceil\cdot\rceil$ is used because there are no arrows so the right hand side is encoded using $\lceil\cdot\rceil$.

This encoding encodes GE on the full level of completeness:

Proposition 3.4. *Let $\Gamma \cup \{C\}$ be a set of GE formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$
\Gamma \vdash_{\mathrm{GE}} C \quad \textit{and} \quad [\mathcal{L}_{\mathrm{GE}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil C\rceil]
$$

Proof. The proof is similar to the earlier proofs, we show a single case:

($\Rightarrow_\mathsf{GE}$):

$$
\dfrac{\Gamma \vdash_{\mathrm{GE}} A_1 \Rightarrow A_2 \qquad \Gamma \vdash_{\mathrm{GE}} A_1 \qquad \Gamma, A_2 \vdash_{\mathrm{GE}} C}{\Gamma \vdash_{\mathrm{GE}} C}\ \Rightarrow_\mathsf{GE}
$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{GE}} \cup \lfloor\Gamma\rfloor$)

$$\frac{\Gamma \vdash_{\mathrm{GE}} A_1 \Rightarrow A_2 \quad \Gamma \vdash_{\mathrm{GE}} A_1 \quad \Gamma, A_2 \vdash_{\mathrm{GE}} C}{\Gamma \vdash_{\mathrm{GE}} C} \Rightarrow_{\mathsf{GE}} \qquad \frac{\Gamma, A_1 \vdash_{\mathrm{GE}} A_2}{\Gamma \vdash_{\mathrm{GE}} A_1 \Rightarrow A_2} \Rightarrow_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{GE}} A_1 \wedge A_2 \quad \Gamma, A_1, A_2 \vdash_{\mathrm{GE}} C}{\Gamma \vdash_{\mathrm{NJ}} C} \wedge_{\mathsf{GE}} \qquad \frac{\Gamma \vdash_{\mathrm{GE}} A_1 \quad \Gamma \vdash_{\mathrm{GE}} A_2}{\Gamma \vdash_{\mathrm{GE}} A_1 \wedge A_2} \wedge_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{GE}} A_1 \vee A_2 \quad \Gamma, A_1 \vdash_{\mathrm{GE}} C \quad \Gamma, A_2 \vdash_{\mathrm{GE}} C}{\Gamma \vdash_{\mathrm{GE}} C} \vee_{\mathsf{GE}} \qquad \frac{\Gamma \vdash_{\mathrm{GE}} A_i}{\Gamma \vdash_{\mathrm{GE}} A_1 \vee A_2} \vee_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{GE}} \forall x A \quad \Gamma, A[t/x] \vdash_{\mathrm{GE}} C}{\Gamma \vdash_{\mathrm{GE}} C} \forall_{\mathsf{GE}} \qquad \frac{\Gamma \vdash_{\mathrm{GE}} A[c/x]}{\Gamma \vdash_{\mathrm{GE}} \forall x A} {\forall_{\mathsf{I}}}^{c}$$

$$\frac{\Gamma \vdash_{\mathrm{GE}} \exists x A \quad \Gamma, A[c/x] \vdash_{\mathrm{GE}} C}{\Gamma \vdash_{\mathrm{GE}} C} {\exists_{\mathsf{GE}}}^{c} \qquad \frac{\Gamma \vdash_{\mathrm{GE}} A[t/x]}{\Gamma \vdash_{\mathrm{GE}} \exists x A} \exists_{\mathsf{I}}$$

$$\frac{}{\Gamma, A \vdash_{\mathrm{GE}} A} \mathsf{I} \qquad \frac{}{\Gamma \vdash_{\mathrm{GE}} \top} \top_{\mathsf{I}} \qquad \frac{\Gamma \vdash_{\mathrm{GE}} \bot}{\Gamma \vdash_{\mathrm{GE}} C} \bot_{\mathsf{E}}$$

Figure 8: The proof system GE. c is not free in Γ or C. $i \in \{1, 2\}$.

$$\frac{\Gamma \vdash_{\mathrm{GEA}} A_1 \Rightarrow A_2 \downarrow \quad \Gamma \vdash_{\mathrm{GEA}} A_1 \uparrow \quad \Gamma, A_2 \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)}{\Gamma \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)} \Rightarrow_{\mathsf{GE}} \qquad \frac{\Gamma, A_1 \vdash_{\mathrm{GEA}} A_2 \uparrow}{\Gamma \vdash_{\mathrm{GEA}} A_1 \Rightarrow A_2 \uparrow} \Rightarrow_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{GEA}} A_1 \wedge A_2 \downarrow \quad \Gamma, A_1, A_2 \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)}{\Gamma \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)} \wedge_{\mathsf{GE}} \qquad \frac{\Gamma \vdash_{\mathrm{GEA}} A_1 \uparrow \quad \Gamma \vdash_{\mathrm{GEA}} A_2 \uparrow}{\Gamma \vdash_{\mathrm{GEA}} A_1 \wedge A_2 \uparrow} \wedge_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{GEA}} A_1 \vee A_2 \downarrow \quad \Gamma, A_1 \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow) \quad \Gamma, A_2 \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)}{\Gamma \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)} \vee_{\mathsf{GE}} \qquad \frac{\Gamma \vdash_{\mathrm{GEA}} A_i \uparrow}{\Gamma \vdash_{\mathrm{GEA}} A_1 \vee A_2 \uparrow} \vee_{\mathsf{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{GEA}} \forall x A \downarrow \quad \Gamma, A[t/x] \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)}{\Gamma \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)} \forall_{\mathsf{GE}} \qquad \frac{\Gamma \vdash_{\mathrm{GEA}} A[c/x] \uparrow}{\Gamma \vdash_{\mathrm{GEA}} \forall x A \uparrow} {\forall_{\mathsf{I}}}^{c}$$

$$\frac{\Gamma \vdash_{\mathrm{GEA}} \exists x A \downarrow \quad \Gamma, A[c/x] \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)}{\Gamma \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)} {\exists_{\mathsf{GE}}}^{c} \qquad \frac{\Gamma \vdash_{\mathrm{GEA}} A[t/x] \uparrow}{\Gamma \vdash_{\mathrm{GEA}} \exists x A \uparrow} \exists_{\mathsf{I}}$$

$$\frac{}{\Gamma, A \vdash_{\mathrm{GEA}} A \downarrow} \mathsf{I} \quad \frac{\Gamma \vdash_{\mathrm{GEA}} A \downarrow}{\Gamma \vdash_{\mathrm{GEA}} A \uparrow} \mathsf{M} \quad \frac{\Gamma \vdash_{\mathrm{GEA}} A \uparrow}{\Gamma \vdash_{\mathrm{GEA}} A \downarrow} \mathsf{S} \quad \frac{}{\Gamma \vdash_{\mathrm{GEA}} \top \uparrow} \top_{\mathsf{I}} \quad \frac{\Gamma \vdash_{\mathrm{GEA}} \bot \downarrow}{\Gamma \vdash_{\mathrm{GEA}} C \uparrow\!(\downarrow)} \bot_{\mathsf{E}}$$

Figure 9: The proof system GEA (annotated GE). c is not free in Γ or C. $i \in \{1, 2\}$.

$(\Rightarrow_{\mathsf{GE}})$	$\lceil A \Rightarrow B \rceil \supset (\lceil A \rceil \supset \lfloor B \rfloor)$	$(\Rightarrow_{\mathsf{I}})$	$\lceil A \Rightarrow B \rceil \subset (\lfloor A \rfloor \supset \lceil B \rceil)$
$(\wedge_{\mathsf{GE}})$	$\lceil A \wedge B \rceil \supset (\lfloor A \rfloor \wedge^+ \lfloor B \rfloor)$	$(\wedge_{\mathsf{I}})$	$\lceil A \wedge B \rceil \subset (\lceil A \rceil \wedge^- \lceil B \rceil)$
$(\vee_{\mathsf{GE}})$	$\lceil A \vee B \rceil \supset (\lfloor A \rfloor \vee \lfloor B \rfloor)$	$(\vee_{\mathsf{I}})$	$\lceil A \vee B \rceil \subset (\lceil A \rceil \vee \lceil B \rceil)$
$(\forall_{\mathsf{GE}})$	$\lceil \forall x A \rceil \supset \forall x \lfloor A \rfloor$	$(\forall_{\mathsf{I}})$	$\lceil \forall x A \rceil \subset \forall x \lceil A \rceil$
$(\exists_{\mathsf{GE}})$	$\lceil \exists x A \rceil \supset \exists x \lfloor A \rfloor$	$(\exists_{\mathsf{I}})$	$\lceil \exists x A \rceil \subset \exists x \lceil A \rceil$
(I)	$\lfloor A \rfloor \supset \lceil A \rceil$		
$(\perp_{\mathsf{E}})$	$\lfloor \perp \rfloor \supset \mathbf{false}$	$(\top_{\mathsf{I}})$	$\lceil \top \rceil \subset \mathbf{true}$

Figure 10: Generalized elimination rules, $\mathcal{L}_{\text{GE}}$.

$(\Rightarrow_{\mathsf{GE}})$	$\lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \supset (\lfloor B \rfloor \wedge^+ \mathbf{true}))$	$(\Rightarrow_{\mathsf{I}})$	$\lceil A \Rightarrow B \rceil \subset (\lfloor A \rfloor \supset \lceil B \rceil)$
$(\wedge_{\mathsf{GE}})$	$\lfloor A \wedge B \rfloor \supset (\lfloor A \rfloor \wedge^+ \lfloor B \rfloor)$	$(\wedge_{\mathsf{I}})$	$\lceil A \wedge B \rceil \subset (\lceil A \rceil \wedge^- \lceil B \rceil)$
$(\vee_{\mathsf{GE}})$	$\lfloor A \vee B \rfloor \supset (\lfloor A \rfloor \vee \lfloor B \rfloor)$	$(\vee_{\mathsf{I}})$	$\lceil A \vee B \rceil \subset (\lceil A \rceil \vee \lceil B \rceil)$
$(\forall_{\mathsf{GE}})$	$\lfloor \forall x A \rfloor \supset \forall x (\lfloor A \rfloor \wedge^+ \mathbf{true})$	$(\forall_{\mathsf{I}})$	$\lceil \forall x A \rceil \subset \forall x \lceil A \rceil$
$(\exists_{\mathsf{GE}})$	$\lceil \exists x A \rceil \supset \exists x \lfloor A \rfloor$	$(\exists_{\mathsf{I}})$	$\lceil \exists x A \rceil \subset \exists x \lceil A \rceil$
(M)	$\lfloor A \rfloor \supset \lceil A \rceil$	(S)	$\lfloor A \rfloor \subset \lceil A \rceil$
$(\perp_{\mathsf{E}})$	$\lfloor \perp \rfloor \supset \mathbf{false}$	$(\top_{\mathsf{I}})$	$\lceil \top \rceil \subset \mathbf{true}$

Figure 11: Generalized elimination rules (annotated), $\mathcal{L}_{\text{GEA}}$.

$$
\dfrac{\dfrac{[\mathcal{K}] \longrightarrow [\lceil A_1 \Rightarrow A_2 \rceil]}{[\mathcal{K}] -_{\lceil A_1 \Rightarrow A_2 \rceil}\!\!\rightarrow}\,\mathsf{R}_\mathsf{R}, []_\mathsf{R} \quad \dfrac{[\mathcal{K}] \longrightarrow [\lceil A_1 \rceil]}{[\mathcal{K}] -_{\lceil A_1 \rceil}\!\!\rightarrow}\,\mathsf{R}_\mathsf{R}, []_\mathsf{R} \quad \dfrac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [\lceil C \rceil]}{[\mathcal{K}] \xrightarrow{\lfloor A_2 \rfloor} [\lceil C \rceil]}\,\mathsf{R}_\mathsf{L}, []_\mathsf{L}}{\dfrac{[\mathcal{K}] \xrightarrow{\lceil A_1 \Rightarrow A_2 \rceil \supset (\lceil A_1 \rceil \supset \lfloor A_2 \rfloor)} [\lceil C \rceil]}{[\mathcal{K}] \longrightarrow [\lceil C \rceil]}\,\mathsf{L}_\mathsf{F}, 2 \times \forall_\mathsf{L}}\,2 \times \supset_\mathsf{L}
$$

□

For GEA we have that $\lfloor \cdot \rfloor$ atoms are negative and $\lceil \cdot \rceil$ atoms are also negative, corresponding to the situation for the (annotated) NJ. The encodings of the rules are given in Figure 11 and the sequent $\Gamma \vdash_{\text{GEA}} A \downarrow$ is encoded as $[\mathcal{L}_{\text{GEA}}, \lfloor \Gamma \rfloor] \longrightarrow [\lfloor A \rfloor]$, and $\Gamma \vdash_{\text{GEA}} A \uparrow$ is encoded as $[\mathcal{L}_{\text{GEA}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil A \rceil]$,

The encoded rules are besides from $\Rightarrow_{\mathsf{GE}}$ and $\forall_{\mathsf{GE}}$ the same as for NJ. Those two rules are encoded using a positive conjunction and **true**. We need that construction because the right side of the implication needs to be positive to loose focus. That is also why we use the positive conjunction for the $\wedge_{\mathsf{GE}}$ encoding. We note that the generalized rules from NJ $\vee_{\mathsf{GE}}$ and $\exists_{\mathsf{GE}}$ already have positive connectives and therefore their encoding is the same.

