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LINEARIZATION OF A COUPLED SYSTEM OF NONLINEAR ELASTICITY AND FLUID

LORENA BOCIU AND JEAN-PAUL ZOLÉSIO

Abstract. We model the coupled system formed by an incompressible, irrotational fluid and a nonlinear elastic body. We work with large displacement, small deformation elasticity (or St Venant elasticity), which makes the problem very interesting from the physical point of view. The elastic body is three-dimensional $\Omega \in \mathbb{R}^3$, and thus it can not be reduced to its boundary Γ (like in the case of a membrane or a shell). In this paper, we study the static problem, which contrary to common belief, it is more subtle than the dynamical one (since in real life, evolution is more plausible than equilibrium).

1. Introduction

1.1. **The model and the problem.** We consider a model of fluid-structure interaction on a bounded domain $\mathcal{D} \in \mathbb{R}^3$. We assume that $\mathcal{D}$ is comprised of two open domains $\mathcal{D} = \Omega \cup \Omega^C$, and has smooth boundary $\partial\mathcal{D} = \Gamma' \cup \Gamma_{in} \cup \Gamma_{out}$. The elastic body occupies domain Ω with sufficiently smooth boundary $\Gamma \cup \Gamma'$, and is described by a nonlinear elastic equation in terms of the displacement u. The fluid occupies domain Ω^C with boundary $\Gamma \cup \Gamma_{in} \cup \Gamma_{out}$, and is described by a Navier-Stokes equation in terms of the velocity of the fluid v and the pressure p. The interaction takes place on the common boundary Γ and is realized via suitable transmission boundary conditions. We assume that there is a flux $\vec{f}$ coming into $\mathcal{D}$ through Γ_{in}, that will determine the velocity of the fluid v (see Figure 1).

One specific example of the above mentioned model is a 3D tube with elastic walls through which a fluid is flowing. From the physical point of view, this is a very important model with a lot of applications in mathematical biology, more precisely, the study of arterial diseases (the tube represents the artery, the elastic body is the wall of the artery and the fluid is the blood).

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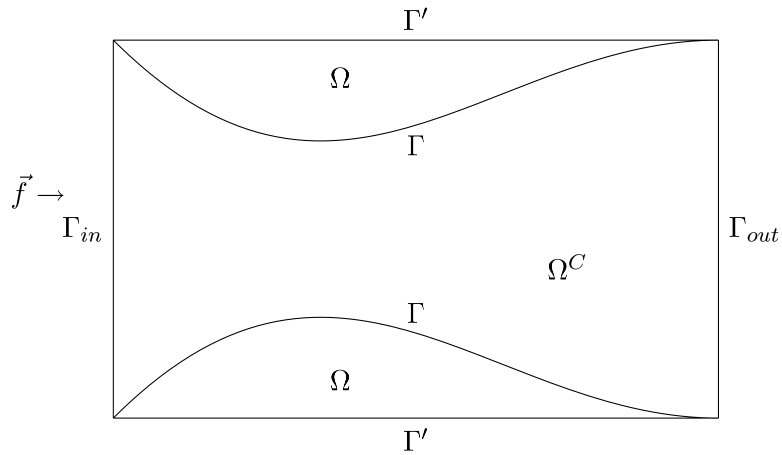

We begin the description of our model by introducing the notation and the basic assumptions on the two equations present in the system.

• **Nonlinear, 3D elasticity**: We work with large displacement, small deformation elasticity (or St Venant elasticity [11]), which makes the problem difficult from the mathematical point of view, and very interesting from the physical point of view. The elastic body is three-dimensional $\Omega \in \mathbb{R}^3$, and thus it can not be reduced to its boundary Γ (like in the case of a membrane or a shell).

At rest, the elastic body occupies a reference configuration $\overline{\mathcal{O}} \in \mathbb{R}^3$, where $\mathcal{O}$ is a bounded, open, connected set in $\mathbb{R}^3$ with sufficiently smooth boundary $\mathcal{S} \cup \Gamma'$. When subjected to applied forces, the elastic body occupies a deformed configuration $\Omega = \varphi(\overline{\mathcal{O}})$, with smooth boundary $\Gamma \cup \Gamma'$ (where Γ' is fixed). The deformation of the reference configurations is given by the map $\varphi : \overline{\mathcal{O}} \to \mathbb{R}^3$, that is smooth enough, injective (except possibly on the boundary of the set $\mathcal{O}$), and orientation-preserving (i.e. $\det\nabla\varphi(x) > 0$, for all $x \in \overline{\mathcal{O}}$).

Together with the deformation φ, we introduce the displacement $u : \overline{\mathcal{O}} \to \mathbb{R}^3$, defined as usual as $\varphi = I + u$, where I denotes identity map $I : \overline{\mathcal{O}} \to \mathbb{R}^3$.

It is well known that a body occupying a deformed configuration $\overline{\Omega}$, and subjected to zero applied body forces in its interior Ω and to applied surface forces on the boundary Γ, is in static equilibrium if the fundamental stress principle of Euler and Cauchy is satisfied:

$$\begin{cases} -div\mathcal{T} = 0 & \text{in } \Omega \\ \mathcal{T}n^\varphi = g^\varphi & \text{on } \Gamma \end{cases} \tag{1.1}$$

where $\hat{g}$ represents the density of the applied surface force, n^φ is the unit outer normal vector along Γ, and the tensor $\mathcal{T}$ is the Cauchy stress tensor. The above equilibrium equations over $\overline{\Omega}$ are equivalent to the equilibrium equations over the reference configuration $\overline{\mathcal{O}}$:

$$\begin{cases} -div\mathcal{P} = 0 & \text{in } \mathcal{O} \\ \mathcal{P}n = g & \text{on } \mathcal{S} \end{cases} \tag{1.2}$$

where n denotes the unit outer normal vector along $\mathcal{S}$, $gda = \hat{g}da^{\varphi}$, and $\mathcal{P} : \overline{\mathcal{O}} \to \mathbb{M}^3$ is the Piola transform of the Cauchy stress tensor field, defined by

$$\mathcal{P}(x) = \mathcal{T}(x^{\varphi})\mathrm{Cof}\nabla T(x) = \det(\nabla\varphi(x))\mathcal{T}(x^{\varphi})(\nabla\varphi)^{-*} \tag{1.3}$$

