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# THE REMOTE POINT PROBLEM, SMALL BIAS SPACES, AND EXPANDING GENERATOR SETS

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**ABSTRACT.** Using  $\varepsilon$ -bias spaces over  $\mathbb{F}_2$ , we show that the Remote Point Problem (RPP), introduced by Alon et al [APY09], has an  $\text{NC}^2$  algorithm (achieving the same parameters as [APY09]). We study a generalization of the Remote Point Problem to groups: we replace  $\mathbb{F}_2^n$  by  $\mathcal{G}^n$  for an arbitrary fixed group  $\mathcal{G}$ . When  $\mathcal{G}$  is Abelian we give an  $\text{NC}^2$  algorithm for RPP, again using  $\varepsilon$ -bias spaces. For nonabelian  $\mathcal{G}$ , we give a deterministic polynomial-time algorithm for RPP. We also show the connection to construction of expanding generator sets for the group  $\mathcal{G}^n$ . All our algorithms for the RPP achieve essentially the same parameters as [APY09].

## 1. Introduction

Valiant, in his celebrated work [V77] on circuit lower bounds for computing linear transformations  $A : \mathbb{F}^n \rightarrow \mathbb{F}^m$  for a field  $\mathbb{F}$ , initiated the study of rigid matrices. If explicit rigid matrices of certain parameters can be constructed it would result in superlinear lower bounds for logarithmic depth linear circuits over  $\mathbb{F}$ . This problem and the construction of such rigid matrices has remained elusive for over three decades.

Alon, Panigrahy and Yekhanin [APY09] recently proposed a problem that appears to be of intermediate difficulty. Given a subspace  $L$  of  $\mathbb{F}_2^n$  by its basis and a number  $r \in [n]$  as input, the problem is to compute in deterministic polynomial time a point  $v \in \mathbb{F}_2^n$  such that  $\Delta(u, v) \geq r$  for all  $u \in L$ , where  $\Delta(u, v)$  is the Hamming distance. They call this the *Remote Point Problem*. The point  $v$  is said to be  $r$ -far from the subspace  $L$ .

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Alon et al [APY09] give a nice polynomial time-bounded (in  $n$ ) algorithm for computing a  $v \in \mathbb{F}_2^n$  that is  $c \log n$ -far from a given subspace  $L$  of dimension  $n/2$  and  $c$  is a fixed constant. For  $L$  such that  $\dim(L) = k < n/2$  they give a polynomial-time algorithm for computing a point  $v \in \mathbb{F}_2^n$  that is  $\frac{cn \log k}{k}$ -far from  $L$ .

**Results of this paper.** In [AS09a] we recently investigated the problem of proving circuit lower bounds in the presence of help functions. Specifically, one of the problems we consider is proving lower bounds for constant-depth Boolean circuits which can take a given set of (arbitrary) help functions  $\{h_1, h_2, \dots, h_m\}$  at the input level, where  $h_i : \{0, 1\}^n \rightarrow \{0, 1\}$  for each  $i$ . Proving explicit lower bounds for this model would allow us to separate EXP from the polynomial-time many-one closure of nonuniform  $\text{AC}^0$ . We show that it suffices to find a polynomial-time solution to the Remote Point Problem for parameters  $k = 2^{(\log \log n)^c}$  and  $r = \frac{n}{2^{(\log \log n)^d}}$  for all constants  $c$  and  $d$ . Unfortunately, the parameters of the Alon et al algorithm are inadequate for our application.

However, motivated by this connection, in the present paper we carry out a more detailed study of the Remote Point Problem as an algorithmic question. We briefly summarize our results.

1. The first question we address is whether we can give a deterministic parallel (i.e. NC) algorithm for the problem — Alon et al’s algorithm is inherently sequential as it is based on the method of conditional probabilities and pessimistic estimators.

It turns out an element of an  $\varepsilon$ -bias space for suitably chosen  $\varepsilon$  is a solution to the Remote Point Problem which gives us an NC algorithm quite easily.

2. Since the RPP for  $\mathbb{F}_2^n$  can be solved using small bias spaces, it naturally leads us to address the problem in a more general group-theoretic setting.

