Convex hulls of bounded curvature

Jean-Daniel Boissonnat, Sylvain Lazard

Convex hulls of bounded curvature

Jean-Daniel Boissonnat⋆ Sylvain Lazard⋆

Abstract

In this paper, we consider the problem of computing a convex hull of bounded curvature of a set $\mathcal{S}$ of points in the plane, *i.e.* a set containing $\mathcal{S}$ and whose boundary is a curve of bounded curvature of minimal length. We prove that, if the radius of the smallest disk that contains $\mathcal{S}$ is greater than 1, such a hull is unique. We show that the computation of a convex hull of bounded curvature reduces to convex programming or to solving a set of algebraic systems.

1 Introduction

The convex hull of a set of points in the plane is defined as the smallest set, or equivalently, the set of smallest perimeter that contains all the points. We consider in this paper convex hulls of bounded curvature. A curve is said of bounded curvature if it is C^1 and if its curvature is upper bounded by 1 everywhere it is defined. We define a convex hull of bounded curvature of a set $\mathcal{S}$ of points in the plane as a set containing $\mathcal{S}$ and whose boundary is a curve of bounded curvature of minimal length.

Convex sets of bounded curvature have been considered in the context of non-holonomic motion planning [ART95, BL96] but we are not aware of any previous work devoted to the construction of such hulls.

In the sequel, the boundary of a region $\mathcal{R}$ will be denoted by $\partial\mathcal{R}$. A polygon whose vertices are $M_1, \ldots, M_n$ such that $M_1, \ldots, M_n$ appear in this order on the boundary of the polygon will simply be called polygon $M_1 \ldots M_n$. When necessary, the suffix i of a vertex M_i of a polygon $M_1 \ldots M_n$ will be considered modulo n. Two polygons are said to be *geometrically equal* if they define the same region; notice that two polygons that are geometrically equal have the same non-flat vertices but may have different flat vertices.

In the sequel, $\mathcal{S}$ denotes a set of points in the plane, $\mathcal{T}$ a convex hull of bounded curvature of $\mathcal{S}$. $\mathcal{P}$ is the usual convex hull of $\mathcal{S}$ and $P_1, \ldots, P_n$ denote the vertices of $\mathcal{P}$. D_i denotes the closed disk of unit radius centered at P_i, for any $i \in \{1, \ldots, n\}$. $\mathcal{Q}$ is a polygonal region whose perimeter is minimal and that intersects all the disks $D_1, \ldots, D_n$ (observe that it is not required that the boundary of $\mathcal{Q}$ intersects all the disks $D_1, \ldots, D_n$). As we will see, $\mathcal{Q}$ plays a central role in the characterization and in the computation of $\mathcal{T}$.

⋆ INRIA, BP 93
06902 Sophia-Antipolis Cedex, France
E-mail : `boissonn, lazard@sophia.inria.fr`

If the radius r of the smallest disk that contains $\mathcal{S}$ is smaller or equal to 1, any disk of unit radius containing $\mathcal{S}$ is plainly a convex hull of bounded curvature of $\mathcal{S}$. Notice that if $r < 1$ there exists an infinite number of such disks, and, if $r = 1$ such a disk is unique. We assume in the sequel that $r > 1$.

2 Properties of $\mathcal{T}$

Lemma 1 *$\mathcal{T}$ is convex and contains $\mathcal{P}$.*

Proof: $\mathcal{T}$ is convex because otherwise its convex hull has shorter perimeter and its boundary is a curve of bounded curvature. Thus, $\mathcal{T}$ contains $\mathcal{P}$ because $\mathcal{P}$ is the smallest convex that contains $\mathcal{S}$. □

We easily deduce from this lemma that the convex hull of bounded curvature of $\mathcal{S}$ is equal to the convex hull of bounded curvature of the vertices of the convex hull of $\mathcal{S}$.

Lemma 2 *$\partial\mathcal{T}$ consists of line segments and arcs of unit circles passing through the vertices of $\mathcal{P}$.*

Proof: Since the radius of the smallest disk that contains $\mathcal{S}$ is strictly greater than 1, $\partial\mathcal{T}$ is not reduced to a unit circle and passes through some points of $\mathcal{S}$, which, by the remark above, are vertices of $\mathcal{P}$. Let $\mathcal{S}'$ be the set of vertices of $\mathcal{P}$. Clearly, any arc of $\partial\mathcal{T}$ joining two oriented points (A, α) and (B, β) in $\mathbb{R}^2 \setminus \mathcal{S}'$ is a locally shortest path of bounded curvature[1] joining these two oriented points. Then, according to [BCL94] and [PBGM62, Theorem.25], any arc of $\partial\mathcal{T}$ in $\mathbb{R}^2 \setminus \mathcal{S}'$ is a curve C^1 of one of the two types CSC or CCC where C denotes a unit circular arc and S a line segment. The paths of type CCC cannot appear in $\partial\mathcal{T}$ because $\partial\mathcal{T}$ is convex. Thus, any circular arc that appears in $\partial\mathcal{T}$ is followed and preceded by a line segment and must pass through a vertex of $\mathcal{P}$.

□

Notice that not all the vertices of $\mathcal{P}$ necessarily belong to $\partial\mathcal{T}$: Figure 1 shows $\partial\mathcal{T}$ when $\mathcal{P}$ is a square; when we add to $\mathcal{P}$ a fifth vertex A that belongs to $\mathcal{T}$, $\mathcal{T}$ is still the convex hull of bounded curvature of these five vertices yet not all five vertices belong to $\partial\mathcal{T}$.