From the constitutive equations, we have that $\mathcal{P}(x) = \nabla\varphi(x)\Sigma(u(x))$, where Σ defines the second Piola-Kirchhoff stress tensor. In terms of the displacement u, Σ is given by

$$\Sigma(\sigma(u)) = \lambda(\mathrm{tr}\sigma(u))I + 2\mu\sigma(u) \tag{1.4}$$

where λ and μ are the Lame constants of the material, and the Green-St Venant strain tensor $\sigma(u)$ is given by

$$\sigma(u) = \frac{1}{2}(Du^* + Du + Du^*Du) \tag{1.5}$$

Therefore equations (1.2) can be rewritten as

$$\begin{cases} -div[(I + \nabla u)\Sigma(\sigma(u))] = 0 & \text{in } \mathcal{O} \\ (I + \nabla u)\Sigma(\sigma(u))n = g & \text{on } \mathcal{S} \end{cases} \tag{1.6}$$

The advantage of the equilibrium equations over the reference configuration (1.2) or (1.6) over (1.1) is the fact that they are written in terms of the Lagrange variable x that is attached to the reference configuration, instead of the Euler variable $x^{\varphi} = \varphi(x)$, which is precisely one of the unknowns.

Nevertheless, we want to stress the fact that equations (1.1) play a critical role when dealing with elastic body - fluid systems, where the coupling is taking place on the boundary interface between the two media. This interface is precisely the boundary Γ of the deformed configuration of the elastic body Ω and thus the coupling requires the continuity of the velocities and the normal stress tensors across Γ. Therefore, we need a relationship between the Cauchy stress tensor $\mathcal{T}$ and the strain tensor $\sigma(u)$, that will provide us with the correct matching of the two dynamics on the common interface.

Recalling the relations between $\mathcal{P}$, $\mathcal{T}$, and $\Sigma(u)$ we obtain that

$$\mathcal{T} = \Big(\frac{1}{\det(\nabla\varphi)}\nabla\varphi \cdot \Sigma(\sigma(u)) \cdot (\nabla\varphi)^*\Big) \circ \varphi^{-1} \tag{1.7}$$

• **Potential fluid**: We assume that the fluid present in the system is incompressible and irrotational. This means that the fluid is a potential fluid $v = \nabla\phi$, where v is the velocity of the fluid and ϕ (the velocity potential of the fluid) satisfies the Laplace equation $\Delta\phi = 0$ (due to the incompressibility condition which translates into $\nabla \cdot u = 0$).

Now if p represents the pressure of the fluid and η the viscosity, then the flow is described by the following Navier-Stokes equation

$$v \cdot \nabla v - \eta \Delta v + \nabla p = \rho \vec{g} \tag{1.8}$$

where ρ is the density, and $\vec{g}$ is the gravitational acceleration. Due to the fluid being irrotational (vorticity curl $v = 0$), the convective acceleration reduces to $v \cdot \nabla v = \nabla\Big(\dfrac{\|v\|^2}{2}\Big)$ and thus (1.8) becomes

$$\nabla(\frac{1}{2}\|\nabla\phi\|^2 + p - \rho g z) = 0$$

This provides us with the formula for the pressure p of the fluid:

$$p = c + \frac{1}{2}\|\nabla\phi\|^2 - \rho g z \tag{1.9}$$

1.2. **Fluid-structure interaction: the mathematical model.** Now we couple the two dynamics described above and we require continuity of both the velocities and the normal stress tensors across the common boundary Γ. Recall that $\mathcal{D} = \Omega \cup \Omega^C$, with boundary $\partial\mathcal{D} = \Gamma' \cup \Gamma_{in} \cup \Gamma_{out}$, the elastic body occupies domain Ω, and the fluid occupies Ω^C (see Figure 1).

We obtain the following PDE model in variables (ϕ, u):

$$\begin{cases} \Delta\phi = 0 & , \Omega^C \\ -div(\mathcal{T}) = 0 & , \Omega \\ \nabla\phi \cdot n = 0 & , \Gamma \\ \mathcal{T} \cdot n = (c + \frac{1}{2}\|\nabla\phi\|^2 - \rho g z) n & , \Gamma \\ u = 0 & , \Gamma' \end{cases} \tag{1.10}$$

where n is the unit outer normal vector along Γ, and f is the flux coming in $\mathcal{D}$ through Γ_{in}. Recall that $\mathcal{T}$ is the Cauchy stress tensor, given by

$$\mathcal{T} = \Big(\frac{1}{\det(\nabla\varphi)} \nabla\varphi \cdot \Sigma(\sigma(u)) \cdot (\nabla\varphi)^*\Big) \circ \varphi^{-1}$$

Thus the two equations on Γ present in (1.10) are equivalent to

$$\begin{cases} \nabla\phi \cdot n = 0 & , \text{on } \Gamma \\ \frac{1}{\det(\nabla\varphi)} \nabla\varphi \cdot \Sigma(\sigma(u)) \cdot (\nabla\varphi)^* \cdot n \circ \varphi = [(c + \frac{1}{2}\|\nabla\phi\|^2 - \rho g z) n] \circ \varphi & , \text{on } \Gamma \end{cases} \tag{1.11}$$

1.2.1. *Boundary conditions on* Γ_{in}. Now we describe the flow (which so far we called f) coming into the domain through Γ_{in}. Let $c(x)$ be a given, smooth function defined on Γ_{in} such that

$$\begin{cases} c(x) = 0 & \text{on } \partial\Gamma_{in}, \\ \frac{\partial}{\partial n}\phi = c(x) & \text{on } \Gamma_{in} \end{cases} \tag{1.12}$$

Then it follows that

$$0 = \int_{\Omega^c} \operatorname{div}(\nabla\phi)dx = \int_{\Gamma_{in}} \frac{\partial\phi}{\partial n_{in}}\, d\Gamma_{in} + \int_{\Gamma_{out}} \frac{\partial\phi}{\partial n_{out}}\, d\Gamma_{out} \tag{1.13}$$

At this point, we choose $\alpha \in \mathbb{R}$ verifying

$$\begin{cases} \alpha = \frac{\partial\phi}{\partial n_{out}} \quad \text{on} \quad \Gamma_{out}, \\ \\ \int_{\Gamma_{out}} \alpha \; d\Gamma_{out} = -\int_{\Gamma_{in}} c(x) \; d\Gamma_{in} \end{cases} \tag{1.14}$$

2. Description of the Static Problem

2.1. **Parameter of Perturbation s.** As previously mentioned, in this paper we are interested in the static problem associated with system (1.10). In a subsequent paper, we will consider the dynamical case. Contrary to common belief, the steady problem is more subtle than the dynamical one, since in real life, evolution is more plausible than equilibrium.

