New classification techniques for ordinary differential equations

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Abstract

The goal of the present paper is to propose an enhanced ordinary differential equations solver by exploitation of the powerful equivalence method of Élie Cartan. This solver returns a target equation equivalent to the equation to be solved and the transformation realizing the equivalence. The target ODE is a member of a dictionary of ODE, that are regarded as well-known, or at least well-studied. The dictionary considered in this article are ODE in a book of Kamke. The major advantage of our solver is that the equivalence transformation is obtained without integrating differential equations. We provide also a theoretical contribution revealing the relationship between the change of coordinates that maps two differential equations and their symmetry pseudo-groups.

Key words: Cartan's equivalence method, $\mathcal{D}$-groupoids, ODE-solver.

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1. Introduction

Current symbolic ODE solvers make use of a combination of Lie symmetry methods and pattern matching techniques. While pattern matching techniques are used when the ODE matches a recognizable pattern (that is, for which a solving method is already implemented), symmetry methods are reserved for the non-classifiable cases (Cheb-Terrab et al., 1997, 1998).

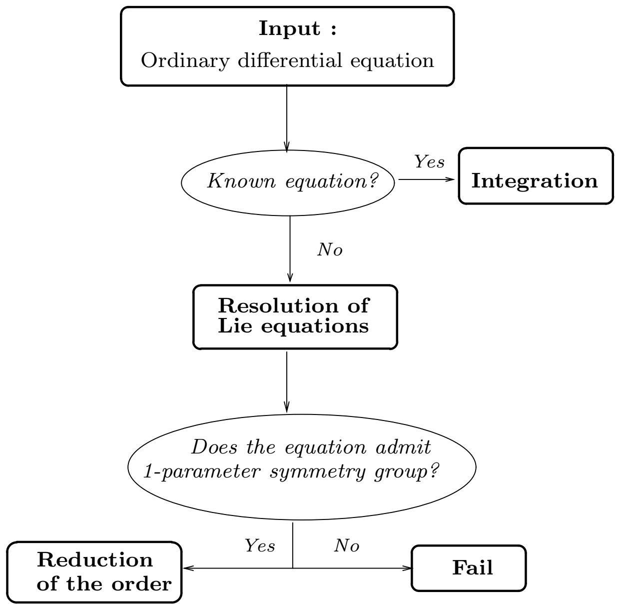

Fig. 1. General flowchart of typical ODE-solver.

Nevertheless, in practice, these solvers often fail to return closed form solutions. This is the case for instance of the equation

$$y'' + y^3 y'^4 + \frac{y'^2}{y} + \frac{1}{2} y = 0, \tag{1}$$

which admits only one 1-parameter symmetry group. Using this information, present solvers return a complicated first order ODE and a quadrature which is quite useless for practical applications.

More dramatically, when applied, to the following equation, these solvers output no result

$$y'' - \frac{2\,x^4 y' - 6\,y^2 x - 1}{x^5} = 0. \tag{2}$$

This failure is due to the fact that the above equation does not match any recognizable pattern and has zero-dimensional point symmetry (pseudo-)group. Thus neither symmetry methods nor classification methods works. The same goes for the equation

$$y''y + y'^2 - 4\,y^6 + 12\,xy^4 - \left(4\,x + 12\,x^2\right) y^2 + 4\,x^3 + 4\,x^2 - 2\,\alpha = 0 \tag{3}$$

Our solver is designed to handle such differential equations. It returns a target equation equivalent to the equation to be solved and the equivalence transformation. The target

ODE is a member of a dictionary of ODE, that are regarded as well-known, or at least well-studied. The dictionary considered in the article are ODE in a book of Kamke. However other ODE could be added to the dictionary without difficulty and this is an advantage of the method.

For the equation (1), we obtain the Rayleigh equation (number 6.72, page 559 in Kamke's book)

$$y'' + {y'}^4 + y = 0$$

and the change of coordinates

$$(x,\, y) \to (x,\, \frac{y^2}{2}).$$

For the equation (2), we obtain the first Painlevé equation (number 6.3, page 542 in Kamke's book)

$$y'' = 6y^2 + x$$

and the change of coordinates

$$(x,\, y) \to (\frac{1}{x},\, y).$$

The equation (3) is mapped to the second Painlevé equation (number 6.6 in Kamke's enumeration)

$$y'' = 2y^3 + yx + \alpha$$

under the transformation

$$(x,\, y) \to (2x,\, y^2 - x).$$

To incorporate such changes of variables, one needs to understand the *equivalence problem* : Given two ODE

$$E_f : y'' = f(x, y, y'), \;\; E_{\bar{f}} : \bar{y}'' = \bar{f}(\bar{x}, \bar{y}, \bar{y}')$$

and an allowed Lie pseudo-group of transformations $\Gamma\Phi$ acting on the variables x and y (for real-life reasons, we restrict ourselves to second order ODE. However, what follows remains true for any order). We shall say that E_f and $E_{\bar{f}}$ are equivalent under the action of the pseudo-group $\Gamma\Phi$ if there exists a change of coordinates $\varphi \in \Gamma\Phi$ that maps one equation to another. This will be denoted by

$$E_{\bar{f}} = \varphi_*(E_f) \text{ and } \varphi \in \Gamma\Phi, \tag{4}$$

or in abridged form $E_f \sim_\Phi E_{\bar{f}}$. As we shall see, the system (4) is PDE's system. One can consider the equivalence of an equation with its self : A *symmetry* σ of the equation E_f is a solution of the self-equivalence obtained by setting $E_f = E_{\bar{f}}$ in the a PDE's system (4). The solutions of this self-equivalence problem form a Lie pseudo-group, the symmetry pseudo-group, which will be denoted by $S_{E_f,\Phi}$.

In practice, one distinguishes two possible situations in the computation of the change of coordinates. First, the input equation (the equation to be solved) and the target equation are known. This is an online computation. In the second situation, considered here in the construction of our solver, only the target equation (an equation from Kamke's list) is known and we look for the change of coordinate that maps the generic equation to the target one. So, assume that E_f is a generic differential equation and $E_{\bar{f}}$ is fixed equation that falls within the effective differential algebra i.e. $E_{\bar{f}}$ is given by equalities between differential polynomials with rational coefficients.

Crucial in the construction of our solver, is the establishment of the relationship between the change of coordinates and symmetry pseudo-groups. In particular, one might ask under which conditions the change of coordinates can be obtained *without* integrating differential equation ? To the authors's knowledge, this is the first time in equivalence problem theory that such questions are investigated. The answer, which constitutes the theoretical contribution of the paper, can be be summarized as follows (we emphasis on the fact that what follows remains true for any order) :

(*i*) The number of constants appearing in the change of coordinates $\varphi \in \Gamma\Phi$, mapping E_f to $E_{\bar{f}}$, is exactly the dimension of $S_{E_{\bar{f}},\Phi}$. This implies that when this dimension vanishes the change of coordinate can be obtained without integrating differential equations. Also, we have $\dim(S_{E_f,\Phi}) = \dim(S_{E_{\bar{f}},\Phi})$.

(*ii*) In the particular case when $\dim(S_{E_{\bar{f}},\Phi}) = 0$, the transformation φ is algebraic in f and its partial derivatives. The degree of this transformation φ is exactly $\mathrm{card}(S_{E_{\bar{f}},\Phi})$. In this case, the symmetry pseudo-groups $S_{E_f,\Phi}$ and $S_{E_{\bar{f}},\Phi}$ have finite cardinals. However, they need not to have the same cardinal.

The simple fact $\dim(S_{E_f,\Phi}) = \dim(S_{E_{\bar{f}},\Phi})$ allows us to construct a powerful hashing function which significantly restricts the space of research in kamke's list. For this reason, we use 7 possible types of transformations $\Phi_1, \cdots, \Phi_7$ (see table 1 page 14). We pre-calculate to each target equation a *signature index*, that is, the dimensions of the 7 symmetry pseudo-groups associated to the 7 types of transformations (this calculation is done without any integration). If two differential equations are equivalent then their signature indices *match*.

