From spider robots to half disk robots

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Abstract

We study the problem of computing the set $\mathcal{F}$ of accessible and stable placements of a spider robot. The body of this robot is a single point and the legs are line segments attached to the body. The robot can only put its feet on some regions, called the foothold regions. Moreover, the robot is subject to two constraints: Each leg has a maximal extension R (accessibility constraint) and the body of the robot must lie above the convex hull of its feet (stability constraint). We present an efficient algorithm to compute $\mathcal{F}$. If the foothold regions are polygons with n edges in total, our algorithm computes $\mathcal{F}$ in $O(n^2 \log n)$ time and $O(n^2\alpha(n))$ space where α is the inverse of the Ackerman's function. $\Omega(n^2)$ is a lower bound for the size of $\mathcal{F}$.

1 Introduction

Although legged robots have already been studied in robotics [9, 10], only a very few papers consider the motion planning problem amidst obstacles [7, 6, 1, 2]. In [7, 6] some heuristic approaches are described while, in [1, 2] efficient and provably correct geometric algorithms are described for a restricted case, namely a point robot (spider) and a finite set of point footholds.

Compared to the classic piano movers problem, legged robots introduce new types of constraints. We assume that the environment is composed of areas of the plane, called *foothold regions*, where the robot can safely put its legs. Then the legged robot must satisfy two different constraints: the accessibility and the stability constraints. A *placement* (position of the boby of the robot) is called *accessible* if the legs of the robot can reach the footholds, and is called *stable* if the center of mass of the robot lies inside the interior of the convex hull of its feet. A placement both feasible and stable is called *admissible*, and the set of admissible placements is clearly the relevant information for a legged robot : we call this set *the free space*.

A first simple instance of a legged robot is the spider robot. The spider robot has been inspired by the Ambler, developped at Carnegie Mellon University [8]. The body of the spider robot is a single point, all its legs are attached to the body and can reach any foothold at distance less than R from the body (see Figure 1). The problem of the spider robot moving in an environment of point footholds has already been

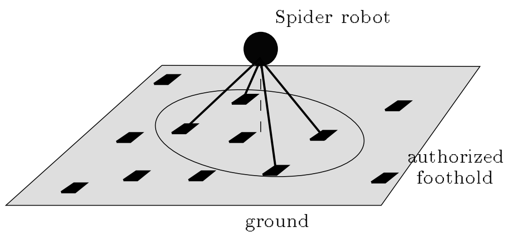

Figure 1: The spider robot

studied in [2] but the method used cannot be generalized to more complex environment. This paper proposes a new method to compute the free space of the spider robot based on a transformation between this problem and the problem of a half-disk moving amidst obstacles. The algorithm is simpler than the one described in [2] and the method can be extended to the case of polygonal foothold regions.

Once the free space has been computed, it can be used to find trajectories and sequences of legs assignments as described in [1].

The paper is organized as follows: Some notations and results of [2] are recalled in the next section. Section 3 shows the transformation between the spider robot problem and the half-disk problem. We present in Section 4 an algorithm that computes the free space of a spider robot for point footholds. The last section shows how the algorithm can be extended to polygonal foothold regions.

Due to the lack of space, some of the proofs are omitted and can be found in [3].

2 Notations and previous results

Let us introduce some notations: $\mathcal{S}$ is the set of discrete footholds $\{s_1, \ldots, s_n\}$. $\mathcal{F}$ and $\delta(\mathcal{F})$ denote respectively the free space of the spider robot using as footholds $\mathcal{S}$ and its boundary. C_i denotes the circle of radius R centered at s_i. $\mathcal{A}$ is the arrangement of the circles C_i for $1 \leq i \leq n$. This arrangement has an important geometric meaning in our problem and we will express the complexity results in term of $|\mathcal{A}|$, the size of $\mathcal{A}$. CH($\mathcal{E}$) denotes the interior of the convex hull of a set $\mathcal{E}$, $\mathcal{C}(\mathcal{E})$ the complementary of $\mathcal{E}$ and $\overline{\mathcal{E}}$ the closure of $\mathcal{E}$.