This encoding encodes GEA on the full level of completeness:

Proposition 3.5. *Let $\Gamma \cup \{C\}$ be a set of GEA formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$\Gamma \vdash_{\text{GEA}} C \downarrow \quad \textit{and} \quad [\mathcal{L}_{\text{GEA}}, \lfloor \Gamma \rfloor] \longrightarrow [\lfloor C \rfloor]$$

$$\dfrac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LJ'}} A_1 \qquad \Gamma, A_1 \Rightarrow A_2, A_2 \vdash_{\mathrm{LJ'}} F}{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LJ'}} F} \Rightarrow_{\mathsf{L}} \qquad \dfrac{\Gamma, A_1 \vdash_{\mathrm{LJ'}} A_2}{\Gamma \vdash_{\mathrm{LJ'}} A_1 \Rightarrow A_2} \Rightarrow_{\mathsf{R}}$$

$$\dfrac{\Gamma, A_1 \wedge A_2, A_i \vdash_{\mathrm{LJ'}} F}{\Gamma, A_1 \wedge A_2 \vdash_{\mathrm{LJ'}} F} \wedge_{\mathsf{L}} \qquad \dfrac{\Gamma \vdash_{\mathrm{LJ'}} A_1 \qquad \Gamma \vdash_{\mathrm{LJ'}} A_2}{\Gamma \vdash_{\mathrm{LJ'}} A_1 \wedge A_2} \wedge_{\mathsf{R}}$$

$$\dfrac{\Gamma, A_1 \vee A_2, A_1 \vdash_{\mathrm{LJ'}} F \qquad \Gamma, A_1 \vee A_2, A_2 \vdash_{\mathrm{LJ'}} F}{\Gamma, A_1 \vee A_2 \vdash_{\mathrm{LJ'}} F} \vee_{\mathsf{L}} \qquad \dfrac{\Gamma \vdash_{\mathrm{LJ'}} A_i}{\Gamma \vdash_{\mathrm{LJ'}} A_1 \vee A_2} \vee_{\mathsf{R}}$$

$$\dfrac{\Gamma, \forall x A, A[t/x] \vdash_{\mathrm{LJ'}} F}{\Gamma, \forall x A \vdash_{\mathrm{LJ'}} F} \forall_{\mathsf{L}} \qquad \dfrac{\Gamma \vdash_{\mathrm{LJ'}} A[c/x]}{\Gamma \vdash_{\mathrm{LJ'}} \forall x A} {\forall_{\mathsf{R}}}^{c}$$

$$\dfrac{\Gamma, \exists x A, A[c/x] \vdash_{\mathrm{LJ'}} F}{\Gamma, \exists x A \vdash_{\mathrm{LJ'}} F} {\exists_{\mathsf{L}}}^{c} \qquad \dfrac{\Gamma \vdash_{\mathrm{LJ'}} A[t/x]}{\Gamma \vdash_{\mathrm{LJ'}} \exists x A} \exists_{\mathsf{R}}$$

$$\dfrac{}{\Gamma, A \vdash_{\mathrm{LJ'}} A} \mathsf{I} \qquad \dfrac{\Gamma \vdash_{\mathrm{LJ'}} A \qquad \Gamma, A \vdash_{\mathrm{LJ'}} F}{\Gamma \vdash_{\mathrm{LJ'}} F} \mathsf{Cut}$$

$$\dfrac{}{\Gamma, \bot \vdash_{\mathrm{LJ'}} \cdot} \bot_{\mathsf{L}} \qquad \dfrac{\Gamma \vdash_{\mathrm{LJ'}} \cdot}{\Gamma \vdash_{\mathrm{LJ'}} C} \mathsf{W}_{\mathsf{R}} \qquad \dfrac{}{\Gamma \vdash_{\mathrm{LJ'}} \top} \top_{\mathsf{R}}$$

Figure 12: The proof system LJ'. F is either $\cdot$ or a proper formula C. c is not free in Γ or F. $i \in \{1, 2\}$.

$$\Gamma \vdash_{\mathrm{GEA}} C \uparrow \quad \textit{and} \quad [\mathcal{L}_{\mathrm{GEA}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil C \rceil]$$

Proof. The proof is similar to the earlier proofs, we show a single case:

($\Rightarrow_{\mathsf{GE}}$):

$$\dfrac{\Gamma \vdash_{\mathrm{GEA}} A_1 \Rightarrow A_2 \downarrow \qquad \Gamma \vdash_{\mathrm{GEA}} A_1 \uparrow \qquad \Gamma, A_2 \vdash_{\mathrm{GEA}} C \uparrow(\downarrow)}{\Gamma \vdash_{\mathrm{GEA}} C \uparrow(\downarrow)} \Rightarrow_{\mathsf{GE}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{GEA}} \cup \lfloor \Gamma \rfloor$ and F either $\lfloor C \rfloor$ or $\lceil C \rceil$)

$$\dfrac{\dfrac{\dfrac{[\mathcal{K}] \longrightarrow [\lfloor A_1 \Rightarrow A_2 \rfloor]}{[\mathcal{K}] -_{\lfloor A_1 \Rightarrow A_2 \rfloor}\!\!\rightarrow} \mathsf{R}_{\mathsf{R}}, []_{\mathsf{R}} \qquad \dfrac{[\mathcal{K}] \longrightarrow [\lceil A_1 \rceil]}{[\mathcal{K}] -_{\lceil A_1 \rceil}\!\!\rightarrow} \mathsf{R}_{\mathsf{R}}, []_{\mathsf{R}} \qquad \dfrac{\dfrac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [F]}{[\mathcal{K}], \lfloor A_2 \rfloor, \mathbf{true} \longrightarrow [F]} \mathbf{true}_{\mathsf{L}}, []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A_2 \rfloor \wedge^{+} \mathbf{true}} [F]} \mathsf{R}_{\mathsf{L}}, \wedge_{\mathsf{L}}^{+}}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \supset (\lceil A_1 \rceil \supset (\lfloor A_2 \rfloor \wedge^{+} \mathbf{true}))} [F]} 2 \times \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [F]} \mathsf{L}_{\mathsf{F}}, 2 \times \forall_{\mathsf{L}}$$

□

3.4 LJ with empty right sides

As we saw in Section 3.1 the version of LJ with non empty right sides could be encoded in a nice way using the meta-level **false**. In this section we turn

$(\Rightarrow_{\mathsf{L}})$	$\lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \supset \lfloor B \rfloor)$	$(\Rightarrow_{\mathsf{R}})$	$\lceil A \Rightarrow B \rceil \subset (\lfloor A \rfloor \supset \lceil B \rceil)$
$(\wedge_{\mathsf{L}})$	$\lfloor A \wedge B \rfloor \supset (\lfloor A \rfloor \wedge^{-} \lfloor B \rfloor)$	$(\wedge_{\mathsf{R}})$	$\lceil A \wedge B \rceil \subset (\lceil A \rceil \wedge^{-} \lceil B \rceil)$
$(\vee_{\mathsf{L}})$	$\lfloor A \vee B \rfloor \supset (\lfloor A \rfloor \vee \lfloor B \rfloor)$	$(\vee_{\mathsf{R}})$	$\lceil A \vee B \rceil \subset (\lceil A \rceil \vee \lceil B \rceil)$
$(\forall_{\mathsf{L}})$	$\lfloor \forall x A \rfloor \supset \forall x \lfloor A \rfloor$	$(\forall_{\mathsf{R}})$	$\lceil \forall x A \rceil \subset \forall x \lceil A \rceil$
$(\exists_{\mathsf{L}})$	$\lfloor \exists x A \rfloor \supset \exists x \lfloor A \rfloor$	$(\exists_{\mathsf{R}})$	$\lceil \exists x A \rceil \subset \exists x \lceil A \rceil$
(I)	$\lfloor A \rfloor \supset \lceil A \rceil$	(Cut)	$\lfloor A \rfloor \subset \lceil A \rceil$
$(\bot_{\mathsf{L}})$	$\lfloor \bot \rfloor \supset \mathbf{empty}$	(W_{R})	$\mathbf{empty} \supset \lceil A \rceil$
		$(\top_{\mathsf{R}})$	$\lceil \top \rceil \subset \mathbf{true}$

Figure 13: Intuitionistic sequent calculus with empty right sides, $\mathcal{L}_{\mathrm{LJ'}}$.

the the version of LJ which uses the empty right sides, we call this version for LJ'. The rules of LJ' are given in Figure 12. Except for making explicit the places where the right side can be empty (the left rules), the rules are identical to the rules from the Nigam and Miller paper. We use F to stand for either a formula or the empty right side.

To encode LJ' we can not use the same encoding of the $\bot_{\mathsf{L}}$ rule because this version is too strong to be used in the system. The reason is that this encoding would allow us to derive $\Gamma, \bot \vdash_{\mathrm{LJ'}} C$ for any C and the object-level rule only allows us to derive the empty right side.

To overcome this problem we introduce a new atomic proposition **empty** (with type o) in the LJF meta-logic. This atom is assigned negative polarity. The rest of the encoding (including the polarities) follows the encoding for LJ, except for the conclusion where the empty conclusion is encoded as **empty**.

To summarize: the sequent $\Gamma \vdash_{\mathrm{LJ'}} C$ is encoded as $[\mathcal{L}_{\mathrm{LJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil C \rceil]$ and $\Gamma \vdash_{\mathrm{LJ'}} \cdot$ as $[\mathcal{L}_{\mathrm{LJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\mathbf{empty}]$. The encoding of the rules for LJ' is given in Figure 13.

The encoding encodes LJ' on the full level of completeness between (open) derivations, as shown by the following proposition:

Proposition 3.6. *Let $\Gamma \cup \{C\}$ be a set of LJ' formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$\Gamma \vdash_{\mathrm{LJ'}} C \quad \textit{and} \quad [\mathcal{L}_{\mathrm{LJ'}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil C \rceil]$$

and

$$\Gamma \vdash_{\mathrm{LJ'}} \cdot \quad \textit{and} \quad [\mathcal{L}_{\mathrm{LJ'}}, \lfloor \Gamma \rfloor] \longrightarrow [\mathbf{empty}]$$

Proof. This proof is very similar to the proof for LJ and we show a couple of cases. Below F in the LJF derivations stands for either **empty** or $\lceil C \rceil$.

