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Global solutions to rough differential equations with unbounded vector fields*

Antoine Lejay[†]

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Abstract

We give a sufficient condition to ensure the global existence of a solution to a rough differential equation whose vector field has a linear growth. This condition is weaker than the ones already given and may be used for geometric as well as non-geometric rough paths with values in any suitable (finite or infinite dimensional) space. For this, we study the properties the Euler scheme as done in the work of A.M. Davie.

Keywords: controlled differential equations, rough paths, Euler scheme, global solution to differential equation, rough differential equation.

AMS Classification: 60H10, 65C30

1 Introduction

Initiated a decade ago, the theory of rough paths imposed itself as a convenient tool to define stochastic calculus with respect to a large class of stochastic processes out of the range of semi-martingales (fractional Brownian motion, ...) and also allows one to define pathwise stochastic differential equations [6, 8, 10, 11, 15, 17, 18].

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[†]Project-team TOSCA, Institut Elie Cartan Nancy (Nancy-Université, CNRS, INRIA), Boulevard des Aiguillettes B.P. 239 F-54506 Vandœuvre-lès-Nancy, France. E-mail: Antoine.Lejay@iecn.u-nancy.fr

The idea to define integrals of differential forms along irregular paths or solutions of differential equations driven by irregular paths is to extend properly such paths in a suitable non-commutative space. These extensions encode in some sense the “iterated integrals” of the paths. Let us denote by x the driving path which lives in a tensor space and Lie group $T(\mathbb{R}^d) := \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$ and that projects onto some continuous path of finite p -variations on \mathbb{R}^d with $p \in [2, 3)$. Such a path x is called a *p -rough path*.

The goal of this article is to study global existence of solutions to the rough differential equation (RDE)

$$y_t = y_0 + \int_0^t f(y_s) dx_s \tag{1}$$

when f is not bounded. What is called a solution to (1) needs to be properly defined. Indeed, there exist two notions of solutions we will deal with (See Definitions 1 and 2 below).

Let us recall here the already existing results. The end of the introduction contains a short discussion about the differences between our result and the ones presented here.

Linear case: The special case of a linear vector field f was studied for example in the articles [1, 6, 9, 12, 16].

The original approach of T. Lyons: In the original approach, uniqueness and continuity of the Itô map $x \mapsto y$, where y is a solution to (1), is proved for a bounded function f which is twice differentiable with bounded derivatives and such that $\nabla^2 f$ is γ -Hölder continuous with $2 + \gamma > p$ [6, 10, 11, 15, 17, 18]. One knows from [2] that these conditions on f are essentially sharp. As pointed out soon after by A.M. Davie [2], existence of a solution to (1) is granted when f is bounded with a bounded derivative which is γ -Hölder continuous, $2 + \gamma > p$ (See also [6, 14] for example). However, in this case, one has to restrict to finite-dimensional spaces and several solutions may exist [2].

The approach of A.M. Davie: Providing an alternative approach to the one of T. Lyons based on a fixed point theorem, A.M. Davie studied in [2] the Euler scheme defined by

$$y_{i+1}^n = y_i^n + f(y_i^n)x_{t_i, t_{i+1}}^1 + f \cdot \nabla f(y_i^n)x_{t_i, t_{i+1}}^2$$

where $x_{s,t} = x_s^{-1} \otimes x_t$ and x^1 (resp. x^2) is the projection of x onto \mathbb{R}^d (resp. $\mathbb{R}^d \otimes \mathbb{R}^d$). With conditions only on the regularity of the vector field, local existence is shown. Global existence is granted if there exist two positive increasing functions A and D on $[1, +\infty)$ with $D(R) \leq R^{1+\gamma}A(R)$, $2 \leq p <$

$\gamma + 2 < 3$ such that

$$|f(y)| \leq D(R), \quad |\nabla f(y) - \nabla f(z)| \leq A(R)|y - z|^\gamma, \quad \forall y, z \text{ s.t. } |y|, |z| \leq R,$$

and

$$\int_1^{+\infty} \left(\frac{1}{A(R)^{p-1} D(R)^{1+p\gamma}} \right)^{\frac{1}{1+\gamma}} dR = +\infty. \quad (2)$$

Besides, if (2) is not satisfied for some functions A and D as above, it is possible to construct a vector field f and a driver x such that the solution to (1) explodes in a finite time.

Applied to functions f with a γ -Hölder continuous derivative ∇f with $2 + \gamma > p$, one may take $A(R)$ equal to a constant. Then global existence is granted for example if $D(R) = R^\delta$ with $\delta < (1 + \gamma)/(1 + \gamma p)$, but an explosion may occur for a function f is in class of functions if $\delta > (1 + \gamma)/(1 + \gamma p)$, which is the case for $\delta = 1$.

The approach by P. Friz and N. Victoir: In [5, 6], P. Friz and N. Victoir provide an alternative construction of the solutions of RDE that relies on sub-Riemannian geodesics and hence of geometric rough paths. The case of geometric rough paths shall be considered using $(p, p/2)$ -rough paths [14]. In [6, Exercise 10.61], they show that it is not necessary that the vector field is bounded, provided that f is Lipschitz continuous, ∇f is γ -Hölder continuous and $f \cdot \nabla f$ is also γ -Hölder continuous.

Using a fixed point approach: In [12], we have also studied the existence of a global solution, by using the approach on fixed point theorem.

With this approach, the RDE (1) is solved up to a finite “short time” horizon. Global existence follows from the convergence of a series related to the sum of the horizons. The complete conditions are cumbersome to write, but if $h(R) = \sup_{|y| \leq R} |f(y)|$, then global existence is granted provided that f has a bounded derivative ∇f which is γ -Hölder continuous with $2 + \gamma > p$ and $h(R) \sim_{R \rightarrow \infty} R^\delta$, $0 < \delta \leq 1/p$ or $h(R) \sim_{R \rightarrow \infty} \log(R)$.

The case of a “regular enough” driver (also called Young case): If $1 \leq p < 2$, then global existence is granted if the vector field f is γ -Hölder continuous with $1 + \gamma > p$ [13].

The one-dimensional approach of H. Doss and H. Sussmann: If $d = 1$, H. Doss [4] and H. Sussmann [20] have shown that a solution to an equation of type (1) may be defined by considering the solution to the ODE $dz_t = f(z_t) dt$ and setting $y_t = z_{x_t}$. Local existence to $dz_t = f(z_t) dt$ is granted provided that f is continuous. Global existence holds under a linear growth condition on f as it follows easily from an application of the Gronwall inequality [19] which provides a global bound on z : If $|f(z)| \leq A + B|z|$, then

$|z_t| \leq (|z_0| + At) \exp(Bt)$ for any $t \geq 0$. This approach may be generalized for a multi-dimensional driver x when f has vanishing Lie brackets.

The situation is more intricate when x lives in a space of dimension bigger than 2. In particular, the asymptotic behavior of ∇f plays a fundamental role.

If x is a p -rough path (a rough path of finite p -variation) with $p \in [2, 3)$, then for any function φ of finite $p/2$ -variations with values in $\mathbb{R}^d \otimes \mathbb{R}^d$, $z = x + \varphi$ is also a p -rough path. In addition, as shown in [14] for the general case, the solution to $y_t = y_0 + \int_0^t f(y_s) dz_s$ is also solution to

$$y_t = y_0 + \int_0^t f(y_s) dx_s + \int_0^t f \cdot \nabla f(y_s) d\varphi_s. \quad (3)$$

Consider any rough path z living above the path 0 on \mathbb{R}^d with $\varphi(t) = ct$ for a matrix c (if c is anti-symmetric, then such a rough path is the limit of a sequence of smooth paths lifted in the tensor space by their iterated integrals). Then y is solution to

$$y_t = y_0 + \int_0^t f \cdot \nabla f(y_s) c ds.$$

Thus, explosion may occur according to the behavior of $f \cdot \nabla f$. In particular, f may grow linearly, but $f \cdot \nabla f$ may grow faster than linearly, and an explosion may occur. Of course, if ∇f is bounded and f grows linearly, then $f \cdot \nabla f$ also grows linearly.

Example 1 (M. Gubinelli). Consider the solution y of the RDE $y_t = a + \int_0^t f(y_s) dx_s$ living in \mathbb{R}^2 and driven by the rough path $x_t = (1, 0, (1 \otimes 1)t)$ with values in $1 \oplus \mathbb{R} \oplus (\mathbb{R} \otimes \mathbb{R})$. This rough path lies above the constant path at $0 \in \mathbb{R}$ and has only a pure area part which is proportional to t . Note that this rough path can only be seen as a p -rough path with $p > 2$ [11]. Then y is also a solution to $y_t = a + \int_0^t (f \cdot \nabla f)(y_s) ds$ (See [14]). The vector field $f \in \mathbb{R}^2 \rightarrow L(\mathbb{R}, \mathbb{R}^2)$ given by

$$f(\xi) = (\sin(\xi_2)\xi_1, \xi_1), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$$

has a linear growth but

$$(f \cdot \nabla f)(\xi) = (\sin^2(\xi_2)\xi_1 + \xi_1^2 \cos(\xi_2), \sin(\xi_2)\xi_1)$$

has a quadratic growth. Take the initial point $a = (a_1, 0)$ with $a_1 > 0$. Then $(y_t)_2 = 0$ and $(y_t)_1 = a_1 + \int_0^t (y_s)_1^2 ds$ so that $(y_t)_1 \rightarrow +\infty$ in finite time equal to $1/a_1$. This proves that explosion may occur in a finite time.

It proves that in the context of RDE, the growth of $f\nabla f$ is important. The previous example involves a pure non-geometric rough path. Yet a slight modification of this example show that this may happen also for geometric rough paths.

Example 2. Let us consider

$$y_t^n = a + \int_0^t \begin{bmatrix} (y_s^n)^1 \sin((y_s^n)^2) \\ 0 \end{bmatrix} d\kappa_s^n + \int_0^t \begin{bmatrix} 0 \\ (y_s^n)^1 \end{bmatrix} d\lambda_s^n \quad (4)$$

with

$$(\kappa^n, \lambda^n)_t = n^{-1/2}[\cos(2\pi nt), \sin(2\pi nt)].$$

The smooth rough path x^n living above (κ^n, λ^n) converges in p -variation with $p > 2$ to the rough path in $1 \oplus \mathbb{R}^2 \oplus (\mathbb{R}^2 \otimes \mathbb{R}^2)$ defined by

$$x_{s,t} = \left(1, 0, \begin{bmatrix} 0 & 2\pi \\ -2\pi & 0 \end{bmatrix} (t-s) \right).$$

This is the standard example which show the discontinuity of the Lévy area [15]. Let us assume that the solutions y^n to (4) is uniformly bounded on $[0, T]$ for some $T > 0$. By the continuity theorem (one may assume that f , ∇f and $\nabla^2 f$ are bounded), the solution y^n to (4) converges to the solution to

$$y_t = a + 2\pi \int_0^t \begin{bmatrix} -(y_s^1)^2 \cos(y_s^2) \\ y_s^1 \sin(y_s^2) \end{bmatrix} ds.$$

Hence, we are in the same situation as Example 1 if $a^2 = 0$ and then $y^2 \equiv 0$. If $f(\xi) = (f_1(\xi), f_2(\xi))$ with $f_1(\xi) = (\xi^1 \sin(\xi^2), 0)$ and $f_2(\xi) = (0, \xi^1)$, then ∇f is not bounded although f grows linearly. Since y^n converges to y which explodes in a finite time $1/a^1$, y^n cannot be uniformly bounded on any time interval $[0, T]$ for T bigger than $1/a^1$. Note however that the p -variation norm of the rough path (x^n) remains bounded. Unlike the situations where the Doss-Sussmann approach may be used, bounds on the solution to RDE driven by a p -rough path with $p > 2$ cannot be derived from the sole information on the linear growth of f .