The algorithm described in [2] used extensively the arrangement $\mathcal{A}$. In a cell Γ of $\mathcal{A}$ the set of footholds that can be reached by the robot is fixed, thus the part of Γ that belongs to $\mathcal{F}$ is exactly the intersection of Γ with the convex hull of the reachable footholds. Therefore the edges of $\delta(\mathcal{F})$ are either circular arcs belonging to $\mathcal{A}$ or portions of line segments joining two footholds; moreover a vertex of $\delta(\mathcal{F})$ which is the intersection of two straight line edges is a foothold (see Figure 2). The complexity of $\mathcal{F}$ has been proved to be $|\mathcal{F}| = \Theta(|\mathcal{A}|)$ [2].

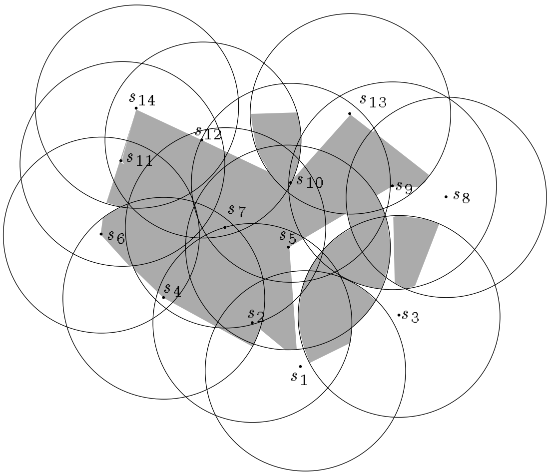

Figure 2: An example of the free space of the spider robot.

An algorithm based on the same basic idea as above is also presented in [2]. It uses sophisticated data structures allowing the offline maintenance of convex hulls. Its time complexity is $O(|\mathcal{A}| \log n)$.

The algorithm described in this paper has the same time complexity, only uses simple data structures and can be extended to the case where the foothold regions are polygons (see Section 5).

3 From spider robot to half-disk robot

Theorem 1 *The spider robot does not admit a stable and accessible placement at point P if and only if there exists a half-disk (of radius R) centered at P which does not contain any foothold of $\mathcal{S}$ (see Figure 3).*

Proof: Easy (see Figure 3). □

The next theorem will establish the connection between the free space of the spider robot and the free space of a half-disk robot moving under translation and rotation amidst n point obstacles.

Definition 2 *Let $HD(P,\theta)$ be the open half-disk of radius R centered at P (see Figure 4) defined by :*

$$\begin{cases} (x - x_P)^2 + (y - y_P)^2 < R^2 \\ (x - x_P)\sin\theta - (y - y_P)\cos\theta < 0 \end{cases}$$

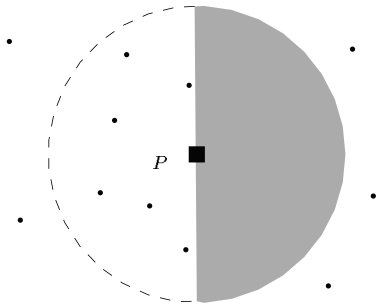

Figure 3: Situation without stable and accessible placement for the spider robot at point P.

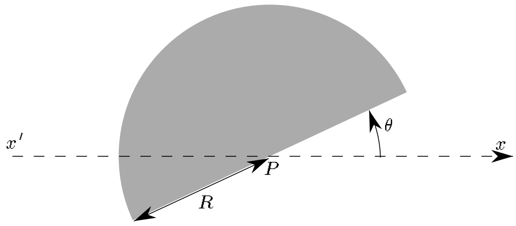

Figure 4: $HD(P,\theta)$

Definition 3 *Let us define the half-disk robot at the position (P,θ) as $HD(P,\theta+\pi)$. We define the free space $\mathcal{L}$ of the half-disk robot moving under translation and rotation amidst the obstacles of $\mathcal{S}$ as the set of positions (P,θ) such that the robot ($HD(P,\theta+\pi)$) does not collide with any obstacle.*

Notice that $\mathcal{L}$ is defined in $\mathbb{R}^2 \times S^1$ (where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$). For simplicity, we will often identify S^1 and the interval $[0, 2\pi]$ of $\mathbb{R}$ and speak of the θ-axis. This identification permits us to define $p_{//\theta}$ the orthogonal projection onto $\mathbb{R}^2$.