Now, we transform the problem of computing $\mathcal{T}$ into a more standard problem in Euclidean geometry (see Figure 1) :

[1] A curve $\mathcal{C}$ is a locally shortest path of bounded curvature joining (A, α) and (B, β) if and only if any curve of bounded curvature joining (A, α) and (B, β) and contained in a sufficiently small neighborhood of $\mathcal{C}$ is longer than $\mathcal{C}$.

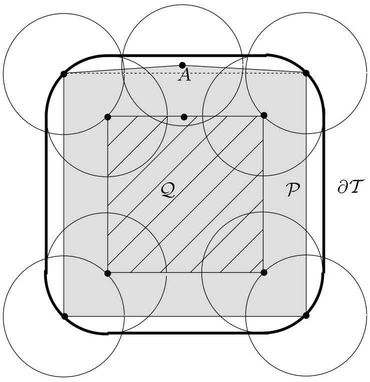

Figure 1: Example where not all the vertices of $\mathcal{P}$ belong to $\partial\mathcal{T}$

Proposition 3 *$\mathcal{T}$ is the Minkowski sum of the disk of unit radius centered at the origin and of a polygonal region $\mathcal{Q}^*$ which is, among the regions that intersect all the disks $D_1, \ldots, D_n$, one whose perimeter is minimal.*

Proof: First, notice that, since $\mathcal{T}$ is convex, the sum of the lengths of the circular arcs of $\partial\mathcal{T}$ is equal to 2π. Hence, the perimeter of $\mathcal{T}$ is equal to 2π plus the sum of the lengths of the line segments of $\partial\mathcal{T}$.

We recall that the eroded region of $\mathcal{T}$ by the unit disk D centered at the origin is $(\mathcal{T}^c \oplus D)^c$ where "$.^c$" denotes the complementation and $\oplus$ the Minkowski sum. In other words, the eroded region of $\mathcal{T}$ by D is $\mathcal{T} \setminus \cup_{P \in \partial\mathcal{T}} D(P)$ where $D(P)$ is the translated of D centered at P. Let $\mathcal{Q}^*$ denote the eroded region of $\mathcal{T}$ by D.

As $\partial\mathcal{T}$ is convex and of bounded curvature, $\mathcal{Q}^*$ is convex, non empty and the Minkowski sum of $\mathcal{Q}^*$ and D is equal to $\mathcal{T}$. Moreover, as $\mathcal{T}$ contains $\mathcal{P}$, $\mathcal{Q}^*$ intersects all the disks $D_1, \ldots, D_n$. The perimeter of $\mathcal{T}$ is equal to 2π plus the perimeter of $\mathcal{Q}^*$. Thus, $\mathcal{Q}^*$ is, among the regions that intersect all the disks $D_1, \ldots, D_n$, one whose perimeter is minimal. ☐

Let $\mathcal{Q}$ denote a polygonal region of minimal perimeter that intersects all the disks $D_1, \ldots, D_n$. We will prove some properties of $\mathcal{Q}$ in Section 3 and show in Section 4 that $\mathcal{Q}$ is unique and therefore equal to $\mathcal{Q}^*$.

3 Properties of $\mathcal{Q}$

Lemma 4 *$\mathcal{Q}$ is convex.*

Proof: $\mathcal{Q}$ is convex because, otherwise, its convex hull has a perimeter strictly smaller than the one of $\mathcal{Q}$ and its convex hull still intersects all the disks $D_1, \ldots, D_n$ and still has a bounded curvature; that contradicts the definition of $\mathcal{Q}$. ☐

Lemma 5 *$\mathcal{Q} \subseteq \mathcal{P}$.*

Proof: Assume for a contradiction that $\mathcal{Q} \not\subseteq \mathcal{P}$. The idea of the proof is to project the part of $\mathcal{Q}$ outside $\mathcal{P}$ onto $\mathcal{P}$. Notice that we cannot simply replace the part of $\mathcal{Q}$ that is outside $\mathcal{P}$ by an arc of $\partial\mathcal{P}$ because the resulting polygon may possibly not intersect all the disks $D_1, \ldots, D_n$ (see Figure 2a).

Precisely, each point of $\mathcal{Q}$ outside $\mathcal{P}$ is projected onto the closest point of $\partial\mathcal{P}$ (see Figure 2b). That transformation shortens $\partial\mathcal{Q}$. Moreover, each point of $\mathcal{Q}$ that belongs to a disk D_i is projected onto a point that belongs to the same disk because $\mathcal{P}$ is convex. Thus, the transformed polygon still intersects all the disks $D_1, \ldots, D_n$. As the perimeter of $\mathcal{Q}$ is minimal, we have a contradiction. ☐

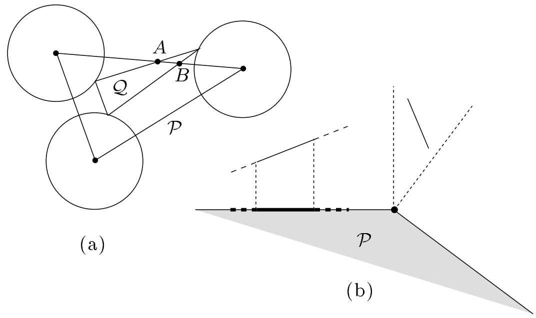

Figure 2: For the proof of Lemma 5

Lemma 6 *$\partial\mathcal{Q}$ intersects all the disks $D_1, \ldots, D_n$.*

Proof: Assume for a contradiction that $\partial\mathcal{Q}$ does not intersect a disk D_{i_0}. As, by hypothesis, $\mathcal{Q} \cap D_{i_0}$ is not empty, P_{i_0} belongs to the interior of $\mathcal{Q}$. Thus, by Lemma 5, P_{i_0} belongs to the interior of $\mathcal{P}$ which contradicts the hypothesis that P_{i_0} is a vertex of $\mathcal{P}$. ☐

Proposition 7 *There exists $M_1^* \in D_1, \ldots, M_n^* \in D_n$ such that the polygon $M_1^* \ldots M_n^*$ is geometrically equal to the polygon $\mathcal{Q}$.*

Proof: By Lemma 6, there exists $M_i \in D_i \cap \partial\mathcal{Q}$, $\forall i \in \{1, \ldots, n\}$.