We assume that the flux entering the domain $\mathcal{D}$ is dependent on a parameter of perturbation s, i.e. $c(x)$ is a given, smooth function defined on Γ_{in} such that

$$\begin{cases} c(x) = 0 & \text{on} \quad \partial\Gamma_{in}, \\ \frac{\partial}{\partial n}\phi_s = (1+s)c(x) & \text{on} \quad \Gamma_{in} \end{cases} \tag{2.1}$$

Then it follows that

$$0 = \int_{\Omega_s^c} \operatorname{div}(\nabla\phi_s)dx = \int_{\Gamma_{in}} \frac{\partial\phi_s}{\partial n_{in}} d\Gamma_{in} + \int_{\Gamma_{out}} \frac{\partial\phi_s}{\partial n_{out}} d\Gamma_{out} \tag{2.2}$$

For any $s \geq 0$, we choose $\alpha_s \in \mathbb{R}$ verifying

$$\begin{cases} \alpha_s = \frac{\partial\phi_s}{\partial n_{out}} \quad \text{on} \quad \Gamma_{out}, \\ \\ \int_{\Gamma_{out}} \alpha \; d\Gamma_{out} = -(1+s)\int_{\Gamma_{in}} c(x) \; d\Gamma_{in}, \quad \text{for all} \quad s \geq 0 \end{cases} \tag{2.3}$$

If the elastic body occupies a reference configuration $\overline{\mathcal{O}} \in \mathbb{R}^3$ with smooth boundary $\mathcal{S} \cup \Gamma'$, then, when subjected to applied forces, it occupies a deformed configuration $\Omega_s = \varphi_s(\overline{\mathcal{O}})$, with smooth boundary $\Gamma_s \cup \Gamma'$ (where Γ' is fixed). The deformation map in this case is dependant on the parameter s: $\varphi_s : \overline{\mathcal{O}} \to \mathbb{R}^3$, but nevertheless is smooth enough, injective, and orientation-preserving. The displacement $u_s : \overline{\mathcal{O}} \to \mathbb{R}^3$ becomes $u_s = \varphi_s - I$, where I is the identity map $I : \overline{\mathcal{O}} \to \mathbb{R}^3$.

Similarly, for the fluid present in the system, its velocity and pressure are now functions of s: $v_s = \nabla\phi_s$, and $p_s = c_s + \frac{1}{2}\|\nabla\phi_s\|^2 - \rho g z$.

Therefore we are interested in the following PDE model:

$$\begin{cases} \Delta\phi_s = 0 & ,\Omega_s^C \\ -div(\mathcal{T}_s) = 0 & ,\Omega_s \\ \nabla\phi_s \cdot n_s = 0 & ,\Gamma_s \\ \mathcal{T}_s \cdot n_s = (c_s + \frac{1}{2}\|\nabla\phi_s\|^2 - \rho g z)n & ,\Gamma_s \\ u = 0 & ,\Gamma' \\ \int_{\Gamma_{out}} \alpha \; d\Gamma_{out} = -(1+s)\int_{\Gamma_{in}} c(x)\; d\Gamma_{in}, \quad \text{for all} \quad s \geq 0 \end{cases} \tag{2.4}$$

where n_s is the unit outer normal vector along Γ_s, $\mathcal{T}_s$ is the Cauchy stress tensor (associated to s), given by

$$\mathcal{T} = \Big(\frac{1}{\det(\nabla\varphi_s)}\nabla\varphi_s \cdot \Sigma(\sigma(u_s)) \cdot (\nabla\varphi_s)^*\Big) \circ \varphi_s^{-1} \tag{2.5}$$

and the boundary conditions are equivalent to

$$\begin{cases} \nabla\phi_s \cdot n_s = 0 & \text{, on } \Gamma_s \\ \frac{1}{\det(\nabla\varphi_s)}\nabla\varphi_s \cdot \Sigma(\sigma(u_s)) \cdot (\nabla\varphi_s)^* \cdot n_s \circ \varphi_s = [(c_s + \frac{1}{2}\|\nabla\phi_s\|^2 - \rho g z)n_s] \circ \varphi_s & \text{, on } \Gamma_s \end{cases} \tag{2.6}$$

Our goal is to study, for any given value of s, the equilibrium problem associated with the coupled system (2.4).

2.2. **The moving boundary Γ_s.** Recall that the deformation map φ maps the reference boundary $\mathcal{S}$ to Γ. Similarly, the deformation φ_s associated with $\hat{f} = f + s$ maps $\mathcal{S}$ to Γ_s.

At this point it is convenient to introduce the map $T_s : \Gamma \to \Gamma_s$ that builds the moving boundary Γ_s:

$$T_s = \varphi_s \circ \varphi^{-1} \tag{2.7}$$

and the speed $V(s,\cdot)$ associated with the flow mapping T_s:

$$V(s,\cdot) = \Big(\frac{\partial}{\partial_s}T_s\Big) \circ T_s^{-1} = \frac{\partial}{\partial_s}\varphi_s \circ \varphi_s^{-1} \tag{2.8}$$

This means that $T_s(V) : X \to x(s)$, where $x(s)$ satisfies the following differential equation

$$\begin{cases} \frac{\partial}{\partial_s}x = V(s, x(s)) \\ x(0) = X \end{cases} \tag{2.9}$$

which is equivalent to $x(s) = X + \displaystyle\int_0^s V(t, x(t))dt$.