With our definition of a half-disk robot, we can rewrite Theorem 1 as:

Theorem 4 *$\mathcal{F} = \mathcal{C}(p_{//\theta}(\mathcal{L}))$, where $\mathcal{L}$ is the free space of the half-disk robot moving amidst the footholds considered as point obstacles (that is $s_1, \ldots, s_n$).*

Definition 5 *$\forall s_i \in \mathcal{S}$ $(1 \le i \le n)$ let us define:*

$$\mathcal{H}_i = \{(P,\theta) \in \mathbb{R}^2 \times S^1 \ / \ P \in HD(s_i,\theta)\}.$$

$$\mathcal{C}_i = C_i \times S^1$$

$\mathcal{H}_i$ will be called the helicoidal volume centered at s_i (see Figure 5). $\mathcal{C}_i$ is the cylinder in $\mathbb{R}^2 \times S^1$ whose basis is the circle C_i.

Notice the typographical difference between the circle C_i defined in $\mathbb{R}^2$ and $\mathcal{C}_i$ defined in $\mathbb{R}^2 \times S^1$

The boundary of $\mathcal{H}_i$ is composed of a helicoidal surface and a cylindrical surface contained in $\mathcal{C}_i$.

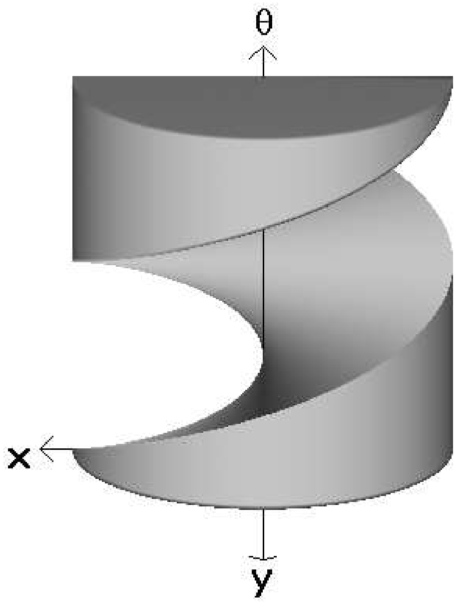

Figure 5: Helicoidal volume $\mathcal{H}_i$

Proposition 6 *The free space of the half-disk robot moving under translation and rotation amidst the obstacles $s_1, \ldots, s_n$ is the complementary in $I\!R^2 \times S^1$ of the union of the n helicoidal volumes centered at the s_i ($1 \leq i \leq n$):*

$$\mathcal{L} = \mathcal{C}\left(\bigcup_{1 \leq i \leq n} \mathcal{H}_i \right)$$

Proof: If Π_{θ_0} denotes the plane $\theta = \theta_0$ of $I\!R^2 \times S^1$, then:

$$\forall \theta \in S^1 \quad \mathcal{L} \cap \Pi_\theta = \mathcal{C}\left(\bigcup_{1 \leq i \leq n} HD(s_i, \theta) \right)$$

$\square$

Let $\mathcal{H}$ be $\bigcup_{1 \leq i \leq n} \mathcal{H}_i$. Theorem 4 and Proposition 6 give $\mathcal{F}$ in terms of $\mathcal{H}$:

Theorem 7 $\mathcal{F} = \mathcal{C}(p_{//\theta}(\mathcal{C}(\mathcal{H})))$ *where $\mathcal{C}$ denotes the complementary in $I\!R^2$ or $I\!R^2 \times S^1$.*

Remark 8 $\mathcal{C}(p_{//\theta}(\mathcal{C}(\mathcal{H})))$ *is the projection ($p_{//\theta}$) of the largest cylinder (whose axis is parallel to the θ-axis) included in $\mathcal{H}$ (see Figure 6). The basis of this cylinder is in fact $\mathcal{F}$.*

4 Computation of $\mathcal{F}$

We know that each arc of the boundary $\delta(\mathcal{F})$ of $\mathcal{F}$ is either a straight line segment belonging to a line joining two footholds or an arc of a circle C_i (of radius R centered at s_i). The circular arcs $\delta(\mathcal{F}) \cap C_i$ are computed first and linked together with the line segments in a second step.