If the points $M_1, \ldots, M_n$ appear in this order on $\partial\mathcal{Q}$, we take $M_i^* = M_i$ ($\forall i \in \{1, \ldots, n\}$). The polygon $M_1^* \ldots M_n^*$ is geometrically equal to $\mathcal{Q}$. Indeed, as $\mathcal{Q}$ is convex and $M_i^* \in \partial\mathcal{Q}$, the polygon $M_1^* \ldots M_n^*$ is convex and included in $\mathcal{Q}$. Thus, the perimeter of the polygon $M_1^* \ldots M_n^*$ is not greater than the perimeter of $\mathcal{Q}$, and it intersects all the disks $D_1, \ldots, D_n$ (because $M_i^* \in D_i$). As, by definition, $\mathcal{Q}$ is a polygon intersecting all the disks $D_1, \ldots, D_n$ whose perimeter is minimal, $\mathcal{Q}$ is geometrically equal to the polygon $M_1^* \ldots M_n^*$.

If the points $M_1, \ldots, M_n$ do not appear in this order on $\partial\mathcal{Q}$, let M_i' be the intersection point between $\partial\mathcal{Q}$ and the line segment P_iM_i which is the closest from P_i (see Figure 3). By construction, $M_i' \in D_i$ and the line segment P_iM_i' does not intersect the interior of $\mathcal{Q}$.

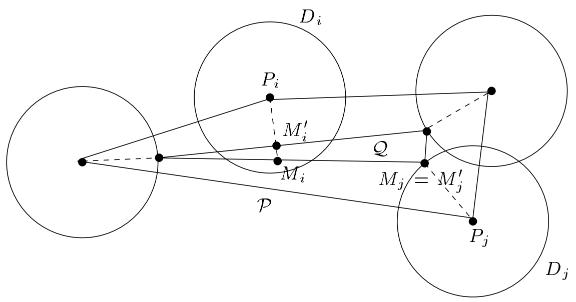

Figure 3: For the proof of Lemma 7

If the points $M'_1, \ldots, M'_n$ appear in this order on $\partial\mathcal{Q}$, by the same argument as above, the polygon $M'_1 \ldots M'_n$ is geometrically equal to $\mathcal{Q}$. Otherwise, there exist two consecutive points M'_i and M'_j on $\partial\mathcal{Q}$ such that the line segments $P_iM'_i$ and $P_jM'_j$ intersect[2], because the segments $P_1M'_1, \ldots, P_nM'_n$ belong to $\mathcal{P}$ ($\mathcal{P}$ is convex and contains $\mathcal{Q}$) and do not intersect the interior of $\mathcal{Q}$. The two segments $P_iM'_i$ and $P_jM'_j$ can intersect only if M'_i or M'_j belongs to the intersection between the two disks D_i and D_j (see Figure 4a). We assume, without loss of generality, that $M'_i \in D_i \cap D_j$. We then define $M''_j = M'_i$. The number of intersection points between the segments $P_1M'_1, \ldots, P_nM'_n$ decreases by 1 when we replace M'_j by $M''_j = M'_i$ (see Figure 4b) : actually, on one hand, the segments $P_iM'_i$ and $P_jM''_j$ do not intersect contrary to the segments $P_iM'_i$ and $P_jM'_j$; on the other hand, the "new" segment $P_jM''_j$ can only be intersected by a segment intersecting the "old" segment $P_jM'_j$ (because M'_i and M'_j are consecutive on $\partial\mathcal{Q}$). Furthermore, we claim that the line segment $P_jM''_j$ does not intersect the interior of $\mathcal{Q}$. Indeed (see Figure 4c), let $\mathcal{H}_i$ be the union of the two half-planes limited by the edges of $\mathcal{Q}$ incident to M'_i that do not contain $\mathcal{Q}$. Let $\mathcal{H}_i^c$ be the complementary of $\mathcal{H}_i$. By construction, $P_iM'_i$ does not intersect the interior of $\mathcal{Q}$, thus $P_iM'_i \subset \mathcal{H}_i$. If $P_jM''_j$ intersects the interior of $\mathcal{Q}$, $P_j \in \mathcal{H}_i^c$ and so $P_jM'_j \subset \mathcal{H}_i^c$. Then $P_iM'_i \cap P_jM'_j = \emptyset$, which contradicts our assumption and proves the claim.

Repeating this procedure for all the pairs of consecutive points on $\partial\mathcal{Q}$ such that the corresponding line segments intersect, we define a list of points $M^*_1, \ldots, M^*_n$ that belong to $\partial\mathcal{Q}$ such that the segments $P_1M^*_1, \ldots, P_nM^*_n$ do not pairwise intersect and do not intersect the interior of $\mathcal{Q}$.