The transformation φ in (*ii*) can be obtained using differential elimination algorithms. This is explained in third section. Unfortunately, such approach is rarely effective due to expressions swell. In order to avoid this, we propose in section 4 a new method to pre-compute the transformation φ in terms of differential invariants (we do this for each target equation $E_{\bar{f}}$ in Kamke's list). These invariants are provided by Cartan's method.

2. Equivalence problem

The equivalence problem is the study of the action of a given pseudo–group of transformations on a given class of differential systems. In the algebraic framework, this action is viewed as the action of a $\mathcal{D}$–groupoid Φ on a diffiety $\mathcal{E}$.

2.1. Equivalence problems and groupoids

Recall that a diffiety (see A.1) is the set of formal Taylor series which are regular solutions of a finite PDE's system. It is a pro-algebraic variety, fibered over an algebraic variety X and which will be denoted by $\pi : \mathcal{E} \longrightarrow X$. The projection of a Taylor series $\jmath_x^\infty f \in \mathcal{E}$, of a function $f : X \to U$, is the expansion point $x \in X$. The coordinate ring of a diffiety is a reduced finitely generated *differential* algebra. The automorphisms group of the diffiety is the set of the contact transformations (see Olver (1993)) from $\mathcal{E}$ to $\mathcal{E}$.

A $\mathcal{D}$-groupoid Φ is a diffiety formed by invertible Taylor series and closed under the composition (see section A.4). A $\mathcal{D}$-groupoid acting on a manifold X, is a subset of the space of infinite invertible jets $\mathrm{J}_*^\infty(X, X)$. The Taylor series of contact transformations of a diffiety $\mathcal{E}$ form a $\mathcal{D}$-groupoid that acts on $\mathcal{E}$ and which will be denoted by $\mathrm{aut}(\mathcal{E})$.

Given a diffiety $\mathcal{E}$ fibred over X and a $\mathcal{D}$-groupoid Φ acting on X. An equivalence problem is the action of Φ on the diffiety $\mathcal{E}$, that is, an injective representation[1], i.e. an injective morphism of $\mathcal{D}$-groupoids

$$\rho : \Phi \longrightarrow \mathrm{aut}(\mathcal{E}).$$

Example 1 (2nd order ODE, $\bar{x} = x + C, \bar{y} = \eta(x, y)$)**.** The infinite dimensional $\mathcal{D}$-groupoid $\Phi := \Phi_3$ acts on the points $(x, y) \in \mathrm{J}^0(\mathbb{C}, \mathbb{C})$. The corresponding Lie defining equations are (we set $\bar{\mathrm{x}} = \xi(x, y)$)

$$\xi_x = 1,\, \xi_y = 0,\, \eta_y \neq 0. \tag{5}$$

The action of Φ on the jets space $\mathrm{J}^0(\mathbb{C}, \mathbb{C})$ is prolonged (see appendix A.5) to an action on the first order jets space $X := \mathrm{J}^1(\mathbb{C}, \mathbb{C})$. In the coordinates $(x, y, p = y')$, this action reads

$$\bar{x} = \xi,\ \bar{y} = \eta,\ \bar{p} = \eta_x + p\eta_y. \tag{6}$$

The equivalence condition (4) is obtained by prolonging the action of Φ on the second order jets space $\mathrm{J}^2(\mathbb{C}, \mathbb{C})$ and by setting $y'' = f(x, y, p)$. Thus, we obtain

$$\begin{aligned} \xi_x = 1,\, \xi_y = 0,\, \xi_p = 0, w\ \eta_y = 1, \\ \bar{x} = \xi,\ \bar{y} = \eta,\ \bar{p} = \eta_x + p\eta_y, \\ \bar{f}(\bar{x}, \bar{y}, \bar{p}) = \eta_{xx} + 2p\eta_{xy} + p^2\eta_{yy} + f(x, y, p)\ \eta_y. \end{aligned} \tag{7}$$

This action of Φ on the equation E_f is viewed as an action on the Taylor series $\jmath^\infty_{\mathrm{x}} f$, in other words, as an action on the trivial diffiety $\mathcal{E} := \mathrm{J}^\infty(X, \mathbb{C})$ (fibred over the manifold X). The coordinates ring of X is $\mathbb{C}[X] := \mathbb{C}[x, y, p]$ and the coordinate ring of $\mathcal{E}$ is the ring of differential polynomials

$$\mathbb{C}[\mathcal{E}] := \mathbb{C}[x, y, p]\{f\} \text{ with } \Delta = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial p} \right\}.$$

3. Differential-algebraic approach

The aim of this section is to use differential elimination to solve the equivalence problem (4) when the target function $\bar{f}(\bar{x}, \bar{y}, \bar{p})$ is a $\mathbb{Q}$-rational function, explicitly known and the symmetry pseudo–group of $E_{\bar{f}}$ is zero-dimensional. The reader can find in the appendix A.1 a brief introduction to differential algebra.

3.1. The self-equivalence problem

The system (4) is fundamental and can be treated by two different approaches : brute-force method based on differential algebra (this section) and geometric approach relying on Cartan's theory of exterior differential systems (the next section). While it is classically known that the existence of at least one transformation $\varphi : X \to X$ can be checked by computing the *integrability conditions* of the system (4), which is completely algorithmic whenever the functions $f, \bar{f} : X \to \mathbb{C}$ are explicitly known (Boulier et al., 1995), there is no general algorithm for computing closed form of φ. We shall show that if the symmetry pseudo–group of $E_{\bar{f}}$ is zero-dimensional, the transformation φ is obtained *without* integrating any differential equations.

[1] Élie Cartan call *prolongement holohédrique* such injective representation.

Definition 1 (Symmetry pseudo–group)**.** To any differential equation $E_{\bar f}$ and any $\mathcal{D}$-groupoid Φ that acts on (x, y), we associate the $\mathcal{D}$-groupoid $\mathcal{S}_{E_{\bar f},\Phi}$ formed by the formal Taylor series solutions of the *self–equivalence* problem

$$E_{\bar f} = \sigma_*(E_{\bar f}) \text{ and } \sigma \in \Gamma\Phi. \tag{8}$$

The symmetry pseudo-group $S_{E_{\bar f},\Phi} := \Gamma\mathcal{S}_{E_{\bar f},\Phi}$ is the set of C^∞-functions $\sigma : X \to X$ that are local solutions of the Lie defining equations (8).

Example 2. Consider the $\mathcal{D}$-groupoid $\Phi := \Phi_3$ from example 1 and the Emden-Fowler equation $E_{\bar f}$ (number 6. 11, page 544 in (Kamke, 1944))

$$y'' = \frac{1}{xy^2}. \tag{9}$$

The Lie defining equations of the symmetry pseudo-group of $E_{\bar f}$ are obtained by setting $f(x, y, p) = \dfrac{1}{xy^2}$ and $\bar f(\bar x, \bar y, \bar p) = \dfrac{1}{\bar x\bar y^2}$ in the equations (7). The characteristic set (see definition 8 page 16) of these equations is

$$C_\sigma = \left\{ \bar p = p\, \frac{\bar y}{y},\ \bar y^3 = y^3,\ \bar x = x \right\}. \tag{10}$$

This PDE's system is particular : it contains only non–differential equations. We have $\dim C_\sigma = 0$ and $\deg C_\sigma = 3$. The symmetry pseudo-group $S_{\bar f} := S_{E_{\bar f},\Phi}$ is actually a group with 3 elements:

$$S_{\bar f} = \left\{ (x, y, p) \to (x, \lambda y, \lambda p) \mid \lambda^3 = 1 \right\}$$

3.2. Equivalence problem with fixed target

Assume that $\bar f(\bar x, \bar y, \bar p)$ is a fixed $\mathbb{Q}$-rational function.

