A polynomial-time algorithm for computing shortest paths of bounded curvature amidst moderate obstacles

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Extended abstract

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Abstract

In this paper, we consider the problem of computing a shortest path of bounded curvature amidst obstacles in the plane. More precisely, given prescribed initial and final configurations (i.e. positions and orientations) and a set of obstacles in the plane, we want to compute a shortest C^1 path joining those two configurations, avoiding the obstacles, and with the further constraint that, on each C^2 piece, the radius of curvature is at least 1. In this paper, we consider the case of moderate obstacles (as introduced by Agarwal et al. [1]) and present a polynomial-time exact algorithm to solve this problem.

1 Introduction

In this paper, we consider the problem of computing a shortest path of bounded curvature amidst obstacles in the plane, SBC path for short. More precisely, given prescribed initial and final configurations (i.e. positions and orientations) and a set of obstacles in the plane, we want to compute a shortest C^1 path joining those two configurations, avoiding the obstacles, and with the further constraint that, on each C^2 piece[1], the radius of curvature is at least 1. This question appears in many applications and goes back to Markov who studied the problem for joining pieces of railways. More recently, a great deal of attention has been paid to this question in the context of non-holonomic robot motion planning [2, 3, 13, 15, 16, 17, 18, 19, 20, 24, 25]. A robot is said to be non-holonomic if some kinematics constraints locally restricts the authorized directions for its velocity. A typical example of a non-holonomic robot is that of a car : assuming no slipping of the wheels on the ground, the velocity of the midpoint between the two rear wheels of the car is always tangent to the car axis. Though the problem considered in this paper is one of the simplest instances of non-holonomic motion planning, it is still far from being well understood.

Even in the absence of obstacles, the problem is not easy. Dubins [10] proved that any SBC path takes one of the following forms CSC or CCC, where C means a circular arc of radius 1 and S a straight line segment. The proof in Dubins's paper is quite long and intricate. Recently, a much simpler proof has been obtained using the Minimum Principle of Pontryagin (a central result in Control Theory) [6, 22] and a complete characterization of SBC paths has also been established [7].

The problem becomes much harder in the presence of obstacles. By basic theorems in Control Theory, there exists a SBC path amidst obstacles and joining two given configurations as soon as there exists a BC path, i.e. a (not necessarily optimal) C^1 path joining the two given configurations, avoiding the obstacles and where the radius of curvature is everywhere (where it is defined) greater than or equal to 1. Moreover, a SBC path is a finite concatenation of subpaths either contained in the boundary of some obstacle or joining two obstacle edges (considering the initial and the final configurations as point obstacles); each subpath joining two obstacle edges is a Dubins' path, i.e. a path of type CSC or CCC. Computing a shortest path seems however a formidable task. Even if we remove the requirement for the path to be a shortest one and look for a BC path (instead of a SBC path), no polynomial-time algorithm is known. In [11], Fortune and Wilfong present an exact algorithm that can decide if a BC path exists but does not generate the path in question. This algorithm runs in time and space that is exponential with respect to the number n of corners of the environment and the number of bits used to specify the positions of the corners. By the remark above, this algorithm can also decide if a SBC path exists.

For computing SBC paths, only approximate algorithms have been proposed in the literature. Jacobs and Canny [12]

[1] As we will see below, the optimal path is piecewise C^2.

discretize the problem and calculate a path that approximates the shortest one in time $O(n^2(\frac{n+L}{\varepsilon})\log n + \frac{(n+L)^2}{\varepsilon^2})$, where ε describes the closeness of the approximation and L is the total edge length of the obstacle boundaries. Very recently, Wang and Agarwal [23] improved on this result and proposed an algorithm whose time complexity is $O(\frac{n^2}{\varepsilon^2}\log n)$, and thus does not depend on L. In another recent paper, Agarwal et al. [1] have considered a restricted class of obstacles, the so-called moderate obstacles : an obstacle is said to be moderate if it is convex and if its boundary is a differentiable curve whose radius of curvature is everywhere greater than or equal to 1. This restriction is quite strong but valid in many practical situations. Under the assumption that all the obstacles are disjoint and moderate, Agarwal et al. show that an approximate SBC path can be computed in $O(n^2 \log n + 1/\varepsilon)$ time.

In this paper, we consider also the case of moderate obstacles (in a more restrictive sense than Agarwal et al.) and present a polynomial-time algorithm to compute a SBC path (assuming that the roots of some polynomials of bounded degree can be computed in constant time). To the best of our knowledge, this is the first polynomial-time exact algorithm for a non trivial instance of the problem.

The paper is organized as follows. In Section 2, we introduce some notations and show that the problem reduces to finding an Euclidean shortest path when the initial and the final positions are sufficiently far away and also sufficiently far from the obstacles. In the following Sections 3, 4 and 5 we show that SBC paths belong to a finite family. In Section 6, we describe an algorithm that computes an optimal path between two given configurations.

2 Preliminaries

First, we give some definitions and notations. Let Ω be a set of obstacles. In this paper, the obstacles are assumed to be disjoint and moderate. An obstacle is said to be *moderate* if it is convex and if its boundary is a differentiable curve made of line segments and circular arcs of unit radius. For convenience and without real loss of generality, we assume that no two edges of the obstacles are parallel. A path that avoids the obstacles (i.e. that does not intersect the interior of the obstacles) is called *free*. In the sequel, a free SBC path is simply called an *optimal path*.

Let $\omega_S = (S, \vec{U}_S)$ and $\omega_T = (T, \vec{U}_T)$ be two configurations (i.e. positions and orientations). Let $\mathcal{P}$ be an optimal path joining ω_S to ω_T. As mentioned in the introduction, $\mathcal{P}$ is a finite concatenation of O, C and S-segments; an O-segment is a maximal portion of $\mathcal{P}$ that coincides with the boundary of an obstacle; a C-segment is a maximal circular arc of unit radius that is not an O-segment; a S-segment is a maximal line segment, possibly on the boundary of some obstacle. To a path, we will associate the sequence of the types (O, C or S) of its segments.

The first and last segments are called *terminal*. A terminal segment is, in general, a C-segment; we denote it by C_t. A C-segment (or a circle of unit radius) is denoted by $\tilde{C}$ if it

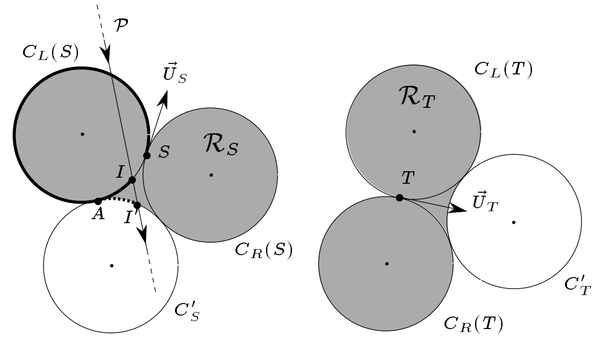


Figure 1: Regions $\mathcal{R}_S$ and $\mathcal{R}_T$

is tangent to at least one obstacle. A C-segment (or a circle of unit radius) is called *anchored* and denoted by $\bar{\tilde{C}}$ either if it is tangent to at least two obstacles, or if it is tangent to at least one obstacle and adjacent to a terminal C-segment, or if it is terminal.

The first theorem shows that, when the initial and the final positions are sufficiently far away and also sufficiently far from the obstacles, the optimal path is an Euclidean shortest path for an augmented set of obstacles.

Let X be a point of $\mathcal{P}$ and $\vec{U}$ the vector tangent to $\mathcal{P}$ at X. Let $C_L(X)$ (resp. $C_R(X)$) be the unit circle tangent to $\mathcal{P}$ at X and lying on the left (resp. right) side of $\mathcal{P}$ (i.e. on the left side of the oriented line passing through X with direction $\vec{U}$). $C_L(X)$ is oriented counterclockwise and $C_R(X)$ is oriented clockwise. An arc of one of these circles will be oriented accordingly.