In the generalization we study we will replace  $\mathbb{F}_2$  with an arbitrary fixed finite group  $\mathcal{G}$  such that  $|\mathcal{G}| \geq 2$ . Hence we will have the  $n$ -fold product group  $\mathcal{G}^n$  instead of the vector space  $\mathbb{F}_2^n$ .

Given elements  $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$  of  $\mathcal{G}^n$ , let  $\Delta(x, y) = |\{i \mid x_i \neq y_i\}|$ . I.e.  $\Delta(x, y)$  is the *Hamming distance* between  $x$  and  $y$ . Furthermore, for  $S \subseteq \mathcal{G}^n$ , let  $\Delta(x, S)$  denote  $\min_{y \in S} \Delta(x, y)$ .

We now define the *Remote Point Problem (RPP) over a finite group  $\mathcal{G}$* . The input is a subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$ , where  $\mathcal{H}$  is given by a generating set, and a number  $r \in [n]$ . The problem is to compute in deterministic polynomial (in  $n$ ) time an element  $x \in \mathcal{G}^n$  such that  $\Delta(x, \mathcal{H}) > r$ . The results we show in this general setting are the following.

- (a) The Remote Point Problem over any *Abelian group*  $\mathcal{G}$  has an  $\text{NC}^2$  algorithm for  $r = O(\frac{n \log k}{k})$  and  $k \leq n/2$ , where  $k = \log_{|\mathcal{G}|} |\mathcal{H}|$ .
- (b) Over an arbitrary group  $\mathcal{G}$  the Remote point problem has a polynomial-time algorithm for  $r = O(\frac{n \log k}{k})$  and  $k \leq n/2$ , where  $k = \log_{|\mathcal{G}|} |\mathcal{H}|$ .

The parallel algorithm stated in part(a) above is based on  $\varepsilon$ -bias space constructions for finite Abelian groups described in Azar et al [AMN98]. The sequential algorithm stated in part(b) above is a group-theoretic generalization of the Alon et al algorithm for  $\mathbb{F}_2^n$  [APY09].

Due to lack of space, some proofs have been omitted. They may be found in the full version which has been published as an ECCC report [AS09b].

## 2. Preliminaries

Fix a finite group  $\mathcal{G}$  such that  $|\mathcal{G}| \geq 2$ . Given any  $x \in \mathcal{G}^n$ , let  $wt(x)$  denote the number of coordinates  $i$  such that  $x_i \neq 1$ , where 1 is the identity of the group  $\mathcal{G}$ . By  $B(r)$ , we will refer to the set of  $x \in \mathcal{G}^n$  such that  $wt(x) \leq r$ . Given a subset  $S$  of  $\mathcal{G}^n$ ,  $B(S, r)$  will denote the set  $S \cdot B(r) = \{sx \mid s \in S, x \in B(r)\}$ . Clearly, for any  $S \subseteq \mathcal{G}^n$  and any  $x \in \mathcal{G}^n$ ,  $x \in B(S, r)$  if and only if  $\Delta(x, S) \leq r$ . We say that  $x$  is  $r$ -close to  $S$  if  $x \in B(S, r)$  and  $r$ -far from  $S$  if  $x \notin B(S, r)$ .

The *Remote Point Problem (RPP)* over  $\mathcal{G}$  is defined to be the following algorithmic problem:

INPUT: A subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$  (given by its generators) and an  $r \in \mathbb{N}$ .  
 OUTPUT: An  $x \in \mathcal{G}^n$  such that  $x \notin B(\mathcal{H}, r)$ .

Clearly, there are inputs to the above problem where no solution can be found. But the input instances of the kind that we will study will clearly have a solution (in fact, a random point of  $\mathcal{G}^n$  will be a solution with high probability).

Given a subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$ , denote by  $\delta(\mathcal{H})$  the quantity  $\log_{|\mathcal{G}|} |\mathcal{H}|$ . We will call  $\delta(\mathcal{H})$  the *dimension of  $\mathcal{H}$  in  $\mathcal{G}^n$* .

