Convexifying Star-Shaped Polygons*

H. Everett[1] S. Lazard[2] S. Robbins[2] H. Schröder[3] S. Whitesides[2]

Extended Abstract

1 Introduction

The reconfiguration problem for chains is to determine whether a chain of n links can be moved from one given configuration to another. The links have fixed lengths and may rotate about their endpoints. Previous work on the reconfiguration of chains (e.g. [1]) has allowed links to pass over one another, so that the links act as distance constraints but not as obstacles.

We study a variant of this problem, which we call *polygon convexification*: the initial configuration of the chain forms a simple polygon, the final configuration is a convex polygon, and the links are *not* allowed to cross. It is unknown whether every polygon can be convexified.

A polygon is *star-shaped* if it contains a point o, possibly lying on the boundary, that can see all the other points in the polygon; more precisely, for any point p in the polygon including its boundary, the line segment $[o,p]$ does not intersect the exterior of the polygon. A polygon is in *general position* if no three of its vertices are collinear. Our main result is that every star-shaped polygon in general position can be convexified.

2 The Algorithm

Let $\mathcal{P}$ be a star-shaped simple polygon whose n vertices are in general position. A *wedge* is a closed region delimited by two rays that share their endpoint. A wedge is called *small* if its wedge angle is smaller than π; it is called *big* otherwise. We present a recursive algorithm that straightens, one by one, the vertices of $\mathcal{P}$ until $\mathcal{P}$ is a convex polygon or a pentagon. We show how to convexify pentagons independently.

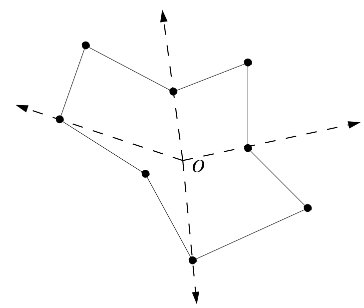

Figure 1: Partition into wedges for n even.

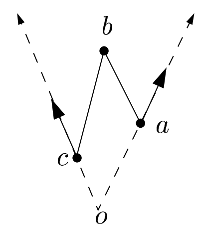

Figure 2: A radial expansion applied to a two-link chain.

Suppose for the moment that n is even and $n > 5$. The first step is to partition the plane into wedges by first finding a point o in the kernel of $\mathcal{P}$ and then constructing rays from o through every second vertex of $\mathcal{P}$. See Figure 1. The result is a set of wedges, each of which contains a two-link chain (a polygonal chain of 3 vertices and 2 edges) whose extremities lie on the rays forming the wedge boundary and whose intermediate vertex lies inside the wedge. Now we apply a "radial expansion" motion (see Figure 2) in which the ex-

---

*This research was conducted at the International Workshop on Wrapping and Folding. We would like to acknowledge the other participants, T. Biedl, E. Demaine, M. Demaine, A. Lubiw, J. O'Rourke, M. Overmars, I. Streinu, and G. Toussaint for useful discussions as well as NSERC and FCAR for financial assistance.

[1]Université du Québec à Montréal, Montréal, Canada.
[2]McGill University, Montréal, Canada.
[3]Loughborough University, UK.

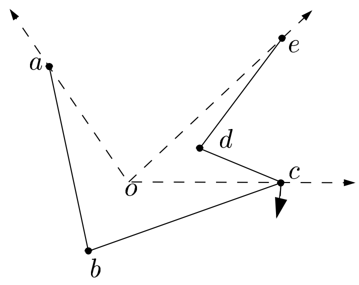

Figure 3: Closing a big wedge. Vertex c is rotated clockwise about o.

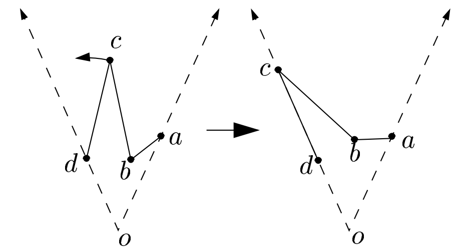

Figure 4: Reconfiguring a three-link chain. Vertex c is rotated counterclockwise around d.

tremities of each two-link chain move along their respective rays at equal constant speed away from o until some intermediate vertex straightens. The straight vertex is kept straight for the remainder of the algorithm, effectively reducing the number of links by one, and the algorithm is called recursively.

We show that during a radial expansion the polygon remains star-shaped. To show this we prove that at no time during a radial expansion does an intermediate vertex leave its wedge. In fact, this will only be true when the wedge angle is less than π so before applying the radial expansion we first close the big wedge, if there is one (since $n > 5$ there exists at most one). To close the big wedge (see Figure 3) we rotate around o, toward the interior of the wedge, an endpoint of the two-link chain contained in that wedge; all the other vertices of $\mathcal{P}$ that lie on the rays emanating from o do not move. (The only vertices that move are the vertex we rotate and its two adjacent intermediate vertices.) The motion stops when an intermediate vertex straightens or when the big wedge angle becomes smaller than π by some small ε. We show that during this motion the polygon remains star-shaped by proving that the two moving intermediate vertices remain inside their respective wedges.

We need to do something slightly different in the case that n is odd and $n > 5$. Firstly, we partition the plane into wedges as before, but this time we have one wedge containing a three-link chain. The partition can be done such that the three-link chain is contained in a small wedge. The strategy is to transform this chain, while leaving the others unchanged, so that either the middle edge is aligned with o or one of the other two edges lies on a wedge boundary. The transformation is accomplished by rotating one of the two intermediate vertices around the extreme vertex to which it is adjacent. Figure 4 shows an example. In the case that the middle edge is aligned with o we repartition the polygon so that this edge lies on a wedge boundary. Thus, after reconfiguring the three-link chain, each wedge contains a two-link chain and exactly one edge is on a wedge boundary. Then, during the radial expansion, this edge stays on the wedge boundary and all other wedges are treated as before.

We have shown that, using the motion described above we can reconfigure a star-shaped polygon that has $n > 5$ vertices into a convex polygon (in which case we stop) or into a pentagon. The case $n = 5$ differs from the case n odd and greater than 5 because when $n = 5$ all the three-link chains may be contained in big wedges. In order to circumvent this difficulty we choose the point o in the kernel of $\mathcal{P}$ to be a vertex of $\mathcal{P}$. Using the transformation described above we close one big wedge until (one vertex straightens or) both wedges are flat. Then, a simple motion can reconfigure the pentagon into a quadrilateral which is easy to convexify.

For the time complexity note that since $O(n)$ radial expansions are needed, each of which takes $O(n)$ time to compute, the entire motion description can be computed in $O(n^2)$ time.

References

[1] W. J. Lenhart and S. H. Whitesides, “Reconfiguring Closed Polygonal Chains in Euclidean d-space,” *Discrete and Computational Geometry* **13** (1995), 123 - 140.