The fact that the segments $P_1M^*_1, \ldots, P_nM^*_n$ do not pairwise intersect, are included in $\mathcal{P}$ and do not intersect the interior of $\mathcal{Q}$, implies that the points $M^*_1, \ldots, M^*_n$ appear in this order on $\partial\mathcal{Q}$. By the same argument as above, the polygon $M^*_1 \ldots M^*_n$ is geometrically equal to $\mathcal{Q}$. □

Remark 8 The points $M^*_1, \ldots, M^*_n$ may not be unique.

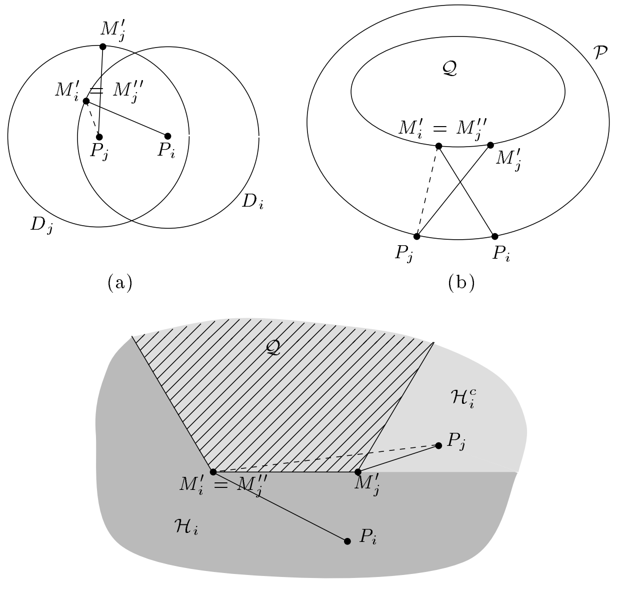

Figure 4: For the proof of Proposition 7

For example in Figure 1, the number of non-flat vertices of $\mathcal{Q}$ is smaller than the number of vertices of $\mathcal{P}$. In that case, one of the M^*_i is a flat vertex of $\mathcal{Q}$ and moving M^*_i inside D_i along $\partial\mathcal{Q}$ does not modify $\partial\mathcal{Q}$.

Proposition 9 *If $M^*_1 \in D_1, \ldots, M^*_n \in D_n$ are the vertices of a polygon geometrically equal to $\mathcal{Q}$, then, for any non-flat*[3] *vertex M^*_i,*

- *M^*_i belongs to the boundary of D_i,*
- *the segments $M^*_{i-1}M^*_i$ and $M^*_iM^*_{i+1}$ do not intersect the interior of D_i,*
- *the line $P_iM^*_i$ is the bisector of the two lines $M^*_{i-1}M^*_i$ and $M^*_iM^*_{i+1}$ that separates M^*_{i-1} and M^*_{i+1}.*

Proof: The first claim of the proposition is a direct consequence of the second one.

Let M^*_i be a non-flat vertex of $\mathcal{Q}$. As M^*_i is not flat, the line segment $M^*_{i-1}M^*_{i+1}$ does not intersect D_i. As the perimeter of $\mathcal{Q}$ is minimal, the polygonal line $M^*_{i-1}M^*_iM^*_{i+1}$ is, among the polygonal lines $M^*_{i-1}MM^*_{i+1}$ such that $M \in D_i$, one of smallest length. The set of points M such that the length of the polygonal line $M^*_{i-1}MM^*_{i+1}$ is equal to a given l is an ellipse whose focuses are M^*_{i-1} and M^*_{i+1}. It follows that M^*_i is the common point of D_i and the ellipse whose focuses are M^*_{i-1} and M^*_{i+1} that is tangent to D_i and does not enclose D_i (see Figure 5). This proves the second claim of the proposition.

[2]We say that two line segments intersect if their relative interior intersect.

[3]If $\mathcal{Q}$ is reduced to a point, $M^*_i = M^*_j$ for all (i,j) and we consider the vertices M^*_i as flat, by convention.

A well known property of the ellipses is that the normal line to an ellipse at a point M is the bisector of the two lines M_{i-1}^*M and MM_{i+1}^* that separates the focuses. In our case, the normal to the ellipse at the point M_i^* is also normal to the boundary of D_i and so passes through P_i. Therefore, $P_iM_i^*$ is the bisector of the two lines $M_{i-1}^*M_i^*$ and $M_i^*M_{i+1}^*$ that separates M_{i-1}^* and M_{i+1}^*. □

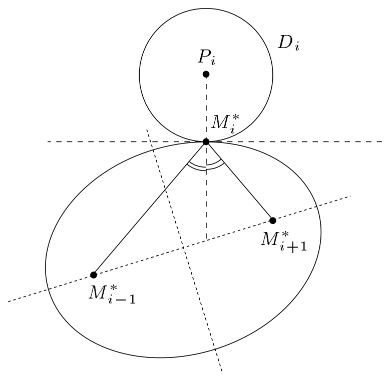

Figure 5: $P_iM_i^*$ is the bisector of the two lines $M_{i-1}^*M_i^*$ and $M_i^*M_{i+1}^*$ that separates M_{i-1}^* and M_{i+1}^*

4 Uniqueness of $\mathcal{Q}$ and of $\mathcal{T}$

We define the following function f :

$$\begin{array}{rlll} f & : & D_1 \times \ldots \times D_n & \to & I\!R \\ & & M_1, \ldots, M_n & \mapsto & \|M_1M_2\| + \|M_2M_3\| + \ldots \\ & & & & +\|M_{n-1}M_n\| + \|M_nM_1\| \end{array}$$

where $\|M_iM_{i+1}\|$ denotes the Euclidean distance between the points M_i and M_{i+1}.