We say that the RPP over  $\mathcal{G}$  has a  $(k(n), r(n))$ -algorithm if there is an efficient algorithm that solves the Remote Point Problem when given as input a subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$  of dimension at most  $k(n)$  and an  $r$  that is bounded by  $r(n)$ . (Here, ‘efficient’ can correspond to polynomial time or some smaller complexity class.)

A simple counting argument shows that there is a valid solution to the RPP over  $\mathcal{G}$  on inputs  $(\mathcal{H}, r)$  where  $\delta(\mathcal{H}) + r \leq n(1 - \frac{H(r/n)}{\log |\mathcal{G}|} - \varepsilon)$ , for any fixed  $\varepsilon > 0$  (where  $H(\cdot)$  denotes the binary entropy function). However, the best known deterministic solution to the RPP – from [APY09] – is a polynomial time  $(k, \frac{cn \log k}{k})$ -algorithm which works over  $\mathbb{F}_2^n$  (i.e, the group  $\mathcal{G}$  involved is the additive group of the field  $\mathbb{F}_2$ ).

### 2.1. Some Group-Theoretic Algorithms

We introduce basic definitions and review some group-theoretic algorithms. Let  $\text{Sym}(\Omega)$  denote the group of all permutations on a finite set  $\Omega$  of size  $m$ . In this section we use  $G, H$  etc. to denote *permutation groups on  $\Omega$* , which are simply subgroups of  $\text{Sym}(\Omega)$ .

Let  $G$  be a subgroup of  $\text{Sym}(\Omega)$ . For a subset  $\Delta \subseteq \Omega$  denote by  $G_{\{\Delta\}}$  the *point-wise stabilizer* of  $\Delta$ . I.e  $G_{\{\Delta\}}$  is the subgroup consisting of exactly those elements of  $G$  that fix each element of  $\Delta$ .

**Theorem 2.1** (Schreier-Sims). [Lu93]

- (1) *If a subgroup  $G$  of  $\text{Sym}(\Omega)$  is given by a generating set as input along with the subset  $\Delta$  there is a polynomial-time (sequential) algorithm for computing a generator set for  $G_{\{\Delta\}}$ .*
- (2) *If a subgroup  $G$  of  $\text{Sym}(\Omega)$  is given by a generating set as input, then there is a polynomial time algorithm for computing  $|G|$ .*
- (3) *Given as input a permutation  $\sigma \in \text{Sym}(\Omega)$  and a generator set for a subgroup  $G$  of  $\text{Sym}(\Omega)$ , we can test in deterministic polynomial time if  $\sigma$  is an element of  $G$ .*

We are also interested in a special case of this problem which we now define. A subset  $\Gamma \subseteq \Omega$  is an *orbit* of  $G$  if  $\Gamma = \{\sigma(i) \mid \sigma \in G\}$  for some  $i \in \Omega$ . Any subgroup  $G$  of  $\text{Sym}(\Omega)$  partitions  $\Omega$  into orbits (called  $G$ -orbits).

For a constant  $b > 0$ , a subgroup  $G$  of  $\text{Sym}(\Omega)$  is defined to be a  *$b$ -bounded permutation group* if every  $G$ -orbit is of size at most  $b$ .

In [MC87], McKenzie and Cook studied the parallel complexity of *Abelian* permutation group problems. Specifically, they gave an  $\text{NC}^3$  algorithm for testing membership in an Abelian permutation group given by a generator set and for computing the order of an Abelian permutation group. When restricted to  $b$ -bounded Abelian permutation groups, the algorithms of [MC87] for these problems are actually  $\text{NC}^2$  algorithms. We formally state their result and derive a consequence.