2.2.1. *Transport of scalar operators.* Let D be a bounded domain in R^N and T be a one-to-one transformation from D onto D and from $\bar{D}$ onto $\bar{D}$. Let $S = T^{-1}$ be the inverse mapping. As $T \circ S = I$, we obtain that $DT \circ S.DS = I$, and therefore

$$DS = (DT)^{-1} \circ S,$$

and

$$(DS) \circ T = (DT)^{-1}.$$

For any $\phi \in H^1(D)$ we have

$$(\nabla\phi) \circ T = (DT)^{-*}.\nabla(\phi \circ T)$$

We also have:

$$\frac{d}{ds}T_s = V(s) \circ T_s \Rightarrow \frac{d}{ds}T_s\Big|_{s=0} = V(0).$$

$$\frac{d}{ds}DT_s(X) = DV(s, T_s(X))DT_s(X), \quad DT_0(X) = I$$

$$\Rightarrow \frac{d}{ds}DT_s\Big|_{s=0} = DV(0) \quad \text{and} \quad \frac{d}{ds}(DT_s)^{-1}\Big|_{s=0} = -DV(0).$$

$$\frac{d}{ds}\mathrm{det}DT_s(X) = \mathrm{tr}DV(s, T_S(X))\mathrm{det}DT_s(X) = \mathrm{div}V(s, T_s(X))\mathrm{det}DT_s(X),$$

$$\Rightarrow \frac{d}{ds}\mathrm{det}(DT_S)\Big|_{s=0} = \mathrm{div}V(0).$$

Now let $\vec{E}$ be a C^1 vector field defined over D. Then we have the following property:

Property 2.1.

$$(divE) \circ T = det(DT)^{-1}\ div(\ det(DT)\,(DT)^{-1}.(E \circ T)\ )$$

Proof. Let $\phi \in C^\infty_{comp}(D)$. Using the change of variable $y = T(x)$ (or $x = S(y)$), we obtain:

$$\int_D (divE) \circ T(x)\,\phi(x)\,dx = \int_D divE(y)\,\phi \circ S(y)\,det(DS)(y)\,dy$$

$$= -\int_D <E(y),\ \nabla(\ \phi \circ S(y)\,det(DS)(y)\,)\ >\ dy$$

$$= -\int_D <E(y),\ \nabla(\ \phi \circ S(y)\,)\,det(DS)(y)\,) + \phi \circ S(y)\,\nabla(\ det(DS)(y)\,)\ >\ dy$$

[Using the identity $\nabla(\phi \circ S) = (DS)^*.(\nabla\phi) \circ S$, we obtain:]

$$= -\int_D <E(y),\ (DS)^*.(\nabla\phi)\circ S\,det(DS)(y)\,) + \phi\circ S(y)\,\nabla(\,det(DS)(y)\,)\ >\ dy$$

[Transposing of the matrix DS^*, we can rewrite as follows:]

$$= -\int_D \{ <det(DS)(y)\,)\,DS.E(y),\ (\nabla\phi)\circ S\ > + <\nabla(\,det(DS)(y)\,)\,E(y),\ \phi\circ S(y)\ >\}dy$$

[Performing the change of variable $y = T(x)$, we obtain:]

$$= -\int_D \{< det(DT)det(DS)\circ T\,DS\circ T.E\circ T, \nabla\phi > + < det(DT)(\nabla det(DS))\circ T\ E\circ T, \phi >\}dx$$

[As $(DS)\circ T = (DT)^{-1} \Rightarrow det(DS)\circ T = det(\,(DS)\circ T\,) = det(\,(DT)^{-1}\,) = (\,detDT\,)^{-1}$, then we have:]

$$= -\int_D \{< (DT)^{-1}.E \circ T, \nabla\phi > + < det(DT)(\nabla det(DS)) \circ T\, E \circ T, \phi >\}dx$$

[Using the fact that $(\nabla det(DS))\circ T = (DT)^{-*}.\nabla(det(DS)\circ T\,) = (DT)^{-*}.\nabla(\,\frac{1}{detDT}\,)$ $= -\frac{1}{(detDT)^2}\,(DT)^{-*}.\nabla(detDT)$, we obtain:]

$$= -\int_D \{< (DT)^{-1}.E\circ T, \nabla\phi > - < det(DT)^{-1}\ (DT)^{-*}.\nabla(detDT).E\circ T, \phi >\}dx$$

$$= \int_D \{div((DT)^{-1}.E \circ T) + det(DT)^{-1}\ < (DT)^{-*}.\nabla(detDT), E \circ T >\}\phi dx$$

$$= \int_D \{div((DT)^{-1}.E \circ T) + det(DT)^{-1}\ < \nabla(detDT), (DT)^{-1}.E \circ T >\}\phi dx$$

$$= \int_D det(DT)^{-1}\{det(DT)\, div((DT)^{-1}.E\circ T) + < \nabla(detDT), (DT)^{-1}.E\circ T >\}\phi dx$$

[For any scalar function a and any vector function $\vec{A}$ we have $div(a\,\vec{A}) = a\,div\vec{A} + < \nabla a, \vec{A} >_{R^3}$. Therefore we have that $det(DT)\,div((DT)^{-1}.E \circ T) + < \nabla(detDT), (DT)^{-1}.E \circ T > = div(det(DT)\,(DT)^{-1}.E \circ T)$, which gives us the desired conclusion:]

$$= \int_D det(DT)^{-1}\ div(det(DT)\,(DT)^{-1}.E \circ T)\ \phi dx$$

□

Similarly, we can prove the following proposition:

Property 2.2. *For any $\phi \in H^1(D)$, we have the following identity:*

$$\Delta\phi \circ T = det(DT)^{-1} div(det(DT)(DT)^{-1}(DT)^{-*}\nabla(\phi \circ T)) \tag{2.10}$$

2.2.2. *Transport of Vector Operators.* Let $\mathcal{T}$ be a $N \times N$ matrix function defined on D . We consider the vector Divergence operator $Div\vec{\mathcal{T}}$ being defined as the vector whose i^{th} component is the (scalar) divergence of the vector composed of the i^{th} line of the matrix $\mathcal{T}$:

$$(Div\vec{\mathcal{T}})_i = div(\mathcal{T}_{i,.}) = \Sigma_{j=1,\dots,N}\ \frac{\partial}{\partial x_j}\mathcal{T}_{i,j}$$

From the previous section we obtain that

$$(\,(Div\vec{\mathcal{T}}) \circ T)_i = (Div\vec{\mathcal{T}})_i \circ T = det(DT)^{-1}\ div(\ det(DT)\,(DT)^{-1}.(\mathcal{T}_{i,.}) \circ T\ )$$

It turns out that $(\mathcal{T}_{i,.}) \circ T$ is the i^{th} column vector of the matrix $\mathcal{T}^* \circ T$ so that $(DT)^{-1}.(\mathcal{T}_{i,.}) \circ T$ is the i^{th} column of the matrix $(DT)^{-1}.\mathcal{T}^* \circ T$, and thus the i^{th} line of the matrix $(DT)^{-1}.\mathcal{T}\circ T$. Therefore we have the following identity:

Property 2.3.