Proposition 10 *f is a convex function.*

Proof: A simple computation, omitted here, yields the proposition. □

Proposition 11 *$f(M_1^*, \ldots, M_n^*)$ is the minimum of f if and only if the polygon $M_1^* \ldots M_n^*$ is, among the regions that intersect all the disks $D_1, \ldots, D_n$, one of minimal perimeter.*

Proof: Let $M_1^* \ldots M_n^*$ be such that $f(M_1^*, \ldots, M_n^*)$ is minimum. Let $\mathcal{Q}$ be a polygon of minimal perimeter that intersects all the disks $D_1, \ldots, D_n$. Clearly, the perimeter of $\mathcal{Q}$ is smaller or equal to the perimeter of the polygon $M_1^* \ldots M_n^*$. By Proposition 7, $\mathcal{Q}$ is geometrically equal to a polygon $M_1^\star \ldots M_n^\star$ where $M_i^\star \in D_i$. As $f(M_1^\star, \ldots, M_n^\star)$ cannot be smaller than the minimum of f, $f(M_1^\star, \ldots, M_n^\star)$, which is equal to the perimeter of $\mathcal{Q}$, is also equal to $f(M_1^*, \ldots, M_n^*)$, which is equal to the perimeter of the polygon $M_1^* \ldots M_n^*$.

Conversely, if $f(M_1, \ldots, M_n)$ is not the minimum of f, the perimeter of the polygon $M_1 \ldots M_n$ is not minimum among the regions that intersect all the disks $D_1, \ldots, D_n$. □

Lemma 12 *Let θ_i be the angle $\angle(\overrightarrow{P_iP_{i-1}}\overrightarrow{P_iM_i})$ where M_i is a point of the boundary of D_i and let $U_i \subset [0, 2\pi]$ be the set of the θ_i such that $M_i \in \mathcal{P}$ $(1 \le i \le n)$. Let g be*

$$\begin{array}{rlll} g & : & U_1 \times \ldots \times U_n & \to & I\!R \\ & & \theta_1, \ldots, \theta_n & \mapsto & \|M_1M_2\| + \|M_2M_3\| + \ldots \\ & & & & +\|M_{n-1}M_n\| + \|M_nM_1\| \end{array}$$

and let $\mathcal{D} \subset [0, 2\pi]^n$ be the open set of the $\Theta = (\theta_1, \ldots, \theta_n) \in U_1 \times \ldots \times U_n$ such that the polygon $M_1 \ldots M_n$ does not intersect the interior of the disks $D_1, \ldots, D_n$.

Then, g is locally strictly convex on $\mathcal{D}$, i.e. for any $\Theta \in \mathcal{D}$, there exists an open neighborhood of Θ such that the restriction of g on this neighborhood is a strictly convex function.

Remark 13 g is not convex on $\mathcal{D}$ because it can be shown that $\mathcal{D}$ is not a convex set. Notice that $\Theta \in \mathcal{D}$ if and only if the polygon $M_1 \ldots M_n$ does not intersect the interior of the disks $D_1, \ldots, D_n$ and if the edge M_iM_{i+1} is neither tangent to D_i nor to D_{i+1}, $\forall i \in \{1, \ldots, n\}$.

Proof: We consider the function

$$\begin{array}{rlll} \tilde{g}_i & : & U_i \times U_{i+1} & \to & I\!R \\ & & \theta_i, \theta_{i+1} & \mapsto & \|M_iM_{i+1}\|. \end{array}$$

We show by computing the Hessian matrix of $\tilde{g}_i$ that $\tilde{g}_i$ is locally strictly convex at any point (θ_i, θ_{i+1}) such that the relative interior of the line segment M_iM_{i+1} does not intersect (and is not tangent to) D_i and D_{i+1}. It follows that $\tilde{g}_i$ is locally strictly convex on the projection $\mathcal{D}_i$ of $\mathcal{D}$ onto $U_i \times U_{i+1}$. We omit here these computations. Let g_i be the function

$$\begin{array}{rlll} g_i & : & U_1 \times \ldots \times U_n & \to & I\!R \\ & & \theta_1, \ldots, \theta_n & \mapsto & \|M_iM_{i+1}\|. \end{array}$$

Since $g = \sum_{1 \le i \le n} g_i$, the fact that $\tilde{g}_i$ is locally strictly convex on $\mathcal{D}_i$ implies that g is locally strictly convex on $\mathcal{D}$. □

Proposition 14 *$\mathcal{Q}$ is unique.*

Proof: Let $\mathcal{Q}$ be a polygon of minimal perimeter that intersects all the disks $D_1, \ldots, D_n$. Since the radius of the smallest disk that contains $\mathcal{S}$ is strictly greater than 1, $\cap_{1 \le i \le n} D_i = \emptyset$; therefore, $\mathcal{Q}$ is not reduced to a point. By Proposition 7, $\mathcal{Q}$ is geometrically equal to a polygon $M_1^* \ldots M_n^*$ such that $M_i^* \in D_i$, $\forall i \in \{1, \ldots, n\}$.

Let $M_{i_1}^*, \ldots, M_{i_q}^*$ be the non-flat vertices of $\mathcal{Q}$ and let $\mathcal{U}$ be the set of all the polygons that intersect the q disks $D_{i_1}, \ldots, D_{i_q}$.

The proof consists of three steps : first, we show that the polygon $M_{i_1}^* \ldots M_{i_q}^*$ is, among the polygons of $\mathcal{U}$, one of minimal perimeter. Secondly, we show that there exists a unique polygon of $\mathcal{U}$ of minimal perimeter. In a third step, we consider the n disks $D_1, \ldots, D_n$.