Although we use some general ideas already used in the context of rough paths theory (See for example [2, 5, 6, 8, 12, 13], ...), we should note that:

- Our conditions on the vector field are weaker than the ones already given in the literature. In particular, we show that for existence, f and ∇f need to be continuous and ∇f needs to be bounded. Yet the Hölder regularity of ∇f plays no role here, while $f\nabla f$ shall be Hölder continuous. Even if the computations are very close to the one of [2, 5], we then give a

simple natural condition on the vector field as well as a simple bound on the solution.

The class of functions f with a bounded derivative ∇f which is γ -Hölder continuous is different from the class of functions f with a bounded derivative ∇f and such that $f\nabla f$ is γ -Hölder continuous.

Example 3. Let us consider $f(x) = \sin(x^2)/x$ for $|x| \geq 1$. The first and second order derivatives of f are

$$\begin{aligned}\nabla f(x) &= 2 \cos(x^2) - \frac{\sin(x)^2}{x^2} \\ \text{and } \nabla^2 f(x) &= -4x \sin(x^2) - \frac{2 \cos(x) \sin(x)}{x} + \frac{\sin(x^2)}{x^3}.\end{aligned}$$

Hence, ∇f is not uniformly γ -Hölder continuous on $\{x; |x| \geq 1\}$ whatever $\gamma \in (0, 1]$. Yet

$$f(x)\nabla f(x) = \frac{2 \cos(x^2) \sin(x^2)}{x} - \frac{\sin^2(x^2)}{x^3}$$

and it is easily checked that $f\nabla f$ has a bounded derivative on $\{x; |x| \geq 1\}$ and is then uniformly Lipschitz continuous (and thus γ -Hölder continuous) on $\{x; |x| \geq 1\}$.

The class of vector fields f with a bounded derivatives ∇f and such that $f\nabla f$ is γ -Hölder continuous enjoys the property to be stable under some change of variables. This is not necessarily the case for vector fields f with bounded first and second-order derivatives ∇f and $\nabla^2 f$.

For a toy example, let us consider the simple case of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that ∇f is bounded and $L = f\nabla f$ has a bounded derivative. Assume that for a smooth path x , $y_t = a + \int_0^t f(y_s) dx_s$ has a solution y which remains in $(1, +\infty)$. Set $z_t = \log(y_t)$. Then z is solution to $z_t = \log(a) + \int_0^t g(z_s) dx_s$ with $g(z) = e^{-z} f(e^z)$. The vector field g has the interesting property to remain bounded on $(0, +\infty)$ and one may then hope to deduce some bounds on the solution y from some bound on the solution z .

Since $L = f\nabla f$ has a bounded derivative, g has a bounded derivative on $(0, +\infty)$ and $g\nabla g$ has also a bounded derivative, which means that $g\nabla g$ is globally Lipschitz. Yet

$$g''(z) = e^{-z} f(e^z) - f'(e^z) + e^z f''(e^z).$$

This means that g' is uniformly Lipschitz on $(0, +\infty)$ — a condition required to deal with RDE with a regularity index $\gamma = 1$ — only if $yf''(y)$ remains bounded, which is a stronger condition than assuming that f'' remains bounded.

This kind of computations may be carried to the multi-dimensional case using polar coordinates and an exponential change of variable of the radial component.

This explains the failure of the attempt carried by some persons, including the author of the present article, to get a global bound by such a change of variable under the sole assumption that ∇f is uniformly γ -Hölder and bounded.

- The condition on the vector field appears naturally when one uses the approach by the Euler scheme proposed by A.M. Davie in [2]. This is not the case when one uses the notion of solution proposed by T. Lyons because the term $f\nabla f$ is somewhat hidden in the cross iterated integral between the solution y and the driver x . The idea used in this article to get a global bound is closely related to the one in [12] where a fixed point approach was used. Yet in this article [12], we did not succeed to obtain a global bound in a general. In addition, the superfluous assumption on the Hölder regularity of ∇f was used and necessary.

- The use of sub-Riemannian geodesics and the reduction to smooth drivers was the core ideas of [5, 6]. Here, there is no need to restrict to geometric rough paths and the results may be used for any finite-dimensional Banach spaces and even infinite-dimensional in some cases. Hence, the structure of the underlying spaces plays no role here.

Example 4. The most natural example of a non-geometric rough path is the one B living above a d -dimensional Brownian path W with $B^1 = W$ and $B_{s,t}^{2,i,j} = \int_s^t (W_r^i - W_s^i) dB_r^j$. Here the integral has to be understood in the Itô sense. A geometric rough path \tilde{B} may be constructed above the Brownian path W with $\tilde{B}^1 = W$ and $\tilde{B}_{s,t}^{2,i,j} = \int_s^t (W_r^i - W_s^i) \circ dW_r^j$, where the stochastic integral is the Stratonovich one. Hence, \tilde{B} and B are linked by $B_{s,t}^{2,i,j} = \tilde{B}_{s,t}^{2,i,j} + \frac{1}{2}(t-s)\delta_{i,j}$. RDE driven by B correspond to Itô SDE while the ones driven by \tilde{B} corresponds to Stratonovich SDE.

When one use the Brownian rough path B , the Euler scheme presented here corresponds indeed to the Milstein scheme. In the very beginning of rough paths theory, the rate of convergence in this case has been studied by J. Gaines and T. Lyons [7] with aim at developing simulation algorithms with adaptative stepsize.

In view of (3), the regularity condition on $f\nabla f$ is also the one which is necessary to deal with a non-geometric p -rough path seen as a $(p, p/2)$ -geometric rough path using the decomposition of the space introduced in [14].

- The notion of solution to a RDE introduced by T. Lyons [15] cannot be used here (See Definition 1), so that we use the notion of solution introduced by A.M. Davie in [2] (see Definition 2) which is similar to the one proposed

by M. Gubinelli [8]. We show in Section 10 that the two notions of solutions coincide when ∇f is γ -Hölder continuous.

- A general conclusion to draw from the cited works on global existence for non bounded vector fields f is that a variety of results could be given according to the behaviour at infinity of f and its derivatives. Hence, the growth of f is not the only factor to look at.

2 Notations and hypotheses

Let ω be a *control*. By this, we mean a function defined from $\Delta^2 := \{0 \leq s \leq t \leq T\}$ to \mathbb{R}_+ which is continuous close to its diagonal and super-additive

$$\omega(s, r) + \omega(r, t) \leq \omega(s, t), \quad 0 \leq s < r < t \leq T.$$

For the sake of simplicity, we assume here that ω is continuous and that $\omega(s, t) > 0$ as soon as $t > s$.

Let x be a path with values in $T(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$. We set $x_{s,t} = x^{-1} \otimes x_t$. The part of x in \mathbb{R}^d is denoted by x^1 and the part in $\mathbb{R}^d \otimes \mathbb{R}^d$ is denoted by x^2 . Then x is a *rough path of finite p -variation controlled by ω* when the quantity

$$\|x\| := \sup_{0 \leq s < t \leq T} \max \left\{ \frac{|x_{s,t}^1|}{\omega(s, t)^{1/p}}, \frac{|x_{s,t}^2|}{\omega(s, t)^{2/p}} \right\}$$

is finite for a fixed p .

If $\omega(s, t) = t - s$, then we work indeed with paths that are $1/p$ -Hölder continuous.

Throughout all this article, we consider only the case where $p \in [2, 3)$. The case $p \leq 2$ is covered for example by [13].

For a path y with values in \mathbb{R}^m , we set

$$\|y\| := \sup_{0 \leq s < t \leq T} \frac{|x_{s,t}^1|}{\omega(s, t)^{1/p}}.$$

A vector field is an application f which is linear from \mathbb{R}^m to $L(\mathbb{R}^d, \mathbb{R}^m)$, the space of linear applications from \mathbb{R}^d to \mathbb{R}^m . With indices, we set $f(x) = e_i f_j^i(x) \widehat{e}_j^*$, where $\{e_i\}_{i=1, \dots, m}$ is the canonical basis of \mathbb{R}^m , and $\{\widehat{e}_j^*\}_{j=1, \dots, d}$ is the dual of the canonical basis of \mathbb{R}^d . We set

$$f \cdot \nabla f(x) := \sum_{i=1}^m e_i \sum_{k=1}^d \frac{\partial f_j^i}{\partial x_k}(x) f_\ell^k(x) \widehat{e}_\ell^* \otimes \widehat{e}_j^*,$$

which means that $f \cdot \nabla f$ is an application from \mathbb{R}^m to $L(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^m)$.

Hypothesis 1. The function f is continuously differentiable from \mathbb{R}^m to $L(\mathbb{R}^d, \mathbb{R}^m)$ and is such that ∇f is bounded and $F(x) := f(x) \cdot \nabla f(x)$ is γ -Hölder continuous with norm $H_\gamma(F)$ from \mathbb{R}^m to $L(\mathbb{R}^m \otimes \mathbb{R}^d, \mathbb{R}^m)$.

Note that this hypothesis is slightly different from the one given usually to prove existence to solutions of RDE where it is assumed that ∇f is γ -Hölder continuous.

Hypothesis 2. We assume that $p \in [2, 3)$ and that $\theta := (2 + \gamma)/p$ is greater than 1.

Definition 1 (Solution in the sense of Lyons). We call by a *solution to (1) in the sense of Lyons* the projection onto \mathbb{R}^m of a rough path z of finite p -variations controlled by ω with values in $T(\mathbb{R}^m \oplus \mathbb{R}^d)$ which solves the following equation

$$z_t = z_0 + \int_0^t g(z_s) dz_s, \quad (5)$$

where z projects onto $T(\mathbb{R}^d)$ as x , onto \mathbb{R}^m as y , and g is the differential form $g(y, x) = dx + f(y) dx$. The integral in (5) shall be understood as the “rough integral”, that is as an integral in the sense of rough path.

Note that the definitions of z involves the “cross-iterated integrals” between x and y , and requires that the derivative of f is γ -Hölder continuous.

Under Hypothesis 1, it is not compulsory that ∇f is γ -Hölder continuous, so that we shall use another notion of solution, since we cannot use a fixed point theorem that relies on the definition of a rough integral.

3 Solution in the sense of Davie

The notion of solution of (1) we use is the notion of solution in the sense of Davie, introduced in [2].