Let C'_S (resp. C'_T) be the circle tangent to $C_L(S)$ and $C_R(S)$ ($C_L(T)$ and $C_R(T)$) that does not intersect the ray $(S, \vec{U}_S)$ (the ray $(T, -\vec{U}_T)$) (see Figure 1). Let $\mathcal{R}_S$ (resp. $\mathcal{R}_T$) be the shaded region limited by $C_L(S)$, $C_R(S)$ and C'_S ($C_L(T)$, $C_R(T)$ and C'_T) in Figure 1.

Lemma 1 *If $\mathcal{R}_S$ and $\mathcal{R}_T$ are disjoint and do not intersect the obstacles, $\mathcal{P}$ does not intersect the interior of $\mathcal{R}_S$ nor that of $\mathcal{R}_T$.*

Proof: We assume for a contradiction that $\mathcal{P}$ intersects the interior of $\mathcal{R}_S$. We consider first the case where $\mathcal{P}$ does not intersect the interior of $\mathcal{R}_T$. As $\mathcal{P}$ is a path of bounded curvature, $\mathcal{P}$ intersects $C_L(S)$ or $C_R(S)$. Let I be the last intersection point (along $\mathcal{P}$) between $\mathcal{P}$ and $C_L(S) \cup C_R(S)$; we assume, without loss of generality, that $I \in C_L(S)$. Let I' be the last intersection point (along $\mathcal{P}$) between $\mathcal{P}$ and $\mathcal{R}_S$ and let II' be the part of $\mathcal{P}$ from I to I'. We denote by A the point common to $C_L(S)$ and C'_S (see Figure 1).

First, we assume that $I \neq S$. Let SI be the arc of $C_L(S)$, oriented as $C_L(S)$, that starts at S and ends at I. Let $\mathcal{P}'$ be the concatenation of SI and the part of $\mathcal{P}$ from I to T. $\mathcal{P}'$ is not a path of bounded curvature but it is shorter than $\mathcal{P}$ since the shortest path of bounded curvature from ω_S to I (the orientation at I is not specified) is the arc SI [5]. Let $\mathcal{P}''$ be the path obtained by modifying $\mathcal{P}'$ as follows : if $I' \neq I$, then we replace the arc AI of $C_L(S)$ and II' by the circular arc AI' of C'_S. The path $\mathcal{P}''$ is shorter than $\mathcal{P}'$. Thus $\mathcal{P}''$ is shorter than $\mathcal{P}$, avoids all the moderate obstacles, avoids $\mathcal{R}_S$ by construction and $\mathcal{R}_T$ because $\mathcal{R}_S \cap \mathcal{R}_T = \emptyset$. Hence, the Euclidean shortest path from S to T avoiding Ω, $\mathcal{R}_S$ and $\mathcal{R}_T$ is shorter than $\mathcal{P}$. That yields a contradiction because this Euclidean shortest path is a path of bounded curvature from ω_S to ω_T.

If $I = S$, the orientation of $\mathcal{P}$ at I can only be $\vec{U}_S$ or $-\vec{U}_S$ since I is the last intersection point between $\mathcal{P}$ and $C_L(S) \cup C_R(S)$. But only the latter case can occur since otherwise, $\mathcal{P}$ would not be optimal. As, by definition, I lies before I' along $\mathcal{P}$, the part of $\mathcal{P}$ from $(S, \vec{U}_S)$ to I' is longer than the shortest Dubins' path from $(S, \vec{U}_S)$ to $(S, -\vec{U}_S)$ which is a path of type CCC of length $2\pi + \pi/3$. Let SI' be the concatenation of the arc SA of $C_L(S)$ and the circular arc AI', and let $\mathcal{P}'$ be the concatenation of SI' and the part of $\mathcal{P}$ from I' to T. As, the length of SI' is at most 2π, $\mathcal{P}'$ is shorter than $\mathcal{P}$. We then get a contradiction as above.

Similar arguments hold if $\mathcal{P}$ intersects the interior of $\mathcal{R}_T$. □

Theorem 2 *If $\mathcal{R}_S$ and $\mathcal{R}_T$ are disjoint and do not intersect Ω, $\mathcal{P}$ is the Euclidean shortest path from S to T avoiding Ω and the two additional obstacles $\mathcal{R}_S$ and $\mathcal{R}_T$.*

Proof: It follows from Lemma 1 that $\mathcal{P}$ is also the shortest path from ω_S to ω_T if we consider $\mathcal{R}_S$ and $\mathcal{R}_T$ as some other (moderate) obstacles. On the other hand, the Euclidean shortest path from S to T that avoids Ω, $\mathcal{R}_S$ and $\mathcal{R}_T$ is a path of bounded curvature from $(S, \vec{U}_S)$ to $(T, \vec{U}_T)$. Hence, $\mathcal{P}$ is the Euclidean shortest path from S to T in the presence of the obstacles Ω, $\mathcal{R}_S$ and $\mathcal{R}_T$. □

Corollary 3 *A Dubins' path of type CCC between two configurations ω_S and ω_T is optimal only if the two regions $\mathcal{R}_S$ and $\mathcal{R}_T$ intersect.*

In the rest of the paper, we will assume that Theorem 2 does not apply.

3 Characterization of the C-segments

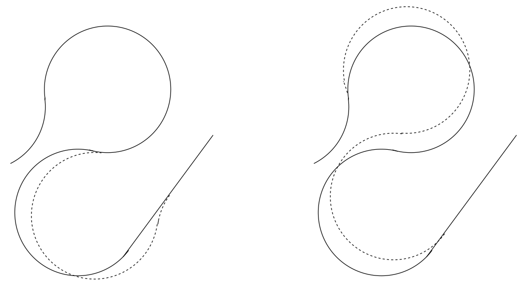
We first recall the following lemma mentionned in the introduction which follows from [10] or [6] :

Lemma 4 *Each subpath of an optimal path which has no point in common with the obstacles except possibly its two end points must be of type CCC or CSC.*

We now recall three lemmas and a theorem established by Agarwal et al. [1].

Lemma 5 *Any non-terminal C-segment of an optimal path is longer than π.*

Lemma 6 *Any optimal path does not contain a subpath of type CCC, except when the first or the last C-segments of this subpath is terminal.*



(a) (b)

Figure 2: Length reducing perturbations for $CCCS$ paths

Lemma 7 *If an optimal path contains a subpath of type SCS, OCS, SCO or OCO, the C-segment is anchored.*

Theorem 8 *Any C-segment appearing in an optimal path belongs to one of the following subpaths :*

$$\bar{\bar{C}},\quad \bar{C}\bar{C},\quad C_tC\bar{C},\quad \bar{C}CC_t.$$

We further restrict the possible types of C-segments that may appear in an optimal path :

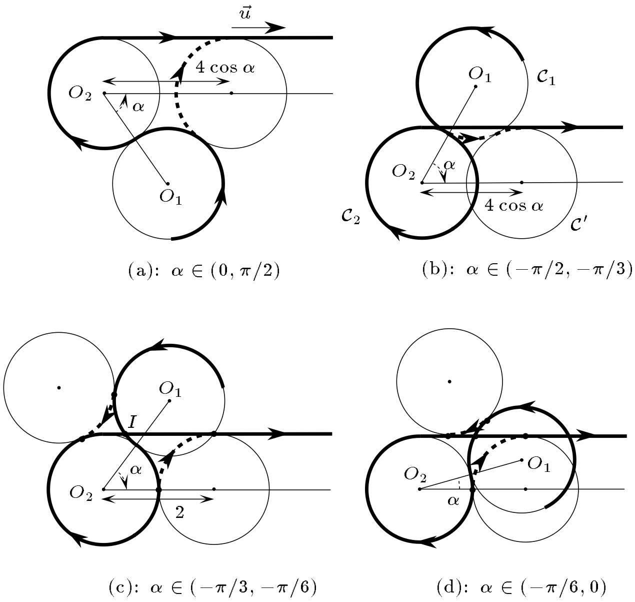
Theorem 9 *Any C-segment of an optimal path belongs to one of the following subpaths :*

$$\bar{\bar{C}},\quad \bar{C}\bar{C},\quad C_tC\bar{\bar{C}},\quad \bar{\bar{C}}CC_t.$$

Proof: Consider first an optimal subpath of type $C_tC\bar{C}O$ where the O-segment is a circular arc. We use the same perturbation that Dubins used to reduce the length of $CCCC$-paths. It follows that the second or the third C-segments of $C_tC\bar{C}O$ must be clamped by some obstacles. Hence, the third C-segment is anchored or both the second and the third C-segments are tangent to some obstacles.