$(\Rightarrow_{\mathsf{L}})$:

$$\frac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LJ}'} A_1 \qquad \Gamma, A_1 \Rightarrow A_2, A_2 \vdash_{\mathrm{LJ}'} F}{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{LJ}'} F} \Rightarrow_{\mathsf{L}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}'} \cup \lfloor \Gamma \rfloor \cup \{\lfloor A_1 \Rightarrow A_2 \rfloor\}$)

$$\frac{\frac{\frac{}{[\mathcal{K}] -_{\lfloor A_1 \Rightarrow A_2 \rfloor}\!\rightarrow} \mathsf{I}_{\mathsf{R}} \qquad \frac{[\mathcal{K}] \longrightarrow [\lceil A_1 \rceil]}{[\mathcal{K}] -_{\lceil A_1 \rceil}\!\rightarrow} \mathsf{R}_{\mathsf{R}}, []_{\mathsf{R}} \qquad \frac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [F]}{[\mathcal{K}] \xrightarrow{\lfloor A_2 \rfloor} [F]} \mathsf{R}_{\mathsf{L}}, []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \supset (\lceil A_1 \rceil \supset \lfloor A_2 \rfloor)} [F]} 2 \times \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [F]} \mathsf{L}_{\mathsf{F}}, 2 \times \forall_{\mathsf{L}}$$

($\perp_{\mathsf{L}}$):

$$\frac{}{\Gamma, \perp \vdash_{\mathrm{LJ}'} \cdot} \perp_{\mathsf{L}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}'} \cup \lfloor \Gamma \rfloor \cup \{\lfloor \perp \rfloor\}$)

$$\frac{\frac{\frac{}{[\mathcal{K}] -_{\lfloor \perp \rfloor}\!\rightarrow} \mathsf{I}_{\mathsf{R}} \qquad \frac{}{[\mathcal{K}] \xrightarrow{\mathbf{empty}} [\mathbf{empty}]} \mathsf{I}_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor \perp \rfloor \supset \mathbf{empty}} [\mathbf{empty}]} \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{empty}]} \mathsf{L}_{\mathsf{F}}$$

(W_{R}):

$$\frac{\Gamma \vdash_{\mathrm{LJ}'} \cdot}{\Gamma \vdash_{\mathrm{LJ}'} C} \mathsf{W}_{\mathsf{R}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}'} \cup \lfloor \Gamma \rfloor$)

$$\frac{\frac{\frac{[\mathcal{K}] \longrightarrow [\mathbf{empty}]}{[\mathcal{K}] -_{\mathbf{empty}}\!\rightarrow} \mathsf{R}_{\mathsf{R}}, []_{\mathsf{R}} \qquad \frac{}{[\mathcal{K}] \xrightarrow{\lceil C \rceil} [\lceil C \rceil]} \mathsf{I}_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\mathbf{empty} \supset \lceil C \rceil} [\lceil C \rceil]} \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil C \rceil]} \mathsf{L}_{\mathsf{F}}, \forall_{\mathsf{L}}$$

$\square$

$$\frac{\Gamma \vdash_{\mathrm{FD}} A_1 \Rightarrow A_2, \Delta \quad \Gamma \vdash_{\mathrm{FD}} A_1, \Delta \quad \Gamma, A_2 \vdash_{\mathrm{FD}} \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \Rightarrow_{\mathsf{GE}}$$

$$\frac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{FD}} \Delta \quad \Gamma, A_1 \vdash_{\mathrm{FD}} \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \Rightarrow_{\mathsf{GI}1} \qquad \frac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{FD}} \Delta \quad \Gamma \vdash_{\mathrm{FD}} A_2, \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \Rightarrow_{\mathsf{GI}2}$$

$$\frac{\Gamma \vdash_{\mathrm{FD}} A_1 \wedge A_2, \Delta \quad \Gamma, A_i \vdash_{\mathrm{FD}} \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \wedge_{\mathsf{GE}} \qquad \frac{\Gamma, A_1 \wedge A_2 \vdash_{\mathrm{FD}} \Delta \quad \Gamma \vdash_{\mathrm{FD}} A_1, \Delta \quad \Gamma \vdash_{\mathrm{FD}} A_2, \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \wedge_{\mathsf{GI}}$$

$$\frac{\Gamma \vdash_{\mathrm{FD}} A_1 \vee A_2, \Delta \quad \Gamma, A_1 \vdash_{\mathrm{FD}} \Delta \quad \Gamma, A_2 \vdash_{\mathrm{FD}} \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \vee_{\mathsf{GE}} \qquad \frac{\Gamma, A_1 \vee A_2 \vdash_{\mathrm{FD}} \Delta \quad \Gamma \vdash_{\mathrm{FD}} A_i, \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \vee_{\mathsf{GI}}$$

$$\frac{}{\Gamma, A \vdash_{\mathrm{FD}} A, \Delta} \mathsf{I} \qquad \frac{\Gamma, \neg A \vdash_{\mathrm{FD}} \Delta \quad \Gamma, A \vdash_{\mathrm{FD}} \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \neg_{\mathsf{GI}1} \qquad \frac{\Gamma, \vdash_{\mathrm{FD}} \neg A, \Delta \quad \Gamma \vdash_{\mathrm{FD}} A, \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \neg_{\mathsf{GI}2}$$

Figure 14: The proof system FD. $i \in \{1, 2\}$.

$$\begin{array}{rlrl}
(\Rightarrow_{\mathsf{GE}}) & \lceil A \Rightarrow B \rceil \vee \lceil A \rceil \vee \lfloor B \rfloor & (\Rightarrow_{\mathsf{GI}1}) & \lfloor A \Rightarrow B \rfloor \vee \lfloor A \rfloor \\
 & & (\Rightarrow_{\mathsf{GI}2}) & \lfloor A \Rightarrow B \rfloor \vee \lceil B \rceil \\
(\wedge_{\mathsf{GE}1}) & \lceil A \wedge B \rceil \vee \lfloor A \rfloor & (\wedge_{\mathsf{GI}}) & \lfloor A \wedge B \rfloor \vee \lceil A \rceil \vee \lceil B \rceil \\
(\wedge_{\mathsf{GE}2}) & \lceil A \wedge B \rceil \vee \lfloor B \rfloor & & \\
(\vee_{\mathsf{GE}}) & \lceil A \vee B \rceil \vee \lfloor A \rfloor \vee \lfloor B \rfloor & (\vee_{\mathsf{GI}1}) & \lfloor A \vee B \rfloor \vee \lceil A \rceil \\
 & & (\vee_{\mathsf{GI}2}) & \lfloor A \vee B \rfloor \vee \lceil B \rceil \\
(\neg_{\mathsf{GI}1}) & \lfloor \neg A \rfloor \vee \lfloor A \rfloor & (\neg_{\mathsf{GI}2}) & \lceil \neg A \rceil \vee \lceil A \rceil \\
(\mathsf{I}) & (\lfloor A \rfloor \wedge^{+} \lceil A \rceil) \supset \mathbf{false} & &
\end{array}$$

Figure 15: Free deduction, $\mathcal{L}_{\mathrm{FD}}$.

3.5 Free deduction

In this section we consider an additive version of the system of free deduction FD [12], the rules are given in Figure 14. This system covers only the propositional fragment of classical logic and uses an explicit negation $\neg$.

FD will be encoded in the same way as LK, so for FD we have that both $\lfloor \cdot \rfloor$ atoms and $\lceil \cdot \rceil$ atoms are positive. The encodings of the rules are given in Figure 15 and the sequent $\Gamma \vdash_{\mathrm{FD}} \Delta$ is encoded as $[\mathcal{L}_{\mathrm{FD}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil] \longrightarrow [\mathbf{false}]$.

The encodings of the rules are similar to the rules for LK, but where LK uses implication FD uses disjunction. The disjunction is used because the principal formula is above the inference line and therefore we must not focus on it. It might be possible to use a construction, like the one for GEA, to loose focus explicitly, but we prefer this version.

This encoding encodes FD on the full level of completeness:

Proposition 3.7. *Let Γ and Δ be sets of FD formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$\Gamma \vdash_{\mathrm{FD}} \Delta \quad \textit{and} \quad [\mathcal{L}_{\mathrm{FD}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil] \longrightarrow [\mathbf{false}]$$

Proof. The proof is similar to the earlier proofs but the rules are quite different so we show a couple of cases:

($\Rightarrow_{\mathsf{GE}}$):

$$\dfrac{\Gamma \vdash_{\mathrm{FD}} A_1 \Rightarrow A_2, \Delta \qquad \Gamma \vdash_{\mathrm{FD}} A_1, \Delta \qquad \Gamma, A_2 \vdash_{\mathrm{FD}} \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \Rightarrow_{\mathsf{GE}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{FD}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil$)

$$\dfrac{\dfrac{\dfrac{[\mathcal{K}, \lceil A_1 \Rightarrow A_2 \rceil] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lceil A_1 \Rightarrow A_2 \rceil \longrightarrow [\mathbf{false}]} []_{\mathsf{L}} \qquad \dfrac{[\mathcal{K}, \lceil A_1 \rceil] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lceil A_1 \rceil \longrightarrow [\mathbf{false}]} []_{\mathsf{L}} \qquad \dfrac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lfloor A_2 \rfloor \longrightarrow [\mathbf{false}]} []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lceil A_1 \Rightarrow A_2 \rceil \vee \lceil A_1 \rceil \vee \lfloor A_2 \rfloor} [\mathbf{false}]} \mathsf{R}_{\mathsf{L}}, 2 \times \vee_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]} \mathsf{L}_{\mathsf{F}}, 2 \times \forall_{\mathsf{L}}$$

($\Rightarrow_{\mathsf{GI1}}$):

$$\dfrac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{FD}} \Delta \qquad \Gamma, A_1 \vdash_{\mathrm{FD}} \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \Rightarrow_{\mathsf{GI1}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{FD}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil$)

$$\dfrac{\dfrac{\dfrac{[\mathcal{K}, \lfloor A_1 \Rightarrow A_2 \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lfloor A_1 \Rightarrow A_2 \rfloor \longrightarrow [\mathbf{false}]} []_{\mathsf{L}} \qquad \dfrac{[\mathcal{K}, \lfloor A_1 \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lfloor A_1 \rfloor \longrightarrow [\mathbf{false}]} []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \vee \lfloor A_1 \rfloor} [\mathbf{false}]} \mathsf{R}_{\mathsf{L}}, \vee_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]} \mathsf{L}_{\mathsf{F}}, 2 \times \forall_{\mathsf{L}}$$

($\neg_{\mathsf{GI1}}$):

$$\dfrac{\Gamma, \neg A \vdash_{\mathrm{FD}} \Delta \qquad \Gamma, A \vdash_{\mathrm{FD}} \Delta}{\Gamma \vdash_{\mathrm{FD}} \Delta} \neg_{\mathsf{GI1}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{FD}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil$)

$$\dfrac{\dfrac{\dfrac{[\mathcal{K}, \lfloor \neg A \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lfloor \neg A \rfloor \longrightarrow [\mathbf{false}]} []_{\mathsf{L}} \qquad \dfrac{[\mathcal{K}, \lfloor A \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}], \lfloor A \rfloor \longrightarrow [\mathbf{false}]} []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor \neg A \rfloor \vee \lfloor A \rfloor} [\mathbf{false}]} \mathsf{R}_{\mathsf{L}}, \vee_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]} \mathsf{L}_{\mathsf{F}}, \forall_{\mathsf{L}}$$

$\square$

$$\frac{\Gamma, A_1, A_1 \Rightarrow A_2, A_2 \vdash_{\mathrm{KE}} \Delta}{\Gamma, A_1, A_1 \Rightarrow A_2 \vdash_{\mathrm{KE}} \Delta} \Rightarrow_{\mathsf{L1}} \quad \frac{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{KE}} A_1, A_2, \Delta}{\Gamma, A_1 \Rightarrow A_2 \vdash_{\mathrm{KE}} A_2, \Delta} \Rightarrow_{\mathsf{L2}} \quad \frac{\Gamma, A_1 \vdash_{\mathrm{KE}} A_1 \Rightarrow A_2, A_2, \Delta}{\Gamma, \vdash_{\mathrm{KE}} A_1 \Rightarrow A_2, \Delta} \Rightarrow_{\mathsf{R}}$$

$$\frac{\Gamma, A_1 \wedge A_2, A_1, A_2 \vdash_{\mathrm{KE}} \Delta}{\Gamma, A_1 \wedge A_2 \vdash_{\mathrm{KE}} \Delta} \wedge_{\mathsf{L}} \quad \frac{\Gamma, A_1 \vdash_{\mathrm{KE}} A_1 \wedge A_2, A_2, \Delta}{\Gamma, A_1 \vdash_{\mathrm{KE}} A_1 \wedge A_2, \Delta} \wedge_{\mathsf{R1}} \quad \frac{\Gamma, A_2 \vdash_{\mathrm{KE}} A_1 \wedge A_2, A_1, \Delta}{\Gamma, A_2 \vdash_{\mathrm{KE}} A_1 \wedge A_2, \Delta} \wedge_{\mathsf{R2}}$$

$$\frac{\Gamma, A_1 \vee A_2, A_2 \vdash_{\mathrm{KE}} A_1, \Delta}{\Gamma, A_1 \vee A_2 \vdash_{\mathrm{KE}} A_1, \Delta} \vee_{\mathsf{L1}} \quad \frac{\Gamma, A_1 \vee A_2, A_1 \vdash_{\mathrm{KE}} A_2, \Delta}{\Gamma, A_1 \vee A_2 \vdash_{\mathrm{KE}} A_2, \Delta} \vee_{\mathsf{L2}} \quad \frac{\Gamma \vdash_{\mathrm{KE}} A_1 \vee A_2, A_1, A_2, \Delta}{\Gamma \vdash_{\mathrm{KE}} A_1 \vee A_2, \Delta} \vee_{\mathsf{R}}$$

$$\frac{\Gamma, \neg A \vdash_{\mathrm{KE}} A, \Delta}{\Gamma, \neg A \vdash_{\mathrm{KE}} \Delta} \neg_{\mathsf{L}} \quad \frac{\Gamma, A \vdash_{\mathrm{KE}} \neg A, \Delta}{\Gamma \vdash_{\mathrm{KE}} \neg A, \Delta} \neg_{\mathsf{R}}$$

$$\frac{}{\Gamma, A \vdash_{\mathrm{KE}} A, \Delta} \mathsf{I} \quad \frac{\Gamma, A \vdash_{\mathrm{KE}} \Delta \qquad \Gamma \vdash_{\mathrm{KE}} A, \Delta}{\Gamma \vdash_{\mathrm{KE}} \Delta} \mathsf{Cut}$$

Figure 16: The proof system KE.