Definition 2 (Solution in the sense of Davie). A solution of (1) in the sense of Davie is a continuous path y from $[0, T]$ to \mathbb{R}^m of finite p -variation controlled by ω such that for some constant L ,

$$|y_t - y_s - \mathfrak{D}(s, t)| \leq L\omega(s, t)^\theta, \quad \forall (s, t) \in \Delta^2 \quad (6)$$

with

$$\mathfrak{D}(s, t) = f(y_s)x_{s,t}^1 + F(y_s)x_{s,t}^2.$$

The next propositions, which assume the existence of a solution in the sense of Davie, will be proved below in Section 4

Proposition 1. *Let y be a solution of (1) in the sense of Davie under Hypotheses 1 and 2. Then $\mathfrak{D}(s, t)$ is an almost rough path whose associated rough path is y and for a family of partitions $\{\{t_i^n\}_{i=0, \dots, n}\}_{n \in \mathbb{N}^*}$ of $[0, t]$ whose meshes decrease to 0,*

$$y_t = y_0 + \lim_{n \rightarrow \infty} \sum_{i=0}^n \mathfrak{D}(t_i^n, t_{i+1}^n). \quad (7)$$

Proposition 2 (Boundedness of the solution). *Let y be a solution to (1) in the sense of Davie under Hypotheses 1 and 2. Then $\|y\|$ and $\|y\|_\infty$ are bounded by some constants that depend only on $\|\nabla f\|_\infty$, $H_\gamma(F)$, $|f(y_0)|$, $|y_0|$, $\|x\|$, $\omega(0, T)$, γ and p . More precisely, there exist some constants C_1 depending only on $\|\nabla f\|_\infty$, $H_\gamma(F)$, $\|x\|$, p and γ and C_2 depending only on $|f(0)|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, $\|x\|$, p , γ such that*

$$\sup_{t \in [0, T]} |y_t| \leq R(T)|y_0| + C_2(R(T) - 1)$$

with

$$R(T) \leq \exp(C_1 \max\{\omega(0, T)^{1/p}, \omega(0, T)\}).$$

4 Proofs of Propositions 1 and 2

Let us set for $0 \leq s \leq r \leq t \leq T$,

$$\mathfrak{D}(s, r, t) := \mathfrak{D}(s, t) - \mathfrak{D}(s, r) - \mathfrak{D}(r, t).$$

Then the main idea of the proofs (as well as the proofs of the other theorems) are the following: First, we find a function $C(\|y\|, L, T)$ such that $|\mathfrak{D}(s, r, t)| \leq C(\|y\|, L, T)\omega(s, t)^\theta$ for all $(s, t) \in \Delta^2$. From the sewing Lemma (See for example [11, Theorem 5, p. 89]), after having shown that y is the rough path corresponding to the almost rough path $(\mathfrak{D}(s, t))_{(s, t) \in \Delta^2}$, we get that for some universal constant M ,

$$|y_t - y_s - \mathfrak{D}(s, t)| \leq MC(\|y\|, L, T)\omega(s, t)^\theta.$$

Then, after having estimated $\|\mathfrak{D}(s, t)\| \leq C'(\|y\|, L, T)\omega(s, t)^{1/p}$, we get an inequality of type

$$\|y\| \leq MC(\|y\|, L, T)\omega(0, T)^{\theta-1/p} + C'(\|y\|, L, T).$$

A careful examination of the functions $C(\|y\|, L, T)$ and $C'(\|y\|, L, T)$ shows that indeed L itself depends on y_0 , $\|y\|$ and T and that

$$\|y\| \leq A(y_0, T) + B(T)\|y\|$$

with $B(T)$ decreasing to 0 as T decreases to 0. Then, choosing T small enough implies that $\|y\|$ is bounded in small time and then for any time T using the arguments presented in Appendix. This idea is the central one used for example in [12, 13].

Let us set

$$\begin{aligned} B_1 &:= \sup_{t \in [0, T]} |F(y_t)|, \quad B_2(s, t) := y_t - y_s - f(y_s)x_{s,t}^1, \\ B_3(a, b) &:= f(b) - f(a), \quad B_4(a, b) := F(b) - F(a), \\ \text{and } B_5(s, t) &:= f(y_t) - f(y_s) - F(y_s)x_{s,t}^1 \end{aligned}$$

as well as

$$\mu := \omega(0, T)^{1/p}.$$

Remark 1. The quantity μ will be used to denote the “short time” horizon and is the central quantity for getting our estimates.

Since F is γ -Hölder continuous,

$$B_1 \leq |F(y_0)| + H_\gamma(F)\|y\|^\gamma \mu^\gamma.$$

If y is a solution in the sense of Davie with a constant L ,

$$|B_2(s, t)| \leq L\omega(s, t)^\theta + B_1\|x\|\omega(s, t)^{2/p}.$$

Since f is Lipschitz continuous,

$$|B_3(a, b)| \leq \|\nabla f\|_\infty |b - a|.$$

In addition,

$$|B_4(a, b)| \leq H_\gamma(F)|b - a|^\gamma.$$

Finally,

$$\begin{aligned}
f(y_t) - f(y_s) &= \int_0^1 \nabla f(y_s + \tau y_{s,t}) y_{s,t} \, d\tau \\
&= \int_0^1 \nabla f(y_s + \tau y_{s,t}) f(y_s) x_{s,t}^1 \, d\tau + \int_0^1 \nabla f(y_s + \tau y_{s,t}) B_2(s, t) \, d\tau \\
&= - \int_0^1 \nabla f(y_s + \tau y_{s,t}) (f(y_s + \tau y_{s,t}) - f(y_s)) x_{s,t}^1 \, d\tau \\
&\quad + \int_0^1 F(y_s + \tau y_{s,t}) x_{s,t}^1 \, d\tau + \int_0^1 \nabla f(y_s + \tau y_{s,t}) B_2(s, t) \, d\tau \\
&= \int_0^1 F(y_s + \tau y_{s,t}) x_{s,t}^1 \, d\tau + \int_0^1 \nabla f(y_s + \tau y_{s,t}) (B_2(s, t) - B_3(y_s, y_s + \tau y_{s,t})) x_{s,t}^1 \, d\tau \\
&= F(y_s) x_{s,t}^1 + \int_0^1 B_4(y_s, y_s + \tau y_{s,t}) x_{s,t}^1 \, d\tau \\
&\quad + \int_0^1 \nabla f(y_s + \tau y_{s,t}) (B_2(s, t) - B_3(y_s, y_s + \tau y_{s,t})) x_{s,t}^1 \, d\tau. \quad (8)
\end{aligned}$$

This proves that

$$\begin{aligned}
|B_5(s, t)| &\leq H_\gamma(F) \|y\|^\gamma \|x\| \omega(s, t)^{(1+\gamma)/p} + \|\nabla f\|_\infty^2 \|x\| \|y\| \omega(s, t)^{2/p} \\
&\quad + \|\nabla f\|_\infty L \omega(s, t)^\theta + \|\nabla f\|_\infty |F(y_0)| \|x\| \omega(s, t)^{2/p} \\
&\quad + H_\gamma(F) \|y\|^\gamma \omega(0, T)^{\gamma/p} \|x\| \omega(s, t)^{2/p}.
\end{aligned}$$

Lemma 1. For any $0 \leq s \leq r \leq t \leq T$,

$$|\mathfrak{D}(s, r, t)| \leq (C_3(\mu) \|y\|^\gamma + C_4(\mu) \|y\| + C_5(\mu) L + C_6(\mu, y_0)) \omega(s, t)^\theta$$

with

$$C_3(\mu) := H_\gamma(F) (\|x\|^2 (1 + \mu^{1+\gamma}) + \|x\|),$$

$$C_4(\mu) := \|\nabla f\|_\infty^2 \|x\|^2 \mu,$$

$$C_5(\mu) := \|\nabla f\|_\infty \|x\| \mu$$

$$\text{and } C_6(\mu, y_0) := \|\nabla f\|_\infty |F(y_0)| \|x\|^2 \mu^{1-\gamma} \leq |f(y_0)| \|\nabla f\|_\infty^2 \|x\|^2 \mu^{1-\gamma}.$$

Proof. With $x_{s,t}^2 = x_{s,r}^2 + x_{r,t}^2 + x_{s,r}^1 \otimes x_{r,t}^1$,

$$\mathfrak{D}(s, r, t) = (f(y_s) - f(y_r)) x_{r,t}^1 + F(y_s) x_{s,r}^1 \otimes x_{r,t}^1 + (F(y_s) - F(y_r)) x_{r,t}^2.$$

Hence

$$|\mathfrak{D}(s, r, t)| \leq |B_5(s, r)| \|x\| \omega(s, t)^{1/p} + |B_4(y_s, y_r)| \|x\| \omega(s, t)^{2/p}.$$

This proves the result. \square

Lemma 2. For any $(s, t) \in \Delta^2$,

$$|\mathfrak{D}(s, t)| \leq (C_7(\mu, y_0) + C_8(\mu)\|y\|^\gamma + C_9(\mu)\|y\|)\omega(s, t)^{1/p}$$

with

$$C_7(\mu, y_0) := (|f(y_0)| + |F(y_0)|\mu)\|x\| \leq |f(y_0)|(1 + \|\nabla f\|_\infty\mu)\|x\|,$$

$$C_8(\mu) := H_\gamma(F)\|x\|\mu^\gamma,$$

$$\text{and } C_9(\mu) := \|\nabla f\|_\infty\|x\|\mu.$$

Proof. This follows from

$$\begin{aligned} |\mathfrak{D}(s, t)| &\leq |f(y_0)|\|x\|\omega(s, t)^{1/p} + \|y\|\|x\|\|\nabla f\|_\infty\omega(s, t)^{2/p} \\ &\quad + |F(y_0)|\|x\|\omega(s, t)^{2/p} + \|y\|^\gamma\|x\|H_\gamma(F)\omega(s, t)^{(1+\gamma)/p}. \end{aligned}$$

This proves the Lemma. \square

We have now all the required estimates to prove Propositions 1 and 2.

Proof of Proposition 1. It follows from Lemma 1 that $\{\mathfrak{D}(s, t)\}_{(s, t) \in \Delta^2}$ is an almost rough path. From the sewing lemma (See for example [11, Theorem 5, p. 89]), there exists a path $\{z_t\}_{t \in [0, T]}$ as well as a constant M depending only on θ such that

$$|z_t - z_s - \mathfrak{D}(s, t)| \leq M\omega(s, t)^\theta, \quad \forall (s, t) \in \Delta^2. \quad (9)$$

This function is unique in the class of functions satisfying (9) and with (6), z is equal to y . Equality (7) follows from the very construction of z . \square

Proof of Proposition 2. We assume first that y is a solution in the sense of Davie with constant L . Let us note that

$$|y_t - y_s| \leq |y_t - y_s - \mathfrak{D}(s, t)| + |\mathfrak{D}(s, t)| \leq L\omega(s, t)^\theta + |\mathfrak{D}(s, t)|.$$

With Lemma 2, if $\mu = \omega(0, T)^{1/p}$ is small enough such that

$$C_8(\mu) \leq \frac{1}{4} \text{ and } C_9(\mu) \leq \frac{1}{4}, \quad (10)$$

then

$$\|y\| \leq L\mu^{1+\gamma} + C_7(\mu, y_0) + \frac{1}{4}\|y\| + \frac{1}{4}\|y\|^\gamma.$$

Since $\gamma \leq 1$, if $\|y\| \geq 1$, then $\|y\|^\gamma \leq \|y\|$ and then for μ small enough (note that the choice of μ does not depend on y_0),

$$\|y\| \leq 2 \max\{1, L\mu^{1+\gamma} + C_7(\mu, y_0)\}. \quad (11)$$

Since $C_7(\mu, y_0)$ decreases with μ , the boundedness of $\|y\|$ and $\|y\|_\infty$ also hold, with a different constant, for any time T by applying Proposition 6 in Appendix.