Example 3. Consider again the equivalence problem of example 1 :

$$\begin{gathered} \bar p - \bar y_x - p\bar y_y = 0, \\ \bar y_{xx} + 2p\bar y_{xy} + p^2\bar y_{yy} + f\ \bar y_y - \bar f(\bar x,\ \bar y,\ \bar p) = 0, \\ \bar x_x - 1 = 0,\ \bar x_y = 0,\ \bar x_p = 0,\ \bar y_p = 0,\ \bar y_y \neq 0. \end{gathered} \tag{11}$$

These equations constitute a quasi-linear characteristic set w.r.t. the elimination ranking

$$\Theta f \succ \Theta\bar p \succ \Theta\bar y \succ \Theta\bar x.$$

Consequently, the associated differential ideal is prime. This fact can be generalized to any $\mathcal{D}$-groupoid Φ defined by quasi-linear characteristic set (see proposition 8).

Proposition 1. The PDE's system (4) (where $\bar f(\bar x, \bar y, \bar p)$ is a fixed rational function) is a quasi–linear characteristic set C w.r.t the ranking $\Theta f \succ \Theta\bar p \succ \Theta\{\bar y, \bar x\}$.

3.3. Brute-force method

Using Rosenfeld-Gröbner we compute a new characteristic set C of the PDE's system (4) w.r.t. the new elimination ranking $\Theta\{\bar p,\ \bar y,\ \bar x\} \succ \Theta\{f\}$, which eliminates the indeterminates $\{\bar p,\ \bar y,\ \bar x\}$. We make the partition of C as in the formula (A.4)

$$C = C_f \sqcup C_\varphi \tag{12}$$

where $C_f := C \cap \mathbb{Q}[x,y,p]\{f\}$ and $C_\varphi := C \setminus C_f$.

Proposition 2. The transformation φ does exist for *almost any* function f satisfying the PDE's system associated to the characteristic set C_f. The function $\bar{\mathrm{x}} = \varphi(\mathrm{x})$ is solution of the PDE's system associated to C_φ.

Definition 2. When $\dim C_\varphi = 0$, the algebraic system associated to C_φ is called the *necessary form of the change of coordinates* $\bar{\mathrm{x}} = \varphi(\mathrm{x})$.

Example 4. Consider the equivalence problem of example 1. Suppose that the target $E_{\bar{f}}$ is the Airy equation

$$\bar{y}'' = \bar{x}\bar{y}.$$

In this case, Rosenfeld-Gröbner returns C_φ and C_f resp. given by (13) and (14)

$$\begin{aligned}
\bar{y}_{xx} &= -f\bar{y}_y + pf_p\bar{y}_y - \tfrac{1}{2}p^2 f_{pp}\bar{y}_y + \bar{y}f_y - \tfrac{1}{2}\,\bar{y}f_{xp} - \tfrac{1}{2}\bar{y}f_{pp}f + \tfrac{1}{4}\bar{y}{f_p}^2 - \tfrac{1}{2}\bar{y}pf_{yp}\\
\bar{y}_{xy} &= -\tfrac{1}{2}f_p\bar{y}_y + \tfrac{1}{2}pf_{p,p}\bar{y}_y\\
\bar{y}_{yy} &= -\tfrac{1}{2}f_{pp}\bar{y}_y,\\
\bar{y}_p &= 0,\\
\bar{x} &= f_y - \tfrac{1}{2}f_{xp} - \tfrac{1}{2}f_{pp}f + \tfrac{1}{4}{f_p}^2 - \tfrac{1}{2}\,pf_{yp}
\end{aligned} \tag{13}$$

$$\begin{aligned}
f_{xxp} &= 2\,f_{xy} + f_pf_{xp} - 2 + p^2 f_{yyp} - f_{pp}f_x + \cdots\\
f_{xyp} &= 2\,f_{yy} - pf_{yyp} - f_{ypp}f - f_{pp}f_y + f_pf_{yp}\\
f_{xpp} &= f_{yp} - pf_{ypp}\\
f_{ppp} &= 0.
\end{aligned} \tag{14}$$

We have $\dim C_\varphi = 3$ which means that the transformation $\bar{\mathrm{x}} = \varphi(\mathrm{x})$, when f satisfies C_f, depends on 3 arbitrary constants.

Example 5. Assume now that the target equation $E_{\bar{f}}$ is

$$\bar{y}'' = \bar{y}^3.$$

Rosenfeld-Gröbner returns C_φ and C_f resp. given by (15) and (16)

$$\begin{aligned}
\bar{y}^2 &= \tfrac{1}{12}\,(4\,f_y - 2\,f_{xp} - 2\,f_{pp}f - 2\,pf_{yp} + {f_p}^2),\\
\bar{x} &= x,
\end{aligned} \tag{15}$$

$$
\begin{aligned}
f_{xxxp} &= \left(4\,f_y - 2\,f_{xp} - 2\,f_{pp}f - 2\,p f_{yp} + {f_p}^2\right)^{-1} \times \\
&\quad \left(24\,p^2 {f_{yp}}^2 f_y - 24\,p^2 f_{yy} f_p f_{yp} + \cdots + 12\,p^2 {f_{x,yp}}^2 + 12\,f^2 {f_{yp}}^2 - 8\,{f_{pp}}^3 f^3\right) \\
f_{xxyp} &= (4\,f_y - 2\,f_{xp} - 2\,f_{pp}f - 2\,p f_{yp} + {f_p}^2)^{-1} \times \\
&\quad (-4\,p f_{ypp} f f_p f_{yp} - 4\,{f_{xp}}^2 f_{yp} + \cdots + 6\,p f_p f_{y,yp} f_{pp} f + 2\,p^3 {f_{yyp}}^2) \\
f_{xyyp} &= \ (4\,f_y - 2\,f_{xp} - 2\,f_{pp}f - 2\,p f_{yp} + {f_p}^2)^{-1} \times \\
&\quad (2\,{f_{ypp}}^2 f^2 - 2\,f_{pp} f_{ypp} f p f_{yp} + \cdots + 4\,{f_{yp}}^2 f_{xp} + 4\,{f_{yp}}^3 p + 16\,f_{ypp} {f_y}^2) \\
f_{xpp} &= f_{yp} - p f_{ypp} \\
f_{ppp} &= 0.
\end{aligned}
\tag{16}
$$

Consequently $\dim C_\varphi = 0$ and $\deg C_\varphi = 2$. Thus, φ is an algebraic transformation of degree 2, given by equations (15).

3.4. From equivalence problem with determined target $E_{\bar f}$ to the self-equivalence problem

Consider the characteristic set $C = C_f \sqcup C_\varphi$ associated to the equivalence problem with determined target $E_{\bar f}$ defined by (12) and computed w.r.t the elimination ranking $\Theta\{\bar p,\, \bar y,\, \bar x\} \succ \Theta\{f\}$. On the other hand, consider the characteristic set C_σ associated to the self-equivalence problem (8), computed w.r.t the orderly ranking on $\{\bar p,\, \bar y,\, \bar x\}$. By definition, we have

$$
\begin{aligned}
C &\subset \mathbb{Q}(x, y, p)\{\bar x,\, \bar y,\, \bar p,\, f\} \\
C_\sigma &\subset \mathbb{Q}(x, y, p)\{\bar x,\, \bar y,\, \bar p\}
\end{aligned}
$$

We obtain C_σ from C by requiring that the two functions f and $\bar f$ are equal. The set C is *specialized* by substituting the symbol f by the value $\bar f(x,\, y,\, p)$. After specialization, the differential system C_f constraining the function f is automatically satisfied since there exists at least one solution $\bar{\mathrm{x}} = \sigma(\mathrm{x})$ of the problem, namely $\sigma = \mathrm{Id}$.