Consider now an optimal subpath of type $C_tC\bar{C}S$ and the two types of perturbation shown in Figure 2. Perturbation (a) has been used by Dubins to shorten the path and it can be shown, using the same kind of argument, that perturbation (b) shortens also the path. Thus, if the subpath is optimal, either the third C-segment is anchored or both the second and the third C-segments are tangent to some obstacles. Since a subpath of type $CCCC$ cannot be optimal, we have shown that a subpath of type C_tCC is either $C_tC\bar{\bar{C}}$ or $C_t\bar{C}\bar{C}$. □

As, for a given set of obstacles, the number of anchored circles is finite, the number of the subpaths in Theorem 9 is finite except for the subpaths of type $\bar{C}\bar{C}$. The two following sections will show that the number of these subpaths is also finite. First, in Section 4, we will show that any non-terminal subpath of type $\bar{C}\bar{C}$ of an optimal path is necessarily contained in a subpath of type $XS\bar{C}\bar{C}SX'$ where $X, X' \in \{O, \bar{\bar{C}}\}$. Then, in Section 5, we will show that, given X, X' and two obstacle edges, the number of subpaths of an optimal path of type $XS\bar{C}\bar{C}SX'$ where the C-segments are tangent to the given obstacle edges is finite.



(a): $\alpha \in (0, \pi/2)$ (b): $\alpha \in (-\pi/2, -\pi/3)$

(c): $\alpha \in (-\pi/3, -\pi/6)$ (d): $\alpha \in (-\pi/6, 0)$

Figure 3: Shortcuts used in Lemma 11

4 Characterization of the subpaths of type $\bar{C}\bar{C}$

The section is devoted to the proof of the following theorem :

Theorem 10 *Any non terminal subpath of type $\bar{C}\bar{C}$ of an optimal path is necessarily contained in a subpath of type $XS\bar{C}\bar{C}SX'$ where $X, X' \in \{O, \bar{\bar{C}}\}$. The length of S-segment may be zero.*

In the sequel, we will use the following notations. For a given subpath $\mathcal{P}$, C_i will denote the i-th C-segment of $\mathcal{P}$, $\mathcal{C}_i$ will denote the circle supporting C_i and O_i the center of $\mathcal{C}_i$ ($i \in \{1, 2, 3\}$).

We first establish two lemmas and a proposition.

Lemma 11 *In a subpath of type CCS of an optimal path where the first C-segment is not terminal, the length of the S-segment is smaller than $4\cos\alpha$. Here $\alpha = \angle(\overrightarrow{O_2O_1}, \overrightarrow{u})$ and $\vec{u}$ is the direction of the S-segment (see Figure 3).*

Proof: We omit the proof in this abstract but Figure 3 shows the different kinds of shortcuts we use if the length of the S-segment is greater than $4\cos\alpha$. Because the obstacles are moderate, the obstacles cannot intersect the shortcuts. $\square$

We consider now subpaths of type $CCSC$.

Lemma 12 *In a subpath of type $CCSC$ of an optimal path where the first and the last C-segments are not terminal, the two C-segments adjacent to the line segment have the same orientation (clockwise or counterclockwise).*

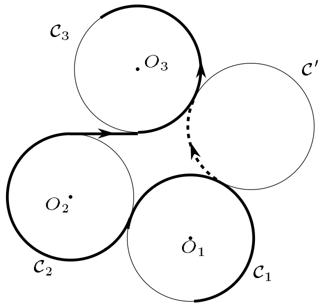
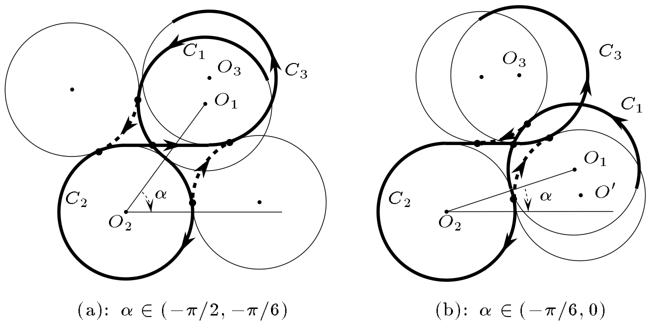


Figure 4: Shortcut used in Lemma 12 if $\mathcal{C}_1$ does not intersect $\mathcal{C}_3$

Proof: We omit the proof but Figure 4 shows the shortcut we use when $\mathcal{C}_1$ does not intersect $\mathcal{C}_3$ and Figure 5 shows the shortcut we use when $\mathcal{C}_1$ intersects $\mathcal{C}_3$. $\square$



(a): $\alpha \in (-\pi/2, -\pi/6)$ (b): $\alpha \in (-\pi/6, 0)$

Figure 5: Shortcut used in Lemma 12 if $\mathcal{C}_1$ intersect $\mathcal{C}_3$

Proposition 13 *An optimal path cannot contain a subpath of type $CCSCC$, except when the first or the last C-segment of this subpath is terminal.*

Proof: By Lemma 5, the lengths of the first and the last C-segments are greater than π if they are not terminal. According to the two previous lemmas, the length of the S-segment is less than $4\cos\alpha$ and the two C-segments adjacent to the line segment have the same orientation. Then, as shown in Figure 6, it is possible to shorten the subpath. We omit the details of the proof. $\square$

The proof of Theorem 10 now follows. Indeed, let us consider a subpath of type $XS\bar{C}\bar{C}SX'$, $X \notin \{O, \bar{C}\}$. As X is not terminal, Lemma 6 implies that the length of the first S-segment of the subpath cannot be zero. Therefore, by Theorem 9, X is necessarily a C-segment tangent to some obstacle and following another C-segment X_1. Then, by Proposition 13, X_1 is terminal and therefore, X is anchored by definition. This contradicts our assumption and ends the proof of the theorem.

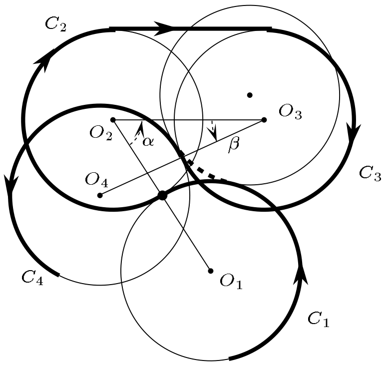


Figure 6: Shortcut used in Proposition 13

5 Bounding the number of subpaths of type $\bar{C}\bar{C}$

This section is devoted to the proof of the following theorem :

Theorem 14 *Let $\mathcal{X}$ (resp. $\mathcal{X}'$) be a circular edge of some obstacle or an anchored circle, and let $\mathcal{O}$ and $\mathcal{O}'$ be two obstacle edges. Let $X \subset \mathcal{X}$, $X' \subset \mathcal{X}'$, $\bar{C}$ a C-segment tangent to $\mathcal{O}$ and $\bar{C}'$ a C-segment tangent to $\mathcal{O}'$. The shortest subpath of type $XS\bar{C}\bar{C}'SX'$ belong to a finite family.*

Let $\mathcal{P}$ denote a subpath of type $XS\bar{C}\bar{C}'SX'$. The number of subpaths $\mathcal{P}$ where $\bar{C}$ or $\bar{C}'$ are anchored is finite; so we assume that neither $\bar{C}$ nor $\bar{C}'$ are anchored. Without loss of generality, we assume in the sequel that the path $\mathcal{P}$ is oriented counterclockwise on $\bar{C}$ and clockwise on $\bar{C}'$ as shown in Figure 7. We only consider the case where both obstacle edges $\mathcal{O}$ and $\mathcal{O}'$ are circular arcs; the other cases are simpler and omitted here.