Now, if y is a solution in the sense of Davie with a constant L , it is also a solution in the sense of Davie with the constant

$$L' := \sup_{(s,t) \in \Delta^2, s \neq t} \frac{|y_t - y_s - \mathfrak{D}(s,t)|}{\omega(s,t)^\theta}.$$

From the sewing Lemma, there exists a universal constant M depending only on θ such that

$$L' \leq M \sup_{(s,t) \in \Delta^2, s \neq t, r \in (s,t)} \frac{|\mathfrak{D}(s,r,t)|}{\omega(s,t)^\theta}.$$

From Lemma 1 and the inequalities $a^\gamma \leq 1 + a$ for $a \geq 0$ and $\gamma \in [0, 1]$ as well as $|f(y_0)| \leq |f(0)| + \|\nabla f\|_\infty |y_0|$,

$$\begin{aligned} L' &\leq M(C_3(\mu)\|y\|^\gamma + C_4(\mu)\|y\| + C_5(\mu)L' + C_6(\mu, y_0)) \\ &\leq C_{10}(\mu) + C_{13}(\mu)\|y\| + MC_5(\mu)L' + C_{12}(\mu)|y_0|, \end{aligned}$$

with

$$\begin{aligned} C_{10}(\mu) &:= MC_3(\mu) + |f(0)|C_{11}(\mu), \\ C_{11}(\mu) &:= M\|\nabla f\|_\infty^2 \|x\|^2 \mu^{1-\gamma}, \\ C_{12}(\mu) &:= \|\nabla f\|_\infty C_{11}(\mu), \\ \text{and } C_{13}(\mu) &:= M(C_3(\mu) + C_4(\mu)). \end{aligned}$$

If

$$MC_5(\mu) \leq 1/2, \tag{12}$$

then

$$L' \leq 2C_{10}(\mu) + 2C_{13}(\mu)\|y\| + 2C_{12}(\mu)|y_0|.$$

With (11), under conditions (10) and (12) on μ ,

$$\|y\| \leq 2 + 4C_{10}(\mu)\mu^{1+\gamma} + 4C_{13}(\mu)\mu^{1+\gamma}\|y\| + 4C_{12}(\mu)|y_0|\mu^{1+\gamma} + C_7(\mu, y_0),$$

Under the additional condition that

$$4C_{13}(\mu)\mu^{1+\gamma} \leq \frac{1}{2}, \tag{13}$$

we get that

$$\|y\| \leq C_{14}(\mu) + C_{15}(\mu)|y_0|$$

with

$$\begin{aligned} C_{14}(\mu) &:= 4 + 8C_{10}(\mu)\mu^{1+\gamma} + 2|f(0)|(1 + \|\nabla f\|_\infty\mu)\|x\|, \\ C_{15}(\mu) &:= 8C_{12}(\mu)\mu^{1+\gamma} + 2\|\nabla f\|_\infty(1 + \|\nabla f\|_\infty\mu)\|x\|. \end{aligned}$$

Due to the dependence of the constants with respect to μ , conditions (10), (12) and (13) hold true if $\mu \leq K$ for a constant K depending only on $\|\nabla f\|_\infty$, $H_\gamma(F)$, $\|x\|$, p and γ .

Indeed, (11) also holds for $y_{[S,S']}$ on any time interval $\omega(S, S')$ provided that $\omega(S, S')^{1/p}$ is small enough. The result follows from Proposition 7 in Appendix. \square

5 Existence of a solution

The existence of a solution is proved thanks to the Euler scheme, which allows one to study to define a family of paths that is uniformly bounded with the uniform norm.

Let us fix a partition $\{t_i\}_{i=0,\dots,n}$ of $[0, T]$ and let us set $x_{i,j} := x_{t_i}^{-1} \otimes x_{t_j}$ and $\omega_{i,j} := \omega(t_i, t_j)$.

Let us consider the Euler scheme

$$y_{i+1} = y_i + f(y_i)x_{i,i+1}^1 + F(y_i)x_{i,i+1}^2, \quad i = 0, \dots, n-1$$

as well as the family $\{y_{i,j}\}_{0 \leq i < j \leq n}$ defined by

$$y_{i,j} := f(y_i)x_{i,j}^1 + F(y_i)x_{i,j}^2.$$

We set

$$\|y\|_{\star,0,n} := \sup_{0 \leq i < j \leq n} \frac{|y_j - y_i|}{\omega_{i,j}^{1/p}}. \quad (14)$$

Lemma 3. *If $\mu := \omega_{0,n}^{1/p}$ is small enough so that*

$$C_5(\mu) = \|\nabla f\|_\infty\|x\|\mu \leq \frac{1-K}{4} \text{ with } K := 2^{1-\theta} < 1 \quad (15)$$

and

$$L := 4 \frac{C_6(\mu, y_0) + C_3(\mu)\|y\|_{\star,0,n}^\gamma + C_4(\mu)\|y\|_{\star,0,n}}{1-K}$$

then

$$|y_{i,k} - y_k - y_i| \leq L\omega_{i,k}^\theta \quad (16)$$

for all $0 \leq i \leq k \leq n$.

Proof. The proof of this Lemma follows along the lines the one of Lemma 2.4 in [2] and relies on an induction on $k - i$.

Clearly, (16) is true for $k = i + 1$. Fix $m > 1$ and let us assume that (16) holds for any $i < k$ such that $k - i < m$.

Let us choose $i < k$ such that $k - i = m$, $m \geq 2$. Let j be the index such that $\omega_{i,j} \leq \omega_{i,k}/2$ and $\omega_{i,j+1} > \omega_{i,k}/2$. Since ω is super-additive, $\omega_{j+1,k} \leq \omega_{i,k}/2$ and then

$$\omega_{i,j}^\theta + \omega_{j+1,k}^\theta \leq 2^{1-\theta} \omega_{i,k}^\theta = K \omega_{i,k}^\theta. \quad (17)$$

We set

$$y_{i,j,k} := y_{i,k} - y_{i,j} - y_{j,k}.$$

For j as above, since $y_{j,j+1} - y_j - y_{j+1} = 0$, and using (17),

$$\begin{aligned} |y_{i,k} - y_k + y_i| &\leq |y_{i,j,k}| + |y_{i,j} - y_j + y_i| + |y_{j,k} - y_k + y_j| \\ &\leq |y_{i,j,k}| + |y_{j,j+1,k}| + |y_{j+1,k} - y_k + y_{j+1}| + |y_{i,j} - y_i + y_j| \\ &\quad + |y_{j,j+1} - y_{j+1} + y_j| \leq |y_{i,j,k}| + |y_{j,j+1,k}| + LK \omega_{i,k}^\theta. \end{aligned}$$

Since $x_{i,k}^2 = x_{i,j}^2 + x_{j,k}^2 + x_{i,j}^1 \otimes x_{j,k}^1$,

$$y_{i,j,k} = (f(y_j) - f(y_i))x_{j,k}^1 + (F(y_j) - F(y_i))x_{j,k}^2 + F(y_i)x_{i,j}^1 \otimes x_{j,k}^1.$$

Using the same computations as (8),

$$\begin{aligned} y_{i,j,k} &= (f(y_i) - f(y_j))x_{i,j}^1 - F(y_i)x_{i,j}^1 \otimes x_{j,k}^1 + (F(y_i) - F(y_j))x_{j,k}^2 \\ &= \int_0^1 (F(y_i + \tau(y_j - y_i)) - F(y_i))x_{i,j}^1 \otimes x_{j,k}^1 d\tau \\ &\quad + \int_0^1 \nabla f(y_i + \tau(y_j - y_i))(y_j - y_i - y_{i,j} + F(y_i)x_{i,j}^2)x_{i,j}^1 d\tau \\ &\quad - \int_0^1 \int_0^1 \tau \nabla f(y_i + \tau(y_j - y_i)) \nabla f(y_i + \rho\tau(y_j - y_i))(y_j - y_i) d\rho d\tau x_{i,j}^1 \otimes x_{j,k}^1 \\ &\quad + (F(y_i) - F(y_j))x_{j,k}^2. \quad (18) \end{aligned}$$

We then face the same estimates as the one in the proof of Lemma 1, where we replace the fact that y is a solution in the sense of Davie with a constant L by our induction hypothesis on $|y_{i,j} - y_j + y_i|$. Then

$$|y_{i,j,k}| \leq (C_3(\mu) \|y\|_{*,0,n}^\gamma + C_4(\mu) \|y\|_{*,0,n} + C_5(\mu)L + C_6(\mu, y_0)) \omega_{i,k}^\theta.$$

The results follows from our choice of μ and L . \square

The next lemma is the equivalent of Proposition 2 for the Euler scheme.

Lemma 4. For n such that $t_n = T$, $\|y\|_{\star,0,n}$ is bounded by a constant that depends only on $\omega(0, T)$, $\|x\|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, γ , p and $|f(0)|$.

Proof. Using Lemma 2,

$$|y_{i,j}| \leq (C_7(\mu, y_0) + C_8(\mu)\|y\|_{\star,0,n}^\gamma + C_9(\mu)\|y\|_{\star,0,n})\omega_{i,j}^{1/p}$$

and then with Lemma 3,

$$\begin{aligned} \|y\|_{\star,0,n} &\leq (C_7(\mu, y_0) + C_8(\mu)\|y\|_{\star,0,n}^\gamma + C_9(\mu)\|y\|_{\star,0,n}) \\ &\quad + 4\mu^{1+\gamma} \frac{C_6(\mu, y_0) + C_3(\mu)\|y\|_{\star,0,n}^\gamma + C_4(\mu)\|y\|_{\star,0,n}}{1 - K}. \end{aligned}$$

In addition to (15), we choose μ small enough so that

$$C_8(\mu) + 4(1 - K)^{-1}\mu^{1+\gamma}C_3(\mu) \leq \frac{1}{4} \quad (19)$$

and

$$C_9(\mu) + 4(1 - K)^{-1}\mu^{1+\gamma}C_3(\mu) \leq \frac{1}{4}. \quad (20)$$

Since $\gamma \leq 1$, $\|y\|_{\star,0,n}^\gamma \leq \|y\|_{\star,0,n}$ when $\|y\|_{\star,0,n} \leq 1$. Hence,

$$\|y\|_{\star,0,n} \leq \max\{1, 2C_7(\mu, y_0) + 8\mu^{1+\gamma}C_6(\mu, y_0)\}.$$

This proves that for a choice of μ (or equivalently T or n) small enough depending only on $\|x\|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, γ and p , then $\|y\|_{\star,0,n}$ is bounded by a constant that depends only on $\|x\|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, γ , p , $|f(y_0)|$ and $|F(y_0)|$. However, $|F(y_0)|$ and $|f(y_0)|$ are bounded by some constants that depends only on $|f(0)|$ and $\|\nabla f\|_\infty$.

The result is proved by finding a sequence $n_0 = 0 < n_1 < \dots < n_N$ such that $\omega_{n_i, n_{i+1}} \leq \mu^p$ with $t_{n_N} = T$ and μ satisfying (15), (19) and (20). Since ω is continuous close to its diagonal, there exists such a finite number N of intervals, and this number depends only on the choice of μ , and then on $\|x\|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, γ and p (and not on y_0 nor $f(0)$). Finally, it is easily shown that

$$\|y\|_{\star,0,n_N} \leq N^{p-1} \sum_{i=1}^{N-1} \|y\|_{\star, n_i, n_{i+1}},$$

which proves the result by applying the result on the successive time intervals $[t_{n_i}, t_{n_{i+1}}]$ and replacing y_0 by y_{t_i} . \square

Finally, in order to interpolate the Euler scheme and to get a good control, we shall add an hypothesis on ω , which is trivially satisfied in the case of $\omega(s, t) = t - s$, that is for Hölder continuous rough paths.

Hypothesis 3. We assume that there exists a continuous, increasing function φ such that

$$\frac{\omega(s, t)}{\varphi(t) - \varphi(s)} \text{ and } \frac{\varphi(t) - \varphi(s)}{\omega(s, t)}$$

are bounded for $0 \leq s < t \leq T$.

Proposition 3. *Under Hypotheses 1, 2 and 3, there exists at least a solution in the sense of Davie to (1).*

Proof. From the family $\{y_i\}$, we construct a path from $[0, T]$ to \mathbb{R}^m by

$$y_t = y_i + \frac{\varphi(t) - \varphi(t_i)}{\varphi(t_{i+1}) - \varphi(t_i)} y_{i,i+1}, \quad t \in [t_i, t_{i+1}]. \quad (21)$$

From standard computations, there exists a constant C_{16} which depends only on p and the lower and upper bounds of $\omega(s, t)/(\varphi(t) - \varphi(s))$ such that

$$\|y\| \leq C_{16} \|y\|_{\star, 0, n}.$$

With Lemma 4, we have a uniform bound in $\|y\|_{\star, 0, n}$, and then on the constant L when μ (or n) is small enough. Hence, for any partition satisfying Hypothesis 3, the path y has a p -variation which is bounded by a constant that does not depend on the choice of the partition.