$(\Rightarrow_{\mathsf{L1}})$	$\lfloor A \Rightarrow B \rfloor \supset (\lfloor A \rfloor \supset \lfloor B \rfloor)$	$(\Rightarrow_{\mathsf{R}})$	$\lceil A \Rightarrow B \rceil \supset (\lfloor A \rfloor \wedge^+ \lceil B \rceil)$
$(\Rightarrow_{\mathsf{L2}})$	$\lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \subset \lceil B \rceil)$		
$(\wedge_{\mathsf{L}})$	$\lfloor A \wedge B \rfloor \supset (\lfloor A \rfloor \wedge^+ \lfloor B \rfloor)$	$(\wedge_{\mathsf{R1}})$	$\lceil A \wedge B \rceil \supset (\lfloor A \rfloor \supset \lceil B \rceil)$
		$(\wedge_{\mathsf{R2}})$	$\lceil A \wedge B \rceil \supset (\lceil A \rceil \subset \lfloor B \rfloor)$
$(\vee_{\mathsf{L1}})$	$\lfloor A \vee B \rfloor \supset (\lceil A \rceil \supset \lfloor B \rfloor)$	$(\vee_{\mathsf{R}})$	$\lceil A \vee B \rceil \supset (\lceil A \rceil \wedge^+ \lceil B \rceil)$
$(\vee_{\mathsf{L2}})$	$\lfloor A \vee B \rfloor \supset (\lfloor A \rfloor \subset \lceil B \rceil)$		
$(\neg_{\mathsf{L}})$	$\lfloor \neg A \rfloor \supset \lceil A \rceil$	$(\neg_{\mathsf{R}})$	$\lceil \neg A \rceil \supset \lfloor A \rfloor$
(I)	$(\lfloor A \rfloor \wedge^+ \lceil A \rceil) \supset \mathbf{false}$	(Cut)	$\lfloor A \rfloor \vee \lceil A \rceil$

Figure 17: Classical tableaux, $\mathcal{L}_{\mathrm{KE}}$.

3.6 Tableaux

In this section we consider a system of tableaux KE [2], the rules are given in Figure 16. Like FD this system covers only the propositional fragment, again including an explicit negation.

KE will be encoded in the same way as LK, so for KE we have that both $\lfloor \cdot \rfloor$ atoms and $\lceil \cdot \rceil$ atoms are positive. The encodings of the rules are given in Figure 17 and the sequent $\Gamma \vdash_{\mathrm{KE}} \Delta$ is encoded as $[\mathcal{L}_{\mathrm{KE}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil] \longrightarrow [\mathbf{false}]$.

The rules that correspond to LK rules have the same encoding ($\Rightarrow_{\mathsf{R}}$,$\wedge_{\mathsf{L}}$,$\vee_{\mathsf{R}}$), the rules which are split into two rules have two encodings each using an implication to focus on either A_1 or A_2 in the rule, corresponding to the formula that needs to be under the inference line.

This encoding encodes KE on the full level of completeness:

Proposition 3.8. *Let Γ and Δ be sets of KE formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$\Gamma \vdash_{\mathrm{KE}} \Delta \quad \textit{and} \quad [\mathcal{L}_{\mathrm{KE}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil] \longrightarrow [\mathbf{false}]$$

Proof. The proof is similar to the previous proofs, we show a couple of cases:

$(\vee_{\mathsf{L}1})$:

$$\frac{\Gamma, A_1 \vee A_2, A_2 \vdash_{\mathrm{KE}} A_1, \Delta}{\Gamma, A_1 \vee A_2 \vdash_{\mathrm{KE}} A_1, \Delta}\ \vee_{\mathsf{L}1}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{KE}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil \cup \{\lfloor A_1 \vee A_2 \rfloor, \lceil A_1 \rceil\}$)

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor A_1 \vee A_2 \rfloor}\!\!\rightarrow}\ \mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}] -_{\lceil A_1 \rceil}\!\!\rightarrow}\ \mathsf{I_R} \qquad \dfrac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}] \xrightarrow{\lfloor A_2 \rfloor} [\mathbf{false}]}\ \mathsf{R_L}, []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \vee A_2 \rfloor \supset (\lceil A_1 \rceil \supset \lfloor A_2 \rfloor)} [\mathbf{false}]}\ 2 \times \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]}\ \mathsf{L_F}, 2 \times \forall_{\mathsf{L}}$$

$(\neg_{\mathsf{R}})$:

$$\frac{\Gamma, A \vdash_{\mathrm{KE}} \neg A, \Delta}{\Gamma \vdash_{\mathrm{KE}} \neg A, \Delta}\ \neg_{\mathsf{R}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{KE}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil \cup \{\lceil \neg A \rceil\}$)

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lceil \neg A \rceil}\!\!\rightarrow}\ \mathsf{I_R} \qquad \dfrac{[\mathcal{K}, \lfloor A \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}] \xrightarrow{\lfloor A \rfloor} [\mathbf{false}]}\ \mathsf{R_L}, []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lceil \neg A \rceil \supset \lfloor A \rfloor} [\mathbf{false}]}\ \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{false}]}\ \mathsf{L_F}, \forall_{\mathsf{L}}$$

□

3.7 Analytic cut

In this section we consider the Smullyan's proof system for analytic cuts AC [20], the rules are given in Figure 18. As in Nigam and Miller's paper the side condition on cut is dropped. Only propositional formulas are included and we use an explicit negation.

AC will be encoded in the same way as LK, so for AC we have that both $\lfloor \cdot \rfloor$ atoms and $\lceil \cdot \rceil$ atoms are positive. The encodings of the rules are given in Figure 19 and the sequent $\Gamma \vdash_{\mathrm{AC}} \Delta$ is encoded as $[\mathcal{L}_{\mathrm{AC}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil] \longrightarrow [\mathbf{false}]$.

$$\frac{}{\Gamma, A_1, A_1 \Rightarrow A_2 \vdash_{\mathrm{AC}} A_2, \Delta} \Rightarrow_{\mathsf{L}} \quad \frac{}{\Gamma \vdash_{\mathrm{AC}} A_1, A_1 \Rightarrow A_2, \Delta} \Rightarrow_{\mathsf{R1}} \quad \frac{}{\Gamma, A_2 \vdash_{\mathrm{AC}} A_1 \Rightarrow A_2, \Delta} \Rightarrow_{\mathsf{R2}}$$

$$\frac{}{\Gamma, A_1 \wedge A_2 \vdash_{\mathrm{AC}} A_1, \Delta} \wedge_{\mathsf{L1}} \quad \frac{}{\Gamma, A_1 \wedge A_2 \vdash_{\mathrm{AC}} A_2, \Delta} \wedge_{\mathsf{L2}} \quad \frac{}{\Gamma, A_1, A_2 \vdash_{\mathrm{AC}} A_1 \wedge A_2, \Delta} \wedge_{\mathsf{R}}$$

$$\frac{}{\Gamma, A_1 \vee A_2 \vdash_{\mathrm{AC}} A_1, A_2, \Delta} \vee_{\mathsf{L}} \quad \frac{}{\Gamma, A_1 \vdash_{\mathrm{AC}} A_1 \vee A_2, \Delta} \vee_{\mathsf{R1}} \quad \frac{}{\Gamma, A_2 \vdash_{\mathrm{AC}} A_1 \vee A_2, \Delta} \vee_{\mathsf{R2}}$$

$$\frac{}{\Gamma, \neg A, A \vdash_{\mathrm{AC}} \Delta} \neg_{\mathsf{L}} \quad \frac{}{\Gamma \vdash_{\mathrm{AC}} \neg A, A, \Delta} \neg_{\mathsf{R}}$$

$$\frac{}{\Gamma, A \vdash_{\mathrm{AC}} A, \Delta} \mathsf{I} \quad \frac{\Gamma, A \vdash_{\mathrm{AC}} \Delta \qquad \Gamma \vdash_{\mathrm{AC}} A, \Delta}{\Gamma \vdash_{\mathrm{AC}} \Delta} \mathsf{Cut}$$

Figure 18: The proof system AC.

$(\Rightarrow_{\mathsf{L}})$	$\lfloor A \Rightarrow B \rfloor \supset ((\lfloor A \rfloor \wedge^+ \lceil B \rceil) \supset \mathbf{false})$	$(\Rightarrow_{\mathsf{R}})$	$\lceil A \Rightarrow B \rceil \supset ((\lceil A \rceil \vee \lfloor B \rfloor) \supset \mathbf{false})$
$(\wedge_{\mathsf{L}})$	$\lfloor A \wedge B \rfloor \supset ((\lceil A \rceil \vee \lceil B \rceil) \supset \mathbf{false})$	$(\wedge_{\mathsf{R}})$	$\lceil A \wedge B \rceil \supset ((\lfloor A \rfloor \wedge^+ \lfloor B \rfloor) \supset \mathbf{false})$
$(\vee_{\mathsf{L}})$	$\lfloor A \vee B \rfloor \supset ((\lceil A \rceil \wedge^+ \lceil B \rceil) \supset \mathbf{false})$	$(\vee_{\mathsf{R}})$	$\lceil A \vee B \rceil \supset ((\lfloor A \rfloor \vee \lfloor B \rfloor) \supset \mathbf{false})$
$(\neg_{\mathsf{L}})$	$\lfloor \neg A \rfloor \supset (\lfloor A \rfloor \supset \mathbf{false})$	$(\neg_{\mathsf{R}})$	$\lceil \neg A \rceil \supset (\lceil A \rceil \supset \mathbf{false})$
(I)	$(\lfloor A \rfloor \wedge^+ \lceil A \rceil) \supset \mathbf{false}$	(Cut)	$\lfloor A \rfloor \vee \lceil A \rceil$

Figure 19: Smullyan's analytic cut, $\mathcal{L}_{\mathrm{AC}}$.

The encoded rules are pretty verbose, which is because every left and right rule must be able to end the proof and therefore prove **false**. The right side of the implication in the encodings of the rules is basically the double negation of the right side for LK. We note that some of the rules can be joined into one formula in the encoding.