We first observe that $\mathcal{P}$ is optimal when some mechanical device is at equilibrium. This leads to an algebraic system of equations whose solutions correspond to potential equilibriums of the mechanical device. We then show that this system has a finite number of solutions. This is done as follows. Let $E_1 = 0, \ldots, E_r = 0$ be the equations of the system where $E_1, \ldots, E_r$ are polynomials in the variables $x_1, \ldots, x_r$. The resultant $R(x_1, \ldots, x_{r-1})$ with respect to x_r of two of the E_i, say E_1 and E_2, vanishes at $\tilde{x}_1, \ldots, \tilde{x}_{r-1}$ iff there exists $\tilde{x_r}$ such that $(\tilde{x}_1, \ldots, \tilde{x}_{r-1}, \tilde{x}_r)$ is a common root of E_1 and E_2 [4]. By cascading resultants on the polynomials $E_1, \ldots, E_r$, we eliminate successively the indeterminates $x_r, \ldots, x_2$ and compute a univariate polynomial $R(x_1)$. If $(\tilde{x}_1, \ldots, \tilde{x}_r)$ is a solution of the system, $R(\tilde{x}_1) = 0$ (the converse is not necessarily true). Then we show that the univariate polynomial $R(x_1)$ is not identically zero and that any value of the indeterminate x_1 determines the other indeterminates. It immediately follows that the considered algebraic system of equations has a finite number of roots. Unfortunately, computing such a polynomial $R(x_1)$ exceeds the capabilities of the current computer algebra systems. However, we can compute the leading monomial of $R(x_1)$ which is sufficient to show that $R(x_1) \not\equiv 0$.

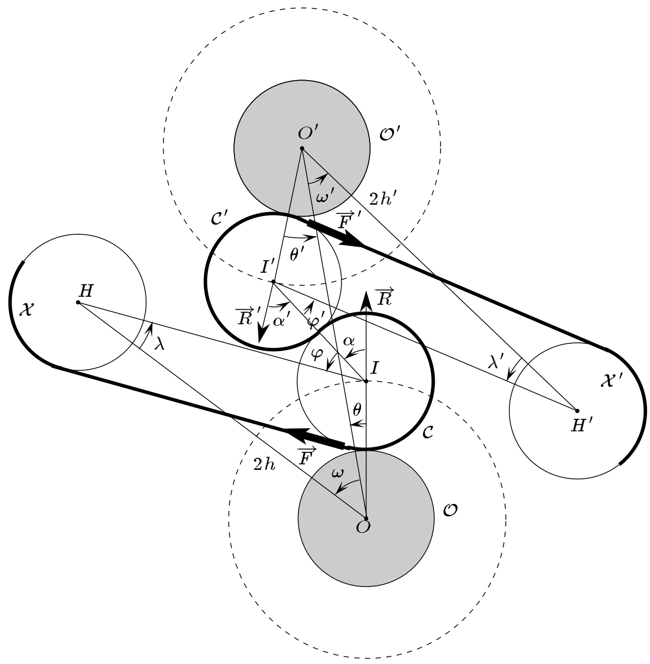


Figure 7: Mechanical device where both obstacle edges are circular edges

The mechanical device consists of four fixed objects and one moving object D. The fixed objects are the two obstacles $\mathcal{O}$ and $\mathcal{O}'$ and the two disks of unit radius supporting $\mathcal{X}$ and $\mathcal{X}'$. The moving object D is the union of two tangent disks (corresponding to the circles $\mathcal{C}$ and $\mathcal{C}'$). We consider a rubber band of thickness zero attached on $\mathcal{X}$ and on $\mathcal{X}'$ and passing around $\mathcal{C}$ and $\mathcal{C}'$ (see Figure 7). The case we are interested in is when both mobile disks are tangent to the obstacles. The moving object D is subject to four forces $\vec{F}$, $\vec{F'}$, $\vec{R}$ and $\vec{R'}$ (see Figure 7). $\vec{F}$ and $\vec{F'}$ are the two forces, of equal norm F, exerted by the rubber band. $\vec{R}$ and $\vec{R'}$ are the reactions of the obstacles $\mathcal{O}$ and $\mathcal{O}'$ onto the mobile D.

We introduce the following notations (see Figure 7). Let $\mathcal{C}$ and $\mathcal{C}'$ be the circles supporting $\bar{C}$ and $\bar{C}'$. I and I' are the centers of $\mathcal{C}$ and $\mathcal{C}'$. H and H' are the centers of the circles supporting $\mathcal{X}$ and $\mathcal{X}'$. O and O' are the centers of the circles (of unit radius) supporting $\mathcal{O}$ and $\mathcal{O}'$. Let $2d$ be the length of OO' and let $2h$ and $2h'$ be the lengths of OH and $O'H'$ respectively. In addition, let $\alpha = \angle(\vec{R}, \overrightarrow{II'})$, $\alpha' = \angle(\vec{R'}, \overrightarrow{I'I})$, $\varphi = \angle(\overrightarrow{II'}, \vec{F})$, $\varphi' = \angle(\overrightarrow{I'I}, \vec{F'})$, $\omega = \angle(\overrightarrow{OO'}, \overrightarrow{OH})$ and $\omega' = \angle(\overrightarrow{O'O}, \overrightarrow{O'H'})$.

We first establish a lemma that holds regardless of the nature of the obstacle edges $\mathcal{O}$ and $\mathcal{O}'$.

Lemma 15 *The moving object D is at an equilibrium only if :*

$$\sin(\alpha' - \alpha) + \sin\alpha' \sin(\alpha + \varphi) - \sin\alpha \sin(\alpha' + \varphi') = 0 \quad (1)$$

Proof: The moving object D is at an equilibrium iff the sum of the forces $\overrightarrow{F}$, $\overrightarrow{F'}$, $\overrightarrow{R}$, $\overrightarrow{R'}$ and the sum of their moments is zero. Straightforward computations yield Equation 1. □

We now establish a system of four equations in four indeterminates whose solutions correspond to potential equilibriums of the mechanical device :

Lemma 16 *The moving object D is at an equilibrium only if :*

$$\begin{cases} \sin(\alpha' - \alpha) + \sin\alpha' \sin(\alpha + \varphi) - \sin\alpha \sin(\alpha' + \varphi') = 0 \\ 2\cos\alpha + 2\cos\alpha' + 2\cos(\alpha' - \alpha) + 3 - d^2 = 0 \\ h\sin(\varphi - \omega) + h\sin(\alpha' + \varphi - \omega) + \\ \qquad h\sin(\alpha + \varphi - \omega) - d\sin(\alpha + \varphi) + \delta d = 0 \\ h'\sin(\varphi' - \omega') + h'\sin(\alpha + \varphi' - \omega') + \\ \qquad h'\sin(\alpha' + \varphi' - \omega') - d\sin(\alpha' + \varphi') + \delta' d = 0 \end{cases} \quad (2)$$

where δ (resp. δ') is zero if the path $\mathcal{P}$ has the same orientation on $\mathcal{X}$ and $\mathcal{C}$ ($\mathcal{X}'$ and $\mathcal{C}'$) and 1 otherwise.