$$(Div\vec{\mathcal{T}}) \circ T = det(DT)^{-1} \; Div(\; det(DT) \, (DT)^{-1}.(\mathcal{T} \circ T) \,)$$

2.2.3. *Boundary change of variable.* Let $\Gamma = \partial\Omega$ be a C^1 manifold: there exists a covering $\Gamma \subset \cup_{i=1,\dots,M}\mathcal{O}_i$, open subsets and charts $c_i : \mathcal{O}_i \to B$ where B is the unit ball of R^N such that $c_i(\Gamma \cap \mathcal{O}_i) \subset B_0 = \{x = (x', 0) \in B\}$ and $c_i(\Omega \cap \mathcal{O}_i) \subset B + 0 = \{x = (x', z) \in B \; s.t.z > 0 \; \}$. Let $r_i \in C^\infty(D)$ ith compact support in $\mathcal{O}_i$ such that $0 \le r_i \le 1$, $\Sigma r_i = 1$ in a neibourhood of the boundary Γ. For any $f \in L^1(\Gamma)$ we have

$$\int_\Gamma f \, d\Gamma = \Sigma \int_{\Gamma \cap \mathcal{O}_i} r_i f \, d\Gamma$$

$$= \Sigma \int_{B_0} r_i o c^{-1} f o c^{-1} \, ||cof(D(c_i^{-1})).n_0|| \, dx'$$

Where for any square matrix A, the cofactor's matrix is

$$cof(A) = detA \quad A^{-*}, \;\; cof(A^{-1}) = \frac{1}{detA} \;\; A^*$$

We get $D(c_i^{-1}) = \; (Dc_i)^{-1} \circ c_i^{-1}$ then

$$cof(D(c_i^{-1})) = cof((Dc_i)^{-1}) \circ c_i^{-1} = (\, \frac{1}{detDc_i} \, (Dc_i)^* \;\;) \circ c_i^{-1}$$

It can be easily verified that if T is a smooth enough transformation we have, with $\Sigma = T(\Gamma)$,

$$\int_{T(\Gamma)} f \, d\Sigma = \int_\Gamma f \circ T \;\omega \; d\Gamma$$

Where, n being the unitary normal field on Γ,

$$\omega = ||cof(DT).n|| = \; |det(DT)| \, ||(DT)^{-*}.n||$$

Also, we have the following lemma ([15]):

Lemma 2.1. *If the mapping $s \to T_s(V)$ is in $C^1([0,\tau]; C^k(\overline{D}, \mathbb{R}^n))$, then*

$$s \to n_s \circ T_s = \frac{DT_s^{-*} n}{\|DT_s^{-*} n\|} \;\; is \; in \quad C^1([0,\tau]; C^k(\Gamma))$$

where n and n_s are the outward normal fields respectively to Ω and Ω_s, on Γ and Γ_s. Moreover, its derivative is given by:

$$\frac{d}{ds}(n_s \circ T_s) =< DV \cdot n_s, n_s > \circ T_s n_s \circ T_s - DV^* \circ T_s n_s \circ T_s$$

2.2.4. *Shape derivatives.* Assume that the transformation under consideration $T_s(V)$ is the flow mapping of a Lipschitz-continuous vector field $V(s,x)$. Then we get

$$\forall x \in \Gamma, \omega(s,x) = det(DT_s(V))\,||(DT_s(V))^{-*}.n||$$

and

$$\forall x \in \Gamma,\ \frac{\partial}{\partial s}\omega(s,x)|_{s=0} =\ H(x)\ \ < V(0,x), n(x) >$$

Where H is the mean curvature of Γ, $H = Trace(D^2 b_\Omega)|_\Gamma = (\Delta b_\Omega)|_\Gamma$ and $v =< V(0,x), n(x) >$ is the so-called normal speed of the moving boundary Γ_s.

3. Material Derivatives

3.1. **Existence results for the derivative.** Recall that $\mathcal{O}$ is the reference domain whose boundary is S and let $\mathcal{O}^c$ be its complementary. The mapping $I + u_s$ is invertible from $\mathcal{O}$ onto Ω_s as soon as $\det(I + Du_s) > 0$ over $\mathcal{O}$. We transport the harmonic problem whose solution is $\phi_s \in H^1(\Omega_s)$. Let

$$\phi^s :=\ \phi_s \circ (I + u_s) \in H^1(\hat{\Omega})$$

From the Transport Lemma we know that

$$div(\ A(s).\nabla\phi^s\ )\ = 0\ \ in\ \ \mathcal{O}^c \tag{3.1}$$

where

$$A(s) = \det(I + Du_s)\,(I + Du_s)^{-1}.(I + Du_s)^{-*}$$

Moreover, we have the following boundary condition

$$< \hat{n},\ A(s).\nabla\phi^s >=\ 0\ \ \text{ on }\ S \tag{3.2}$$

Concerning the elastic boundary condition on Γ_s, we have:

$$\mathcal{T}_s.n_s =\ \frac{1}{2}|\nabla_{\Gamma_s}\phi_s|^2 +\ f\ \text{ on }\ \Gamma_s \tag{3.3}$$

where the forcing term f, may be due to the gravity acceleration and can take the form $f(x) = \rho g x_3$ in R^3. The change of variable leads to

$$\mathcal{T}_s\circ(I+u_s).n_s\circ(I+u_s) =\ \frac{1}{2\det(I + Du_s)}\ < A(s).\nabla\phi^s, \nabla\phi^s > + f\circ(I+u_s)\ \text{ on }\ \Gamma_s \tag{3.4}$$

The stress tensor is the matrix

$$\mathcal{T}_s \circ (I + u_s) = (\frac{1}{\det(I + Du_s)}(I + Du_s).\Sigma(\sigma(u_s)).(I + D^*u_s)\ \ ) \tag{3.5}$$

Where

$$\Sigma(\sigma(u_s)) = C_{\lambda,\mu}..(Du_s + D^*u_s + \frac{1}{2}Du_s.D^*u_s) \tag{3.6}$$

$$n_s \circ (I + u_s) = (I + Du_s)^{-*}.\nabla(b_{\Omega_s} \circ (I + u_s)\,)$$

Equation (3.1), (3.2), (3.3), (3.4), and (3.5) above form a system that we can rewrite as

$$\mathcal{F}(s,\ (u_s, \phi^s)\,)\ =\ 0,$$

where the mapping

$$\mathcal{F} : [0,\ s_1[\times (H^1(\hat{\Omega}, R^N) \times H^1(\hat{\Omega}) \to H^{-1}(\hat{\Omega}, R^N) \times H^{-1}(\hat{\Omega})$$

verifies the Implicit Function theorem assumptions so that the derivative $u' := \frac{\partial}{\partial s} u_s|_{s=0}$ exists in $H^1(\hat{\Omega}, R^N)$ and also $\dot{\phi} := \frac{\partial}{\partial s}\phi^s|_{s=0}$ exists in $H^1(\hat{\Omega}, R)$.