Now, let y^n be a family of paths constructed along an increasing family of partitions Π^n whose meshes decrease to 0.

Then there exists a subsequence $\{y^{n_k}\}_{k \geq 1}$ of $\{y^n\}_{n \in \mathbb{N}^*}$ which converges in q -variation for $q > p$ to some path y of finite p -variation.

For any (s, t) in $\cap_{n \geq 0} \Pi^n$,

$$|y_t - y_s - f(y_s)x_{s,t}^1 - F(y_s)x_{s,t}^2| \leq L\omega(s, t)^\theta.$$

Since $\cap_{n \geq 0} \Pi^n$ is dense in $[0, T]$ and y is continuous, this proves that y is the solution to (16) in the sense of Davie, at least when T is small enough.

The passage from a solution on $[0, T]$ with T small enough to a global solution is done by using the arguments of Lemma 7, Lemma 8 and Proposition 6. \square

6 Distance between two solutions and uniqueness

We now consider a more stringent assumption than Hypothesis 1.

Hypothesis 4. The function f is twice continuously differentiable from \mathbb{R}^m to $L(\mathbb{R}^d, \mathbb{R}^m)$ and is such that $\nabla f, \nabla^2 f$ are bounded and $F(x) := f(x) \cdot \nabla f(x)$ is such that ∇F is γ -Hölder continuous with constant $H_\gamma(\nabla F)$ from \mathbb{R}^m to $L(\mathbb{R}^m \otimes \mathbb{R}^d, \mathbb{R}^m)$.

We consider two rough paths u and v of finite p -variation controlled by ω , $p \in [2, 3)$, as well as two vector fields f and g satisfying Hypothesis 4. Let y and z be respectively some solutions to $y_t = y_0 + \int_0^t f(y_s) du_s$ and $z_t = z_0 + \int_0^t g(z_s) dv_s$.

We have seen that y and z remain in a ball of radius R that depends only on $\|\nabla f\|_\infty, \|\nabla g\|_\infty, \|\nabla F\|_\infty, \|\nabla G\|_\infty$ (since F and $G = g \cdot \nabla g$ are Lipschitz continuous), $\|u\|, \|v\|, y_0, z_0, |f(0)|, |g(0)|, \omega(0, T), \gamma$ and p .

Definition 3. We say that a constant C satisfies Condition (S) if it depends only on the above quantities, as well as $H_\gamma(\nabla F), H_\gamma(\nabla G), \|\nabla^2 f\|_\infty$ and $\|\nabla^2 g\|_\infty$.

We then set for some functions h, h' ,

$$\delta_R(h, h') = \sup_{z \in B(0, R)} |h(z) - h'(z)|.$$

We also set

$$\delta(u, v) = \|u - v\| \text{ and } \delta(y, z) = \sup_{0 \leq s < t \leq T} \frac{|y_{s,t} - z_{s,t}|}{\omega(s, t)^{1/p}}.$$

Theorem 1. *Under Hypotheses 2 and 4, there exists some constant C_{17} satisfying Condition (S) such that for all $(s, t) \in \Delta^2$,*

$$\begin{aligned} |y_{s,t} - z_{s,t}| &\leq C_{17} \omega(s, t)^{1/p} (|y_0 - z_0| \\ &\quad + \delta(u, v) + \delta_R(f, g) + \delta_R(F, G) + \delta_R(\nabla f, \nabla g) + \delta_R(\nabla F, \nabla G)). \end{aligned} \quad (22)$$

The proof of the following corollary is then immediate from the previous estimate.

Corollary 1. *Under Hypotheses 2 and 4, there exists a unique solution in the sense of Davie to (1).*

Theorem 1 proves that the Itô map which sends x to the unique solution to (1) is locally Lipschitz continuous in y_0, f and x .

The next lemma is the main estimate of the proof and show why extra regularity shall be assumed on F . This lemma is already well-known (See Lemma 3.5 in [2]) but we recall its proof which is straightforward.

Lemma 5. *Let h be a function of class \mathcal{C}^1 from \mathbb{R}^m to \mathbb{R} such that ∇h is γ -Hölder and bounded. Then for all a, b, c, d in \mathbb{R}^m ,*

$$\begin{aligned} & |h(a) - h(b) + h(c) - h(d)| \\ & \leq |a - b - c + d| \|\nabla h\|_\infty + H_\gamma(\nabla h)(|a - b|^\gamma + |c - d|^\gamma)|b - d|. \end{aligned}$$

Proof. The result follows from

$$\begin{aligned} & |h(a) - h(b) + h(c) - h(d)| \\ & = \left| \int_0^1 \nabla h(c + \tau(a - c))(a - c) \, d\tau - \int_0^1 \nabla h(d + \tau(b - d))(b - d) \, d\tau \right| \\ & = \left| \int_0^1 \nabla h(c + \tau(a - c))(a - c - b + d) \, d\tau \right| \\ & \quad + \left| \int_0^1 (\nabla h(c + \tau(a - c)) - \nabla h(d + \tau(b - d)))(b - d) \, d\tau \right| \\ & \leq |a - b - c + d| \|\nabla h\|_\infty + H_\gamma(\nabla h) \int_0^1 |(1 - \tau)(c - d) + \tau(a - b)|^\gamma |b - d| \, d\tau, \end{aligned}$$

since $(x + y)^\gamma \leq x^\gamma + y^\gamma$ for $x, y \geq 0$ and $\gamma \in [0, 1]$. \square

Let us denote by \mathfrak{d} the operator which, applied to an expression involving y, f and u , takes the difference between this expression and the similar expression with y replaced by z, f replaced by g and u replaced by v . For example,

$$\mathfrak{d}(f(y_s)x_{s,t}^1) = f(y_s)u_{s,t}^1 - g(z_s)v_{s,t}^1.$$

If $\alpha(y, f, u)$ and $\beta(y, f, u)$ are two expressions, then

$$\mathfrak{d}(\alpha(y, f, u)\beta(y, f, u)) = \mathfrak{d}(\alpha(y, f, u))\beta(y, f, u) + \alpha(z, g, v)\mathfrak{d}(\beta(y, f, u)). \quad (23)$$

Proof of Theorem 1. In the proof, we assume without loss of generalities that $\gamma < 1$.

Let us choose a constant A such that

$$|\mathfrak{d}(\mathfrak{D}(s, r, t))| \leq A\omega(s, t)^\theta \quad (24)$$

and

$$|\mathfrak{d}(\mathfrak{D}(s, t))| \leq A\omega(s, t)^{1/p}. \quad (25)$$

From the sewing lemma on the difference of two almost rough paths (See for example [11, Theorem 6, p. 95]), there exists some universal constant M (depending only on θ) such that

$$|\mathfrak{d}(y_{s,t} - \mathfrak{D}(s, t))| \leq MA\omega(s, t)^\theta. \quad (26)$$

Our aim is to obtain some estimate on A .

Let us note first that

$$\begin{aligned} |\mathfrak{d}F(y_s)u_{s,t}^2| &\leq (|F(y_s) - F(z_s)| + |F(z_s) - G(z_s)|\|u\|\omega(s,t)^{2/p} \\ &\quad + (|G(z_0)| + \|\nabla G\|_\infty\|z\|\mu)\delta(u,v)\omega(s,t)^{2/p}. \end{aligned}$$

Since $|y_s - z_s| \leq \delta(y, z)\mu + |y_0 - z_0|$, with Lemma 5,

$$|F(y_s) - F(z_s)| \leq \|\nabla F\|_\infty(\delta(y, z)\mu + |y_0 - z_0|),$$

then for some constant C_{18} satisfying Condition (S),

$$|\mathfrak{d}F(y_s)u_{s,t}^2| \leq C_{18}(\delta_R(F, G) + |y_0 - z_0| + \mu\delta(y, z) + \delta(u, v))\omega(s, t)^{2/p}.$$

With (26),

$$\begin{aligned} |\mathfrak{d}B_2(s, t)| &= |\mathfrak{d}(y_{s,t} - f(y_s)u_{s,t}^1)| = |\mathfrak{d}(y_{s,t} - \mathfrak{D}(s, t) + F(y_s)u_{s,t}^2)| \\ &\leq MA\omega(s, t)^\theta + |\mathfrak{d}F(y_s)u_{s,t}^2| \\ &\leq MA\omega(s, t)^\theta + C_{18}(\delta_R(F, G) + |y_0 - z_0| + \mu\delta(y, z) + \delta(u, v))\omega(s, t)^{2/p}. \end{aligned}$$

Since f is differentiable, for $\tau \in [0, 1]$,

$$f(\tau y_t + (1 - \tau)y_s) - f(y_s) = \int_0^1 \nabla f(y_s + \rho(\tau y_t + (1 - \tau)y_s))\tau y_{s,t} \, d\rho. \quad (27)$$

Hence

$$\begin{aligned} |f(\tau y_t + (1 - \tau)y_s) - f(y_s) - g(\tau y_t + (1 - \tau)y_s) + g(y_s)| \\ \leq \delta_R(\nabla f, \nabla g)\|y\|\omega(s, t)^{1/p}. \quad (28) \end{aligned}$$

With (28) and (27) for $\tau \in [0, 1]$,

$$\begin{aligned} \mathfrak{d}(f(\tau y_t + (1 - \tau)y_s) - f(y_s)) &\leq \delta_R(\nabla f, \nabla g)\|y\|\omega(s, t)^{1/p} \\ &\quad + 2\|\nabla^2 f\|_\infty(\delta(y, z)\mu + |y_0 - z_0|)\|y\|\omega(s, t)^{1/p} + \delta(y, z)\|\nabla f\|_\infty\omega(s, t)^{1/p}. \end{aligned}$$

With Lemma 5 and (27)-(28) applied to F and G , for $\tau \in [0, 1]$,

$$\begin{aligned} |\mathfrak{d}B_4(y_s, y_s + \tau y_{s,t})| &= \mathfrak{d}(F(\tau y_t + (1 - \tau)y_s) - F(y_s)) \\ &\leq \delta_R(\nabla F, \nabla G)\|y\|\omega(s, t)^{1/p} + 2\|\nabla F\|_\infty\delta(y, z)\omega(s, t)^{1/p} \\ &\quad + 2H_\gamma(\nabla F)(\|y\|^\gamma + \|z\|^\gamma)\omega(s, t)^{\gamma/p}(\mu\delta(y, z) + |y_0 - z_0|). \end{aligned}$$

Besides,

$$\mathfrak{d}(\nabla f(y_s + \tau y_{s,t})) \leq \delta_R(\nabla f, \nabla g) + \|\nabla^2 f\|_\infty(\delta(y, z)\mu + |y_0 - z_0|).$$

In addition,

$$|\mathfrak{d}(F(y_0))| \leq \delta_R(F, G) + \|\nabla F\|_\infty |y_0 - z_0|.$$

Combining all these estimates using (23) with the ones given in Section 4 in some lengthy computations,

$$\begin{aligned} |\mathfrak{d}(\mathfrak{D}(s, r, t))| &\leq C_{19}\omega(s, t)^\theta (MA\mu + \mu\delta(y, z) + \delta(y, z)\mu^{1-\gamma} \\ &\quad + |z_0 - y_0| + \delta(u, v) + \delta_R(F, G) + \delta_R(\nabla F, \nabla G) + \delta_R(\nabla f, \nabla g)), \end{aligned} \quad (29)$$

where C_{19} satisfies Condition (S) (Note that C_{19} decreases with T).