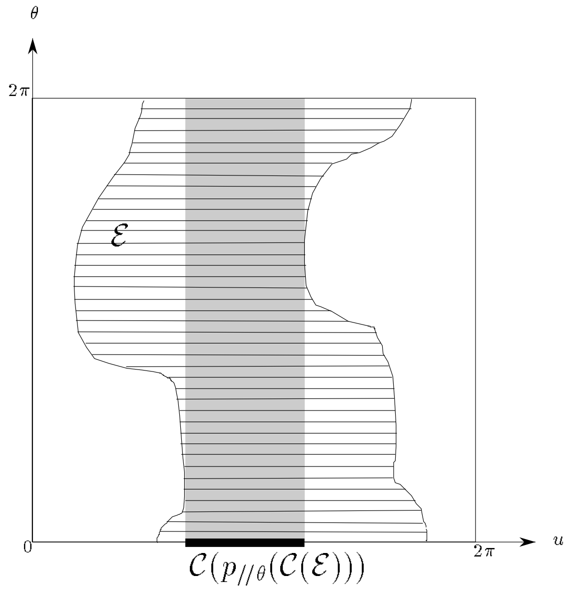

Figure 6: $\mathcal{C}(p_{//\theta}(\mathcal{C}(\mathcal{E})))$

4.1 Computation of $\delta(\mathcal{F}) \cap C_i$

We compute the contribution of each circle C_i to $\delta(\mathcal{F})$ in turn. Let $\mathcal{C}_{i_0}$ be the torus $C_{i_0} \times S^1$ for some $i_0 \in \{1, \ldots, n\}$. The natural parameters of $\mathcal{C}_{i_0}$ will be denoted u and θ. The portion $\mathcal{S}_{i_0}$ of the boundary of the helicoidal volume $\mathcal{H}_{i_0}$ which is included in $\mathcal{C}_{i_0}$ is shown in dark grey in Figure 7

Let $\mathcal{Z}_i$ denotes the intersection $\mathcal{H}_i \cap \mathcal{C}_{i_0}$. Figure 7 shows an example of such a region $\mathcal{Z}_i$ through the natural parametrization.

According to Theorem 7 and Remark 8, the largest vertical strip Σ_{i_0} included in $\cup_{i \neq i_0} \overline{\mathcal{Z}_i} \cup \mathcal{S}_{i_0}$ (*ie.* $\cup_i \overline{\mathcal{H}_i} \cap \mathcal{C}_{i_0}$) projects, along a direction parallel to the θ-axis, onto the portion of C_{i_0} which contributes to the closure of $\mathcal{F}$. The largest vertical strip Σ'_{i_0} included in $\cup_{i \neq i_0} \mathcal{Z}_i$ (*ie.* $\cup_i \mathcal{H}_i \cap \mathcal{C}_{i_0}$) projects, along a direction parallel to the θ-axis, onto the portion of C_{i_0} which contributes to the interior of $\mathcal{F}$. It follows that the contribution of $\mathcal{C}_{i_0}$ to $\delta(\mathcal{F})$ is the vertical projection onto $\mathcal{C}_{i_0}$ of the vertical strip $\Sigma_{i_0} \setminus \Sigma'_{i_0}$ (see Figure 9).

The main problem is to compute the union of the regions $\mathcal{Z}_i$ traced on $\mathcal{C}_{i_0}$ by the $\mathcal{H}_i$ in $O(k_{i_0} \log k_{i_0})$ time where k_{i_0} is the number of helicoidal volumes $\mathcal{H}_i$ intersecting $\mathcal{C}_{i_0}$. We distinguish three classes of regions $\mathcal{H}_i \cap \mathcal{C}_{i_0}$ and we will compute the union of the regions of each class separately. The different classes of regions depend on the distance between s_i and s_{i_0}. The k_{i_0} helicoidal volumes $\mathcal{H}_i$ that intersect $\mathcal{C}_{i_0}$ can be found in $O(k_{i_0})$ time once the Delaunay triangulation of the footholds has been computed [4].