1) In order to show that the polygon $M^*_{i_1} \ldots M^*_{i_q}$ is a polygon of $\mathcal{U}$ of minimal perimeter, we first show that any perturbation of a vertex $M^*_{i_j}$ that keeps $M^*_{i_j}$ inside D_{i_j} strictly increases the perimeter of the polygon.

Let $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) be the ellipse whose focuses are $M^*_{i_j-1}$ and $M^*_{i_j+1}$ (resp. $M^*_{i_{j-1}}$ and $M^*_{i_{j+1}}$) that contains $M^*_{i_j}$ (see Figure 6). As $\mathcal{Q}$ is a polygon of minimal perimeter, by the proof of Proposition 9, $\mathcal{E}_1$ is tangent to D_{i_j} at $M^*_{i_j}$ and lies outside D_{i_j}. Moreover, the line L normal to $\mathcal{E}_1$ at $M^*_{i_j}$ is the bisector of the two line segments $M^*_{i_j} M^*_{i_j-1}$ and $M^*_{i_j} M^*_{i_j+1}$ that separates $M^*_{i_j-1}$ and $M^*_{i_j+1}$. Since $M^*_{i_j-1}$ and $M^*_{i_j+1}$ are flat vertices, they belong to the line segments $M^*_{i_j} M^*_{i_{j-1}}$ and $M^*_{i_j} M^*_{i_{j+1}}$ respectively. It follows that the line normal to $\mathcal{E}_2$ at $M^*_{i_j}$ is L, which implies that $\mathcal{E}_2$ is tangent to $\mathcal{E}_1$ and to D_{i_j} at $M^*_{i_j}$. Furthermore, $\mathcal{E}_1$ is plainly inside $\mathcal{E}_2$, which implies that $\mathcal{E}_2$ lies outside D_{i_j}.

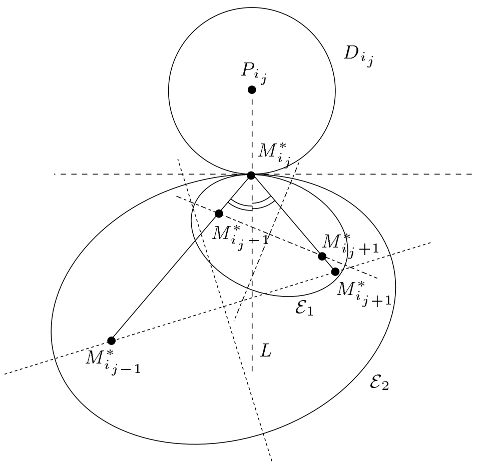

Figure 6: For the proof of Proposition 14

Hence, $\forall M \in D_{i_j}$ such that $M \neq M^*_{i_j}$, M belongs to the open region outside the ellipse $\mathcal{E}_2$ and therefore, $\|M^*_{i_{j-1}} M\| + \|M M^*_{i_{j+1}}\| > \|M^*_{i_{j-1}} M^*_{i_j}\| + \|M^*_{i_j} M^*_{i_{j+1}}\|$. Therefore, any perturbation of a vertex $M^*_{i_j}$ that keeps $M^*_{i_j}$ inside D_{i_j} strictly increases the perimeter of the polygon $M^*_{i_1} \ldots M^*_{i_q}$.

It follows that $(M^*_{i_1}, \ldots, M^*_{i_q})$ realizes a local minimum of the function

$$\begin{array}{rccl} \hat{f} & : & D_{i_1} \times \ldots \times D_{i_q} & \rightarrow \mathbb{R} \\ & & M_{i_1}, \ldots, M_{i_q} & \mapsto \|M_{i_1} M_{i_2}\| + \ldots \\ & & & \quad +\|M_{i_{q-1}} M_{i_q}\| + \|M_{i_q} M_{i_1}\|. \end{array}$$

Indeed, let $\vec{u} \in \mathbb{R}^{2n}$ be a sufficiently small vector such that $(M^*_{i_1}, \ldots, M^*_{i_q}) + \vec{u} \in D_{i_1} \times \ldots \times D_{i_q}$. The vector $\vec{u}$ is the sum of q vectors $(\ldots, \vec{0}, \vec{u}_{i_j}, \vec{0}, \ldots) \in \mathbb{R}^2 \times \ldots \times \mathbb{R}^2$ such that $M^*_{i_j} + \vec{u}_{i_j} \in D_{i_j}$. Let $D\hat{f}(M^*)$ be the differential at $M^* = (M^*_{i_1}, \ldots, M^*_{i_q})$. As shown above, $D\hat{f}(M^*).(\ldots, \vec{0}, \vec{u}_{i_j}, \vec{0}, \ldots) \geq 0$. It follows that $D\hat{f}(M^*).\vec{u} \geq 0$. As $\hat{f}$ is convex (see Lemma 10) the claim is proved.

As $\hat{f}$ is convex, $\hat{f}(M^*_{i_1}, \ldots, M^*_{i_q})$ is the global minimum of $\hat{f}$. Therefore, by Proposition 11, the polygon $M^*_{i_1} \ldots M^*_{i_q}$ is, among the polygons of $\mathcal{U}$, one of minimal perimeter.

2) We now show that there is only one polygon in $\mathcal{U}$ of minimal perimeter. As $\hat{f}$ is convex, the set of points for which the function $\hat{f}$ is minimum is connected. Thus, in order to prove the uniqueness of the polygon, it is sufficient to show that there exists an open neighborhood of $(M^*_{i_1}, \ldots, M^*_{i_q})$ such that $(M^*_{i_1}, \ldots, M^*_{i_q})$ is the only point of that neighborhood for which the function $\hat{f}$ is minimum.