Proof: The first equation of System 2 is given by Lemma 15. Let $\theta = \angle(\overrightarrow{OI}, \overrightarrow{OO'})$ and $\theta' = \angle(\overrightarrow{O'I'}, \overrightarrow{O'O})$ (see Figure 7). Considering the polygon $OII'O'$, we have :

$$\alpha' - \theta' = \alpha - \theta \; [2\pi] \quad (3)$$

$$\cos\theta + \cos(\alpha - \theta) + \cos\theta' = d \quad (4)$$

$$\sin\theta - \sin(\alpha - \theta) + \sin\theta' = 0 \quad (5)$$

Considering in turn $((Eq4)^2 + (Eq5)^2)$, $((Eq4)\sin\theta - (Eq5)\cos\theta)$, $((Eq4)\cos\theta + (Eq5)\sin\theta)$, $((Eq4)\sin\theta' - (Eq5)\cos\theta')$ and $((Eq4)\cos\theta' + (Eq5)\sin\theta')$, we obtain :

$$2\cos\alpha + 2\cos\alpha' + 2\cos(\alpha' - \alpha) + 3 - d^2 = 0 \quad (6)$$

$$d\sin\theta = \sin\alpha - \sin(\alpha' - \alpha) \quad (7)$$

$$d\cos\theta = 1 + \cos\alpha + \cos(\alpha' - \alpha) \quad (8)$$

$$d\sin\theta' = \sin\alpha' + \sin(\alpha' - \alpha) \quad (9)$$

$$d\cos\theta' = 1 + \cos\alpha' + \cos(\alpha' - \alpha) \quad (10)$$

Equation 6 is the second equation of System 2. We show how to compute the two other equations of System 2 for the possible orientations of X and X'.

• $\mathcal{P}$ has the same orientation on $\mathcal{X}$ and $\mathcal{C}$ (see Figure 7). Let $\lambda = \angle(\overrightarrow{HO}, \overrightarrow{HI})$ and consider the triangle (OHI) :

$$\frac{\sin(\alpha + \varphi)}{2h} = \frac{\sin\lambda}{2}$$

As $\lambda = \alpha + \varphi - \theta - \omega \; [2\pi]$, we get :

$$h\sin(\alpha + \varphi - \theta - \omega) - \sin(\alpha + \varphi) = 0$$

That equation can be expanded with respect to θ, and simplified thanks to Equations 7 and 8 :

$$h\sin(\varphi - \omega) + h\sin(\alpha' + \varphi - \omega) + h\sin(\alpha + \varphi - \omega) - d\sin(\alpha + \varphi) = 0$$

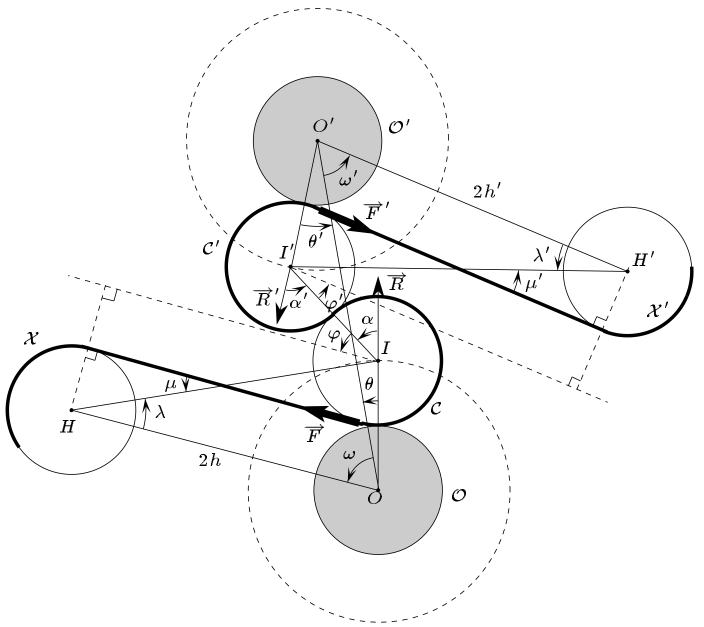


Figure 8: For the proof of Theorem 14

• $\mathcal{P}$ has the same orientation on $\mathcal{X}'$ and $\mathcal{C}'$.
Similarly as above, we obtain the following equations :

$$h'\sin(\alpha' + \varphi' - \theta' - \omega') - \sin(\alpha' + \varphi') = 0$$

$$h'\sin(\varphi' - \omega') + h'\sin(\alpha + \varphi' - \omega') + \\ h'\sin(\alpha' + \varphi' - \omega') - d\sin(\alpha' + \varphi') = 0$$

• $\mathcal{P}$ has opposite orientation on $\mathcal{X}$ and $\mathcal{C}$ (see Figure 8). Let $\mu = \angle(\overrightarrow{F}, \overrightarrow{IH})$ and consider the triangle (OHI) :

$$\frac{\sin(\alpha + \varphi + \mu)}{2h} = \frac{\sin\lambda}{2} = \frac{\sin(\theta + \omega)}{IH}$$

As $\lambda = \alpha + \varphi + \mu - \theta - \omega \; [2\pi]$, we get :

$$\begin{aligned} 2\sin(\theta + \omega) &= IH\sin(\alpha + \varphi - \theta - \omega)\cos\mu \\ &\quad + IH\cos(\alpha + \varphi - \theta - \omega)\sin\mu \\ 2h\sin(\theta + \omega) &= IH\sin(\alpha + \varphi)\cos\mu + IH\cos(\alpha + \varphi)\sin\mu \end{aligned}$$

Eliminating $\cos\mu$ from these two equations gives :

$$\begin{aligned} & 2\sin(\theta + \omega)\sin(\alpha + \varphi) - 2h\sin(\theta + \omega)\sin(\alpha + \varphi - \theta - \omega) = \\ & \qquad IH\cos(\alpha + \varphi - \theta - \omega)\sin\mu\sin(\alpha + \varphi) \\ & \qquad - IH\cos(\alpha + \varphi)\sin\mu\sin(\alpha + \varphi - \theta - \omega) \end{aligned}$$

We simplify this equation into :

$$\begin{aligned} & 2\sin(\theta + \omega)\sin(\alpha + \varphi) - 2h\sin(\theta + \omega)\sin(\alpha + \varphi - \theta - \omega) = \\ & \qquad IH\sin\mu\sin(\theta + \omega) \end{aligned}$$

As $\sin\mu = 2/IH$ (see Figure 8), we have :

$$\sin(\theta + \omega)(h\sin(\alpha + \varphi - \theta - \omega) - \sin(\alpha + \varphi) + 1) = 0$$

We can assume that $\sin(\theta+\omega) \neq 0$ because otherwise I lies on the straight line OH and the number of such paths is less than four. Thus, the moving object is at an equilibrium only if :

$$h\sin(\alpha+\varphi-\theta-\omega)-\sin(\alpha+\varphi)+1=0$$

Similarly as above, we expand that equation with respect to θ and simplify it, thanks to Equations 7 and 8 :

$$h\sin(\varphi-\omega)+h\sin(\alpha'+\varphi-\omega)+ \\ h\sin(\alpha+\varphi-\omega)-d\sin(\alpha+\varphi)+d=0$$

• $\mathcal{P}$ have opposite orientations on $\mathcal{X}'$ and $\mathcal{C}'$.
Similarly as above, we obtain the following equations :

$$h'\sin(\alpha'+\varphi'-\theta'-\omega')-\sin(\alpha'+\varphi')+1=0$$

$$h'\sin(\varphi'-\omega')+h'\sin(\alpha+\varphi'-\omega')+ \\ h'\sin(\alpha'+\varphi'-\omega')-d\sin(\alpha'+\varphi')+d=0$$

That ends the proof of the lemma. □

We now show that System 2 has a finite number of roots $(\alpha,\alpha,\varphi,\varphi')$ (in $(S^1)^4$). It then follows that the moving object D has a finite number of equilibriums.

We expand each equation of System 2 and apply the variable substitution $x=\tan(\alpha/2)$, $y=\tan(\alpha'/2)$, $z=\tan(\varphi/2)$ and $t=\tan(\varphi'/2)$. This yields an algebraic system consisting of four equations where x,y,z,t are the four indeterminates and $\sin\omega$, $\cos\omega$, d, h, h', l' are considered as six independent parameters. Let $E_i=0$ ($i\in\{1,\ldots,4\}$) denote the algebraic equation obtained from the i-th equation of System 2. We compute[2] the resultant E_{14} of E_1 and E_4 with respect to the inderminate t. E_{14} can be written as $16(1+y^2)E'_{14}$. We compute[3] the resultant Q of E'_{14} and E_2 with respect to the inderminate y. We also compute the resultant T of E_2 and E_3 with respect to the inderminate y. Q and T are two polynomials where x and z are the indeterminates.