Consider a region $\mathcal{Z}_i$, which is the intersection of $\mathcal{H}_i$ and $\mathcal{C}_{i_0}$ and which satisfies $\sqrt{2}\, R \leq |s_{i_0}, s_i| < 2R$

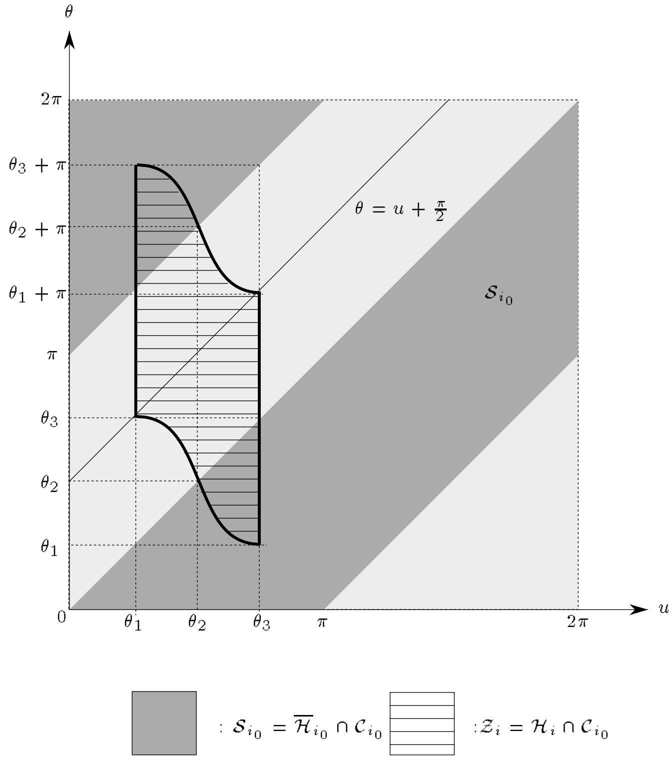

Figure 7: Intersection of $\mathcal{H}_i$ with $\mathcal{C}_{i_0}$ for $|s_{i_0}, s_i| = \sqrt{2}\,R$, $\theta_1 = \pi/4$, $\theta_2 = \pi/2$, $\theta_3 = 3\pi/4$.

(see Figure 8 and 7). The two curves corresponding to the upper and the lower part of the boundary of such a region are separated by the line $\theta = u + \frac{\pi}{2}$ (proof omitted). So the union of the regions $\mathcal{Z}_i$ can be obtained by computing separately the upper envelope of the curves above $\theta = u + \frac{\pi}{2}$ and the lower envelope of the curves below this line. Because two of these curves intersect at most once (proof omitted), the upper and lower envelopes can be computed in $O(k_{i_0} \log k_{i_0})$ time and $O(k_{i_0}\alpha(k_{i_0}))$ space where α is the pseudo inverse of the Ackerman's function [5]. The computation of the union of the regions $\mathcal{Z}_i$ for $R \leq |s_{i_0}, s_i| < \sqrt{2}\,R$ and for $0 < |s_{i_0}, s_i| < R$ (see Figure 9) can be done in a similar way (proof omitted). The overall union of the three partial unions of the three classes can be done in $O(k_{i_0}\alpha(k_{i_0}))$ time because each envelope is a graph of a function of u and any arc of the first envelope intersects any arc of the other at most once (proof omitted).

Since $\mathcal{A}$ is the arrangement of the circles of radius R centered at the footholds, $|\mathcal{A}| = \sum_{i=1}^{n} k_i$. The above considerations yield an algorithm for computing $\bigcup_{1 \leq i \leq n}(\delta(\mathcal{F}) \cap C_i)$ that is $\delta(\mathcal{F}) \cap \mathcal{A}$ in $O(|\mathcal{A}| \log n)$ time. Moreover, labelling an arc of $\delta(\mathcal{F})$ either by i if the arc belongs to the circle C_i or by (i, j) if the arc

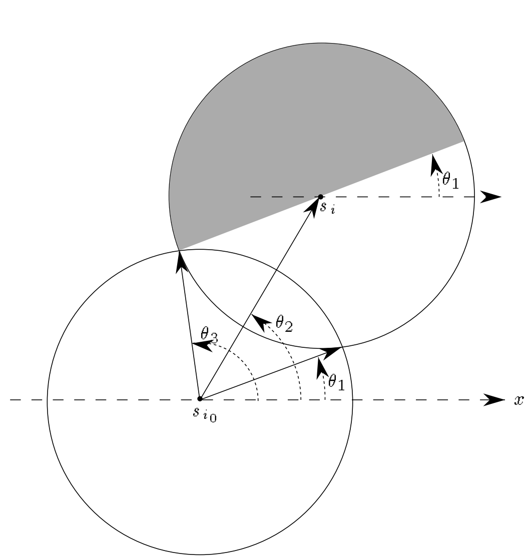

Figure 8: Definition of θ_1, θ_2 and θ_3 $(0 < |s_{i_0}, s_i| < 2R)$.