For any point $(M_{i_1}, \ldots, M_{i_q})$ in a sufficiently small neighborhood of $(M^*_{i_1}, \ldots, M^*_{i_q})$, the polygon $M_{i_1} \ldots M_{i_q}$ does not have any flat vertex since the polygon $M^*_{i_1} \ldots M^*_{i_q}$ does not have any. Thus, by Propositions 9 and 11, the function $\hat{f}$ is minimum at $(M_{i_1}, \ldots, M_{i_q})$ only if each vertex M_{i_j} belongs to the boundary C_{i_j} of D_{i_j}. Therefore, in order to prove that there exists a unique polygon of minimal perimeter in $\mathcal{U}$, it is sufficient to show that for any $(M_{i_1}, \ldots, M_{i_q}) \neq (M^*_{i_1}, \ldots, M^*_{i_q})$ in $C_{i_1} \times \ldots \times C_{i_q}$ and in a sufficiently small neighborhood of $(M^*_{i_1}, \ldots, M^*_{i_q})$, $\hat{f}(M_{i_1}, \ldots, M_{i_q}) > \hat{f}(M^*_{i_1}, \ldots, M^*_{i_q})$.

Similarly as in Lemma 12, let θ_{i_j} be the angle $\angle(\overrightarrow{P_{i_j} P_{i_{j-1}}} \overrightarrow{P_{i_j} M_{i_j}})$ and let $\theta^*_{i_j}$ be the angle $\angle(\overrightarrow{P_{i_j} P_{i_{j-1}}} \overrightarrow{P_{i_j} M^*_{i_j}})$. Let U_{i_j} be the set of θ_{i_j} such that M_{i_j} belongs to the polygon $P_{i_1} \ldots P_{i_q}$. Let $\mathcal{D}$ be the open set of the $(\theta_{i_1}, \ldots, \theta_{i_q}) \in U_{i_1} \times \ldots \times U_{i_q}$ such that the interior of the polygon $M_{i_1} \ldots M_{i_q}$ does not intersect the circles $C_{i_1} \ldots, C_{i_q}$. Let g be

$$\begin{array}{rccl} g & : & U_{i_1} \times \ldots \times U_{i_q} & \rightarrow \mathbb{R} \\ & & \theta_{i_1}, \ldots, \theta_{i_q} & \mapsto \|M_{i_1} M_{i_2}\| + \ldots \\ & & & \quad +\|M_{i_{q-1}} M_{i_q}\| + \|M_{i_q} M_{i_1}\|. \end{array}$$

As shown above, the polygon $M^*_{i_1} \ldots M^*_{i_q}$ is of minimal perimeter among the regions that intersect the disks $D_{i_1}, \ldots, D_{i_q}$. By Lemma 5, $(\theta^*_{i_1}, \ldots, \theta^*_{i_q}) \in U_{i_1} \times \ldots \times U_{i_q}$ and, as the polygon $M^*_{i_1} \ldots M^*_{i_q}$ does not have any flat vertex, $(\theta^*_{i_1}, \ldots, \theta^*_{i_q}) \in \mathcal{D}$ by Proposition 9. Thus, by Lemma 12, g is locally strictly convex at $(\theta^*_{i_1}, \ldots, \theta^*_{i_q})$. As $\hat{f}(M^*_{i_1}, \ldots, M^*_{i_q})$ is the minimum of $\hat{f}$, it is the minimum of the restriction of $\hat{f}$ to $C_{i_1} \times \ldots \times C_{i_q}$ and therefore $g(\theta^*_{i_1}, \ldots, \theta^*_{i_q})$ is the minimum of g. Thus for any $(\theta_{i_1}, \ldots, \theta_{i_q}) \neq (\theta^*_{i_1}, \ldots, \theta^*_{i_q})$ in a sufficiently small neighborhood of $(\theta^*_{i_1}, \ldots, \theta^*_{i_q})$, $g(\theta_{i_1}, \ldots, \theta_{i_q}) > g(\theta^*_{i_1}, \ldots, \theta^*_{i_q})$. Hence, for any $(M_{i_1}, \ldots, M_{i_q}) \neq (M^*_{i_1}, \ldots, M^*_{i_q})$ in $C_{i_1} \times \ldots \times C_{i_q}$ and in a sufficiently small neighborhood of $(M^*_{i_1}, \ldots, M^*_{i_q})$, the perimeter of the polygon $M_{i_1} \ldots M_{i_q}$ is strictly greater than the perimeter of the polygon $M^*_{i_1} \ldots M^*_{i_q}$. This shows that the polygon of minimal perimeter that intersects the q disks $D_{i_1}, \ldots, D_{i_q}$ is unique.

3)Let $\mathcal{Q}'$ be a polygon intersecting all the disks $D_1, \ldots, D_n$ and whose perimeter is minimal. We show that $\mathcal{Q}' = \mathcal{Q}$. Plainly, $\mathcal{Q}'$ intersects the q disks $D_{i_1}, \ldots, D_{i_q}$. As shown above, the polygon of minimal perimeter that intersects the q disks $D_{i_1}, \ldots, D_{i_q}$ is unique and equal to $\mathcal{Q}$. Thus, either $\mathcal{Q}' = \mathcal{Q}$ or the perimeter of $\mathcal{Q}'$ is strictly greater than the one of $\mathcal{Q}$. As the perimeters of $\mathcal{Q}'$ and of $\mathcal{Q}$ are equal, $\mathcal{Q}' = \mathcal{Q}$. □

Propositions 3 and 14 yield the following proposition :