This encoding encodes AC on the full level of completeness:

Proposition 3.9. *Let Γ and Δ be sets of AC formulas then there is a bijective correspondence between the open derivations of the following sequents:*

$$\Gamma \vdash_{\mathrm{AC}} \Delta \quad \textit{and} \quad [\mathcal{L}_{\mathrm{AC}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil] \longrightarrow [\mathbf{false}]$$

Proof. Again the proof is similar and we show a couple of cases. We note that in those cases where two rules are joined into one formula each rule corresponds exactly to one branch in the LJF derivation (one of the different available macro-rules):

$(\Rightarrow_{\mathsf{L}})$:

$$\frac{}{\Gamma, A_1, A_1 \Rightarrow A_2 \vdash_{\mathrm{AC}} A_2, \Delta} \Rightarrow_{\mathsf{L}}$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{AC}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil \cup \{\lfloor A_1 \Rightarrow A_2 \rfloor, \lfloor A_1 \rfloor, \lceil A_2 \rceil\}$)

$$
\dfrac{
\dfrac{}{[\mathcal{K}] - \lfloor A_1 \Rightarrow A_2 \rfloor \rightarrow}\,\mathsf{I_R}
\qquad
\dfrac{\dfrac{}{[\mathcal{K}] - \lfloor A_1 \rfloor \rightarrow}\,\mathsf{I_R} \quad \dfrac{}{[\mathcal{K}] - \lceil A_2 \rceil \rightarrow}\,\mathsf{I_R}}{[\mathcal{K}] - \lfloor A_1 \rfloor \wedge^{+} \lceil A_2 \rceil \rightarrow}\,\wedge^{+}_{\mathsf{R}}
\qquad
\dfrac{}{[\mathcal{K}] \xrightarrow{\mathbf{false}} [\mathbf{false}]}\,\mathsf{R_L}, \mathbf{false}_{\mathsf{L}}
}{
\dfrac{[\mathcal{K}] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \supset ((\lfloor A_1 \rfloor \wedge^{+} \lceil A_2 \rceil) \supset \mathbf{false})} [\mathbf{false}]}{[\mathcal{K}] \longrightarrow [\mathbf{false}]}\,\mathsf{L_F}, 2 \times \forall_{\mathsf{L}}
}\,2 \times \supset_{\mathsf{L}}
$$

$(\Rightarrow_{\mathsf{R}1})$:

$$
\dfrac{}{\Gamma \vdash_{\mathrm{AC}} A_1, A_1 \Rightarrow A_2, \Delta}\Rightarrow_{\mathsf{R}1}
$$

corresponds to (with $\mathcal{K} = \mathcal{L}_{\mathrm{AC}} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil \cup \{\lceil A_1 \Rightarrow A_2 \rceil, \lceil A_1 \rceil\}$)

$$
\dfrac{
\dfrac{}{[\mathcal{K}] - \lceil A_1 \Rightarrow A_2 \rceil \rightarrow}\,\mathsf{I_R}
\qquad
\dfrac{\dfrac{}{[\mathcal{K}] - \lceil A_1 \rceil \rightarrow}\,\mathsf{I_R}}{[\mathcal{K}] - \lceil A_1 \rceil \vee \lfloor A_2 \rfloor \rightarrow}\,\vee_{\mathsf{R}}
\qquad
\dfrac{}{[\mathcal{K}] \xrightarrow{\mathbf{false}} [\mathbf{false}]}\,\mathsf{R_L}, \mathbf{false}_{\mathsf{L}}
}{
\dfrac{[\mathcal{K}] \xrightarrow{\lceil A_1 \Rightarrow A_2 \rceil \supset ((\lceil A_1 \rceil \vee \lfloor A_2 \rfloor) \supset \mathbf{false})} [\mathbf{false}]}{[\mathcal{K}] \longrightarrow [\mathbf{false}]}\,\mathsf{L_F}, 2 \times \forall_{\mathsf{L}}
}\,2 \times \supset_{\mathsf{L}}
$$

□

4 Relative Completeness

This section is concerned with the different relative completeness theorems. We show how to prove correspondence between the different system using the formalizations from the previous section. The proofs illustrate how easy/hard it is to come with a new system and show that it is complete with respect to intuitionistic/classical logic.

There are several ways to prove relative completeness. In this work we use three different methods.

1. LJF method: we exploit the completeness theorems from the previous section and do the proof by induction on the focused derivations of the encoded sequents in LJF.

2. Intuitionistic method: We use the completeness theorems and Corollary 2.2 to map the encoded sequents to intuitionistic logic and reason directly in intuitionistic logic, if we can show that each set of rules follows from the other this approach gives the simplest proof.

   If we can show that the rules in one system follows from the rules in the other we call it proof by *rule implication* and if we can show both directions we call it proof by *rule equivalence*.

3. Object-level method: we reason directly in the object-level systems using induction on the derivations. This method does not use the encodings so we use this method as little as possible.

Most of the proofs in this section uses option two. The cases where we need the other methods are in the cases where cut elimination are needed, where we use a transformation on the formulas or when we try to relate structurally different systems (e.g. LJ and LJ' or intuitionistic and classical logic).

4.1 Intuitionistic systems

In this section we compare the different intuitionistic systems. A general comment about the encoded systems is that their relative completeness with regard to each other is in most of the cases trivial, as the systems mostly have the same rules, save for polarities, which does not effect provability.

The first proposition shows correspondence between LJ and NJ:

Proposition 4.1. *Let $\Gamma \cup \{C\}$ be a set of formulas then:*

$$\Gamma \vdash_{\mathrm{LJ}} C \quad \textit{if and only if} \quad \Gamma \vdash_{\mathrm{NJ}} C \uparrow$$

Proof. The idea behind the proof is to use the completeness results from Section 3 and then reason in intuitionistic logic.

Using Proposition 3.1 and Proposition 3.3 we need to show that (keep in mind the different polarities on the left and the right side):

$$[\mathcal{L}_{\mathrm{LJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil C \rceil] \quad \text{if and only if} \quad [\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil C \rceil]$$

which by Corollary 2.2 is the same as showing that (where $\wedge^+$ and $\wedge^-$ is replaced with $\wedge$):

$$\mathcal{L}_{\mathrm{LJ}}, \lfloor \Gamma \rfloor \vdash_{\mathrm{I}} \lceil C \rceil \quad \text{if and only if} \quad \mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor \vdash_{\mathrm{I}} \lceil C \rceil$$

And if we can show that:

$$\begin{array}{ll} \forall F \in \mathcal{L}_{\mathrm{NJ}} & \mathcal{L}_{\mathrm{LJ}} \vdash_{\mathrm{I}} F \\ \forall F \in \mathcal{L}_{\mathrm{LJ}} & \mathcal{L}_{\mathrm{NJ}} \vdash_{\mathrm{I}} F \end{array}$$

the result follows because we can cut each formula in the logic encoding, as the following derivation shows (for the "only if"-direction):

$$
\dfrac{
\begin{array}{c}
\mathcal{L}_{\mathrm{NJ}} \vdash_{\mathrm{I}} F_1^{\mathrm{LJ}} \\
\vdots \\
\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor \vdash_{\mathrm{I}} F_1^{\mathrm{LJ}}
\end{array}
\qquad
\dfrac{
\begin{array}{c}
\mathcal{L}_{\mathrm{NJ}} \vdash_{\mathrm{I}} F_2^{\mathrm{LJ}} \\
\vdots \\
\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor, F_1^{\mathrm{LJ}} \vdash_{\mathrm{I}} F_2^{\mathrm{LJ}}
\end{array}
\qquad
\begin{array}{c}
\lfloor \Gamma \rfloor, \mathcal{L}_{\mathrm{LJ}} \vdash_{\mathrm{I}} \lceil C \rceil \\
\vdots \\
\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor, \mathcal{L}_{\mathrm{LJ}} \vdash_{\mathrm{I}} \lceil C \rceil \\
\vdots \\
\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor, F_1^{\mathrm{LJ}}, F_2^{\mathrm{LJ}} \vdash_{\mathrm{I}} \lceil C \rceil
\end{array}
}{
\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor, F_1^{\mathrm{LJ}} \vdash_{\mathrm{I}} \lceil C \rceil
}
}{
\mathcal{L}_{\mathrm{NJ}}, \lfloor \Gamma \rfloor \vdash_{\mathrm{I}} \lceil C \rceil
}
$$

What remains is to show that all the formulas in the first logic follows from the rules in the second logic and vice-versa.

In this case this is highly trivial as both systems has the same rules. □

For the systems with generalized elimination rules the situation is the same as in the Nigam and Miller paper: for GE we can only prove one direction of equivalence to LJ, because of the missing cut rule. If we remove the cut rule from LJ we can prove full correspondence, which is witnessed by the following two propositions:

Proposition 4.2. *Let $\Gamma \cup \{C\}$ be a set of formulas then:*

$$\textit{If} \quad \Gamma \vdash_{\mathrm{GE}} C \quad \textit{then} \quad \Gamma \vdash_{\mathrm{LJ}} C$$

Proof. Like as the proof for relative completeness of LJ and NJ we take a detour to intuitionistic logic using Propositions 3.1, 3.3 and Corollary 2.2. Therefore it comes down to showing that:

$$\text{If} \quad \mathcal{L}_{\mathrm{GE}}, \lfloor \Gamma \rfloor \vdash_{\mathrm{I}} \lceil C \rceil \quad \text{then} \quad \mathcal{L}_{\mathrm{LJ}}, \lfloor \Gamma \rfloor \vdash_{\mathrm{I}} \lceil C \rceil$$

which follows if we can show that:

$$\forall F \in \mathcal{L}_{\mathrm{GE}} \quad \mathcal{L}_{\mathrm{LJ}} \vdash_{\mathrm{I}} F$$

We will show one of these equivalences and leave the rest for the reader. In the proof of these equivalences we will not give a full derivation in $\vdash_{\mathrm{I}}$ but rather in a less verbose way, it should be clear how to convert our proof into a derivation.

($\Rightarrow_{\mathsf{GE}}$): We need to show:

$$\mathcal{L}_{\mathrm{LJ}} \vdash_{\mathrm{I}} \forall A \forall B \lceil A \Rightarrow B \rceil \supset (\lceil A \rceil \supset \lfloor B \rfloor)$$

So assume $\lceil A \Rightarrow B \rceil$ by (Cut) we have that:

$$\lfloor A \Rightarrow B \rfloor$$

and the result follows from $\Rightarrow_L$.

□

Proposition 4.3. *Let $\Gamma \cup \{C\}$ be a set of formulas, and let LJ^{f} be the proof system LJ without the cut rule then:*

$$\Gamma \vdash_{\mathrm{LJ^f}} C \quad \textit{if and only if} \quad \Gamma \vdash_{\mathrm{GE}} C$$

Proof. We use the same method as the former proofs in this section, strictly speaking we have not proved completeness for $\mathrm{LJ^f}$ but completeness is proven in exactly the same way as for LJ. When the cut rule is removed the result follows like for LJ and NJ. We show a single case:

$(\Rightarrow_{\mathsf{L}})$: We need to show:

$$\mathcal{L}_{\mathrm{LJ}} \vdash_{\mathrm{I}} \forall A \forall B \lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \supset \lfloor B \rfloor)$$

So assume $\lfloor A \Rightarrow B \rfloor$ by (I) we have that:

$$\lceil A \Rightarrow B \rceil$$

and the result follows from $(\Rightarrow_{\mathsf{GE}})$.

□

As a side remark note that because LJ satisfies cut elimination we have that provability in GE is equivalent to provability in LJ. But in this work we are more interested in the method than the (well-known) result.

For GEA we can prove the full correspondence:

Proposition 4.4. *Let $\Gamma \cup \{C\}$ be a set of formulas then:*

$$\Gamma \vdash_{\mathrm{LJ}} C \quad \textit{if and only if} \quad \Gamma \vdash_{\mathrm{GEA}} C \uparrow$$

Proof. Again we use the same method and the only rules that are different are $\Rightarrow_{\mathsf{GE}}$ and $\forall_{\mathsf{GE}}$ which are clearly equivalent to the corresponding LJ rules. □

Last we look at how to connect the two versions of sequent calculus LJ and LJ'.

Proposition 4.5. *Let $\Gamma \cup \{C\}$ be a set of formulas then:*

$$\Gamma \vdash_{\mathrm{LJ}} C \quad \textit{if and only if} \quad \Gamma \vdash_{\mathrm{LJ'}} C$$

Proof. This proof is different from the other proofs of relative completeness because instead of going all the way to intuitionistic logic, we stay in LJF and prove the correspondence there. The reason why we can not use equivalence between the rules is because the encoded sequents are of a different format and therefore not directly related.