belongs to the straight line segment $[s_i, s_j]$, the labels of each arc of $\delta(\mathcal{F}) \cap \mathcal{A}$ can be found without increasing the complexity. An arc of $\delta(\mathcal{F}) \cap \mathcal{A}$ corresponds to a vertical strip $\Sigma_{i_0} \setminus \Sigma'_{i_0}$. It is limited either by a vertical line segment corresponding to some intersection $C_{i_0} \cap C_i$ or by an intersection between two arcs corresponding to some intersection $C_{i_0} \cap [s_i, s_j]$. Assuming general position, there is no triple intersection between the circles C_i and the straight line segments $[s_i, s_j]$, thus an end point of an arc of $\delta(\mathcal{F}) \cap \mathcal{A}$ is either an intersection $C_{i_0} \cap C_i$ or an intersection $C_{i_0} \cap [s_i, s_j]$. Hence the labels of the edges of $\delta(\mathcal{F})$ adjacent to the arcs of $\delta(\mathcal{F}) \cap \mathcal{A}$ can be found with no overcost during the construction. These considerations yield the following theorem:

Theorem 9 *We can compute $\delta(\mathcal{F}) \cap \mathcal{A}$ and the labels of the edges of $\delta(\mathcal{F})$ adjacent to the arcs of $\delta(\mathcal{F}) \cap \mathcal{A}$ in $O(|\mathcal{A}| \log n)$ time and $O(|\mathcal{A}|\alpha(n))$ space.*

4.2 Computation of the arcs of $\delta(\mathcal{F})$ issued from a foothold

The above section has shown how to compute all the vertices of $\mathcal{F}$ wich are incident to at least one arc of circle. It remains to find the vertices of $\mathcal{F}$ incident to two straight edges. Such vertices are known to be footholds of $\mathcal{S}$ (see Section 2), thus we are looking for all the footholds in $\delta(\mathcal{F})$ and the labels of arcs of $\delta(\mathcal{F})$ issued from each such foothold. Consider a foothold s_{i_0}, and let CH be the interior of the convex hull of the footholds contained in the disk $D(s_{i_0}, R)$. We assume CH to be non empty, otherwise $s_{i_0} \notin \delta(\mathcal{F})$.

- If s_{i_0} belongs to CH, then s_{i_0}belongs to the interior of $\mathcal{F}$ because $s_{i_0} \in \mathcal{F}$ and $\mathcal{F}$ is an open set. Hence in this case $s_{i_0} \notin \delta(\mathcal{F})$.

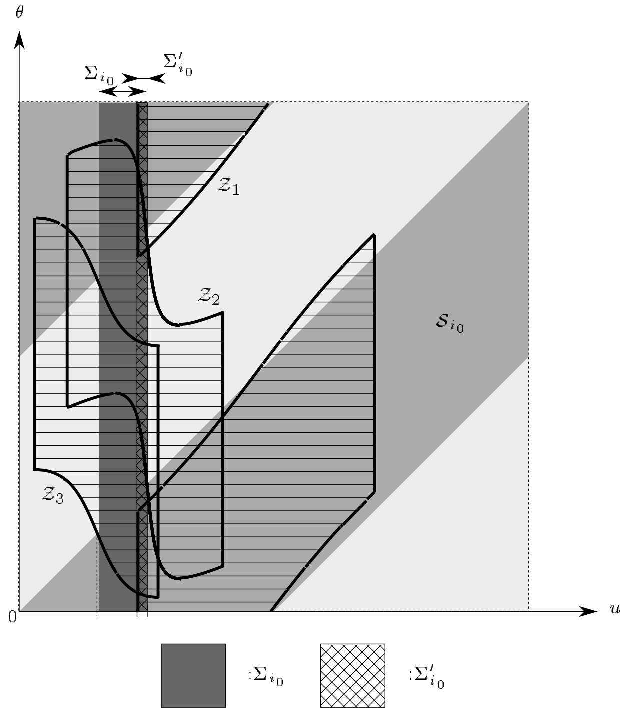

Figure 9: Contribution of C_{i_0} to $\delta(\mathcal{F})$ ($0 < |s_1, s_{i_0}| < R$, $R \leq |s_2, s_{i_0}| < \sqrt{2}\,R$, $\sqrt{2}\,R \leq |s_3, s_{i_0}| < 2R$).