**Theorem 2.2** ([MC87]). *There is an  $\text{NC}^2$  algorithm for membership testing in a  $b$ -bounded Abelian permutation group  $G$  given by a generator set.*

We now consider problems over  $\mathcal{G}^n$ , for a fixed finite group  $\mathcal{G}$ . We know from basic group theory that every group  $\mathcal{G}$  is a permutation group acting on itself. I.e. every  $\mathcal{G}$  can be seen as a subgroup of  $\text{Sym}(\mathcal{G})$ , where  $\mathcal{G}$  acts on itself by left (or right) multiplication. Therefore,  $\mathcal{G}^n$  can be easily seen as a permutation group on the set  $\Omega = \mathcal{G} \times [n]$  and hence,  $\mathcal{G}^n$  can be considered a subgroup of  $\text{Sym}(\Omega)$ . Furthermore, notice that each subset  $\mathcal{G} \times \{i\}$  is an orbit of this group  $\mathcal{G}^n$ . Hence,  $\mathcal{G}^n$  is a  $b$ -bounded permutation group contained in  $\text{Sym}(\Omega)$ , where  $b = |\mathcal{G}|$ . Finally, if  $\mathcal{G}$  is an Abelian group, then so is this subgroup of  $\text{Sym}(\Omega)$ . We have the following lemma as an easy consequence of Theorem 2.2.

**Lemma 2.3.** *Let  $\mathcal{G}$  be Abelian. There is an  $\text{NC}^2$  algorithm that takes as input a generator set for some subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$  and an  $x \in \mathcal{G}^n$ , and accepts iff  $x \in \mathcal{H}$ .*

Given any  $y = (y_1, y_2, \dots, y_i) \in \mathcal{G}^i$  with  $1 \leq i \leq n$  and any  $S \subseteq \mathcal{G}^n$ , let  $S_y$  denote the set  $\{x \in S \mid x_j = y_j \text{ for } 1 \leq j \leq i\}$ .

**Lemma 2.4.** *Let  $\mathcal{G}$  be any fixed finite group. There is a polynomial time algorithm that takes as input a subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$ , where  $\mathcal{H}$  is given by generators, and a  $y \in \mathcal{G}^i$  with  $1 \leq i \leq n$ , and computes  $|\mathcal{H}_y|$ .*

*Proof.* Let  $\mathcal{K} = \{(x_1, x_2, \dots, x_n) \in \mathcal{H} \mid x_1 = x_2 = \dots = x_n = 1\}$ , where 1 denotes the identity element of  $\mathcal{G}$ . Clearly,  $\mathcal{K}$  is a subgroup of  $\mathcal{H}$ . The set  $\mathcal{H}_y$ , if nonempty, is simply a coset of  $\mathcal{K}$  and thus, we have  $|\mathcal{H}_y| = |\mathcal{K}|$ . To check if  $\mathcal{H}_y$  is nonempty, we consider the map  $\pi_i : \mathcal{G}^n \rightarrow \mathcal{G}^i$  that projects its input onto its first  $i$  coordinates; note that  $\mathcal{H}_y$  is nonempty iff the subgroup  $\pi_i(\mathcal{H})$  contains  $y$ , which can be checked in polynomial time by point (3) of Theorem 2.1 (here, we are identifying  $\mathcal{G}^n$  with a subgroup of  $\text{Sym}(\mathcal{G} \times [n])$  as above). If  $y \notin \pi_i(\mathcal{H})$ , the algorithm outputs 0. Otherwise, we have  $|\mathcal{H}_y| = |\mathcal{K}|$  and it suffices to compute  $|\mathcal{K}|$ . But  $\mathcal{K}$  is simply the point-wise stabilizer of the set  $\mathcal{G} \times [i]$  in  $\mathcal{H}$ , and hence  $|\mathcal{K}|$  can be computed in polynomial time by points (1) and (2) of Theorem 2.1. ■

### 3. Expanding Cayley Graphs and the Remote Point Problem

Fix a group  $\mathcal{G}$  such that  $|\mathcal{G}| \geq 2$ , and consider an instance of the RPP over  $\mathcal{G}$ . The main idea that we develop in this section is that if we have a (symmetric) expanding generator set  $S$  for the group  $\mathcal{G}^n$  with appropriate expansion parameters then for a subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$  such that  $\delta(\mathcal{H}) \leq k$  some element of  $S$  will be  $r$ -far from  $H$ , for suitable  $k$  and  $r$ .