So using Propositions 3.1 and 3.6 what we wish to show in one direction is:

$$\text{If}\quad [\mathcal{L}_{\mathrm{LJ}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil C\rceil] \quad\text{then}\quad [\mathcal{L}_{\mathrm{LJ'}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil C\rceil]$$

and to go in the other direction we generalize slightly, and want to show:

$$\begin{array}{rl} \text{If}\quad [\mathcal{L}_{\mathrm{LJ'}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil C\rceil] & \text{then}\quad [\mathcal{L}_{\mathrm{LJ}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil C\rceil] \\ \text{If}\quad [\mathcal{L}_{\mathrm{LJ'}}, \lfloor\Gamma\rfloor] \longrightarrow [\mathbf{empty}] & \text{then}\quad [\mathcal{L}_{\mathrm{LJ}}, \lfloor\Gamma\rfloor] \longrightarrow [\lceil\bot\rceil] \end{array}$$

The first direction is proved by induction on the focused derivations and the second direction is proved by mutual induction on focused derivations. We note that the only way the sequents can be provable is by focusing on one of the formulas in $\mathcal{L}_{\mathrm{LJ}}$ so those are the only cases we consider.

We only show the cases where $\bot$ is involved as the rest follows straightforwardly and are very similar to what one would do if one was working with the derivations in the object-level systems directly. First we consider the first direction.

(Focus on $(\bot_{\mathsf{L}})$):
The derivation on the left must be (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor\Gamma\rfloor \cup \{\lfloor\bot\rfloor\}$):

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor\bot\rfloor}\!\!\rightarrow}\,\mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}] \xrightarrow{\mathbf{false}} [\lceil C\rceil]}\,\mathsf{R_L}, \mathbf{false}_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor\bot\rfloor\supset\mathbf{false}} [\lceil C\rceil]}\,\supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil C\rceil]}\,\mathsf{L_F}$$

With $\mathcal{K}' = \mathcal{L}_{\mathrm{LJ'}} \cup \lfloor\Gamma\rfloor \cup \{\lfloor\bot\rfloor\}$ we construct the needed derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\mathcal{K}'] -_{\lfloor\bot\rfloor}\!\!\rightarrow}\,\mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}'] \xrightarrow{\mathbf{empty}} \mathbf{empty}}\,\mathsf{I_L}}{[\mathcal{K}'] \xrightarrow{\lfloor\bot\rfloor\supset\mathbf{empty}} [\mathbf{empty}]}\,\supset_{\mathsf{L}}}{[\mathcal{K}'] \longrightarrow [\mathbf{empty}]}\,\mathsf{L_F}}{[\mathcal{K}'] -_{\mathbf{empty}}\!\!\rightarrow}\,\mathsf{R_R}, []_{\mathsf{R}} \qquad \dfrac{}{[\mathcal{K}'] \xrightarrow{\lceil C\rceil} [\lceil C\rceil]}\,\mathsf{I_L}}{[\mathcal{K}'] \xrightarrow{\mathbf{empty}\supset\lceil C\rceil} [\lceil C\rceil]}\,\supset_{\mathsf{L}}}{[\mathcal{K}'] \longrightarrow [\lceil C\rceil]}\,\mathsf{L_F}, \forall_{\mathsf{L}}$$

Next we consider the second direction.

(Focus on $(\bot_{\mathsf{L}})$):
The derivation on the left must be (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ'}} \cup \lfloor\Gamma\rfloor \cup \{\lfloor\bot\rfloor\}$):

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor\bot\rfloor}\!\rightarrow}\,\mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}] \xrightarrow{\mathbf{empty}} [\mathbf{empty}]}\,\mathsf{I_L}}{[\mathcal{K}] \xrightarrow{\lfloor\bot\rfloor\supset\mathbf{empty}} [\mathbf{empty}]}\,\supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\mathbf{empty}]}\,\mathsf{L_F}$$

With $\mathcal{K}' = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor\Gamma\rfloor \cup \{\lfloor\bot\rfloor\}$ we construct the needed derivation:

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}'] -_{\lfloor\bot\rfloor}\!\rightarrow}\,\mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}'] \xrightarrow{\mathbf{false}} [\lceil\bot\rceil]}\,\mathsf{R_L}, \mathbf{false}_{\mathsf{L}}}{[\mathcal{K}'] \xrightarrow{\lfloor\bot\rfloor\supset\mathbf{false}} [\lceil\bot\rceil]}\,\supset_{\mathsf{L}}}{[\mathcal{K}'] \longrightarrow [\lceil\bot\rceil]}\,\mathsf{L_F}$$

(Focus on ($\mathsf{W_R}$)):

The derivation on the left must be (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}'} \cup \lfloor\Gamma\rfloor$):

$$\dfrac{\dfrac{\dfrac{[\mathcal{K}] \longrightarrow [\mathbf{empty}]}{[\mathcal{K}] -_{\mathbf{empty}}\!\rightarrow}\,\mathsf{R_R}, []_{\mathsf{R}} \qquad \dfrac{}{[\mathcal{K}] \xrightarrow{\lceil C\rceil} [\lceil C\rceil]}\,\mathsf{I_L}}{[\mathcal{K}] \xrightarrow{\mathbf{empty}\supset\lceil C\rceil} [\lceil C\rceil]}\,\supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil C\rceil]}\,\mathsf{L_F}, \forall_{\mathsf{L}}$$

With $\mathcal{K}' = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor\Gamma\rfloor$ we get by IH:

$$[\mathcal{K}'] \longrightarrow [\lceil\bot\rceil]$$

from which the result follows by the following derivation:

$$\dfrac{\dfrac{\dfrac{[\mathcal{K}'] \longrightarrow [\lceil\bot\rceil]}{[\mathcal{K}'] -_{\lceil\bot\rceil}\!\rightarrow}\,\mathsf{R_R}, []_{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{}{[\mathcal{K}', \lfloor\bot\rfloor] -_{\lfloor\bot\rfloor}\!\rightarrow}\,\mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}', \lfloor\bot\rfloor] \xrightarrow{\lceil C\rceil} [\lceil C\rceil]}\,\mathsf{I_L}}{[\mathcal{K}', \lfloor\bot\rfloor] \longrightarrow [\lceil C\rceil]}\,\mathsf{L_F}, \supset_{\mathsf{L}}}{[\mathcal{K}'] \xrightarrow{\lfloor\bot\rfloor} [\lceil C\rceil]}\,\mathsf{R_L}, []_{\mathsf{L}}}{[\mathcal{K}'] \xrightarrow{\lceil\bot\rceil\supset\lfloor\bot\rfloor} [\lceil C\rceil]}\,\supset_{\mathsf{L}}}{[\mathcal{K}'] \longrightarrow [\lceil C\rceil]}\,\mathsf{L_F}, \forall_{\mathsf{L}}$$

□

4.2 Classical systems

In this section we consider the classical systems. Because only LK deals with quantifiers, $\perp$ and $\top$ we shall restrict our view to propositional formulas without $\perp$ and $\top$. So in the following Γ, Δ and C will represent propositional formulas without $\perp$ and $\top$.

Another remark is that the LK does not have an explicit negation and therefore we must use a transformation to relate the later systems to LK. Because of the absence of negation we can not use a proof of equivalence between the rules like for the intuitionistic cases. So instead we introduce LK with negation (called LK$\neg$) prove that LK$\neg$ is equivalent to LK and then relate LK$\neg$ to FD, KE and AC.

The rules for LK$\neg$ are the same as for the propositional part of LK except $\perp_{\mathsf{L}}$ and $\top_{\mathsf{R}}$ plus the following two rules:

$$\frac{\Gamma, \neg A \vdash_{\mathrm{LK}\neg} A, \Delta}{\Gamma, \neg A \vdash_{\mathrm{LK}\neg} \Delta}\,\neg_{\mathsf{L}} \qquad \frac{\Gamma, A \vdash_{\mathrm{LK}\neg} \neg A, \Delta}{\Gamma \vdash_{\mathrm{LK}\neg} \neg A, \Delta}\,\neg_{\mathsf{R}}$$

For the encoding we get a complete encoding of $\mathcal{L}_{\mathrm{LK}\neg}$ by adding the following (and removing the $\perp, \top, \forall, \exists$ part):

$$(\neg_L) \quad \lfloor \neg A \rfloor \supset \lceil A \rceil \qquad (\neg_R) \quad \lceil \neg A \rceil \supset \lfloor A \rfloor$$

For the correspondence the function φ is defined as in the Nigam and Miller paper as:

$$\begin{aligned} \varphi(A) &= A & &A \text{ atomic} \\ \varphi(\neg C) &= \varphi(C) \Rightarrow \top & & \\ \varphi(C_1 \star C_2) &= \varphi(C_1) \star \varphi(C_2) & &\star \in \{\Rightarrow, \wedge, \vee\} \end{aligned}$$

φ is extended to sets in the obvious way.

Using φ we can prove correspondence between LK and LK$\neg$:

Lemma 4.6. *Let Γ and Δ be sets of formulas then:*

$$\varphi(\Gamma) \vdash_{\mathrm{LK}} \varphi(\Delta) \quad \textit{if and only if} \quad \Gamma \vdash_{\mathrm{LK}\neg} \Delta$$

Proof. As the rules for LK and LK$\neg$ are almost the same the only things to prove is the cases with the negation, which can be on either the left or the right side. The proof goes by induction on the height of the derivations.

For the "only if"-direction the interesting cases are $\Rightarrow_{\mathsf{L}}$ and $\Rightarrow_{\mathsf{R}}$ where the principal formulas are $A \Rightarrow \perp$.

$(\Rightarrow_{\mathsf{L}})$:

$$\frac{\varphi(\Gamma), \varphi(A) \Rightarrow \bot \vdash_{\mathrm{LK}} \varphi(A), \varphi(\Delta) \qquad \varphi(\Gamma), \varphi(A) \Rightarrow \bot, \bot \vdash_{\mathrm{LK}} \varphi(\Delta)}{\varphi(\Gamma), \varphi(A) \Rightarrow \bot \vdash_{\mathrm{LK}} \varphi(\Delta)}$$

By IH we have that:

$$\Gamma, \neg A \vdash_{\mathrm{LK}\neg} A, \Delta$$

which gives the result by the $\neg_{\mathsf{L}}$ rule.

($\Rightarrow_{\mathsf{R}}$):

$$\frac{\varphi(\Gamma), \varphi(A) \vdash_{\mathrm{LK}} \bot, \varphi(A) \Rightarrow \bot, \varphi(\Delta)}{\varphi(\Gamma) \vdash_{\mathrm{LK}} \varphi(A) \Rightarrow \bot, \varphi(\Delta)}$$

We can then obtain a proof of the same height of the following (removing the $\bot$ on the right - seen by straightforward induction):

$$\varphi(\Gamma), \varphi(A) \vdash_{\mathrm{LK}} \varphi(A) \Rightarrow \bot, \varphi(\Delta)$$

and then by IH we have that:

$$\Gamma, A \vdash_{\mathrm{LK}\neg} \neg A, \Delta$$

which gives the result by the $\neg_{\mathsf{R}}$ rule.

For the "if"-direction we consider the cases $\neg_{\mathsf{L}}$ and $\neg_{\mathsf{R}}$:

($\neg_{\mathsf{L}}$):

$$\frac{\Gamma, \neg A \vdash_{\mathrm{LK}\neg} A, \Delta}{\Gamma, \neg A \vdash_{\mathrm{LK}\neg} \Delta} \neg_{\mathsf{L}}$$

By IH we have that:

$$\varphi(\Gamma), \varphi(A) \Rightarrow \bot \vdash_{\mathrm{LK}} \varphi(A), \varphi(\Delta)$$

and the result follows from $\Rightarrow_{\mathsf{L}}$ and $\bot_{\mathsf{L}}$.