- If s_{i_0} is a vertex of CH, then in a neighborhood of s_{i_0}, CH and $\mathcal{F}$ are identical: Suppose the spider robot put its legs on the vertices of CH then there exists a neighborhood V of s_{i_0}such that any placement of the spider robot in $V \cap$CH is admissible and any placement in $V \cap \delta(CH)$ correspond to an unstable equilibrium of the spider robot.

Let k_{i_0} be the number of footholds contained in the disk $D(s_{i_0}, R)$. These footholds can be found in $O(k_{i_0})$ time once the Delaunay triangulation of the footholds has been computed [4]. We can decide if s_{i_0} belongs to the interior of CH, or else, find the two vertices of CH adjacent to s_{i_0} in $O(k_{i_0})$ time and space. As the sum of the k_i for $\{1, \ldots, n\}$ is the size of $\mathcal{A}$, we have the following theorem:

Theorem 10 *The footholds of $\delta(\mathcal{F})$ ($\mathcal{S} \cap \delta(\mathcal{F})$) and the labels of the arcs of $\delta(\mathcal{F})$ issued from these footholds can be found in $O(|\mathcal{A}|)$ time and space.*

4.3 Conclusion

Theorem 11 *The free space of the spider robot can be computed in $O(|\mathcal{A}| \log n)$ time and $O(|\mathcal{A}|\alpha(n))$ space.*

Proof: omitted. □

5 Generalization to polygonal foothold regions

Now we consider the case where the set of footholds is no longer a set of points but a set $\mathcal{S}$ of polygonal regions. Clearly $\mathcal{S}$ is contained in the free space. Let $\mathcal{F}$ and $\mathcal{F}_s$ denote the free space of the spider robot using as foothold regions the polygonal foothold regions $\mathcal{S}$ and the boundary of $\mathcal{S}$ respectively. Suppose the spider robot admits an admissible placement outside $\mathcal{S}$ with its legs inside some polygonal footholds ; then the placement remains admissible if it retracts its legs on the boundary of these polygonal regions. Hence $\mathcal{F} = \mathcal{F}_s \cup \mathcal{S}$. We first show how to compute $\mathcal{F}_s$.

All we have done in Section 3 remains valid if the foothold regions are line segments provided that C_i $\mathcal{H}_i$ and $\mathcal{A}$ are replaced by $\mathcal{E}C_i$, $\mathcal{EH}_i$ and $\mathcal{EA}$. If the environment consists of a set of n segments e_i (the edges of the polygons) then we define the generalized circle $\mathcal{E}C_i$ and the generalized helicoidal volume $\mathcal{EH}_i$ as (see Figures 10 and 11):

$$\mathcal{E}C_i = \{P \in \mathbb{R}^2 \;/\; |P, s| = R,\ s \in e_i\}$$

$$\mathcal{EH}_i = \{(P, \theta) \in \mathbb{R}^2 \times S^1 \;/\; P \in HD(s, \theta),\ s \in e_i\}.$$

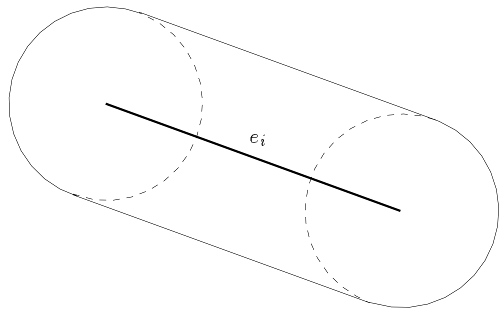

Figure 10: $\mathcal{E}C_i$

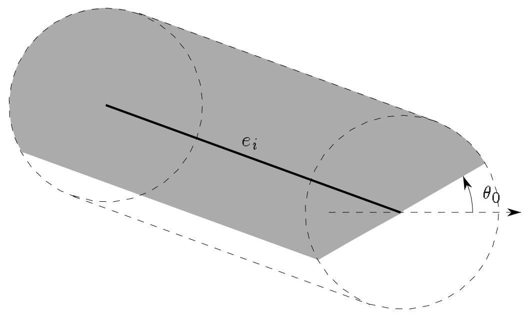

Figure 11: $\mathcal{EH}_i \cap P_{\theta_0}$

$\mathcal{EA}$ is the arrangement of the n generalized circles $\mathcal{E}C_i$. Notice that $|\mathcal{EA}| = \Theta(n^2)$.