We review some definitions related to expander graphs (e.g. see the survey of Hoory, Linial, and Wigderson [HLW06]). An undirected multigraph  $G = (V, E)$  is an  $(n, d, \alpha)$ -graph for  $n, d \in \mathbb{N}$  and  $\alpha > 0$  if  $|V| = n$ , the degree of each vertex is  $d$ , and the second largest value  $\lambda(G)$  from among the absolute values of eigenvalues of  $A(G)$  – the adjacency matrix of the graph  $G$  – is bounded by  $\alpha d$ .

A *random walk* of length  $t \in \mathbb{N}$  on an  $(n, d, \alpha)$ -graph  $G = (V, E)$  is the output of the following random process: a vertex  $v_0 \in V$  of picked uniformly at random, and for  $0 \leq i < t$ , if  $v_i$  has been picked, then  $v_{i+1}$  is obtained by selecting a neighbour  $v_{i+1}$  uniformly at random (i.e a random edge out of  $v_i$  is picked, and  $v_{i+1}$  is chosen to be the other endpoint of the edge); the output of the process is  $(v_0, v_1, \dots, v_t)$ . We now state an important result regarding random walks on expanders (see [HLW06, Theorem 3.6] for details).

**Lemma 3.1.** *Let  $G = (V, E)$  be an  $(n, d, \alpha)$ -graph and  $B \subseteq V$  with  $|B| \leq \beta n$ . Then, the probability that a random walk  $(v_0, v_1, \dots, v_t)$  is entirely contained inside  $B$  (i.e,  $v_i \in B$  for each  $i$ ) is bounded by  $(\beta + \alpha)^t$ .*

Let  $\mathcal{H}$  be a group and  $S$  a *symmetric* multiset of elements from  $\mathcal{H}$ . I.e. there is a bijection of multisets  $\varphi : S \rightarrow S$  such that  $\varphi(s) = s^{-1}$  for each  $s \in S$ . We define the Cayley graph  $C(\mathcal{H}, S)$  to be the (multi)graph  $G$  with vertex set  $\mathcal{H}$  and edges of the form  $(x, xs)$  for each  $x \in \mathcal{H}$  and each  $s \in S$ ; since  $S$  is symmetric, we consider  $C(\mathcal{H}, S)$  to be an undirected graph by identifying the edges  $(x, xs)$  and  $(xs, (xs)\varphi(s))$ , for each  $x$  and  $s$ .

We now show a lemma that will help relate generators of expanding Cayley graphs on  $\mathcal{G}^n$  and the RPP over  $\mathcal{G}$ . In what follows, let  $S$  be a symmetric multiset of elements from  $\mathcal{G}^n$ ; let  $G$  denote the Cayley graph  $C(\mathcal{G}^n, S)$ ; and let  $N, D$  denote  $|\mathcal{G}|^n$  and  $|S|$  (counted with repetitions) respectively.

**Lemma 3.2.** *Assume  $S$  as above is such that  $G$  is an  $(N, D, \alpha)$ -graph, where  $\alpha \leq \frac{1}{n^d}$ , for some fixed  $d > 0$ . Then, given any subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$  such that  $\delta(\mathcal{H}) \leq 2n/3$ , we have  $\frac{|S \cap \mathcal{H}|}{|S|} \leq \frac{1}{n^{d/2}}$  for large enough  $n$  (where the elements of  $S \cap \mathcal{H}$  are counted with repetitions).*

*Proof.* Let  $S' = S \cap \mathcal{H}$  and let  $\eta = |S'|/|S|$ . We want an upper bound on  $\eta$ . Consider a random walk  $(x_0, x_1, \dots, x_t)$  of length  $t$  on the graph  $G$  (the exact value of  $t$  will be fixed later). Let  $\mathcal{B}$  denote the following event: there is a  $y \in \mathcal{G}^n$  such that all the vertices  $x_0, x_1, \dots, x_t$  are all contained in the coset  $y\mathcal{H}$  of  $\mathcal{H}$ . Let  $p$  denote the probability that  $\mathcal{B}$  occurs.