Proposition 15 *$\mathcal{T}$ is unique.*

5 Results and algorithms

We sum up the results of Propositions 3, 10, 11, 14 and 15 in the following theorem :

Theorem 16 *Let $\mathcal{S}$ be a finite set of points such that the radius of the smallest disk that contains $\mathcal{S}$ is strictly greater than 1. Let $\mathcal{P} = P_1, \ldots, P_n$ be its convex hull and $D_1, \ldots, D_n$ the closed disks of unit radius centered at $P_1, \ldots, P_n$. $\mathcal{S}$ has a unique convex hull of bounded curvature of which is equal to the Minkowski sum of the disk of unit radius centered at the origin and of any polygon $M_1^* \ldots M_n^*$ such that $f(M_1^*, \ldots, M_n^*)$ is the minimum of the convex function*

$$\begin{array}{rcl} f \ : \ D_1 \times \ldots \times D_n & \to & \mathbb{R} \\ M_1, \ldots, M_n & \mapsto & \|M_1M_2\| + \|M_2M_3\| + \ldots \\ & & + \|M_{n-1}M_n\| + \|M_nM_1\|. \end{array}$$

According to Theorem 16, the main problem in computing $\mathcal{T}$ is the computation of a point for which the function f is minimum. The minimization of f can be viewed as the minimization of the function :

$$\begin{array}{rcl} F \ : \ \mathbb{R}^2 \times \ldots \times \mathbb{R}^2 & \to & \mathbb{R} \\ M_1, \ldots, M_n & \mapsto & \|M_1M_2\| + \|M_2M_3\| + \ldots \\ & & + \|M_{n-1}M_n\| + \|M_nM_1\| \end{array}$$

under the n constraints $M_i \in D_i$, $1 \leq i \leq n$. Interior point algorithms can be used to compute, in polynomial time, a point that approximate the minimum of F under these constraints (see [NN94]).

We can also compute $\mathcal{Q}$ exactly : if $M_{i_1}^*, \ldots, M_{i_q}^*$ are the non-flat vertices of $\mathcal{Q}$, then, by Proposition 9, $(M_{i_1}^*, \ldots, M_{i_q}^*)$ is a solution of the system

$$\left\{ \begin{array}{l} \left(\dfrac{\overrightarrow{M_{i_j}M_{i_{j-1}}}}{\|M_{i_j}M_{i_{j-1}}\|} + \dfrac{\overrightarrow{M_{i_j}M_{i_{j+1}}}}{\|M_{i_j}M_{i_{j+1}}\|} \right) \times \overrightarrow{P_{i_j}M_{i_j}} = 0 \\ M_{i_j} \in C_{i_j} \\ i_j \in \{i_1, \ldots, i_q\} \subseteq \{1, \ldots, n\} \end{array} \right.$$

That system can be transformed into an algebraic system of q equations (of degree 6) in q indeterminates $\tan(\theta_{i_j}/2)$ ($j \in \{i_1, \ldots, i_q\}$) where θ_{i_j} is the polar angle of $\overrightarrow{P_iM_i}$. Then, $(M_{i_1}^*, \ldots, M_{i_q}^*)$ is, among all the solutions of the system, the one for which the perimeter of the polygon $M_{i_1}^* \ldots M_{i_q}^*$ is minimal. That system can be solved in time $O(2^{O(n)})$ (see [LL91]).

By considering all the possible sets of suffix $\{i_1, \ldots, i_q\} \subseteq \{1, \ldots, n\}$ ($q \in \{1, \ldots, n\}$) we can compute the solution $(M_{i_1}^*, \ldots, M_{i_q}^*)$ such that the polygon $M_{i_1}^* \ldots M_{i_q}^*$ intersects all the disks $D_1, \ldots, D_n$ and for which the perimeter is minimal. Hence, we can compute the convex hull of bounded curvature of n points in exponential time in n.

6 Open questions

The work reported here raises many questions. We mention a few of them we plan to consider in near future : Does there exists a polynomial time algorithm for this problem? Can these results be generalized to higher dimensions? Can similar results be obtained for convex hulls of bounded curvature and of minimal area? Can similar results be obtained for convex hulls whose boundaries are curves C^2 for which the derivative of the curvature is bounded?

References

[ART95] Agarwal (P. K.), Raghavan (P.) et Tamaki (H.). – Motion planning for a steering-constrained robot through moderate obstacles. *In : Proc. 27th Annu. ACM Sympos. Theory Comput.*, pp. 343–352.

[BCL94] Boissonnat (J.-D.), Cérézo (A.) et Leblond (J.). – Shortest paths of bounded curvature in the plane. *Journal of Intelligent Systems*, vol. 10, 1994, pp. 1–16.

[BL96] Boissonnat (J-D.) et Lazard (S.). – A polynomial-time algorithm for computing a shortest path of bounded curvature amidst moderate obstacles. *In : Proc. 12th Annu. ACM Sympos. Comput. Geom.* – Philadelphia, PA, USA, 1996.

[LL91] Lakshman (Y.N.) et Lazard (D.). – On the complexity of zero-dimensional algebraic systems. *In : Effective Methods in Algebraic Geometry (MEGA)*, éd. par Mora (T.) et Traverso (C.). pp. 217–225. – Birkhäuser.

[NN94] Nesterov (Y.) et Nemirovskii (A.). – *Interior-point polynomial algorithms in convex programming.* – SIAM Studies in Applied Mathematics, 1994.

[PBGM62] Pontryagin (L. S.), Boltyanskii (V. G.), Gamkrelidze (R. V.) et Mishchenko (E. F.). – *The mathematical theory of optimal processes.* – Interscience Publishers, 1962.