Lemma 1. The two characteristic sets C_φ and C_σ have the same dimension and the same degree in the zero-dimensional case.

Proof. For each equation E_f equivalent to the target equation $E_{\bar f}$, denote by $\Phi_{f,\bar f}$ the diffiety defined by C_φ. The $\mathcal{D}$-groupoid $\mathcal{S}_{E_{\bar f},\Phi}$ acts simply transitively (see Figure 2) on the diffiety $\Phi_{f,\bar f}$, i.e.

$$
\forall \varphi_0, \varphi \in \Gamma\Phi_{f,\bar f}, \quad \exists! \sigma \in \Gamma\mathcal{S}_{E_{\bar f},\Phi}, \quad \varphi = \sigma \circ \varphi_0.
$$

$$
\begin{tikzcd}
\jmath_{\mathrm{x}} f \arrow[r, "\varphi_0"] \arrow[dr, "\varphi"'] & \jmath_{\bar{\mathrm{x}}_0} \bar f \arrow[d, "\sigma"] \\
 & \jmath_{\bar{\mathrm{x}}} \bar f
\end{tikzcd}
$$

Fig. 2. Simply transitive action of $\mathcal{S}_{E_{\bar f},\Phi}$ on $\Phi_{f,\bar f}$ where $\bar{\mathrm{x}}_0 = \varphi_0(\mathrm{x})$ and $\bar{\mathrm{x}} = \varphi(\mathrm{x})$

Every $\varphi_0 \in \Gamma\Phi_{f,\bar{f}}$, define a bijective correspondence $\mathcal{S}_{E_{\bar{f}},\Phi} \to \Phi_{f,\bar{f}}$

$$\jmath^{\infty}_{\bar{\mathrm{x}}_0}\sigma \longrightarrow \jmath^{\infty}_{\mathrm{x}}\varphi = (\jmath^{\infty}_{\bar{\mathrm{x}}_0}\sigma) \circ (\jmath^{\infty}_{\mathrm{x}}\varphi_0), \quad (\sigma \in S_{E_{\bar{f}},\Phi}).$$

In fact, according to the Taylor series composition formulae, the one-to-one correspondence between the two algebraic varieties $\mathcal{S}_{E_{\bar{f}},\Phi}$ and $\Phi_{f,\bar{f}}$ is bi-rational. Consequently, these two varieties have the same dimension and the same degree in the zero-dimensional case. The same goes for the two characteristic sets C_φ and C_σ defining these varieties. □

We are now in position to announce the main theorem of the paper (recall that $\bar{f}$ is assumed $\mathbb{Q}$-rational function of its arguments and by definition, the degree of an algebraic transformation $\bar{\mathrm{x}} = \varphi(\mathrm{x})$ is the generic number of points $\bar{\mathrm{x}}$ when $\mathrm{x} = (x, y, p) \in \mathbb{C}^3$ is determined):

Theorem 3. *The following conditions are equivalent*

(1) $\dim(C_\varphi) = 0$*,*

(2) $\dim(C_\sigma) = 0$*,*

(3) $\dim(S_{E_{\bar{f}},\Phi}) = 0$*,*

(4) $\mathrm{card}(S_{E_{\bar{f}},\Phi}) < \infty$*.*

In this case, $\mathrm{card}\, S_{E_{\bar{f}},\Phi} = \deg(C_\varphi) = \deg \varphi$*.*

Remark 1. When the transformation $\bar{\mathrm{x}} := \varphi(\mathrm{x})$ is locally bijective, but not globally, $S_{E_f,\Phi}$ and $S_{E_{\bar{f}},\Phi}$ need not to have the same degree. Indeed, consider again the groupoid Φ_3 and the equations

$$y'' = \frac{6y^4 + x - 2{y'}^2}{2y} \text{ and } \bar{y}'' = 6\bar{y}^2 + \bar{x}$$

which are equivalent under $(\bar{x} = x,\ \bar{y} = y^2)$. The corresponding symmetry group are respectively given by

$$S_{E_f,\Phi} = \{(x, y) \to (x, \lambda y) \mid \lambda^2 = 1\} \text{ and } S_{E_{\bar{f}},\Phi} = \{\mathrm{Id}\}.$$

They have the same dimension but different cardinality.

3.5. Expression swell

In practice, the above brute–force method, which consists of applying Rosenfeld-Gröbner to the PDE's system (4), is rarely effective due to expressions swell. Most of the examples treated here and in (Dridi, 2007), using our algorithm ChgtCoords, cannot be treated with this approach.

It seems that the problem lies in the fact that we can not separate the computation of C_φ from that of C_f which contains, very often, long expressions (observe that, since C_φ is computed in terms of f then there is no need to compute the equivalence conditions C_f).

An other disadvantage of the above method is that we have to restart computation from the very beginning if the target equation is changed. In the next section, we propose our algorithm ChgtCoords to compute the transformation φ alone and in terms of differential invariants. These invariants are provided by Cartan method for a generic f which means that we have not to re-apply Cartan method if the target equation is changed and a big part of calculations is generic. Furthermore, the computation of φ in terms of differential invariants significantly reduces the size of the expressions.

4. Cartan's method based approach

We refer the reader to (Cartan, 1953; Hsu and Kamran, 1989; Olver, 1995; Neut, 2003; Dridi, 2007) for an expanded tutorial presentation on Cartan's equivalence method and application to second order ODE.

When applied, Cartan's method furnishes a finite set of fundamental invariants and a certain number of invariant derivations generating the field of the differential invariants.

Example 6. Consider the equivalence problem of example 1. The PDE's system (4) reads ($p = y'$)

$$\underbrace{\begin{pmatrix} \mathrm{d}\bar{p} - \bar{f}(\bar{x},\bar{y},\bar{p})\mathrm{d}\bar{x} \\ \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x} \\ \mathrm{d}\bar{x} \end{pmatrix}}_{\omega_{\bar{f}}} = \underbrace{\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S(\mathrm{a})} \underbrace{\begin{pmatrix} \mathrm{d}p - f(x,y,p)\mathrm{d}x \\ \mathrm{d}y - p\mathrm{d}x \\ \mathrm{d}x \end{pmatrix}}_{\omega_f}$$

with $\det(S(\mathrm{a})) \neq 0$. In accordance with Cartan, this system is lifted to the new linear Pfaffian system

$$S(\bar{\mathrm{a}})\ \omega_{\bar{f}} = S(\mathrm{a})\ \omega_f.$$

After two normalizations and one prolongation, Cartan's method yields (an e-structure with) three fundamental invariants

$$I_1 = -\frac{1}{4}(f_p)^2 - f_y + \frac{1}{2}D_x f_p, \quad I_2 = \frac{f_{ppp}}{2a^2}, \quad I_3 = \frac{f_{yp} - D_x f_{pp}}{2a}, \tag{17}$$

and the invariant derivations

$$\begin{aligned} X_1 &= \frac{1}{a}\frac{\partial}{\partial p}, \quad X_3 = D_x - \frac{1}{2}f_p a\frac{\partial}{\partial a}, \quad X_4 = a\frac{\partial}{\partial a}, \\ X_2 &= \frac{1}{a}\frac{\partial}{\partial y} + \frac{1}{2}\frac{f_p}{a}\frac{\partial}{\partial p} - \frac{1}{2}f_{pp}\frac{\partial}{\partial a}, \end{aligned} \tag{18}$$

where $a = a_3$ and as usual $D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + f(x,y,p)\frac{\partial}{\partial p}$ denotes the Cartan vector field.