Now, let R be the resultant of Q and T with respect to z. R is a uni-variate polynomial in the indeterminate x and we want to show that $R(x)\not\equiv 0$. Let

$$Q=q_0+q_1\,z+\ldots+q_n\,z^n,\ \ q_n\neq 0,$$

$$T=t_0+t_1\,z+\ldots+t_m\,z^m,\ \ t_m\neq 0,$$

where the q_i and the t_i are uni-variate polynomials in the variable x. The resultant R of Q and T with respect to z is the determinant of the $(n+m)\times(n+m)$ Sylvester matrix of Q and T with respect to z [4] :

$$\begin{vmatrix} q_0 & q_1 & \cdot & \cdot & \cdot & \cdot & \cdot & q_n & & & \\ & q_0 & \cdot & \cdot & \cdot & \cdot & \cdot & q_{n-1} & q_n & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & q_0 & \cdot & \cdot & \cdot & \cdot & \cdot & q_n \\ t_0 & t_1 & \cdot & \cdot & t_{m-1} & t_m & & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & & & & t_0 & \cdot & \cdot & \cdot & t_m \end{vmatrix}$$

The resultant $R(x)$ is too big to be computed with existing computer algebra systems but we are able to compute its leading monomial. The polynomials q_i appear in the first m rows of the Sylvester determinant and the polynomials t_i appear in the last n rows. Thus the degree of R with respect to x is at most

$$m\max_{i\in\{0,\ldots,n\}} degree(q_i(x))+n\max_{i\in\{0,\ldots,m\}} degree(t_i(x))=168.$$

We replace in the Sylvester determinant each $q_i(x)$ by its monomial of highest degree if $degree(q_i(x)) = \max_{i\in\{0,\ldots,n\}} degree(q_i(x))$ and by 0 otherwise, and each $t_i(x)$ by its monomial of highest degree if $degree(t_i(x)) = \max_{i\in\{0,\ldots,m\}} degree(t_i(x))$ and by 0 otherwise. We then compute the determinant and obtain :

$$2^{56}\,d^8\,(d^2-1)^{32}\,h^{16}\,(h'^2\sin^2\omega'+(h'\cos\omega'-d)^2)^4 \\ \times(d^2h'^2\sin^2\omega'+(dh'\cos\omega'-1)^2)^4\,x^{168}$$

When this monomial is not zero, its degree is 168 and so it is the leading monomial of $R(x)$. Hence $R(x)\equiv 0$ only if $d=1$ or $h=0$ or $(\omega',h')=(0,d)$ or $(\omega',h')=(0,1/d)$. $h\neq 0$ since, otherwise $\mathcal{P}$ is of type $X\bar{C}\bar{C}'SX'$ where X is a circular arc; the proof of Theorem 9 shows that $\mathcal{P}$ is optimal only if either $\bar{C}$ or $\bar{C}'$ is anchored which contradicts the assumption made at the beginning of the proof. If $(\omega',h')=(0,d)$ or $(\omega',h')=(0,1/d)$ we replace ω' and h' by their value in Q and T. Then, we apply the same procedure as above and show that the leading monomial of $R(x)$ is

$$2^{72}\,d^{16}\,(d^2-1)^{32}\,h^{16}\,x^{152}$$

This leading monomial does not depend on whether $h'=d$ or $h'=1/d$ and on the orientation of $\mathcal{P}$ on $\mathcal{X}$ and $\mathcal{X}'$. Since the obstacles are disjoint, $d\neq 1$ and $R(x)\not\equiv 0$ (in the case $d=1$, it can also be shown that the number of roots of the system of equations is finite).

Hence, the number of roots of System 2 is finite since any given value of x determines at most two triplets of values for the other indeterminates y, z and t (see Figure 7). It follows that the number of possible equilibriums of our mechanical device is finite. That ends the proof of Theorem 14.

Remark 17 As we have seen, the number of optimal subpaths of type $XS\bar{C}\bar{C}'SX'$ is finite as soon as the number of roots of an algebraic system of four equations in four indeterminates is finite. We have shown that this is indeed the case for any choice of the parameters of the system (which represent the position of the edges $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{O}$ and $\mathcal{O}'$). A weaker result can be obtained by simply choosing pseudo

[2] We used AXIOM on a Sun Sparc-10 4×72Mhz with 512MB of main memory. Notice that several polynomials considered here are two big to be computed with MAPLE.

[3] This computation takes roughly eleven hours and the process exceeds 130MB.

random values for the parameters. The fact that the number of roots of the system is finite for a pseudo random choice of the parameters implies that, with probability close to 1, the number of roots of the system is finite for almost all set of parameters. Moreover, by computing a Gröbner basis of the system for a pseudo random set of parameters, we can show that the number of roots of the system is at most 36 with probability close to 1 (instead of 336 as given by the computations above).

6 The algorithm

Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the disjoint moderate obstacles. We denote by $\mathcal{S}_O$ the set of the obstacle edges and by n its size.

Let S and F be the initial and the final point of the optimal path that we want to compute. By Theorems 9 and 10, any C-segment is either an anchored C-segment, or is adjacent to a terminal C-segment and to an anchored C-segment, or belongs to a subpath of type $XS\bar{C}\bar{C}SX'$ where $X, X' \in \{O, \bar{\bar{C}}\}$ (the lengths of the S-segments being possibly zero).

The algorithm computes first the set $\mathcal{S}_{\bar{\bar{C}}}$ of all the maximal free anchored arcs of circle. A *maximal free anchored arc* is a maximal arc of an anchored circle that does not intersect the interior of the obstacles. It will be simply called a free anchored arc in the sequel. We will also say for short that an arc (or a subpath) intersects an obstacle iff it intersects the interior of the obstacle.

To each obstacle and for a given r, we associate a *grown obstacle* which is the Minkowski sum of the obstacle and of a disk of radius r. Let $\mathcal{A}_r$ be the arrangement of the boundaries of these grown obstacles. A point is said of *level* i in $\mathcal{A}_r$ if it belongs to the interior of i grown obstacles. The vertices of level 0 are simply the vertices of the boundary of the union of the grown obstacles. Because the obstacles are disjoint, there are $O(n)$ such vertices by a result of Kedem et al. [14]. The same bound holds for the number of vertices of the k first levels for any constant k by the random sampling theorem of Clarkson and Shor [8].

Lemma 18 *The number of free anchored arcs is $O(n)$ and these arcs can be computed in $O(n \log n)$ time.*

Proof: A circle of unit radius is intersected by at most five obstacles. Indeed, the obstacles are disjoint and moderate which implies that each one contains a circle of unit radius. The claim follows since there are at most five pairwise disjoint circles of unit radius that may intersect a given circle of unit radius. It follows that any point is of level at most five in $\mathcal{A}_1$. Hence, $\mathcal{A}_1$ has linear size and can be computed in $O(n \log n)$ time by standard techniques.

Since the centers of the anchored circles are the vertices of $\mathcal{A}_1$ and each anchored circle is intersected by at most five obstacles, the lemma is proved. $\square$

Once the free anchored arcs have been computed, we compute the set $\mathcal{S}_S$ of all the free line segments that are tangent to two arcs of $\mathcal{S}_{\bar{\bar{C}}} \cup \mathcal{S}_O$.

Lemma 19 *$\mathcal{S}_S$ has size $O(n^2)$ and can be computed in $O(n^2 \log n)$ time.*

Proof: First, we compute all the free line segments tangent to two obstacles. Let us consider each obstacle in turn, say $\mathcal{O}_1$ for concreteness. Let D_θ be an oriented line of orientation θ tangent to $\mathcal{O}_1$. The set of commun points between $\mathcal{O}_1$ and D_θ is either a single point or a line segment; let P_θ be either this single point or the middle point of this line segment. If D_θ intersects $\mathcal{O}_i$ ($i = 2, \ldots, m$), we call P_i the point common to D_θ and $\mathcal{O}_i$ that is closer to P_θ. Let $f_i(\theta)$ be the algebraic length of $P_\theta P_i$ on the oriented line D_θ.