An edge of $\mathcal{F}_s$ corresponding to an accessibility limit of the spider robot is an arc of an generalized circle $\mathcal{E}C_i$ $(1 \leq i \leq n)$. By arguments similar to those used in the proof of Theorem 9, we obtain :

Theorem 12 *We can compute $\delta(\mathcal{F}_s) \cap \mathcal{EA}$ and the labels of the edges of $\delta(\mathcal{F}_s)$ adjacent to the arcs of $\delta(\mathcal{F}_s) \cap \mathcal{EA}$ in $O(|\mathcal{EA}| \log n)$ time and $O(|\mathcal{EA}|\alpha(n))$ space.*

The arcs of $\delta(\mathcal{F}_s)$ corresponding to an unstable equilibrium of the spider robot are the 2-contact curves drawn by the midpoint of a ladder, of length $2R$, moving amidst the foothold regions, considered as obstacles. In the case where the footholds are points, the 2-contact curves are straight line segments. When the foothold regions are straight line segments the 2-contact curves of the ladder are either straight line segments, arcs of ellipses or arcs of conchoid (see Figure 12). We can compute the 2-contact arcs of a ladder moving under translation and rotation amidst straight line segment obstacles in $O(|\mathcal{EA}| \log n)$ time and $O(|\mathcal{EA}|)$ space (see [11]) and we can determine in constant time, for each arc, the portion that actually corresponds to a stability limit of the spider robot. Thus we can compute the boundary of the free space of the spider robot in the case where the foothold regions are n line segments in $O(|\mathcal{EA}| \log n)$ time and $O(|\mathcal{EA}|\alpha(n))$ space.

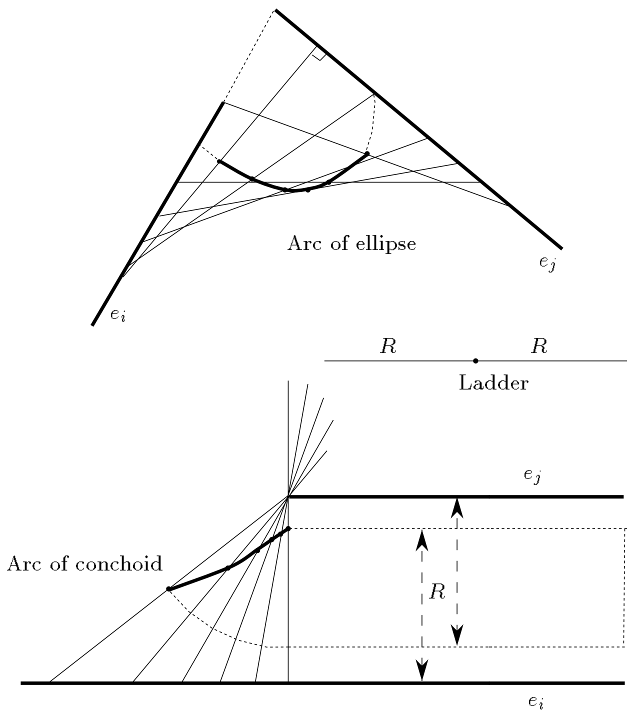

Figure 12: Arc of ellipse and conchoid as limit of stability

As we have seen above, adding the interior of the polygons to $\mathcal{F}_s$ gives the whole free space $\mathcal{F}$ of the spider robot moving on these polygons. This does not increase the geometric complexity of the free space nor the complexity of the computation. Thus we have the following theorem:

Theorem 13 *Given a set of polygonal foothold regions with n edges in total, we can compute the free space $\mathcal{F}$ of the spider robot in $O(|\mathcal{EA}| \log n)$ time and $O(|\mathcal{EA}|\alpha(n))$ space, where α is the pseudo inverse of the Ackerman's function.*

This result is almost optimal since, as shown in [2], $\Omega(|\mathcal{EA}|)$ is a lower bound for the size of $\mathcal{F}$.

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