When $\dim(\mathcal{S}_{E_{\bar{f}},\Phi}) = 0$, the additional parameters a and $\bar{a}$ can be (post)normalized by fixing some invariant to some suitable value.

Proposition 3 (Olver (1995))**.** The symmetry groupoid $\mathcal{S}_{\bar{f},\Phi}$ is zero-dimensional if and only if there exist exactly three functionally independent specialized invariants $I_1[\bar{f}], I_2[\bar{f}], I_3[\bar{f}] : X \to \mathbb{C}$.

Note that the invariants $I_1[\bar{f}], \cdots, I_3[\bar{f}]$ are functionally independent if and only if $\mathrm{d}I_1[\bar{f}] \wedge \cdots \wedge \mathrm{d}I_3[\bar{f}] \neq 0$ and if the function $\bar{f}$ is rational, then the specialized invariants $I[\bar{f}] : X \to \mathbb{C}$ are algebraic functions in (x, y, p).

4.1. Computation of φ

Suppose that the symmetry groupoid $\mathcal{S}_{E_{\bar{f}},\Phi}$ is zero-dimensional. Then, according to theorem 3, there exists 3 invariants $F_k := I_k[\bar{f}]$, $1 \leq k \leq 3$ such that the algebraic (non differential) system

$$\{F_1(\bar{\mathrm{x}}) = I_1, F_2(\bar{\mathrm{x}}) = I_2, F_3(\bar{\mathrm{x}}) = I_3\} \tag{19}$$

is locally invertible and has a finite number of solutions

$$\bar{\mathrm{x}} = F^{-1}(I_1, \ldots, I_3). \tag{20}$$

The specialization of $I_1, \ldots, I_3$ on the source function f yields

$$\bar{\mathrm{x}} = F^{-1}(I_1[f], \ldots, I_3[f]). \tag{21}$$

The main idea here is that the inversion (20) is done by computing a (non differential) characteristic set C for the system (19) w.r.t. the ranking $\{\bar{x}, \bar{y}, \bar{p}\} \succ \{I_1, I_2, I_3\}$. Now, the most simple situation happens when $\deg(C) = 1$. In this case, the necessary form of the change of coordinates φ is the rational transformation defined by C.

Example 7. Consider the equivalence problem of the example 1 and the target equation $E_{\bar{f}}$ introduced by G. Reid (Reid et al., 1993)

$$\bar{y}'' = \frac{\bar{y}'}{\bar{x}} + \frac{4\bar{y}^2}{\bar{x}^3}.$$

The following invariants are functionally independent (where $I_{i;j\cdots k} = X_k \cdots X_j(I_i)$)

$$\bar{I}_1 = \frac{3}{4\bar{x}^2} + 8\,\frac{\bar{y}}{\bar{x}^3}, \quad \bar{I}_{1;3} = \frac{-3\,\bar{x} - 48\,\bar{y} + 16\,\bar{p}\bar{x}}{2\bar{x}^4}, \quad \bar{I}_{1;23} = -20\frac{1}{\bar{a}\bar{x}^4}, \quad \bar{I}_{1;31} = 8\frac{1}{\bar{a}\bar{x}^3}.$$

We normalize the parameter $\bar{a}$ by setting $\bar{I}_{1,23} = -20$. The characteristic set C is

$$\begin{cases} \bar{p} = -\dfrac{3}{32} + \dfrac{3}{512} {I_{1;31}}^2 I_1 + \dfrac{1}{4096} I_{1;3} {I_{1;31}}^3, \\[2ex] \bar{y} = -\dfrac{3}{256} I_{1;31} + \dfrac{1}{4096} I^3_{I_{1;31}}, \\[2ex] \bar{x} = \dfrac{1}{8} I_{1;31}, \end{cases}$$

which gives the sought necessary form of φ. As a byproduct we deduce that the symmetry group $S_{E_{\bar{f}},\Phi} = \{\mathrm{Id}\}$.

When $\deg(C)$ is strictly bigger than 1, we have two cases. First, $\deg(C) = \deg(S_{E_{\bar{f}},\Phi})$ and then φ is the algebraic transformation defined by C. Second, $\deg(C) > \deg(S_{E_{\bar{f}},\Phi})$. In this case, to obtain the transformation φ, we have to look for 3 other functionally independent invariants such that the new characteristic set C has degree equal to $\deg(S_{E_{\bar{f}},\Phi})$.

Example 8. Consider the equivalence problem of example 1 and the target equation number 6.9 in (Kamke, 1944):

$$\bar{y}'' = \bar{y}^3 + \bar{x}\bar{y}.$$

The corresponding symmetry group is

$$S_{E_{\bar{f}},\Phi} = \left\{(x, y) \to (x,\, \lambda y)\,|\lambda^2 = 1\right\}.$$

One can verify that I_1, $I_{1;13}$ and $I_{1;133}$, when specialized on the considered equation, are functionally independent. The associated characteristic set C is

$$
\begin{cases}
\bar{p} &= -\dfrac{\left(4\bar{x}^2 + 2I_1\bar{x} - 3I_{1;33} - 2{I_1}^2\right)}{3(I_{1;3}+1)}\bar{y}, \\
\bar{y}^2 &= -\dfrac{1}{3}\bar{x} - \dfrac{1}{3}I_1, \\
\bar{x}^3 &= -\dfrac{3}{2}I_1\bar{x}^2 + \dfrac{3}{4}I_{1;33}\bar{x} - \dfrac{3}{4}I_{1;3} - \dfrac{3}{8}{I_{1;3}}^2 + \dfrac{3}{4}I_{1;33}I_1 + \dfrac{1}{2}{I_1}^3 - \dfrac{3}{8},
\end{cases}
$$

which has degree 6, strictly bigger than the degree of the symmetry group. However, if we consider the invariants $K_1 := I_{1;233}/I_{1;31}$, $K_2 := I_{1;234}/I_{1;31}$ and $K_3 := I_{1:231}/I_{1;31}^2$, we obtain

$$
\begin{cases}
\bar{p} &= -K_1\bar{y}, \\
\bar{y}^2 &= \dfrac{1}{6}K_3, \\
\bar{x} &= -\dfrac{1}{6}K_3 + K_1.
\end{cases}
$$

That is, the necessary form of φ since this new set has degree two.

4.2. Heuristic of degree reduction

We have to remark here that searching invariants giving the required degree (as in example above) is not an easy task, although algorithmic. This is simply because the algebra of invariants can be very large. For this reason we provide an important heuristic which enables us to obtain the desired degree even for a "bad" choice of invariants.

Example 9. Consider the Emden-Fowler equation (9) and the $\mathcal{D}$-groupoid of transformations Φ_3. We have already computed the corresponding symmetry groupoid. The specialization of the invariants I_1, $I_{1;13}$ and $I_{1;133}$ gives three functionally independent functions. As explained above, we obtain the following characteristic set computed w.r.t. the ranking $\bar{p} \succ \bar{y} \succ \bar{x} \succ I_1 \succ I_{1;3} \succ I_{1;33}$

$$
\begin{cases}
\bar{p} &= \left(\dfrac{3}{8}I_1 - \dfrac{1}{4}\dfrac{I_{1;33}}{I_1} + \dfrac{1}{3}\dfrac{{I_{1;3}}^2}{{I_1}^2}\right)\bar{x}\bar{y} - \dfrac{1}{6}\dfrac{I_{1;3}}{I_1}\bar{y}, \\
\bar{y}^3 &= \left(-\dfrac{9}{4} - 2\dfrac{{I_{13}}^2}{{I_1}^3} + \dfrac{3}{2}\dfrac{I_{1;33}}{{I_1}^2}\right)\bar{x} - \dfrac{I_{1;3}}{{I_1}^2}, \\
\bar{x}^2 &= 4\left(\dfrac{I_{1;3}I_1}{9{I_1}^3 - 8{I_{1;3}}^2 + 6I_{1;33}I_1}\right)\bar{x} + 8\dfrac{{I_1}^2}{9{I_1}^3 - 8{I_{1;3}}^2 + 6I_{1;33}I_1}.
\end{cases}
\tag{22}
$$