($\neg_{\mathsf{R}}$):

$$\frac{\Gamma, A \vdash_{\mathrm{LK}\neg} \neg A, \Delta}{\Gamma \vdash_{\mathrm{LK}\neg} \neg A, \Delta} \neg_{\mathsf{R}}$$

By IH we have that:

$$\varphi(\Gamma), \varphi(A) \vdash_{\mathrm{LK}} \varphi(A) \Rightarrow \bot, \varphi(\Delta)$$

and the result follows from the following derivation:

$$\dfrac{\dfrac{\overline{\varphi(\Gamma), \varphi(A) \vdash_{\mathrm{LK}} \varphi(A), \bot, \varphi(A) \Rightarrow \bot, \varphi(\Delta)}}{\varphi(\Gamma) \vdash_{\mathrm{LK}} \varphi(A), \varphi(A) \Rightarrow \bot, \varphi(\Delta)} \qquad \varphi(\Gamma), \varphi(A) \vdash_{\mathrm{LK}} \varphi(A) \Rightarrow \bot, \varphi(\Delta)}{\varphi(\Gamma) \vdash_{\mathrm{LK}} \varphi(A) \Rightarrow \bot, \varphi(\Delta)}$$

□

Using LK¬ in the proofs we can prove the correspondence from LK to FD, KE and AC. For FD we run into the same problem as for GE, namely that we can only prove one direction of the equivalence because of the missing cut rule. If we remove the cut rule from LK we can prove the correspondence in the other way (note that we use the cut rule in the first direction so we do not get a full equivalence for the cut-free LK).

Proposition 4.7. *Let* Γ *and* Δ *be sets of formulas then:*

$$\textit{If} \quad \Gamma \vdash_{\mathrm{FD}} \Delta \quad \textit{then} \quad \varphi(\Gamma) \vdash_{\mathrm{LK}} \varphi(\Delta)$$

Proof. We use that provability in LK is equivalent to provability in LK¬ so we need to prove that:

$$\text{If} \quad \Gamma \vdash_{\mathrm{FD}} \Delta \quad \text{then} \quad \Gamma \vdash_{\mathrm{LK}\neg} \Delta$$

which we prove by proving that (using the corresponding completeness propositions):

$$\forall F \in \mathcal{L}_{\mathrm{FD}} \quad \mathcal{L}_{\mathrm{LK}\neg} \vdash_{\mathrm{I}} F$$

$(\Rightarrow_{\mathsf{E}})$: We need to show:

$$\mathcal{L}_{\mathrm{LK}\neg} \vdash_{\mathrm{I}} \forall A \forall B \lceil A \Rightarrow B \rceil \vee \lceil A \rceil \vee \lfloor B \rfloor$$

By using (Cut) for LK¬ we get that:

$$\lfloor A \Rightarrow B \rfloor \vee \lceil A \Rightarrow B \rceil$$

In the second disjunct we are done so assume the first disjunct then we can use $(\Rightarrow_{\mathsf{L}})$ to prove that:

$$\lceil A \rceil \supset \lfloor B \rfloor \tag{1}$$

and then we can use (Cut) again to show that

$$\lfloor A \rfloor \vee \lceil A \rceil$$

again we are done in the second disjunct and in the first disjunct the result follows from the implication in (1).

($\Rightarrow_{\mathsf{I1}}$): We need to show:

$$\mathcal{L}_{\mathrm{LK}\neg} \vdash_{\mathrm{I}} \forall A \forall B \lfloor A \Rightarrow B \rfloor \vee \lfloor A \rfloor$$

By using (Cut) for LK¬ we get that:

$$\lfloor A \Rightarrow B \rfloor \vee \lceil A \Rightarrow B \rceil$$

In the first disjunct we are done, so assume the second disjunct then we can use ($\Rightarrow_{\mathsf{R}}$) to prove that:

$$\lfloor A \rfloor \wedge \lceil B \rceil$$

which proves the result.

($\neg_{\mathsf{I1}}$): We need to show:

$$\mathcal{L}_{\mathrm{LK}\neg} \vdash_{\mathrm{I}} \forall A \lfloor \neg A \rfloor \vee \lfloor A \rfloor$$

By using (Cut) for LK¬ we get that:

$$\lfloor \neg A \rfloor \vee \lceil \neg A \rceil$$

In the first disjunct we are done, so assume the second disjunct then we can use ($\neg_{\mathsf{R}}$) to prove that:

$$\lfloor A \rfloor$$

which proves the original proposition.

□

Proposition 4.8. *Let Γ and Δ be sets of formulas and let LK^{f} be LK without the cut rule then:*

$$\textit{If} \quad \varphi(\Gamma) \vdash_{\mathrm{LK}^{\mathrm{f}}} \varphi(\Delta) \quad \textit{then} \quad \Gamma \vdash_{\mathrm{FD}} \Delta$$

Proof. We take the detour through $\mathrm{LK}^{\mathrm{f}}\neg$ (LK¬ without cut) and into intuitionistic logic, so we need to prove that:

$$\forall F \in \mathcal{L}_{\mathrm{LK}^{\mathrm{f}}\neg} \quad \mathcal{L}_{\mathrm{FD}} \vdash_{\mathrm{I}} F$$

Again we show a couple of cases:

($\Rightarrow_{\mathsf{L}}$): We need to show:

$$\mathcal{L}_{\mathrm{FD}} \vdash_{\mathrm{I}} \forall A \forall B \lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \vee \lfloor B \rfloor)$$

Assume $\lfloor A \Rightarrow B \rfloor$ by ($\Rightarrow_{\mathsf{E}}$) we get that:

$$\lceil A \Rightarrow B \rceil \vee \lceil A \rceil \vee \lfloor B \rfloor$$

In the first disjunct we can use the assumption and (I) to conclude **false**, and in the second disjunction we are done.

$(\Rightarrow_\mathsf{R})$: We need to show:

$$\mathcal{L}_{\mathrm{FD}} \vdash_{\mathrm{I}} \forall A \forall B \lceil A \Rightarrow B \rceil \supset (\lfloor A \rfloor \wedge \lceil B \rceil)$$

Assume $\lceil A \Rightarrow B \rceil$ by $(\Rightarrow_{\mathsf{I1}})$ and $(\Rightarrow_{\mathsf{I2}})$ we get that:

$$\lfloor A \Rightarrow B \rfloor \vee \lfloor A \rfloor$$
$$\lfloor A \Rightarrow B \rfloor \vee \lceil B \rceil$$

If we in either formula have $\lfloor A \Rightarrow B \rfloor$ then we again can use (I) to conclude **false**. So we must have both $\lfloor A \rfloor$ and $\lceil B \rceil$ and we are done.

$(\neg_\mathsf{L})$: We need to show:

$$\mathcal{L}_{\mathrm{FD}} \vdash_{\mathrm{I}} \forall A \lfloor \neg A \rfloor \supset \lceil A \rceil$$

Assume $\lfloor \neg A \rfloor$ by $(\neg_{\mathsf{I2}})$ we get that:

$$\lceil \neg A \rceil \vee \lceil A \rceil$$

and we can either conclude **false** from (I) or the conclusion, so we are done.

□

Again because of cut elimination for LK we have that provability in FD is equivalent to provability in LK.

For KE and AC we can prove the full equivalence.

Proposition 4.9. *Let* Γ *and* Δ *be sets of formulas then:*

$$\varphi(\Gamma) \vdash_{\mathrm{LK}} \varphi(\Delta) \quad \textit{if and only if} \quad \Gamma \vdash_{\mathrm{KE}} \Delta$$

Proof. As for FD we use LK¬ and intuitionistic logic to prove equivalence, that means we must show:

$$\forall F \in \mathcal{L}_{\mathrm{LK}\neg} \quad \mathcal{L}_{\mathrm{KE}} \vdash_{\mathrm{I}} F$$
$$\forall F \in \mathcal{L}_{\mathrm{KE}} \quad \mathcal{L}_{\mathrm{LK}\neg} \vdash_{\mathrm{I}} F$$

We first consider the first direction, and show a single case:

$(\Rightarrow_\mathsf{L})$: We need to show:

$$\mathcal{L}_{\mathrm{KE}} \vdash_{\mathrm{I}} \forall A \forall B \lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \vee \lfloor B \rfloor)$$

Assume $\lfloor A \Rightarrow B \rfloor$ by using $(\Rightarrow_{\mathsf{L1}})$ we get that:

$$\lfloor A \rfloor \supset \lfloor B \rfloor \tag{2}$$

By (Cut) be get that:

$$\lfloor A \rfloor \vee \lceil A \rceil$$

If the first disjunct is the case the result follows from (2) if the second disjunct is the case the result follows immediately.

Next we consider the second direction, and show a single case:

($\Rightarrow_{\mathsf{L1}}$): We need to show:

$$\mathcal{L}_{\mathrm{LK}\neg} \vdash_{\mathrm{I}} \forall A \forall B \lfloor A \Rightarrow B \rfloor \supset (\lfloor A \rfloor \supset \lfloor B \rfloor)$$

Assume $\lfloor A \Rightarrow B \rfloor$ and $\lfloor A \rfloor$ by using ($\Rightarrow_{\mathsf{L}}$) one the first we get that:

$$\lceil A \rceil \vee \lfloor B \rfloor$$

If the first disjunct is the case then we can derive **false** from (I), for the second disjunct the result follows immediately. □

Proposition 4.10. *Let* Γ *and* Δ *be sets of formulas then:*

$$\varphi(\Gamma) \vdash_{\mathrm{LK}} \varphi(\Delta) \quad \textit{if and only if} \quad \Gamma \vdash_{\mathrm{AC}} \Delta$$

Proof. This proofs follows all the other proofs in this section and again it comes down to showing that:

$$\begin{array}{ll} \forall F \in \mathcal{L}_{\mathrm{LK}\neg} & \mathcal{L}_{\mathrm{AC}} \vdash_{\mathrm{I}} F \\ \forall F \in \mathcal{L}_{\mathrm{AC}} & \mathcal{L}_{\mathrm{LK}\neg} \vdash_{\mathrm{I}} F \end{array}$$

We first consider the first direction, and show a couple of cases:

($\Rightarrow_{\mathsf{L}}$): We need to show:

$$\mathcal{L}_{\mathrm{AC}} \vdash_{\mathrm{I}} \forall A \forall B \lfloor A \Rightarrow B \rfloor \supset (\lceil A \rceil \vee \lfloor B \rfloor)$$

Assume $\lfloor A \Rightarrow B \rfloor$ by using ($\Rightarrow_{\mathsf{L}}$) we get that:

$$(\lfloor A \rfloor \wedge \lceil B \rceil) \supset \textbf{false} \tag{3}$$

By (Cut) be get that:

$$\lfloor A \rfloor \vee \lceil A \rceil$$

If the second disjunct is the case the result follows immediately. So assume $\lceil A \rceil$, by using (Cut) again we get that:

$$\lfloor B \rfloor \vee \lceil B \rceil$$

If the first disjunct is the case the result follows immediately. So by assuming $\lceil B \rceil$ we can derive **false** from (3).

($\Rightarrow_{\mathsf{R}}$): We need to show:

$$\mathcal{L}_{\text{AC}} \vdash_{\text{I}} \forall A \forall B \lceil A \Rightarrow B \rceil \supset (\lfloor A \rfloor \wedge \lceil B \rceil)$$

Assume $\lceil A \Rightarrow B \rceil$ by using ($\Rightarrow_{\mathsf{R}}$) we get that:

$$(\lceil A \rceil \vee \lfloor B \rfloor) \supset \textbf{false} \tag{4}$$

By (Cut) be get that:

$$\lfloor A \rfloor \vee \lceil A \rceil$$

If the second disjunct is the case then we can derive **false** from (4). So assume $\lfloor A \rfloor$, by using (Cut) again we get that:

$$\lfloor B \rfloor \vee \lceil B \rceil$$

If the first disjunct is the case we can again derive **false** from (4). So by assuming $\lceil B \rceil$ we get the conclusion.

Next we consider the second direction, and show a couple of cases:

($\Rightarrow_{\mathsf{L}}$): We need to show:

$$\mathcal{L}_{\text{LK}\neg} \vdash_{\text{I}} \forall A \forall B \lfloor A \Rightarrow B \rfloor \supset ((\lfloor A \rfloor \wedge \lceil B \rceil) \supset \textbf{false})$$

Assume $\lfloor A \Rightarrow B \rfloor$, $\lfloor A \rfloor$ and $\lceil B \rceil$ by using ($\Rightarrow_{\mathsf{L}}$) we get that:

$$\lceil A \rceil \vee \lfloor B \rfloor$$

and for each disjunct we can prove **false** from an assumption and (I).

($\Rightarrow_{\mathsf{R}}$): We need to show:

$$\mathcal{L}_{\text{LK}\neg} \vdash_{\text{I}} \forall A \forall B \lceil A \Rightarrow B \rceil \supset ((\lceil A \rceil \vee \lfloor B \rfloor) \supset \textbf{false})$$

Assume $\lceil A \Rightarrow B \rceil$ and $\lceil A \rceil \vee \lfloor B \rfloor$ by using ($\Rightarrow_{\mathsf{R}}$) we get that:

$$\lfloor A \rfloor \wedge \lceil B \rceil \tag{5}$$

and for each disjunct in the assumption we can prove **false** from a conjunct in (5) and (I).