Let $\mathcal{E}^+$ (resp. $\mathcal{E}^-$) be the lower (upper) envelope of the functions f_i that are positive (negative). As the obstacles are pairwise disjoint, $f_i(\theta) \neq 0$ and $f_i(\theta) \neq f_j(\theta)$ for all θ and $i \neq j$. It follows that $\mathcal{E}^+$ and $\mathcal{E}^-$ can be computed in $O(n \log n)$ time. A line segment joining P_θ to P_i is free iff $(\theta, f_i(\theta))$ belongs to $\mathcal{E}^+$ or $\mathcal{E}^-$. Moreover, a line segment $P_\theta P_i$ is tangent to $\mathcal{O}_i$ iff P_i is an end-point of f_i. Hence, computing the free line segments tangent to $\mathcal{O}_1$ and another obstacle reduces to compute $\mathcal{E}^+$ and $\mathcal{E}^-$.

Repeating the above procedure for all the obstacles, we conclude that all the free line segments tangent to two obstacles can be computed in $O(n^2 \log n)$ time.

We now compute the free line segments tangent to a free anchored arc and to either an obstacle or another anchored free arc. Let $C_1, \ldots, C_p$ be the anchored free arcs. We consider in turn each free anchored arc, say C_1 for concreteness, and apply exactly the same procedure as above to compute the free line segments tangent to C_1 and to an obstacle. As above, these segments can be computed in $O(n \log n)$ time by computing the envelopes $\mathcal{E}^+$ and $\mathcal{E}^-$. It remains to compute the free line segments tangent to C_1 and to the other anchored free arcs $C_2, \ldots, C_p$. We define a function g_i involving C_1 and C_i and similar to the function f_i defined above. To each end point of g_i that lies between $\mathcal{E}^+$ and $\mathcal{E}^-$ corresponds a free line segment tangent to C_1 and C_i. Deciding if such an end point lies between $\mathcal{E}^+$ and $\mathcal{E}^-$ can be done in $O(\log n)$ time by binary search once the envelopes have been computed. As the number of free anchored arcs is $O(n)$ by Lemma 18, the free line segments tangent to C_1 and to another anchored free arc can be computed in $O(n \log n)$ time. Hence, the free line segments tangent to an anchored free arc and to either an obstacle or another anchored free arc can be computed in $O(n^2 \log n)$ time in total. $\square$

We consider now the subpaths of type $C_t C \bar{\bar{C}}$ and $\bar{\bar{C}} C C_t$. We compute the set $\mathcal{S}_C$ of all the circular arcs that avoid the obstacles and are tangent to a terminal circle and to an anchored free arc. As there are $O(n)$ anchored free arcs, this step can easily be done in $O(n^2)$ time.

By Theorem 9, $\mathcal{S}_{\bar{\bar{C}}} \cup \mathcal{S}_O \cup \mathcal{S}_S \cup \mathcal{S}_C$ contains all the arcs potentially taken by an optimal path except the subpaths of type $XS\bar{C}\bar{C}SX'$. We consider, in turn, all the quadruplets $(\mathcal{X}, \mathcal{O}, \mathcal{O}', \mathcal{X}')$ where $\mathcal{X}$ and $\mathcal{X}'$ are obstacle edges or anchored arcs, and where $\mathcal{O}$ and $\mathcal{O}'$ are two obstacle edges. First, we compute the family of potential optimal subpaths of type $XS\bar{C}\bar{C}SX'$ where X (resp. X') is an arc of $\mathcal{X}$ ($\mathcal{X}'$)

and the two C-segments $\bar{C}\bar{C}$ are tangent to respectively $\mathcal{O}$ and $\mathcal{O}'$. In a second step, we will check whether or not these potential optimal subpaths intersect other obstacles.

By solving an algebraic system as described in the proof of Theorem 14, we compute the family of potential optimal subpaths of type $XS\bar{C}\bar{C}SX'$ when none of the two C-segments tangent to $\mathcal{O}$ and $\mathcal{O}'$ is anchored. That step can be performed in constant time for each chosen quadruplet assuming that the roots of a polynomial of bounded degree can be computed in constant time. Hence the total time complexity of this step is $O(n^4)$. We also compute the subpaths of type $XS\bar{\bar{C}}\bar{C}SX'$ and $XS\bar{C}\bar{\bar{C}}SX'$. As the number of anchored C-segments is $O(n)$, the total number of these subpaths is $O(n^4)$ and they can be easily computed in $O(n^4)$ time. It remains to compute the set $\mathcal{S}_{\bar{C}\bar{C}}$ of those subpaths that avoid the obstacles.

Lemma 20 *$\mathcal{S}_{\bar{C}\bar{C}}$ can be computed in $O(n^4 \log n)$ time.*

Proof: We show that we can check in $O(\log n)$ time whether or not a given subpath of type $XS\bar{C}\bar{C}SX'$ intersects the obstacles. We consider successively the case of an arc of circle and the case of a line segment.

As mentionned in the proof of Lemma 18, a circle of unit radius intersects at most five obstacles. We can identify the obstacles that intersect the circle supporting a given arc $\bar{C}$ by locating in $O(\log n)$ time the center of this circle in the arrangement $\mathcal{A}_1$. It then remains to check if the arc $\bar{C}$ (not the whole circle) actually intersects one of the obstacles. Each such test can be done in $O(\log n)$ time since each obstacle has $O(n)$ edges [9].

We describe now how to check if a line segment S of a subpath of type $XS\bar{C}\bar{C}SX'$ intersects the obstacles. By Lemma 11, the length of S is at most 4. It follows that if S intersects an obstacle, each of its end points are contained in the obstacle grown by a disk of radius 4. As the obstacles are disjoint and moderate, a point can only be contained in $g = O(1)$[4] such grown obstacles : hence, arrangement $\mathcal{A}_4$ has linear size and can be computed in $O(n \log n)$ time. We locate one endpoint of S in $\mathcal{A}_4$ and find the at most g obstacles that may intersect S. We consider in turn each of these obstacles and check if S indeed intersects the obstacle. This can be done in $O(\log n)$ time [9]. $\square$

By Theorems 9, $\mathcal{S}_{\bar{C}} \cup \mathcal{S}_O \cup \mathcal{S}_S \cup \mathcal{S}_C \cup \mathcal{S}_{\bar{C}\bar{C}}$ contains all the arcs potentially taken by an optimal path.

Let $\mathcal{G}$ be the weighted graph whose nodes are the tangent points between two arcs of $\mathcal{S}_{\bar{C}} \cup \mathcal{S}_O \cup \mathcal{S}_S \cup \mathcal{S}_C \cup \mathcal{S}_{\bar{C}\bar{C}}$ and whose edges are the arcs of $\mathcal{S}_{\bar{C}} \cup \mathcal{S}_O \cup \mathcal{S}_S \cup \mathcal{S}_C \cup \mathcal{S}_{\bar{C}\bar{C}}$. The final step of the algorithm consists in searching a shortest path in this graph.

Theorem 21 *An optimal path amidst a set of disjoint moderate obstacles with n edges in total can be computed in $O(n^4 \log n)$ time.*

[4] g is the maximal number of disjoint disks of unit radius that can be packed in a disk of radius 6.

Improving the performances of the algorithm

We now show that the time complexity of the algorithm can be reduced in most practical situations. This is a consequence of the fact that the subpaths of type $\bar{C}\bar{C}$ can only be encountered near the endpoints of the path. Theorem 22 is a consequence of a result of Agarwal et al. A complete proof can be found in the full version of this paper.