Comparing with the $\mathcal{D}$-groupoid of symmetries (10) we deduce that, in contrast to $\bar{y}$, the degree of $\bar{x}$ must be reduced to one. This can be done in the following manner. First, observe that the Lie defining equations of Φ_3, more exactly $\bar{x}_p = 0$, implies that $X_1(\bar{x}) = 0$ where $X_1 = \frac{\partial}{a\partial p}$ is the invariant derivation (18). Now, differentiate the last equation of the characteristic set, which we write as $\bar{x}^2 = A\bar{x} + B$, w.r.t the derivation X_1. We find $A_{;1}\bar{x} + B_{;1} = 0$. The coefficient of $\bar{x}$ in this equation, which is invariant, could not vanish (since it is not identically zero when specializing on the Emden–Fowler equation). We obtain $\bar{x} = -\dfrac{B_{;1}}{A_{;1}}$ or explicitly

$$\bar{x} = -2\frac{KI_{1;1} + I_1 K_{;1}}{KI_{1;31} + I_{1;3}K_{;1}} \quad \text{with } K = \frac{I_1}{9{I_1}^3 - 8{I_{1;3}}^2 + 6I_{1;33}I_1}. \tag{23}$$

The necessary form of the change of coordinates φ is then given by (23) and the first two equations of (22).

The above reasoning can be summarized as follows

PROCEDURE ChgtCoords
Input : $E_{\bar{f}}$ and Φ such that $\dim(S_{E_{\bar{f}},\Phi}) = 0$
Output : $\bar{\mathrm{x}} = \varphi(\mathrm{x})$ the necessary form of the change of coordinates
1- Find 3 functionally independent invariants $(I_1[\bar{f}],\ I_2[\bar{f}],\ I_3[\bar{f}])$ defined on X.
2- Compute a char. set C of the algebraic system (20).
3- If $\deg(C) = 1$ then Return C.
4- Compute $S_{E_{\bar{f}},\Phi}$ with ROSENFELD-GRÖBNER.
5- WHILE $\deg(C) \neq \deg(S_{E_{\bar{f}},\Phi})$ DO
Reduce the degree of C.
END DO
6- Return C.

5. The solver

5.1. *Pre-calculation of φ*

5.1.1. *The first step : the adapted $\mathcal{D}$-groupoid*

Let $\Phi_1, \ldots, \Phi_7 \subset \mathrm{J}^\infty_*(\mathbb{C}^2, \mathbb{C}^2)$ denote the $\mathcal{D}$-groupoids defined in the table 1 page 14. It is not difficult to see that $\Phi_1 \subset \Phi_3 \subset \Phi_5$ and $\Phi_2 \subset \Phi_4 \subset \Phi_6$ and finally $\Phi_5, \Phi_6 \subset \Phi_7$.

Definition 4 (Signature index)**.** The *signature index* of E_f is

$$\mathrm{sign}(E_f) := ((\mathrm{d}_1, \mathrm{d}_3, \mathrm{d}_5), (\mathrm{d}_2, \mathrm{d}_4, \mathrm{d}_6), \mathrm{d}_7) \ \text{ where } \mathrm{d}_i := \dim S_{E_f,\Phi_i},\ 1 \le i \le 7.$$

Clearly, $(\mathrm{d}_1 \le \mathrm{d}_3 \le \mathrm{d}_5 \le \mathrm{d}_7)$ and $(\mathrm{d}_2 \le \mathrm{d}_4 \le \mathrm{d}_6 \le \mathrm{d}_7)$. Recall that the calculation of theses dimensions does not require solving differential equations.

Definition 5. We shall say that the signature index $\mathrm{sign}(E_f)$ *matches* the signature index $\mathrm{sign}(E_{\bar{f}})$ if and only if

$$\mathrm{d}_7 = \bar{\mathrm{d}}_7 \text{ and } (s_1 = \bar{s}_1 \text{ or } s_2 = \bar{s}_2)$$

where s_1 and s_2 stand for $(\mathrm{d}_1, \mathrm{d}_3, \mathrm{d}_5)$ and $(\mathrm{d}_2, \mathrm{d}_4, \mathrm{d}_6)$ resp.

	Transformations	Equation number according to Kamke's book
Φ_1	$\bar{x} = x,\ \bar{y} = \eta(x, y)$	1, 2, 4, 7, 10, 21, 23, 24, 30, 31, 32, 40, 42, 43, 45, 47, 50
Φ_3	$\bar{x} = x + C,\ \bar{y} = \eta(x, y)$	11, 78, 79, 87, 90, 91, 92, 94, 97, 98, 105, 106, 156, 172
Φ_5	$\bar{x} = \xi(x),\ \bar{y} = \eta(x, y)$	Null
Φ_2	$\bar{x} = \xi(x, y),\ \bar{y} = y$	81, 89, 133, 134, 135, 237
Φ_4	$\bar{x} = \xi(x, y),\ \bar{y} = y + C$	11, 79, 87, 90, 92, 93, 94, 97, 98, 99, 105, 106, 172, 178
Φ_6	$\bar{x} = \xi(x, y),\ \bar{y} = \eta(y)$	80, 86, 156, 219, 233
Φ_7	$\bar{x} = \xi(x, y),\ \bar{y} = \eta(x, y)$	3, 5, 6, 8, 9, 27, 44, 52, 85, 95, 108, 142, 144, 145, 147, 171, 211, 212, 238

Table 1. Adapted groupoids for certain equations from Kamke's list

Definition 6. Two second order ODE E_f and $E_{\bar{f}}$ are said to be *strongly equivalent* if

$$\exists \Phi \in \{\Phi_1, \cdots, \Phi_7\},\ \exists \varphi \in \Gamma\Phi,\ \varphi_* E_f = E_{\bar{f}},\ \dim S_{E_{\bar{f}},\Phi} = 0.$$

Lemma 2. If E_f and $E_{\bar{f}}$ are strongly equivalent then their signature indices match.

Definition 7 (Adapted $\mathcal{D}$-groupoid)**.** A $\mathcal{D}$-groupoid Φ is said to be *adapted* to the differential equation $E_{\bar{f}}$ if $\dim(S_{E_{\bar{f}},\Phi}) = 0$ and Φ is maximal among $\Phi_1, \cdots, \Phi_7$ satisfying this property.

The table 1 associates to each equation in the third column[2] its adapted groupoids. For instance, the first Painlevé equation (number 3) appears in the last row which means that its adapted $\mathcal{D}$-groupoid is the point transformations $\mathcal{D}$-groupoid Φ_7 . To the Emden–Fowler equation, number 11, we associate the $\mathcal{D}$-groupoids Φ_3 and Φ_4. In the case of homogeneous linear second order ODE (e.g. Airy equation, Bessel equation, Gauß hypergeometric equation) we prove that, generically, the adapted $\mathcal{D}$-groupoid is Φ_4.

5.1.2. The second step

Once the list of adapted $\mathcal{D}$-groupoids Φ is known, we proceed by computing the necessary form of the change of coordinates $\varphi \in \Gamma\Phi$ using ChgtCoords. Doing so, we construct a Maple table indexed by Kamke's book equations and where entries corresponding to the index $E_{\bar{f}}$ are:

1- the signature index of $E_{\bar{f}}$,

2- the list of the adapted $\mathcal{D}$-groupoids Φ of $E_{\bar{f}}$,

3- the necessary form of the change of coordinates $\varphi \in \Gamma\Phi$.