4. Weak Formulation of the Problem

Now we write the variational form associated with system (2.4). $\forall \Psi \in H^1(D)$, $\forall R \in H^1(D, R^N)$, we have

$$\int_{\Omega_s} Tr(\mathcal{T}_s.DR)dx + \int_{\Omega_s^c} < \nabla\phi_s, \nabla\Psi > dx$$
$$= \int_{\Gamma_s} \{\, \frac{1}{2}|\nabla\phi_s|^2 + \rho g\, x_3 \,\} \ < n_s, R > \ d\Gamma_s$$

Our goal is to compute the s derivatives (at $s = 0$). Let $\mathcal{T}' = [\frac{\partial}{\partial s}\mathcal{T}_s]_{s=0}$, $v =< V(0), n >$ on Γ, and $n' = \frac{d}{ds}(\nabla b_{\Omega_s})_{s=0}$. Recall that $n_s = \nabla b_{\Omega_s}$ is the unit normal vector field on Γ_s out going to the (elastic) set Ω_s, and $H = \Delta b_\Omega$ is the mean curvature of Γ. We obtain:

$$\int_\Omega Tr(\mathcal{T}'.DR)dx \ + \ \int_\Gamma Tr(\mathcal{T}.DR)\, v\, d\Gamma$$
$$+ \int_{\Omega^c} < \nabla\phi', \nabla\Psi > dx \ + \ \int_\Gamma < \nabla\phi, \nabla\Psi > \ v\, d\Gamma$$
$$= \int_\Gamma \{ < \nabla\phi', \nabla\phi > < n, R > \ d\Gamma$$
$$+ \int_\Gamma \{\, \frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3 \,\} \ < n', R > \ d\Gamma$$
$$+ \int_\Gamma \frac{\partial}{\partial n}\{\, (\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,) \ < \nabla b_\Omega, R > \}\, v\, d\Gamma$$
$$+ \int_\Gamma H \ (\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,) \ < \nabla b_\Omega, R > \ \ v\, d\Gamma$$

We recall the by part integration formula:

$$\int_\Omega Tr(\mathcal{T}'.DR)dx = -\int_\Omega < \vec{D}iv(\mathcal{T}'), R > dx \ + \ \int_\Gamma < \mathcal{T}'.n, R > \ d\Gamma$$

Then, taking Ψ (repectively R) with compact support in Ω^c (respectively in Ω) we get the following equations:

$$-\Delta\phi' = 0 \ \ in \ \ \Omega^c$$
$$-\vec{D}iv(\mathcal{T}') = 0 \ \ in \ \ \Omega$$

But

$$\frac{\partial}{\partial n}\{\, (\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,) \ < \nabla b_\Omega, R > \}$$
$$=< n,\ \nabla\{\, (\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,) \ < \nabla b_\Omega, R > \} \ \ >$$

$= (< n,\, D^2\phi.\nabla\phi\ > +\rho g\, n_3\,) < n, R > \ + (\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,) < n\,, D^2 b_\Omega.R + D^* R.n >$

Notice that

$$< n, D^2 b_\Omega.R > = < D^2 b_\Omega.n, R > = 0, \ \ as\ \ D^2 b_\Omega.n = 0$$

Therefore, concerning the boundary conditions, and taking $\Psi \in H^1(D)$ and $R = 0$, we obtain:

$$\frac{\partial}{\partial n}\phi' = div_\Gamma(\, v\, \nabla_\Gamma \phi\,)$$

In addition, taking $\Psi = 0$ and $R \in H^1(D, R^N)$ with $DR.n = 0$ on Γ, we have:

$$< n\,,\ D^* R.n > = < DR.n,\ n > = 0.$$

Hence we have the following identity:

$$\frac{\partial}{\partial n}\{\,(\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,)\ < \nabla b_\Omega, R > \}$$

$$= (< n,\ D^2\phi.\nabla\phi\ > +\rho g\, n_3\,) < n, R >$$

Finally, we obtain the following variational problem at the boundary:

$$\forall R \in H^1(\Gamma, R^N)$$

$$\int_\Gamma < \mathcal{T}'.n, R >\ d\Gamma\ + \int_\Gamma Tr(\mathcal{T}.DR)\, v\, d\Gamma$$

$$= \int_\Gamma\ < \nabla\phi', \nabla\phi > < n, R >\ d\Gamma$$

$$+ \int_\Gamma \{\,\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,\}\ < n', R >\ d\Gamma$$

$$+ \int_\Gamma (< n,\ D^2\phi.\nabla\phi\ > +\rho g\, n_3\,) < n, R >\ v\, d\Gamma$$

$$+ \int_\Gamma H\ (\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,)\ < \nabla b_\Omega, R >\ v\, d\Gamma$$

We have to perform a tangential by part integration in the term

$$\int_\Gamma Tr(\mathcal{T}.DR)\, v\, d\Gamma = \int_\Gamma\ \mathcal{T}_{i,j}\frac{\partial}{\partial x_j}R_i\, v\, d\Gamma = \int_\Gamma\ < \mathcal{T}_{i,.}, \nabla R_i >\ v\, d\Gamma$$

But as for all i we have $\frac{\partial}{\partial n}R_i = 0$,

$$\nabla R_i = \nabla_\Gamma R_i\ + \frac{\partial}{\partial n}R_i\ \ \vec{n} = \nabla_\Gamma R_i,$$

and

$$\int_\Gamma Tr(\mathcal{T}.DR)\, v\, d\Gamma = \int_\Gamma\ < \mathcal{T}_{i,.}, \nabla_\Gamma R_i >\ v\, d\Gamma = -\int_\Gamma\ div(v\,\mathcal{T}_{i,.})\ R_i\ d\Gamma$$

$$= -\int_\Gamma\ < \vec{D}iv(v\,\mathcal{T}),\ R\ >\ d\Gamma,$$

then we get the following boundary condition for the stress function :

$$\forall R \in H^1(\Gamma, R^N)$$
$$\int_\Gamma < \mathcal{T}'.n, R > \, d\Gamma \;= + \int_\Gamma < \vec{D}iv(v\,\mathcal{T}),\; R > \; d\Gamma$$
$$+ \int_\Gamma < \nabla\phi', \nabla\phi > < n, R > \, d\Gamma$$
$$+ \int_\Gamma \{ \frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3 \} \; < n', R > \, d\Gamma$$
$$+ \int_\Gamma (< n,\, D^2\phi.\nabla\phi \; > \, + \rho g\, n_3\,) < n, R > \; v\, d\Gamma$$
$$+ \int_\Gamma H \; (\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,) \; < \nabla b_\Omega, R > \; v\, d\Gamma$$