□

4.3 Intuitionistic and Classical systems

In this section we look at how to relate LJ to LK. The result is the well-known fact that intuitionistic provability implies classical provability:

Proposition 4.11. *Let $\Gamma \cup \{C\}$ be a set of formulas then:*

$$\textit{If} \quad \Gamma \vdash_{\mathrm{LJ}} C \quad \textit{then} \quad \Gamma \vdash_{\mathrm{LK}} C$$

Proof. Like the proof for relative completeness of LJ and LJ' we use LJF directly and prove correspondence for focused derivations.

Using Propositions 3.1 and 3.2 what we need to show is (notice that there are different polarities on each side):

$$\text{If} \quad [\mathcal{L}_{\mathrm{LJ}}, \lfloor \Gamma \rfloor] \longrightarrow [\lceil C \rceil] \quad \text{then} \quad [\mathcal{L}_{\mathrm{LK}}, \lfloor \Gamma \rfloor, \lceil C \rceil] \longrightarrow [\mathbf{false}]$$

which we prove by induction on the focused derivations. As for LJ and LJ' the only way the sequents can be provable is by focusing on one of the formulas in $\mathcal{L}_{\mathrm{LJ}}$ so those are the only cases we consider.

(Focus on ($\Rightarrow_{\mathsf{L}}$)):

The derivation on the left must be (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor \Gamma \rfloor \cup \{\lfloor A_1 \Rightarrow A_2 \rfloor\}$):

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor A_1 \Rightarrow A_2 \rfloor}\rightarrow}\,\mathsf{I_R} \quad \dfrac{[\mathcal{K}] \longrightarrow [\lceil A_1 \rceil]}{[\mathcal{K}] -_{\lceil A_1 \rceil}\rightarrow}\,\mathsf{R_R}, []_{\mathsf{R}} \quad \dfrac{[\mathcal{K}, \lfloor A_2 \rfloor] \longrightarrow [\lceil C \rceil]}{[\mathcal{K}] \xrightarrow{\lfloor A_2 \rfloor} [\lceil C \rceil]}\,\mathsf{R_L}, []_{\mathsf{L}}}{[\mathcal{K}] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \supset (\lceil A_1 \rceil \supset \lfloor A_2 \rfloor)} [\lceil C \rceil]}\,2 \times \supset_{\mathsf{L}}}{[\mathcal{K}] \longrightarrow [\lceil C \rceil]}\,\mathsf{L_F}, 2 \times \forall_{\mathsf{L}}$$

Now let $\mathcal{K}'$ be $\mathcal{L}_{\mathrm{LK}} \cup \lfloor \Gamma \rfloor \cup \{\lfloor A_1 \Rightarrow A_2 \rfloor, \lceil C \rceil\}$.

By IH and weakening (in the second case) we get that:

$$[\mathcal{K}', \lceil A_1 \rceil] \longrightarrow [\mathbf{false}]$$
$$[\mathcal{K}', \lfloor A_2 \rfloor] \longrightarrow [\mathbf{false}]$$

We can now construct the needed derivation:

$$\dfrac{\dfrac{\dfrac{}{[\mathcal{K}'] -_{\lfloor A_1 \Rightarrow A_2 \rfloor}\rightarrow}\,\mathsf{I_R} \quad \dfrac{\dfrac{[\mathcal{K}', \lceil A_1 \rceil] \longrightarrow [\mathbf{false}]}{[\mathcal{K}'], \lceil A_1 \rceil \longrightarrow [\mathbf{false}]}\,[]_{\mathsf{L}} \quad \dfrac{[\mathcal{K}', \lfloor A_2 \rfloor] \longrightarrow [\mathbf{false}]}{[\mathcal{K}'], \lfloor A_2 \rfloor \longrightarrow [\mathbf{false}]}\,[]_{\mathsf{L}}}{[\mathcal{K}'] \xrightarrow{\lceil A_1 \rceil \vee \lfloor A_2 \rfloor} [\mathbf{false}]}\,\mathsf{R_L}, \vee_{\mathsf{L}}}{[\mathcal{K}'] \xrightarrow{\lfloor A_1 \Rightarrow A_2 \rfloor \supset (\lceil A_1 \rceil \vee \lfloor A_2 \rfloor)} [\mathbf{false}]}\,\supset_{\mathsf{L}}}{[\mathcal{K}'] \longrightarrow [\mathbf{false}]}\,\mathsf{L_F}, 2 \times \forall_{\mathsf{L}}$$

(Focus on (I)):

The derivation on the left must be (with $\mathcal{K} = \mathcal{L}_{\mathrm{LJ}} \cup \lfloor \Gamma \rfloor \cup \{\lfloor A \rfloor\}$):

$$
\dfrac{\dfrac{}{[\mathcal{K}] -_{\lfloor A \rfloor}\!\!\rightarrow}\ \mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}] \xrightarrow{\lceil A \rceil} [\lceil A \rceil]}\ \mathsf{I_L}}{\dfrac{[\mathcal{K}] \xrightarrow{\lfloor A \rfloor \supset \lceil A \rceil} [\lceil A \rceil]}{[\mathcal{K}] \longrightarrow [\lceil A \rceil]}\ \mathsf{L_F}, \forall_\mathsf{L}}\ \supset_\mathsf{L}
$$

With $\mathcal{K}' = \mathcal{L}_{\mathrm{LK}} \cup \lfloor \Gamma \rfloor \cup \{\lfloor A \rfloor, \lceil A \rceil\})$ the needed derivation follows easily:

$$
\dfrac{\dfrac{\dfrac{}{[\mathcal{K}'] -_{\lfloor A \rfloor}\!\!\rightarrow}\ \mathsf{I_R} \qquad \dfrac{}{[\mathcal{K}'] -_{\lceil A \rceil}\!\!\rightarrow}\ \mathsf{I_R}}{[\mathcal{K}'] -_{\lfloor A \rfloor \wedge^+ \lceil A \rceil}\!\!\rightarrow}\ \wedge^+_\mathsf{R} \qquad \dfrac{}{[\mathcal{K}'] \xrightarrow{\mathbf{false}} [\mathbf{false}]}\ \mathsf{R_L}, \mathbf{false}_\mathsf{L}}{\dfrac{[\mathcal{K}'] \xrightarrow{(\lfloor A \rfloor \wedge^+ \lceil A \rceil) \supset \mathbf{false}} [\mathbf{false}]}{[\mathcal{K}'] \longrightarrow [\mathbf{false}]}\ \mathsf{L_F}, \forall_\mathsf{L}}\ \supset_\mathsf{L}
$$

□

5 Comparison of LJF and LLF

In this section we compare the use of LJF as a framework to the use of LLF as a framework. To compare the frameworks we look at how easy or "natural" the encodings are, how hard the completeness proofs are, and how well can we reason within the framework, for instance how hard the relative completeness theorems are.

We first look at the different encodings, here it is clear that all the different intuitionistic systems have a natural encoding in LJF, which is also seen by the similar encodings, it seems very elegant that just by changing polarities we go from LJ to NJ. Besides the GE encoding only a few of the rule encodings ($\Rightarrow_{\mathsf{GE}}$ and $\forall_{\mathsf{GE}}$ for GEA, $\mathsf{W_R}$ and $\perp_\mathsf{L}$ for LJ') have a different encoding from the rest. Furthermore the encodings are very simple in the sense that the object-level connectives are mostly encoded using the corresponding meta-level connective, one exception here is the use of the atom **empty** in the encoding of LJ', which gives a less elegant encoding.

One the other hand it seems that the encodings of the intuitionistic systems in LLF are more different from each other, than the encodings in LJF are, Although the biggest differences are mostly which exponentials are used. Two observations are interesting. The first is that the encodings of NJ, GE, GEA need an extra formula ($\lfloor \perp \rfloor^\perp$ or $\lceil \perp \rceil$), which we do not need in our encoding. This makes the LJF encoding a little closer to the object-level systems, but for NJ and GEA the extra formula can also be used to describe normal form proofs for the $\Rightarrow, \wedge, \forall, \perp, \top$ fragment. The second observation is that for LJ with empty right sides the LLF encoding can still use the meta-level **false** whereas the LJF encoding needs to create a new meta-level atom. The reason is that the LLF encoding can exploit linear logic's strict structural rules and prevent the $\perp_{\mathsf{L}}$ rule from being used on proper formulas.

The encodings of the classical systems in LJF are not as natural as the encodings of the intuitionistic systems. But for most of the systems, the encodings are still fairly simple and easy to come up with, although the encoding of AC is a little more complicated than the others. The specific encoding of sequents using **false** is also different from the intuitionistic sequents, which make the encoding of the rules different. For LLF the situation is almost the same as for intuitionistic logic, although the different classical encodings have more differences than the intuitionistic encodings have.

A very nice feature of LLF is that the different systems stem from the same generic system with the in- or exclusion of structural rules. We have not found a way to create a generic system for LJF, as there are no control of the structural properties. This means that it would be impossible to encode nicely in LJF, linear systems or systems with multiple conclusions that do not allow weakening. As LLF uses linear logic as a base these systems should be possible to encode in LLF.

The proofs of full completeness are very similar in the two systems in both cases we have to look at the derivations and see how they match the rules. In that aspect we have not met any difficulties in using LJF instead of LLF.

For the LJF encodings, the proofs of relative completeness between the intuitionistic systems and between the classical systems are mostly very easy, we can use proof by rule equivalence, except for the cases where one of the systems are missing a cut rule or when there are syntactic differences. This is very nice as the proofs of rule equivalence are very shallow and suitable for automation. Compared to proving relative completeness using the object-level systems it seems easier to prove relative completeness using the LJF approach, as the needed induction is hidden, whereas in the object-level proof we would need to use a new induction every time.

In the cases where the cut rule is missing in one system, we are not able to prove rule equivalence, it should still be possible to prove relative completeness in those cases, maybe following the work by Miller and Pimentel [9] but we have not investigated that work further in this work.

Because the intuitionistic and classical systems are represented in a different way, we can not prove rule equivalence for LJ and LJ' or rule implication for LJ and LK, and therefore we need to resort to induction over focused derivations, which takes the same or slightly more work than an object-level proof would in this case.

In LLF the proofs of relative completeness usually uses rule equivalence in one direction and induction on focused derivations in the other direction.

This means that it is easier to prove relative completeness for some of the systems in the LJF setting. But when mixing the different encodings in LJF, like we do for LJ and LK, then rule implication can not be used. In LLF the encodings of the sequents are the same, and we think that the proof in LLF could go through using rule implication, therefore making it simpler than the LJF proof.

In conclusion, it seems that while LLF is more general and allowing more systems to be represented in a more streamlined way, LJF is more "specialized" in some way, meaning that the encodings are less general and therefore some proofs are easier because of not needing the extra generality.

6 Summary and related work

In this work we have shown how to encode a wide range of different intuitionistic and classical proof systems, using the focused intuitionistic logic LJF. For each encoding, we prove that the open derivations of sequents in the object-level system are in bijective correspondence with the open derivations of the encoded sequents in LJF. The features of the focused proofs are crucial to reduce the amount of proofs in the meta-logic and the strong equivalences are based on that.

Using the encodings we have proven equivalences of provability between the different intuitionistic systems and between the different classical systems. Only in the cases were cut elimination or a syntactic transformation were needed we have used a different approach.

The methods used in this work are not new and are based on recent work by Nigam and Miller [11]. Our works shows that focusing is the important part and that linear logic is not needed. Furthermore there may be advantages in using LJF, as some of the encodings seems easier and some of the proofs seem shorter. Although in other cases the more streamlined encodings in LLF may be preferred.

Related to the work by Nigam and Miller and therefore also to this work is the work by Miller and Pimentel [8, 9, 10, 16, 17] in which classical linear logic is used as a meta-logic for different proof systems including both representation of and reasoning on the object-level systems. Other people have been using other intuitionistic or dependently typed systems as frameworks for representing different proof systems [4, 6, 13]. Our work differs in that we use focusing to control the amount of proofs.

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