For instance, the entries associated to Rayleigh equation $y'' + {y'}^4 + y = 0$ are:

1- the signature index $((0, 1, 1), (1, 1, 1), 1)$,

2- the $\mathcal{D}$-groupoid Φ_1,

3- the necessary form of the change of coordinates

[2] A more complete list of equations is available upon request.

$$
\begin{cases}
\bar{p} &= -36\dfrac{I_{2;1}}{72+72I_1+{I_{2;1}}^2}\bar{y},\\
\bar{x} &= x,\\
\bar{y}^3 &= \dfrac{-1}{559872I_{2;1}^2}(I_{2;1}^6+216I_1I_{2;1}^4+216I_{2;1}^4+15552I_1^2I_{2;1}^2+31104I_1I_{2;1}^2+373248I_1^3\\
&\quad +15552I_{2;1}^2+1119744I_1^2+1119744I_1+373248)
\end{cases}
$$

with the normalization $I_2/I_{2;1} = 1$. Invariants here are those generated by (17) and (18) plus the essential invariant $\bar{x} = x$.

5.2. Algorithmic scheme of the solver

To integrate a differential equation E_f our solver proceeds as follows

PROCEDURE Newdsolve
Input : E_f
Output : An equation $E_{\bar{f}}$ in Kamke's list and the transformation φ such that $\varphi_*(E_f) = E_{\bar{f}}$
1- Compute the signature index of E_f.
2- Select from the table the list of equations $E_{\bar{f}}$ such that sign($E_{\bar{f}}$) matches sign(E_f).
3- FOR each equation $E_{\bar{f}}$ in the selected list DO
(i) Specialize, on E_f, the necessary form of the change of coordinates associated to $E_{\bar{f}}$. We obtain φ.
(ii) If $\varphi \in \Gamma\Phi$ and $\varphi_*(E_f) = E_{\bar{f}}$ then return $(E_{\bar{f}}, \varphi)$.
END DO.

5.3. Features of the solver

It is worth noticing that the time required to perform steps (i)- (ii) is very small. In fact, it is about one hundredth of a second using Pentium(4) with 256 Mo. Experimented on many examples, the total time needed to solve a given equation does not exceed few seconds in the worse situations.

The second feature of our solver is, contrarily to the symmetry methods, neither the table construction nor the algorithm of the solver involves integration of differential equations. Indeed, even the computation of signature indices is performed without solving the Lie equations.

A. Appendixes

A.1. *Differential algebra*

The reader is assumed to be familiar with the basic notions and notations of differential algebra. Reference books are (Ritt, 1950) and (Kolchin, 1973). We also refer to (Boulier et al., 1995; Hubert, 2000; Boulier, 2006). Let $U = \{u_1, \cdots, u_n\}$ be a set of differential indeterminates. k is a differential field of characteristic zero endowed with the set of derivations $\Delta = \{\partial_1, \cdots, \partial_p\}$. The monoid of derivations

$$\Theta := \left\{\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_p^{\alpha_p} \mid \alpha_1, \ldots, \alpha_p \in \mathbb{N}\right\} \tag{A.1}$$

acts freely on the alphabet U and defines a new (infinite) alphabet ΘU. The differential ring of the polynomials built over ΘU with coefficients in k is denoted $R = \mathrm{k}\{U\}$. Fix an admissible ranking over ΘU. For $f \in R$, $\mathrm{ld}(f) \in \Theta U$ denotes the *leader* (main variable), $I_f \in R$ denotes the *initial* of f and $S_f \in R$ denotes the separant of f. Recall that $S_f = \frac{\partial f}{\partial v}$ where $v = \mathrm{ld}(f)$. Let $C \subset R$ be a finite set of differential polynomials. Denote by $[C]$ the differential ideal generated by C and by $\sqrt{[C]}$ the radical of $[C]$. Let $H_C := \{I_f \mid f \in C\} \cup \{S_f \mid f \in C\}$. As usual, $\mathsf{full\_rem}$ is the Ritt full reduction algorithm (Kolchin, 1973). If $r = \mathsf{full\_rem}\ (f, C)$ then $\exists h \in H_C^\infty$, $hf = r \mod [C]$. Then the *reduced form* is defined by $\mathsf{reduced\_form}(f) := r/h$.

Definition 8 (Characteristic set)**.** The set $C \subset R$ is said to be a *characteristic set* of the differential ideal $\mathfrak{c} := \sqrt{[C]} : H_C^\infty$ if

(1) C is auto-reduced,

(2) $f \in \mathfrak{c}$ if and only if $\mathsf{full\_rem}\ (f, C) = 0$.

Definition 9 (Quasi–linear characteristic set)**.** The characteristic set $C \subset R$ is said to be *quasi–linear* if for each $f \in C$ we have $\deg(f, v) = 1$ where v is the leader of f.

Proposition 4. When the characteristic set C is quasi–linear, the differential ideal $\mathfrak{c} := \sqrt{[C]} : H_C^\infty \subset R$ is prime.

A.2. *Taylor series solutions space*

Let $\mathrm{k} := \mathbb{C}(x_1, \cdots, x_p)$ be the differential field of coefficients endowed with the set of derivations $\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_p}\right\}$. Let C be a characteristic set of a prime differential ideal $\mathfrak{c} \subset R$. We associate to C the system

$$(C = 0, H_C \neq 0) \tag{A.2}$$

of equations $f = 0$, $f \in C$ and inequations $h \neq 0$, $h \in H_C$.

Definition 10 (Taylor series solution)**.** A *Taylor series solution* (with coefficients in $\mathbb{C}$) of the PDE's system (A.2) is a morphism $\mu : R \to \mathbb{C}$ of (non differential) $\mathbb{C}$-algebras such that

$$[C] \subset \ker\mu \text{ and } H_C \cap \ker\mu = \emptyset.$$

The source of the Taylor solution μ is $\mathrm{s}(\mu) := (\mu(x_1), \ldots, \mu(x_p)) \in \mathbb{C}^p$ and the target is $\mathrm{t}(\mu) := (\mu(u_1), \ldots, \mu(u_n)) \in \mathbb{C}^n$. The diffiety associated to the characteristic set C is the set of the formal Taylor solutions of the system (A.2).

The dimension of the solutions space of (A.2) is the number of arbitrary constants appearing in the Taylor series solutions μ when the source point $\mathrm{x} := \mathrm{s}(\mu) \in \mathbb{C}^p$ is determined. Let K be the fractions field $\mathrm{Frac}(R/\mathfrak{c})$. Recall that the *transcendence degree* of a field extension K/k is the greatest number of elements in K which are k-algebraically independent. The degree $[K : \mathrm{k}]$ is the dimension of K as a k-vector space. When $\mathrm{tr}\deg(K/\mathrm{k}) = 0$, the field K is algebraic over k and $[K : \mathrm{k}] < \infty$. If $f \in C$, we denote $\mathrm{rank}(f) := (v, d)$ where $v := \mathrm{ld}\, f$ and $d := \deg(f, v)$. Let

$$\begin{aligned} \mathrm{rank}\, C &:= \{\mathrm{rank}(f) \mid f \in C\} \\ \mathrm{ld}\, C &:= \{\mathrm{ld}(f) \mid f \in C\} \\ \dim C &:= \mathrm{card}\,(\Theta U \setminus \Theta(\mathrm{ld}\, C)) \\ \deg C &:= \prod_{f \in C} \deg(f, \mathrm{ld}\, f). \end{aligned}$$

Proposition 5. $\dim C = \mathrm{tr}\deg(K/\mathrm{k})$ is the dimension of the solutions space of (A.2). If $\dim C = 0$ then the cardinal of the solutions space is finite and equal to $\deg C = [K : \mathrm{k}]$.