Theorem 22 *An optimal path contains at most two non-terminal $\bar{C}\bar{C}$-subpaths (*i.e. subpaths of type $\bar{C}\bar{C}$ where both $\bar{C}$-segments are not terminal*). Moreover, if an optimal path consists of four parts $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ in this order, where $\mathcal{P}_2$ and $\mathcal{P}_3$ are C-segments centered at O_2 and O_3, then $\forall M \in \mathcal{P}_1\ MO_3 \le 3$ or $\forall N \in \mathcal{P}_4,\ NO_2 \le 3$.*

Theorem 23 *Let $\mathcal{P}$ be an optimal path joining S to F. Any subpath of $\mathcal{P}$ of type $XS\bar{C}\bar{C}SX'$, where $X, X' \in \{O, \bar{\bar{C}}\}$, is contained in one of the two disks of radius 9 centered at S or F.*

Proof: This theorem follows from Theorem 22 and we use the notations introduced in this theorem. Let $\mathcal{P}_1$ (resp. $\mathcal{P}_4$) be the portion of $\mathcal{P}$ between S (resp. F) and the first (resp. last) $\bar{C}$ in the considered subpath. From Theorem 22, we have $\forall M \in \mathcal{P}_1\ MO_3 \le 3$ or $\forall N \in \mathcal{P}_4\ NO_2 \le 3$. Assume without loss of generality that $\forall M \in \mathcal{P}_1\ MO_3 \le 3$. Then, S and the whole subpath $XS\bar{C}\bar{C}$ is included in the disk of radius 3 centered at O_3. On the other hand, by Lemma 11, the length of the line segment preceding X' is smaller than 4. Therefore, S and the whole subpath $XS\bar{C}\bar{C}SX'$ is included in a disk of diameter $3 + 2\sqrt{5} + 1 < 9$. Hence the subpath of type $XS\bar{C}\bar{C}SX'$ is included in a disk of radius 9 centered at S. $\square$

It follows that we can improve the procedure that computes the subpaths of type $XS\bar{C}\bar{C}SX'$. Indeed, instead of considering all n^4 quadruplets $(\mathcal{X}, \mathcal{O}, \mathcal{O}', \mathcal{X}')$, we can only consider those that intersect one of the disks of radius 9 centered at S and F. If k is the number of such quadruplets, the time complexity of the algorithm becomes $O(n^2 \log n + k^4 \log k)$. In particular, if the length of any obstacle edge is bounded from below by some positive constant, then $k = O(1)$.

Theorem 24 *Given a set of disjoint moderate obstacles with n edges whose lengths are bounded from below by some positive constant. An optimal path between two configurations amidst those obstacles can be computed in $O(n^2 \log n)$ time.*

7 Final remarks and open questions

The geometric results and the algorithm hold even if the obstacles are not disjoint. However, the time-complexity increases since the number of anchored C-segments may be not linear.

In this paper, we have considered obstacles whose boundaries consist of line segments and circular arcs of unit radius.

It would be interesting to consider more general moderate obstacles (in the sense of Agarwal et al.) and, in particular, obstacles whose boundaries consist of line segments and circular arcs of radii greater than or equal to 1. The system of equations corresponding to the equilibriums of the mechanical device (see Section 5) is very similar to the one in Lemma 16. However, the computations exceed the capabilities of the current computer algebra systems[5] and we have not been able to apply techniques similar to those in Section 5.

Many other questions remain open. We mention three of them we plan to consider in near future : Can similar results be obtained for polygonal robots? Can similar results be obtained if backwards moves are allowed? (preliminary results in that direction can be found in [1, 6, 21]). How crucial is the moderate hypothesis?

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[5] Using AXIOM, the size of the process exceeds 500MB.

References

[1] P. K. Agarwal, P. Raghavan, and H. Tamaki. Motion planning for a steering-constrained robot through moderate obstacles. In *Proc. 27th Annu. ACM Sympos. Theory Comput.*, pages 343–352, 1995.

[2] J. Barraquand and J-C. Latombe. Nonholonomic mobile robots and optimal maneuvering. *Revue d'Intelligence artificielle*, 3:77–103, 1989.

[3] J. Barraquand and J-C. Latombe. Nonholonomic multibody mobile robots : controllability and motion planning in the presence of obstacles. *Algorithmica*, 10:121–155, 1993.

[4] Behnke, Bachmann, and Fladt. *Fundamentals of mathematics, Geometry*, volume 1. MIT Press, 1986.

[5] J-D. Boissonnat and X-N. Bui. Accessibility region for a car that only moves forwards along optimal paths. Research Report 2181, INRIA, BP93, 06902 Sophia-Antipolis, France, 1994.

[6] J.-D. Boissonnat, A. Cérézo, and J. Leblond. Shortest paths of bounded curvature in the plane. *Internat. J. Intell. Syst.*, 10:1–16, 1994.

[7] X-N. Bui, P. Souères, J-D. Boissonnat, and J-P. Laumond. Shortest path synthesis for Dubins nonholonomic robot. In *IEEE Int. Conf. on Robotics and Automation*, pages 2–7, San Diego, 8-13 Mai 1994.

[8] K. L. Clarkson and P. W. Shor. Applications of random sampling in computational geometry, II. *Discrete Comput. Geom.*, 4:387–421, 1989.

[9] D. P. Dobkin and D. L. Souvaine. Computational geometry in a curved world. *Algorithmica*, 5:421–457, 1990.

[10] L. E. Dubins. On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents. *Amer. J. Math.*, 79:497–516, 1957.

[11] S. Fortune and G. Wilfong. Planning constrained motion. In *Proc. 20th Annu. ACM Sympos. Theory Comput.*, pages 445–459, 1988.

[12] P. Jacobs and J. Canny. Planning smooth paths for mobile robots. In *Proc. IEEE Internat. Conf. Robot. Autom.*, pages 2–7, 1989.

[13] P. Jacobs, J-P. Laumond, and M. Taix. Efficient motion planners for nonholonomic mobile robots. *In Proceedings of the IEEE/RSJ International Workshop on Intelligent Robots and Systems*, pages 1229–1235, 1991.

[14] K. Kedem, R. Livne, J. Pach, and M. Sharir. On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles. *Discrete Comput. Geom.*, 1:59–71, 1986.

[15] J-C. Latombe. A fast path-planner for a car-like indoor mobile robot. *In Proceedings of the 9th National Conference on Artificial Intelligence*, pages 659–665, 1991.

[16] J.-C. Latombe. *Robot Motion Planning.* Kluwer Academic Publishers, Boston, 1991.

[17] J-P. Laumond. Finding collision-free smooth trajectories for a non-holonomic mobile robot. *In Proceedings of the International Joint Conference on Artificial Intelligence*, pages 1120–1123, 1987.

[18] J-P. Laumond, M. Taix, P. Jacobs, and R.M. Murray. A motion planner for nonholonomic mobile robots. *IEEE Trans. on Robotics and Automation*, 1993.

[19] Z. Li and J.F. Canny. *Nonholonomic Motion Planning.* Kluwer Academic Publishers, 1992.

[20] Y. Nakamura and R. Mukherjee. Nonholonomic path planning and automation. *In Proc. IEEE Int. Conf. on Robotics and Automation*, pages 1050–1055, 1989.

[21] J. A. Reeds and L. A. Shepp. Optimal paths for a car that goes both forwards and backwards. *Pacific Journal of Mathematics*, 145(2), 1990.

[22] H. J. Sussmann and G. Tang. Shortest paths for the reeds-shepp car: a worked out example of the use of geometric techniques in nonlinear optimal control. Research Report SYCON-91-10, Rutgers University, New Brunswick, NJ, 1991.

[23] H. Wang and P.K. Agarwal. Approximation algorithms for curvature constrained shortest paths. In *Proc. 7th ACM-SIAM Sympos. Discrete Algorithms (SODA '96)*, 1996.

[24] G. Wilfong. Motion planning for an autonomous vehicle. In *Proc. IEEE Internat. Conf. Robot. Autom.*, pages 529–533, 1988.

[25] G. Wilfong. Shortest paths for autonomous vehicles. *In Proceedings of the IEEE International on Robotics and Automation*, pages 15–20, 1989.