We have also

$$\begin{aligned} |\mathfrak{d}(\mathfrak{D}(s, t))| &\leq \delta_R(f, g)(\|y\|\mu + |y_0|)\|u\|\omega(s, t)^{1/p} \\ &\quad + (\|\nabla f\|_\infty + \|\nabla F\|_\infty)(\mu\delta(y, z) + |y_0 - z_0|)\|u\|(1 + \mu)\omega(s, t)^{1/p} \\ &\quad + (\|g\|_\infty + \|G\|_\infty)\delta(u, v)\omega(s, t)^{2/p} + \delta_R(F, G)\|u\|\omega(s, t)^{2/p} \\ &\leq C_{20}(\delta_R(F, G) + \delta_R(f, g) + \delta(u, v) + |y_0 - z_0| + \mu\delta(y, z))\omega(s, t)^{1/p}, \end{aligned}$$

where C_{20} satisfies Condition (S) and decreases when T decreases. With (26),

$$\begin{aligned} |\mathfrak{d}y_{s,t}| &\leq |\mathfrak{d}(y_{s,t} - \mathfrak{D}(s, t))| + |\mathfrak{d}\mathfrak{D}(s, t)| \leq MA\omega(s, t)^\theta \\ &\quad + C_{20}(\delta_R(F, G) + \delta_R(f, g) + \delta(u, v) + |y_0 - z_0| + \mu\delta(y, z))\omega(s, t)^{1/p}. \end{aligned} \quad (30)$$

Let us choose A such that an equality holds in either (24) or (25). If $|\mathfrak{d}(y_{s,t})| = A$, then from (30),

$$A \leq MA\mu^{1+\gamma} + C_{20}(\delta_R(F, G) + \delta_R(f, g) + \delta(u, v) + |y_0 - z_0| + \mu\delta(y, z)).$$

If $|\mathfrak{d}(\mathfrak{D}(s, r, t))| = A$, then from (29),

$$\begin{aligned} A &\leq C_{19}(MA\mu + \mu\delta(y, z) + \delta(y, z)\mu^{1-\gamma} \\ &\quad + |z_0 - y_0| + \delta(u, v) + \delta_R(F, G) + \delta_R(\nabla F, \nabla G) + \delta_R(\nabla f, \nabla g)). \end{aligned}$$

In any case, we may choose μ small enough in function of C_{19} or of M (which depends only on θ) such that

$$\begin{aligned} A &\leq 2C_{19}(\mu\delta(y, z) + \delta(y, z)\mu^{1-\gamma} \\ &\quad + |z_0 - y_0| + \delta(u, v) + \delta_R(F, G) + \delta_R(\nabla F, \nabla G) + \delta_R(\nabla f, \nabla g)) \\ &\quad + 2C_{20}(\delta_R(F, G) + \delta_R(f, g) + \delta(u, v) + |y_0 - z_0| + \mu\delta(y, z)). \end{aligned}$$

Injecting this inequality in (30), we get that

$$\delta(y, z) \leq C_{21}(B + (\mu + \mu^{2-\gamma} + \mu^{2(1-\gamma)})\delta(y, z) + |y_0 - z_0|)$$

with

$$B := \delta_R(F, G) + \delta_R(f, g) + \delta(u, v) + \delta_R(\nabla f, \nabla g) + \delta_R(\nabla F, \nabla G)$$

and C_{21} depends only on C_{19} , C_{20} , M and $\omega(0, T)$.

Again, choosing μ small enough in function of C_{19} , C_{20} , M and $\omega(0, T)$ gives the required bound in $\delta(y, z)$ in short time. As the choice of μ does not depend on $|y_0 - z_0|$ and μ satisfies Condition (S), Proposition 7 may be applied to $y - z$. \square

7 Distance between two Euler schemes

Let us give two rough paths u and v of finite p -variations, as well as a partition $\{t_i\}_{i=0}^n$ of $[0, T]$.

We use the same notations and conventions as in Section 5. Again, we set $\mu := \omega_{0,n}^{1/p}$.

For $0 \leq i < j \leq n$, we set

$$z_{i+1} = z_i + g(z_i)v_{i,i+1}^1 + G(z_i)v_{i,i+1}^2, \quad z_j = z_i + g(z_i)v_{i,j}^1 + G(z_i)v_{i,j}^2,$$

For a family $\{\varepsilon_{i,j}\}_{i,j=1,\dots,n}$ such that $\sup_{0 \leq i \leq j \leq n} |\varepsilon_{i,j}|/\omega_{i,j}^\theta$ is finite, we set

$$\begin{aligned} y_{i+1} &= y_i + f(y_i)u_{i,i+1}^1 + F(y_i)u_{i,i+1}^2 + \varepsilon_{i,i+1}, \quad i = 0, \dots, n-1 \\ y_j &= y_i + f(y_i)u_{i,j}^1 + F(y_i)u_{i,j}^2 + \varepsilon_{i,j}, \quad 0 \leq i < j \leq n. \end{aligned} \quad (31)$$

In addition, define

$$\alpha := \sup_{i=0,\dots,n-1} \frac{|\varepsilon_{i,i+1}|}{\omega_{i,i+1}^\theta}.$$

Here, there are three cases of interest: (a) Both z and y are given by some Euler schemes and then $\varepsilon_{i,j} = 0$ for all $0 \leq i < j \leq n$. (b) The path y is a solution to $y_t = y_0 + \int_0^t f(y_s) du_s$ and then $\varepsilon_{i,j} = y_j - y_i - f(y_i)u_{i,j}^1 - F(y_i)u_{i,j}^2$, while $v = u$ and $g = f$. (c) While $v = u$ and $g = f$, the family $\{y_i\}_{i=0,\dots,n}$ is given by the Euler scheme with respect to a partition $\{t'_i\}_{i=0,\dots,m} \subset \{t_i\}_{i=0,\dots,n}$.

We assume that z and y belong to the ball of radius R and that $\|z\|_{\star,0,n}$ and $\|y\|_{\star,0,n}$ (defined by (14)) are bounded by R' . In any cases, R and R' depend only on $\|\nabla f\|_\infty$, $\|\nabla F\|_\infty$, $\|\nabla g\|_\infty$, $\|\nabla G\|_\infty$, $\|u\|$, $\|v\|$, $|f(0)|$, $|g(0)|$, $|y_0|$, $|z_0|$, $\omega(0, T)$, γ and p .

Definition 4. We say that a constant C satisfies condition (S_e) if it depends only on the quantities listed above as well as $H_\gamma(\nabla F)$, $H_\gamma(\nabla G)$, $\|\nabla^2 f\|_\infty$, $\|\nabla^2 g\|_\infty$ and $\sup_{0 \leq i < j \leq n} |\varepsilon_{i,j}|/\omega_{i,j}^\theta$.

Theorem 2. If f and g satisfies Hypothesis 4, then for some constant C_{22} satisfying Condition (S_e) ,

$$\|z - y\|_{\star,0,n} \leq C_{22}(\alpha + |z_0 - y_0| + \delta_R(f, g) + \delta_R(\nabla f, \nabla g) + \delta_R(\nabla F, \nabla G) + \delta_R(F, G) + \delta(u, v)).$$

Proof. We set

$$A := \max_{0 \leq i < j \leq n} \frac{|y_j - y_i - y_{i,j} - z_j - z_i - z_{i,j}|}{\omega_{i,j}^\theta}. \quad (32)$$

We have

$$|F(y_i) - F(z_i)| \leq \|\nabla F\|_\infty(|y_0 - z_0| + \|z - y\|_{\star,0,n}\mu)$$

and

$$|\nabla f(y_i + \tau(y_j - y_i)) - \nabla f(z_i + \tau(z_j - z_i))| \leq 2\|\nabla^2 f\|_\infty(|y_0 - z_0| + \|z - y\|_{\star,0,n}\mu).$$

Besides, with Lemma 5,

$$|F(y_j) - F(y_i) - F(z_j) + F(z_i)| \leq \|\nabla F\|_\infty \|z - y\|_{\star,0,n} \omega_{i,j}^{1/p} + H_\gamma(\nabla F)(\|y\|^\gamma + \|z\|^\gamma)(|y_0 - z_0| + \mu \|z - y\|_{\star,0,n}) \omega_{i,j}^{\gamma/p}.$$

With (18) and considering the difference between $y_{i,j,k}$ and $z_{i,j,k}$ for $i < j < k$,

$$|y_{i,j,k} - z_{i,j,k}| \leq C_{23} \omega_{i,k}^\theta (|y_0 - z_0| + A\mu + (\mu + \mu^{1-\gamma}) \|z - y\|_{\star,0,n} + B_1).$$

with C_{23} satisfying Condition (S_e) and

$$B_1 := \delta(u, v) + \delta_R(F, G) + \delta_R(\nabla f, \nabla g) + \delta_R(\nabla F, \nabla G).$$

On the other hand, for $i = 0, \dots, n-1$, $z_{i+1} - z_i - z_{i,i+1} = 0$ and

$$|y_{i+1} - y_i - y_{i,i+1}| \leq \alpha \omega_{i,i+1}^\theta.$$

For j given as in the proof of Lemma 3, we have for $K := 2^{1-\theta}$,

$$\begin{aligned} & \frac{|y_k - y_i - y_{i,k} - z_k - z_i - z_{i,k}|}{\omega_{i,k}^\theta} \\ & \leq KA + \alpha + \frac{|y_{i,j,k} - z_{i,j,k}|}{\omega_{i,k}^\theta} + \frac{|y_{j,j+1,k} - z_{j,j+1,k}|}{\omega_{i,k}^\theta} \\ & \leq KA + \alpha + C_{24}(|y_0 - z_0| + A\mu + (\mu + \mu^{1-\gamma})\|z - y\|_{*,0,n} + B_1) \end{aligned}$$

with $C_{24} = 2C_{23}$. From the definition of A (see (32)),

$$A \leq 2\alpha + KA + C_{24}(|y_0 - z_0| + A\mu + (\mu + \mu^{1-\gamma})\|z - y\|_{*,0,n} + B_1). \quad (33)$$

On the other hand, since both f and F are Lipschitz continuous, for some constant C_{25} satisfying Condition (S_e),

$$\begin{aligned} \|z - y\|_{*,0,n} & \leq \sup_{0 \leq i < j \leq n} \frac{|y_j - y_i - y_{i,j} - z_j + z_i + z_{i,j}|}{\omega_{i,j}^{1/p}} + \frac{|y_{i,j} - z_{i,j}|}{\omega_{i,j}^{1/p}} \\ & \leq A\mu^{1+\gamma} + C_{25}(|y_0 - z_0| + \mu\|z - y\|_{*,0,n}) + B_2 \end{aligned}$$

with

$$B_2 := C_{26}(\delta_R(F, G) + \delta_R(f, g) + \delta(u, v)),$$

for some constant C_{26} satisfying Condition (S_e). If μ is small enough so that $C_{25}\mu \leq 1/2$, then

$$\|z - y\|_{*,0,n} \leq 2C_{25}|y_0 - z_0| + 2A\mu^{1+\gamma} + 2B_2. \quad (34)$$

Injecting this in (33),

$$A \leq 2\alpha + KA + C_{27}(|y_0 - z_0| + AC_{28}(\mu)) + B_3$$

with a $C_{28}(\mu)$ satisfying Condition (S_e) for fixed μ and that decreases to 0 as μ decreases to 0, C_{27} satisfying Condition (S_e), and $B_3 = C_{29}(B_1 + B_2)$ for some constant C_{29} satisfying Condition (S_e). For $K + C_{27}C_{28}(\mu) < (1 + K)/2 < 1$, then

$$A \leq \frac{2}{1 - K}(2\alpha + C_{27}|y_0 - z_0| + B_3).$$

Using the inequality on (34), this leads to the required inequality for a value of μ small enough. Thus usual arguments proves now that this is true for any time horizon T up to changing the constants. \square

8 Rate of convergence of the Euler scheme

Let us consider now the solution to $y_t = y_0 + \int_0^t f(y_s) dx_s$ as well as the associated Euler scheme

$$e_{i+1} = e_i + f(e_i)x_{i,i+1}^1 + F(e_i)x_{i,i+1}^2, \quad e_0 = y_0$$

when f satisfies Hypothesis 4.