A.3. Differential elimination

Let $U = U_1 \sqcup U_2$ be a partition of the alphabet U. A ranking which eliminates the indeterminates of U_2 is such that

$$\forall v_1 \in \Theta U_1, \forall v_2 \in \Theta U_2, \quad v_2 \succ v_1. \tag{A.3}$$

Assume that C is a characteristic set of the prime differential ideal $\mathfrak{c} = \sqrt{[C]} : H_C^\infty$ w.r.t. the elimination ranking $\Theta U_2 \succ \Theta U_1$. Let $R_1 := k\{U_1\}$ be the differential polynomials k-algebra generated by the set U_1. Consider the set $C_1 := C \cap R_1$ and the differential ideal $\mathfrak{c}_1 := \mathfrak{c} \cap R_1$.

Proposition 6. C_1 is a characteristic set of $\mathfrak{c}_1$.

Consider the differential field of fractions $K := \mathrm{Frac}(R/\mathfrak{c})$ and denote by $\alpha : R \to K$ the canonical k-algebra morphism. Let K_1 be the differential subfield of K generated by the set $\alpha(R_1)$. Then K_1 is the fraction field associated to the prime differential ideal $\mathfrak{c}_1 := \mathfrak{c} \cap R_1$. The partition of the characteristic set

$$C = C_1 \sqcup C_2 \quad (\text{i.e. } C_2 := C \setminus C_1). \tag{A.4}$$

enables us to study the field extension K/K_1.

Proposition 7. $\mathrm{tr}\deg(K/K_1) = \dim C_2$. If $\dim C_2 = 0$ then $[K : K_1] = \deg C_2$.

A.4. Groupoids

Definition 11 (Groupoid)**.** A *groupoid* is a category in which every arrow is invertible.

Let $(\Phi, X, \circ, \mathrm{s}, \mathrm{t})$ be a category. Each arrow $\varphi \in \Phi$ admits a source $\mathrm{s}(\varphi) \in X$ and a target $\mathrm{t}(\varphi) \in X$ which are *objects* of this category. The composition $\varphi_2 \circ \varphi_1$ of the two arrows φ_1 and φ_2 are defined when $\mathrm{t}(\varphi_1) = \mathrm{s}(\varphi_2)$.

If Φ is a groupoid, for each arrow $\varphi \in \Phi$, there exists a unique inverse arrow φ^{-1} such that $\varphi^{-1} \circ \varphi = \mathrm{Id}_{\mathrm{s}(\varphi)}$ and $\varphi \circ \varphi^{-1} = \mathrm{Id}_{\mathrm{t}(\varphi)}$.

Let X and U be two manifolds and $\mathrm{x} \in X$. The Taylor series up to order q (i.e. the jet of order q) of a function $f : X \to U$, of class C^q, is denoted $\jmath^q_{\mathrm{x}} f$. The Taylor series of f about x is denoted $\jmath_{\mathrm{x}} f$ or $\jmath^\infty_{\mathrm{x}} f$. We shall say that $\mathrm{x} \in X$ is the source and $f(\mathrm{x}) \in U$ is the target of the q-jet $\jmath^q_x f$.

Example 10. For instance, when $X = U = \mathbb{C}$, we have

$$\jmath^q_x f := \Big(x, f(x), f'(x), \ldots, f^{(q)}(x)\Big) \in \mathbb{C}^{q+2}.$$

This jet is said to be *invertible* if $f'(x) \neq 0$. The jet of the function Id about the point x is $(x, x, 1, 0, \ldots, 0)$.

For each integer $q \in \mathbb{N}$ and each $\mathrm{x} \in X$, we set $\mathrm{J}^q_{\mathrm{x}}(X, U) := \bigcup_f \jmath^q_{\mathrm{x}} f$. We denote by $\mathrm{J}^q(X, U) := \bigsqcup_{\mathrm{x} \in X} \mathrm{J}^q_{\mathrm{x}}(X, U)$ the jets space up to order q. We denote by $\mathrm{J}^q_*(X, X)$ the submanifold of $\mathrm{J}^q(X, X)$ formed by the invertible jets. Recall that $\mathrm{J}^q_*(X, X)$ is a groupoid (Olver and Pohjanpelto, 2006) for the composition of Taylor series up to order q according to

$$\jmath^q_{\mathrm{x}}(g \circ f) = \left(\jmath^q_{f(\mathrm{x})} g\right) \circ \left(\jmath^q_{\mathrm{x}} f\right). \tag{A.5}$$

By definition, a $\mathcal{D}$-groupoid (Malgrange, 2001) $\Phi \subset \mathrm{J}^\infty_*(X, X)$ is a sub–groupoid of $\mathrm{J}^\infty_*(X, X)$ formed by the Taylor series solutions (see def. 10) of an algebraic PDE's system called the *Lie defining equations*. This system contains an inequation which expresses the invertibility of the jets.

The set of C^∞-functions $\varphi : X \to X$ that are local solutions of the Lie defining equations of Φ is a *pseudo-group* denoted by $\Gamma\Phi$. We define $\dim \Gamma\Phi = \dim \Phi/X := \dim C$ and, if $\dim C = 0$, $\operatorname{card} \Gamma\Phi = \operatorname{card} \Phi/X := \deg C$ where C is a characteristic set (see sect. 3) of the Lie defining equations of Φ.

A.5. Prolongation algorithm

Our aim, here, is to give an efficient way to prolong the action of $\Phi^{(0)}$ on $\mathrm{J}^0(\mathbb{C}, \mathbb{C})$ on the jets space $\mathrm{J}^n(\mathbb{C}, \mathbb{C})$. For each integer $q \geq 0$, define the differential field

$$\mathrm{k}^{(q)} := \mathbb{Q}(x,\, y,\, y_1, \ldots,\, y_q)$$

and the ring of differential polynomials

$$R^{(q)} := \mathrm{k}^{(q)}\{\bar{x},\, \bar{y},\, \bar{y}_1, \ldots,\, \bar{y}_q\}$$

The differential field $\mathrm{k}^{(q)}$ is the coefficients field of $R^{(q)}$ endowed with the set of derivations $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots, \frac{\partial}{\partial y_q}\right\}$. Let us assume that the Lie defining equations of $\Phi^{(0)}$ are given by a quasi-linear characteristic set $C^{(0)} \subset R^{(0)}$. The $\mathcal{D}$-groupoid $\Phi^{(q)}$ acting on $\mathrm{J}^q(\mathbb{C}, \mathbb{C})$ and prolonging the action of $\Phi^{(0)}$ is characterized by a characteristic set $C^{(q)} \subset R^{(q)}$. The prolongation formulae (Olver, 1993) of the point transformation $(x, y) \to (\xi(x, y),\, \eta(x, y))$ are of the form

$$\bar{y}_q = \eta_q(x,\, y, \ldots,\, y_q),$$

where $\bar{y} = \eta(x, y)$ if $q = 0$. The computation of the characteristic set $C^{(q)}$ is done incrementally using the infinite Cartan field $D_x := \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \cdots$

$$\begin{aligned} \eta_q &:= D_x \eta_{q-1} \cdot (D_x \xi)^{-1} \\ C^{(q)} &:= C^{(q-1)} \cup \left\{ \bar{y}_q - \mathsf{reduced\_form}\left(\eta_q,\, C^{(q-1)}\right) \right\} \end{aligned}$$

Proposition 8. If $C^{(0)}$ is a *quasi-linear* characteristic set of $\Phi^{(0)}$ then $C^{(q)}$ is a *quasi-linear* characteristic set of $\Phi^{(q)}$ w.r.t. the elimination ranking

$$\Theta \bar{y}_q \succ \Theta \bar{y}_{q-1} \succ \cdots \succ \Theta\{\bar{y}, \bar{x}\}.$$

The previous proposition gives an efficient method to prolong a $\mathcal{D}$-groupoid Φ without explicit knowledge of (the form of) the transformations $\varphi \in \Gamma\Phi$.

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