We will first lower bound  $p$ . At each step of the random walk, a random  $s_i \in S$  is chosen and  $x_{i+1}$  is set to  $x_i s_i$ . If these  $s_i$  all happen to belong to  $S'$ , then the cosets  $x_i \mathcal{H}$  and  $x_{i+1} \mathcal{H}$  are the same for all  $i$  and hence, the event  $\mathcal{B}$  does occur. Hence,  $p \geq \eta^t$ .

We now upper bound  $p$ . Fix any coset  $y\mathcal{H}$  of the subgroup  $\mathcal{H}$ . Since the dimension of  $\mathcal{H}$  in  $\mathcal{G}^n$  is bounded by  $2n/3$ , we have  $|y\mathcal{H}| = |\mathcal{H}| \leq |\mathcal{G}|^{2n/3} \leq 2^{-n/3} |\mathcal{G}^n|$ . That is, the coset  $y\mathcal{H}$  is a very small subset of  $\mathcal{G}^n$ . Applying Lemma 3.1, we see that the probability that the random walk  $(x_0, x_1, \dots, x_t)$  is completely contained inside this coset is bounded by  $(2^{-n/3} + n^{-d})^t \leq \frac{2^t}{n^{dt}}$ , for large enough  $n$ . As the total number of cosets of  $\mathcal{H}$  is bounded by  $|\mathcal{G}|^n$ , an application of the union bound tells us that  $p$  is upper bounded by  $|\mathcal{G}|^n \frac{2^t}{n^{dt}} \leq \frac{|\mathcal{G}|^{n+t}}{n^{dt}}$ . Setting  $t = \frac{2n}{d \log_{|\mathcal{G}|} n - 2}$  we see that  $p$  is at most  $\frac{1}{n^{d/2}}$ .

Putting the upper and lower bounds together, we see that  $\eta^t \leq \frac{1}{n^{d/2}}$  and hence,  $\eta \leq \frac{1}{n^{d/2}}$ . This completes the proof.  $\blacksquare$

We follow the structure of the algorithm for the RPP over  $\mathbb{F}_2$  in [APY09]. We first describe their  $(n/2, c \log n)$ -algorithm for the RPP, followed by our own algorithm. We then describe how they extend this algorithm to a  $(k, \frac{cn \log k}{k})$ -algorithm for any  $k \leq n/2$ ; the same procedure works for our algorithm also.

The  $(n/2, c \log n)$ -algorithm proceeds as follows. On an input instance consisting of a subgroup  $V$  (which is a subspace of  $\mathbb{F}_2^n$ ) of dimension at most  $n/2$  and an  $r \leq c \log n$ ,

- (1) The algorithm first computes a collection of  $m = n^{O(c)}$  subspaces  $V_1, V_2, \dots, V_m$ , each of dimension at most  $2n/3$  such that  $B(V, c \log n) \subseteq \bigcup_{i=1}^m V_i$ .
- (2) The algorithm then finds an  $x \in \mathbb{F}_2^n$  such that  $x \notin \bigcup_i V_i$ . (This is done using a method similar to the method of pessimistic estimators introduced by Raghavan [Rag88].)

Our algorithm will proceed exactly as the above algorithm in the first step. The second step of our algorithm will be different (assuming that the group  $\mathcal{G}$  is Abelian). We first state Step 1 of the algorithm of [APY09] in greater generality:

**Lemma 3.3.** *Let  $\mathcal{G}$  be any fixed finite group with  $|\mathcal{G}| \geq 2$ . For any constant  $c > 0$  and large enough  $n$ , the following holds. Given any subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$  such that  $\delta(\mathcal{H}) \leq \frac{n}{2}$ , there is a collection of  $m \leq n^{10c}$  subgroups  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$  such that  $B(\mathcal{H}, c \log n) \subseteq \bigcup_{i=1}^m \mathcal{H}_i$ , and*

$\delta(\mathcal{H}_i) \leq 2n/3$  for each  $i$ . Moreover, there is a logspace algorithm that, when given as input  $\mathcal{H}$  as a set of generators, produces generators for the subgroups  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$ .