We are willing to estimate the difference between y and e . Our key argument is provided by Theorem 2 above.

Proposition 4. *Assume Hypothesis 4 on f . For $\delta := \sup_{0 \leq i < n} \omega_{i,i+1}$ and $p \in [2, 3)$,*

$$\sup_{i=0, \dots, n} |e_i - y_i| \leq C_{30} \delta^{(3-p)/p-\eta}. \quad (35)$$

where C_{30} depends only on y_0 , $|f(0)|$, $\|x\|$, $\|\nabla f\|_\infty$, $\|\nabla^2 f\|_\infty$, $\|\nabla F\|_\infty$, $H_\gamma(\nabla F)$, $\omega(0, T)$, p and γ .

It follows that the rate of convergence is smaller than $(3-p)/p$ and belongs to $(0, 1/2)$. In addition, when p increases to 3, the rate of convergence decreases to 0. This rate is similar to the one given by A.M. Davie [2]. (See also [3] for the convergence of the Milstein scheme for the fractional Brownian motion).

Proof. Let us note first that since y and e remains bounded, if ∇F is γ -Hölder continuous, then it is locally γ' -Hölder continuous for any $\gamma' < \gamma$. Since the constraint $2 + \gamma' > p$ is in force, we set $\gamma' = p - 2 + \eta p$ for some $0 < \eta < (3-p)/p < 1/2$ and $\theta' := (2 + \gamma')/p = 1 + \eta$.

Since F is Lipschitz continuous, then y is a solution in the sense of Davie with $\theta = 3/p$. This way,

$$|y_t - y_s - f(y_s)x_{s,t}^1 - F(y_s)x_{s,t}^2| \leq L\omega(s, t)^{3/p}.$$

It follows that $\{y_i\}_{i=0}^n$ is solution to (31) with

$$\varepsilon_{i,i+1} = y_{i+1} - y_i - f(y_i)x_{i,i+1}^1 - F(y_i)x_{i,i+1}^2.$$

Then

$$\alpha := \sup_{0 \leq i < n} \frac{|\varepsilon_{i,i+1}|}{\omega_{i,i+1}^{\theta'}} \leq L\delta^{3/p-\theta'} = L\delta^{(3-p)/p-\eta}.$$

Applying Theorem 2 with our choice of θ' leads to

$$\|e - y\| \leq C_{31} \delta^{(3-p)/p-\eta},$$

where C_{31} depends only on y_0 , $|f(y_0)|$, $\|x\|$, $\|\nabla f\|_\infty$, $\|\nabla^2 f\|_\infty$, $\|\nabla F\|_\infty$, $H_\gamma(\nabla F)$, $\omega(0, T)$, p and γ . Then (35) is immediate. \square

Let $\{\{t_i^n\}_{i=0,\dots,n}\}_{n \in \mathbb{N}^*}$ be an increasing family of partitions such that the mesh $\sup_{i=0,\dots,n} \omega(t_i^n, t_{i+1}^n)$ converges to 0. Let $\{e_i^n\}_{i=0,\dots,n}\}_{n \in \mathbb{N}^*}$ be the corresponding Euler schemes for a rough path x and a vector field f satisfies Hypothesis 4. With Lemmas 3, 4 and 8,

$$\sup_{n \in \mathbb{N}^*} \sup_{0 \leq i < j \leq n} \frac{|e_i^n - e_j^n - f(e_i^n)x_{t_i^n, t_j^n}^1 - F(e_i^n)x_{t_i^n, t_j^n}^2|}{\omega(t_i^n, t_{j+1}^n)^\theta} < +\infty$$

and with Proposition 3 under Hypothesis 3, the interpolation of the Euler scheme e^n constructed as in (21) has a p -variation norm $\|e^n\|$ which is bounded. Using the same proof as above, we get the following corollary.

Corollary 2. *Under Hypotheses 2, 3 and 4, the family of interpolated Euler schemes $\{e^n\}_{n \in \mathbb{N}^*}$ for a Cauchy sequence of the uniform norm and the q -variation norm for any $q > p$.*

Remark 2. Here the dimension d of the space plays no role so that this argument may be used for an infinite dimensional rough path. This Corollary allows one to define the solution to (1) as the limit of the sequence $\{e^n\}_{n \in \mathbb{N}^*}$.

9 Case of geometric rough paths

We now consider the case where x is a geometric rough path of finite p -variation controlled by ω and the vector field f satisfies Hypothesis 4. This means that there exists a sequence of rough paths x^n converging to x in p -variation such that x^n lives above a piecewise smooth path z^n in \mathbb{R}^d and $x_t^n = 1 + z_t^n + \int_0^t (z_s^n - z_0^n) \otimes z_s^n ds$. Such a path x^n is called a smooth rough path.

In order to simplify the analysis, we assume that $\omega(s, t) = t - s$ and then we are dealing with α -Hölder continuous paths with $\alpha = 1/p$.

The core idea of P. Friz and N. Victoir was to consider a family of smooth rough paths $(\tilde{x}^n)_{n \geq 1}$ such that, given a family of partitions $\{\{t_i^n\}_{i=0,\dots,n}\}_{n \in \mathbb{N}^*}$, \tilde{x}^n converges to x in the β -Hölder norm for any $\beta < \alpha$ and $\tilde{x}_{t_i^n}^n = x_{t_i^n}^n$ for $i = 0, \dots, n$.

For this, they used sub-Riemannian geodesics. In [11], we have provided an alternative construction using some segments and some loops.

Let \tilde{z}^n be the projection of \tilde{x}^n onto \mathbb{R}^d , and let y^n be the solution of the ODE $y_t^n = y_0 + \int_0^t f(y_s^n) d\tilde{z}_s^n$. As \tilde{z}^n is piecewise smooth, one knows that this solution corresponds to the one of the solution of the RDE $y_t^n = y_0 + \int_0^t f(y_s^n) d\tilde{x}_s^n$ in the sense of Davie or in the sense of Lyons (See Section 10 below).

Let \tilde{e}^n be the Euler scheme associated to y^n with the partition $\{t_i^n\}_{i=0,\dots,n}$:

$$\tilde{e}_{i+1}^n = \tilde{e}_i^n + f(\tilde{e}_i^n) \tilde{x}_{t_i^n, t_{i+1}^n}^{1,n} + F(\tilde{e}_i^n) \tilde{x}_{t_i^n, t_{i+1}^n}^{2,n}, \quad i = 0, \dots, n,$$

and e^n the Euler scheme

$$e_{i+1}^n = e_i^n + f(e_i^n) x_{t_i^n, t_{i+1}^n}^{1,n} + F(e_i^n) x_{t_i^n, t_{i+1}^n}^{2,n}, \quad i = 0, \dots, n,$$

The key observation from P. Friz and N. Victoir is that from the very construction of \tilde{x}^n , $\tilde{x}_{t_i^n, t_{i+1}^n}^n = \tilde{x}_{t_i^n, t_{i+1}^n}^n$ and then $\tilde{e}^n = e^n$.

With Proposition 4, for some $\eta \in (0, 3\alpha - 1)$,

$$\|e^n - y^n\|_\star := \sup_{0 \leq i < j \leq n} \frac{|e_j^n - e_i^n - y_j^n + y_i^n|}{\omega_{i,j}^{1/p}} = \|\tilde{e}^n - y^n\| \leq C_{32} \delta^{3\alpha-1-\eta}$$

where C_{32} depends only on $\|x\|$, f , T , α and η , and $\delta := \sup_{i=0,\dots,n+1} \omega(t_i^n, t_{i+1}^n)$.

On the other hand with Theorem 1,

$$\|y^n - y^m\|_\star \leq \|y^n - y^m\| \leq C_{33} \|\tilde{x}^m - \tilde{x}^n\|,$$

where C_{33} depends only on $\|x\|$, f , T , α and η (when \tilde{x}^n is such that $\|\tilde{x}^n\| \leq A\|x\|$ which is the case with the constructions mentioned above).

Since $(\tilde{x}^n)_{n \in \mathbb{N}^*}$ is a Cauchy sequence in the space of β -Hölder continuous functions, $\beta < \alpha$, it follows that (y^n) is a Cauchy sequence and converges to some element y .

We then obtain the convergence of e^n to y in the sense that $\|e^n - y\|_\star$ decreases to 0 as n tends to infinity.

10 Solution in the sense of Davie and solution in the sense of Lyons

The notion of solution in the sense of Davie (Definition 2) is different of the solution in the sense of Lyons (Definition 1), as the iterated integrals of y and the cross-iterated integrals between y and x are not constructed, while they are part of the solution in the sense of Lyons.

However, once a solution in the sense of Davie is constructed, it is easy to construct a rough paths with values in $T_1(\mathbb{R}^d \oplus \mathbb{R}^m)$.

Lemma 6. *A solution y in the sense of Davie — which is a path with values in \mathbb{R}^m — may be lifted to a rough path with values in $T_1(\mathbb{R}^d \oplus \mathbb{R}^m)$, as the rough path \tilde{y} associated to the almost rough path*

$$h_{s,t} := 1 + x_{s,t} + y_{s,t} + f(y_s) \otimes f(y_s) \cdot x_{s,t}^2 + f(y_s) \otimes 1 \cdot x_{s,t}^2 + 1 \otimes f(y_s) \cdot x_{s,t}^2. \quad (36)$$

Besides, the map $y \mapsto \tilde{y}$ is locally Lipschitz continuous.

Proof. It is easily checked that

$$|h_{s,r,t} - h_{s,r} \otimes h_{r,t}| \leq C_{34} \omega(s, t)^\theta$$

with $\theta > 1$ and then that it is an almost rough path. With Proposition 1, the associated rough path \tilde{y} projects onto (x, y) in $\mathbb{R}^d \oplus \mathbb{R}^m$. The local Lipschitz continuity of $y \mapsto \tilde{y}$ follows from the same kind of computation as the one of the proof of Theorem 1. \square

Proposition 5. *Let us assume that f is a vector field with a bounded derivative which is γ -Hölder continuous, such that a solution to (1) exists in the sense of Lyons. Let us assume also that $f \cdot \nabla f$ is γ -Hölder continuous, so that a solution to (1) exists in the sense of Davie, lifted as a rough path with values in $T_1(\mathbb{R}^d \oplus \mathbb{R}^m)$ as above. Then the two solutions coincide.*

Proof. Let y be a solution in the sense of Davie of (1), which is lifted as a rough path \tilde{y} using Lemma 6. Let us consider

$$u_{s,t} := f(y_s) x_{s,t}^1 + \nabla f(y_s) \tilde{y}_{s,t}^\times,$$

where $\tilde{y}_{s,t}^\times$ is the projection onto $\mathbb{R}^m \otimes \mathbb{R}^d$ of $\tilde{y}_{s,t}$ (or roughly speaking, it is the cross-iterated integral between y and x).