Therefore, we obtain

$$\mathcal{T}'.n \;=\; \vec{D}iv(v\,\mathcal{T}) \;\;+\;\; < \nabla\phi', \nabla\phi > \;\; \vec{n} + \{ \,\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\, \}\, \vec{n}'$$
$$+ (< n,\, D^2\phi.\nabla\phi \; > \, + \rho g\, n_3\,)\, v\, \vec{n} + \; H \; (\frac{1}{2}|\nabla\phi|^2 + \rho g\, x_3\,) \;\; v\, \vec{n}$$

4.1. **Calculus of the tangent vector n'.** From [?], [?], [?], we know that in some neighborhood $\mathcal{U}$ of $\Sigma = \cup_{0<s<s_1}\{s\} \times \partial\Omega_s$ the oriented distance function solves the convection equation

$$\frac{\partial}{\partial s} b_{\Omega_s} + \nabla b_{\Omega_s}.V(s) o p_{\Gamma_s} = 0$$

where $p_{\Gamma_s} = I_{d\,-\,b_{\Omega_s}} \nabla b_{\Omega_s}$ is the projection mapping onto Γ_s.

Then we get

$$n' := \frac{\partial}{\partial s}(\nabla b_{\Omega_s})_{s=0} = \nabla(\frac{\partial}{\partial s} b_{\Omega_s})_{s=0} = (\;\nabla(-\nabla b_{\Omega_s}.V(s) o p_{\Gamma_s})\;)_{s=0}$$
$$= -(\,D^2 b_{\Omega_s}.V(s) o p_{\Gamma_s}\;)_{s=0} - (\;\; D^*(V(s) o p_{\Gamma_s}).\nabla b_{\Omega_s}\;\;)_{s=0}$$
$$s = 0,\; x \in \Gamma,\;\; n'(x) = -\,D^2 b_\Omega(x).V(0,x) \;\;-\;\; D^*_\Gamma V(0,x).n(x)$$
$$= -\nabla_\Gamma v(x)$$

where again $v(x) =< V(0,x), n(x) >$ on Γ is the normal speed of the boundary.

As we have

$$\vec{D}iv(\mathcal{T}) = 0 \;\; in \;\; \Omega,$$

assuming the boundary Γ smooth we get that this equation holds true up to the boundary, so that the term $\vec{D}iv(v\mathcal{T})$ simplifies and we get

$$\vec{D}iv(v\mathcal{T}) = v\vec{D}iv(\mathcal{T}) + \; \mathcal{T}.\nabla v.$$

Thus we have the following new expression:

$$\mathcal{T}'.n \;=\; \mathcal{T}.\nabla v \;\;+\;\; < \nabla_\Gamma\phi', \nabla_\Gamma\phi > \;\; \vec{n} - \{ \,\frac{1}{2}|\nabla_\Gamma\phi|^2 + \rho g\, x_3\, \}\, \nabla_\Gamma v$$
$$+ \;(< n,\, D^2\phi.\nabla_\Gamma\phi \; > \, + \frac{1}{2} H \; |\nabla_\Gamma\phi|^2 + \;(1 + H\,)\rho g\, n_3\,)\, v\, \vec{n} \tag{4.1}$$

Regarding the term ∇v on Γ, we have:

$$\nabla v = \nabla(< V(0), \nabla b_\Omega >) = D^2 b_\Omega.V(0) + D^* V(0).\nabla b_\Omega$$

Therefore it follows that

$$< \nabla v, n >=< DV(0).n, n >$$

and

$$\nabla v = \nabla_\Gamma v + < DV(0).n, n > \vec{n}$$

Now using $\mathcal{T}.n = p\vec{n} = (\frac{1}{2}|\nabla_\Gamma \phi|^2 + \rho g\, x_3)\vec{n}$, equation (4.1) simplifies to:

$$\begin{aligned} \mathcal{T}'.n \ &= [\ \mathcal{T} - \{\ \frac{1}{2}|\nabla_\Gamma \phi|^2 + \rho g\, x_3 \ \}\ ].\nabla_\Gamma v \ + \ < \nabla_\Gamma \phi', \nabla_\Gamma \phi > \ \vec{n} \\ &+ (< n,\ D^2\phi.\nabla_\Gamma \phi \ > + \frac{1}{2} H\ |\nabla_\Gamma \phi|^2 + \ (1 + H\ )\rho g\, n_3\ )\, v\, \vec{n} \\ &+ < DV(0).n, n > \ (\frac{1}{2}|\nabla_\Gamma \phi|^2 + \rho g\, x_3)\, \vec{n} \end{aligned} \tag{4.2}$$

4.2. **The stress tensor $\mathcal{T}_s \circ T_s$.** Recall that $T_s = \varphi_s \circ \varphi^{-1}$, where $\varphi_s = I + u_s$, and that $V(s) = \frac{\partial}{\partial s} T_s \circ T_s^{-1} = \frac{\partial}{\partial s}\varphi_s \circ \varphi_s^{-1}$.

Then we get

$$V(s) = \frac{\partial}{\partial s} u_s o (I + u_s)^{-1}.$$

If we let

$$u' := (\ \frac{\partial}{\partial s} u_s\ )_{s=0}$$

then we obtain

$$V(0) = u' \circ (I + u)^{-1}$$

On the boundary Γ we have

$$v =< u' \circ (I + u)^{-1}, n >$$

so that

$$\nabla_\Gamma v = \nabla_\Gamma(< (u' \circ (I + u)^{-1}), n >) = D^*_\Gamma(u' \circ (I + u)^{-1}).n + D^2 b_\Omega.(u' \circ (I + u)^{-1})_\Gamma$$

Remark 4.1. *We design by $K \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ the self adjoint Dirichlet to Neuman operator at the boundary associated with harmoniques functions in Ω. Then we get*

$$\phi'|_\Gamma = K.(div_\Gamma(v\nabla_\Gamma \phi))$$

The stress tensor is the matrix

$$\mathcal{T}_s o T_s = (\frac{1}{det(I + Du_s)}(I + Du_s).\Sigma(\sigma(u_s)).(I + D^* u_s)\ \ )o(I + u)^{-1} \tag{4.3}$$

where

$$\Sigma(\sigma(u_s)) = C_{\lambda,\mu}..(Du_s + D^* u_s + \frac{1}{2} Du_s.D^* u_s) \tag{4.4}$$

In (4.4) we assumed the four entries elasticity tensor to be governed by the *Lamé* coefficients.

Taking derivative w.r.t s in (4.3), we obtain:

$$([\frac{\partial}{\partial s}\mathcal{T}_s o T_s]_{s=0})o(I+u) = -\frac{div(u')}{det(I+Du)} (I+Du).C_{\lambda,\mu}..(\sigma(u)).(I+D^*u)$$

$$+(\frac{1}{det(I+Du)}D(u').C_{\lambda,\mu}..(\sigma(u)).(I+D^*u))$$

$$+(\frac{1}{det(I+Du)}(I+Du).C_{\lambda,\mu}..(\sigma').(I+D^*u))$$

$$+(\frac{1}{det(I+Du)}(I+Du).C_{\lambda,\mu}..(\sigma(u)).D^*(u'))$$

where

$$\sigma' = D(u') + D^*(u') + \frac{1}{2}(D(u').D^*u + Du.D^*(u'))$$

Now we let

$$\dot{\mathcal{T}} = [\frac{\partial}{\partial s}\mathcal{T}_s o T_s]_{s=0}$$
$$= \mathcal{T}' + \mathbf{D}\mathcal{T}.(u' \circ (I+u)^{-1})$$

where $\mathbf{D}$ is a "new coming": it is a three entries tensor, representing the gradient of the matrix $\mathcal{T}$. Its contraction with the vector $(u' \circ (I+u)^{-1})$ gives the matrix $\mathbf{D}\mathcal{T}.(u' \circ (I+u)^{-1})$. Then we have

$$\mathcal{T}' = \{ -\frac{div(u')}{det(I+Du)} (I+Du).C_{\lambda,\mu}..(\sigma(u)).(I+D^*u) \}o(I+u)^{-1}$$

$$+\{ (\frac{1}{det(I+Du)}D(u').C_{\lambda,\mu}..(\sigma(u)).(I+D^*u)) \}o(I+u)^{-1}$$

$$+\{ (\frac{1}{det(I+Du)}(I+Du).C_{\lambda,\mu}..(\sigma').(I+D^*u)) \}o(I+u)^{-1}$$

$$+\{ (\frac{1}{det(I+Du)}(I+Du).C_{\lambda,\mu}..(\sigma(u)).D^*(u')) \}o(I+u)^{-1}$$

$$- \mathbf{D}\mathcal{T}.(u' \circ (I+u)^{-1})$$

5. Linearization around "rest"

The most "popular" framework (among mathematicians) consists in considering $u = 0$. With this assumption, we have the following simplification:

$$\mathcal{T}' = C_{\lambda,\mu}..(\sigma') - \mathbf{D}\mathcal{T}.(u' \circ (I+u)^{-1})$$

Since $u = 0$, we have that $\mathcal{T} = 0$ and thus $\mathbf{D}\mathcal{T} = 0$. Moreover,

$$\sigma' = D(u' \circ (I+u)^{-1}) + D^*(u' \circ (I+u)^{-1})$$

Therefore

$$\mathcal{T}' = C_{\lambda,\mu}..(D(u' \circ (I+u)^{-1}) + D^*(u' \circ (I+u)^{-1}))$$

$$\mathcal{T}' = \lambda\, det(D(u' \circ (I+u)^{-1}))\, I + \mu\, (D(u' \circ (I+u)^{-1}) + D^*(u' \circ (I+u)^{-1}))$$

So now we are in the classical linear elasticity framework: $U := (u' \circ (I + u)^{-1})$ is the linearized displacement, while $\bar{\mathcal{T}} = \mathcal{T}' = \lambda\, det(DU)\, I + \mu\,(DU + D^*U) =$ is the associated stress function.

5.1. **The linearization of the coupled fluid-structure problem around stress less steady structure.** We assume small variations around the rest state $u = 0,\ \mathcal{T} = 0$ for the elastic body occupying the volume Ω in $D \subset R^3$. The fluid speed v is steady, but not zero. We assume this flow to be irrotational so that v derives from an harmonic potential in $\Omega^c = D \setminus \bar{\Omega}$, that is $v_s = \nabla \phi_s$. From (4.2), with the following notation

$$\mathbf{\Phi} = \phi',$$

we obtain the following system for the fluid-structure coupling (with $\bar{\mathcal{T}} = \lambda\, det(DU)\, I\ +\ \mu\,(DU + D^*U)$):

$$\begin{gathered}
\Delta \mathbf{\Phi} = 0 \ \ in\ \ \Omega^c, \\
\vec{D}iv(\bar{\mathcal{T}}) = 0 \ \ in\ \ \Omega, \\
\frac{\partial}{\partial n}\mathbf{\Phi} = -+\ div_\Gamma(\,U.n\,\nabla_\Gamma \phi\,) \ \ on\ \ \Gamma, \\
\bar{\mathcal{T}}.n\ =\ -\{\,\frac{1}{2}|\nabla_\Gamma \phi|^2 + \rho g\, x_3\,\}\,.\nabla_\Gamma(U.n)\ \ +\ <\nabla_\Gamma \mathbf{\Phi}, \nabla_\Gamma \phi>\ \vec{n} \\
+\,(<n,\, D^2\phi.\nabla_\Gamma \phi\ >\ + \frac{1}{2}H\ |\nabla_\Gamma \phi|^2 +\ (1 + H\,)\rho g\, n_3\,)\,U.n\ \ \vec{n} \\
+ < DU.n, n >\ (\frac{1}{2}|\nabla_\Gamma \phi|^2 + \rho g\, x_3)\,\vec{n} \qquad (5.1)
\end{gathered}$$

Note that the coupling on the boundary interface Γ takes into account the mean curvature of the boundary H. This is an important point.

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