Since $\tilde{y}_{s,t}^\times = y_{s,r} \otimes x_{r,t}^1$,

$$\begin{aligned} |u_{s,t} - u_{s,r} - u_{r,t}| &\leq |(f(y_r) - f(y_s)) x_{r,t}^1 - \nabla f(y_s) y_{s,r} \otimes x_{r,t}^1| \\ &\quad + |\nabla f(y_r) - \nabla f(y_s)| |\tilde{y}_{s,r}^\times| \\ &\leq \left| \int_0^1 (\nabla f(y_s + \tau y_{s,r}) - \nabla f(y_s)) y_{s,r} \otimes x_{r,t}^1 d\tau \right| + H_\gamma(\nabla f) \|y\|^\gamma \|z\| \omega_{s,t}^\theta \\ &\leq H_\gamma(\nabla f) (\|y\|^{1+\gamma} \|x\| + \|y\|^\gamma \|z\|) \omega(s, t)^\theta. \end{aligned}$$

Then $\{u_{s,t}\}_{(s,t) \in \Delta^2}$ is an almost rough path in \mathbb{R}^m whose associated rough path v satisfies

$$|v_t - v_s - u_{s,t}| \leq C_{35} \omega(s, t)^\theta.$$

From the very construction, v is the rough integral $v_t = v_0 + \int_0^t g(z_s) dz_s$ with z is a rough path lying above $(x, y, y^\times) \in (\mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^d, \mathbb{R}^m, \mathbb{R}^m \otimes \mathbb{R}^d)$ and $g(y, x) = dx + f(y) dx$. Thus $y = v$ when $v_0 = y_0$. With Lemma 6, this is also true for the iterated integrals. This proves that the a solution in the sense of Davie is also a solution in the sense of Lyons.

If z is a solution in the sense of Lyons, by construction, for a constant L and all $(s, t) \in \Delta^2$,

$$\begin{aligned} |z_t - z_s - f(z_s) x_{s,t}^1 - \nabla f(z_s) z_{s,t}^\times| &\leq L \omega(s, t)^\theta \\ \text{and } |z_{s,t}^\times| &\leq 1 \otimes f(z_s) x_{s,t}^2 \leq L \omega(s, t)^\theta, \end{aligned}$$

where z^\times lives in $\mathbb{R}^m \otimes \mathbb{R}^d$. Hence, it is immediate that a solution in the sense of Lyons is also a solution in the sense of Davie. \square

A From local to global theorems

We present here some results which allow one to pass from local estimates to global estimates and then show the existence for any horizon T provided some uniform estimates.

Lemma 7. *Let $\mu > 0$ be fixed. Then there exists a finite number of times $0 = T_0 < T_1 < \dots < T_{N+1}$ with $T_N \leq T < T_{N+1}$ and $\omega(T_i, T_{i+1}) = \mu$.*

Proof. Let us extend ω on $D_+ := \{(s, t) \mid 0 \leq s \leq t\}$ by

$$\omega(s, t) = \begin{cases} \omega(s, T) + T - s & \text{if } s \leq T \leq t, \\ t - s & \text{if } T \leq s \leq t. \end{cases}$$

Let us note that ω is continuous on D_+ , and that $\omega(s, t) \xrightarrow[t \rightarrow \infty]{} +\infty$. Then for any $\mu > 0$, for any $s \geq 0$, there exists a value $\tau(s)$ such that $\omega(s, \tau(s)) = \mu$, since $\omega(s, s) = 0$.

For $T > 0$ fixed, let us set $T_0 = 0$ and $T_{i+1} = \tau(T_i)$. Then there exists a finite number N such that $\omega(0, T_{N-1}) \leq \omega(0, T) \leq \omega(T_0, T_{N+1})$ and $\omega(T_i, T_{i+1}) = \mu$. This follows from the fact that

$$N\mu = \sum_{i=0}^{N-1} \omega(T_i, T_{i+1}) \leq \omega(0, T_N)$$

and then $\omega(0, T_N) \xrightarrow[N \rightarrow \infty]{} +\infty$. This proves the lemma. \square

For a path y and times $0 \leq S < S' \leq T$, let us set

$$\|y\|_{[S, S']} := \sup_{S \leq s < t \leq S'} \frac{|y_t - y_s|}{\omega(s, t)^{1/p}}.$$

Lemma 8. *Let us assume that a continuous path y on $[S, S'']$ is a solution in the sense of Davie on time interval $[S, S']$ and $[S', S'']$ respectively with constants L_1 and L_2 . Then y is a solution in the sense of Davie on $[S, S'']$ with a new constant L_3 that depends only on $L_1, L_2, H_\gamma(F), \|\nabla f\|_\infty, \|x\|, \|y\|_{[S, S']}, \omega(0, T), \gamma$ and p .*

Proof. First, it is classical that

$$\|y\|_{[S, S'']} \leq 2^{1-1/p} \max\{\|y\|_{[S, S']}, \|y\|_{[S', S'']}\},$$

so that y is of finite p -variation over $[S, S'']$. For $s \in [S, S']$ and $t \in [S', S'']$,

$$|y_t - y_s - \mathfrak{D}(s, t)| \leq |y_t - y_{S'} - \mathfrak{D}(S', t)| + |y_{S'} - y_s - \mathfrak{D}(s, S')| + |\mathfrak{D}(s, S', t)|.$$

Let us note that

$$|\mathfrak{D}(s, S', t)| \leq C_{36} \omega(s, t)^\theta$$

where C_{36} depends only on L_1 , $\|y\|_{[S, S']}$, $H_\gamma(F)$, $\|\nabla f\|_\infty$, $|F(y_s)|$, $\|x\|$, $\omega(0, T)$, γ and p . Hence, since $\omega(s, S')^\theta + \omega(S', t) \leq \omega(s, t)$,

$$|y_t - y_s - \mathfrak{D}(s, t)| \leq (\max\{L_1, L_2\} + C_{36}) \omega(s, t)^\theta.$$

This proves the result. \square

Proposition 6. *Let us assume that a solution to (1) exists on any time interval $[S, S']$ with a condition that $\omega(S, S') \leq C_{37}$ where C_{37} does not depend on S . Then a solution exists for any time T .*

Proof. Using the sequence $\{T_i\}_{i=0, \dots, N}$ of times given by Lemma 7, it is sufficient to solve (1) successively on $[T_i, T_{i+1}]$ with initial condition y_{T_i} and to invoke Lemma 8 to prove the existence of a solution in the sense of Davie in $[0, T]$. \square

Proposition 7. *Let y be a path of finite p -variations such that for some constants A , B and K ,*

$$\text{when } \omega(S, S') \leq K, \quad \|y\|_{[S, S']} \leq B|y_S| + A.$$

Then

$$\sup_{t \in [0, T]} |y_t| \leq R(T)|y_0| + (R(T) - 1) \frac{A}{B} \quad (37)$$

with

$$R(T) = \exp(B(1 + K^{-1})^{1-1/p} \max\{\omega(0, T), \omega(0, T)^{1/p}\}),$$

and the p -variation norm of $\|y\|_{[0, T]}$ depends only on A , B , $\omega(0, T)$, K and p .

Proof. Remark first that

$$\sup_{t \in [S, S']} |y_t| \leq |y_S| (1 + B\omega(S, S')^{1/p}) + A\omega(S, S')^{1/p}.$$

Let us choose an integer N such that $\mu := \omega(0, T)/N \leq K$ and construct recursively T_i with $T_0 = 0$ and $\omega(T_i, T_{i+1}) = \mu$. Then

$$|y_{T_{i+1}}| \leq |y_{T_i}|(1 + \mu^{1/p}B) + A\mu^{1/p}.$$

From a classical result easily proved by an induction,

$$|y_T| \leq \exp(N\mu^{1/p}B)|y_0| + \frac{A}{B}(\exp(N\mu^{1/p}B) - 1).$$

As this is true for any T , surely (37) holds.

Choosing N such that $\omega(0, T)/N \leq K$ and $\omega(0, T)/(N - 1) > K$,

$$N \leq \frac{\omega(0, T)}{K} + 1.$$

This way

$$N\mu^{1/p} \leq \left(\frac{\omega(0, T)}{K} + 1\right)^{1-\frac{1}{p}} \omega(0, T)^{\frac{1}{p}} \leq \begin{cases} \tilde{K}\omega(0, T)^{1/p} & \text{if } \omega(0, T) \leq 1, \\ \tilde{K}\omega(0, T) & \text{if } \omega(0, T) > 1. \end{cases}$$

with $\tilde{K} := (1 + K^{-1})^{1-1/p}$.

Finally,

$$\|y\|_{[0, T]} \leq N^{1-1/p} \max_{i=0, \dots, N-1} \|y\|_{[T_i, T_{i+1}]},$$

which proves the last statement □

References

- [1] S. Aida, *Notes on proofs of continuity theorem in rough path analysis* (2006). Preprint of Osaka University.
- [2] A.M. Davie, *Differential Equations Driven by Rough Paths: an Approach via Discrete Approximation*, Appl. Math. Res. Express. AMRX **2** (2007), Art. ID abm009, 40.
- [3] A. Deya, A. Neuenkirch, and S. Tindel, *A Milstein-type scheme without Levy area terms for SDEs driven by fractional Brownian motion* (2010). arxiv:1001.3344.
- [4] H. Doss, *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. H. Poincaré Sect. B (N.S.) **13** (1977), no. 2, 99–125.
- [5] P. Friz and N. Victoir, *Euler Estimates of Rough Differential Equations*, J. Differential Equations **244** (2008), no. 2, 388–412.
- [6] ———, *Multidimensional Stochastic Processes as Rough Paths. Theory and Applications*, Cambridge University Press, 2010.

- [7] J. G. Gaines and T. J. Lyons, *Variable step size control in the numerical solution of stochastic differential equations*, SIAM J. Appl. Math. **57** (1997), no. 5, 1455–1484, DOI 10.1137/S0036139995286515.
- [8] M. Gubinelli, *Controlling rough paths*, J. Funct. Anal. **216** (2004), no. 1, 86–140.
- [9] Y. Inahama and H. Kawabi, *Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths*, J. Funct. Anal. **243** (2007), no. 1, 270–322.
- [10] A. Lejay, *An introduction to rough paths*, Séminaire de probabilités XXXVII, Lecture Notes in Mathematics, vol. 1832, Springer-Verlag, 2003, pp. 1–59.
- [11] ———, *Yet another introduction to rough paths*, Séminaire de probabilités XLII, Lecture Notes in Mathematics, vol. 1979, Springer-Verlag, 2009, pp. 1–101.
- [12] ———, *On rough differential equations*, Electron. J. Probab. **14** (2009), no. 12, 341–364.
- [13] ———, *Controlled differential equations as Young integrals: a simple approach*, J. Differential Equations **249**, 1777–1798, DOI 10.1016/j.jde.2010.05.006.
- [14] A. Lejay and N. Victoir, *On (p, q) -rough paths*, J. Differential Equations **225** (2006), no. 1, 103–133.
- [15] T.J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana **14** (1998), no. 2, 215–310.
- [16] T. Lyons and Z. Qian, *Flow of diffeomorphisms induced by a geometric multiplicative functional*, Probab. Theory Related Fields **112** (1998), no. 1, 91–119.
- [17] ———, *System Control and Rough Paths*, Oxford Mathematical Monographs, Oxford University Press, 2002.
- [18] T. Lyons, M. Caruana, and T. Lévy, *Differential Equations Driven by Rough Paths*, École d’été des probabilités de Saint-Flour XXXIV — 2004 (J. Picard, ed.), Lecture Notes in Mathematics, vol. 1908, Springer, 2007.
- [19] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Mathematics and its Applications (East European Series), vol. 53, Kluwer Academic Publishers Group, 1991.
- [20] H.J. Sussmann, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. Probability **6** (1978), no. 1, 19–41.