*Proof.* The proof follows exactly as in [APY09]. We reproduce it here for completeness and to analyze the complexity of the procedure.

Let 1 denote the identity element of  $\mathcal{G}$ . For each  $S \subseteq [n]$ , let  $\mathcal{G}^n(S)$  denote the subgroup of  $\mathcal{G}^n$  consisting of those  $x$  such that  $x_i = 1$  for each  $i \notin S$ . Note that  $\delta(\mathcal{G}^n(S)) = |S|$ . Also note that for each  $S \subseteq [n]$ , the group  $\mathcal{G}^n(S)$  is a normal subgroup; in particular, this implies that the set  $\mathcal{K} \cdot \mathcal{G}^n(S)$  is a subgroup of  $\mathcal{G}^n$  whenever  $\mathcal{K}$  is a subgroup of  $\mathcal{G}^n$ .

Partition the set  $[n]$  into  $\ell \leq 10c \log n$  sets of size at most  $\lceil \frac{n}{10c \log n} \rceil$  each – we will call these sets  $S_1, S_2, \dots, S_\ell$ . For each  $A \subseteq [\ell]$  of size  $\lceil c \log n \rceil$ , let  $\mathcal{K}_A$  denote the subgroup  $\mathcal{G}^n(\bigcup_{i \in A} S_i)$ . Note that the number of such subgroups is at most  $2^\ell \leq n^{10c}$ . Also, for each  $A$  as above,  $\delta(\mathcal{K}_A) = |\bigcup_{i \in A} S_i| \leq \left( \frac{n}{10c \log n} + 1 \right) (c \log n + 1) < \frac{n}{9}$ , for large enough  $n$ .

Consider any  $x \in B(c \log n)$  (i.e, an element  $x$  of  $\mathcal{G}^n$  s.t  $wt(x) \leq c \log n$ ). We know that  $x \in \mathcal{G}^n(S)$  for some  $S$  of size at most  $c \log n$ . Hence, it can be seen that  $x \in \mathcal{G}^n(\bigcup_{i \in A} S_i)$  for some  $A$  of size  $\lceil c \log n \rceil$ ; this shows that  $B(c \log n) \subseteq \bigcup_A \mathcal{K}_A$ . Therefore, we see that  $B(\mathcal{H}, c \log n) = \mathcal{H}B(c \log n) \subseteq \bigcup_A \mathcal{H}\mathcal{K}_A$ .

For each  $A \subseteq [\ell]$  of size  $\lceil c \log n \rceil$ , let  $\mathcal{H}_A$  denote the subgroup  $\mathcal{H}\mathcal{K}_A$  (note that this is indeed a subgroup, since  $\mathcal{K}_A$  is a normal subgroup). Moreover, the cardinality of this subgroup is bounded by  $|\mathcal{H}| \cdot |\mathcal{K}_A| \leq |\mathcal{G}|^{n/2} |\mathcal{G}|^{n/9} < |\mathcal{G}|^{2n/3}$ ; hence,  $\delta(\mathcal{H}_A) \leq 2n/3$ . Thus, the collection of subgroups  $\{\mathcal{H}_A\}_A$  satisfies all the properties mentioned in the statement of the lemma. That a set of generators for this subgroup can be computed in deterministic logspace – for some suitable choice of  $S_1, S_2, \dots, S_\ell$  – is a routine check from the definition of the subgroups  $\{\mathcal{K}_A\}_A$ . This completes the proof of the lemma.  $\blacksquare$

Using Lemma 3.3, we are able to efficiently “cover”  $B(\mathcal{H}, c \log n)$  for any small subgroup  $\mathcal{H}$  of  $\mathcal{G}^n$  by a union of small subgroups. Therefore, to find a point that is  $c \log n$ -far from  $\mathcal{H}$ , it suffices to find a point  $x \in \mathcal{G}^n$  not contained in any of the covering subgroups. To do this, we note that if  $S$  is a multiset containing elements from  $\mathcal{G}^n$  such that  $C(\mathcal{G}^n, S)$  is a Cayley graph with good expansion, then  $S$  must contain such an element. This is formally stated below.

**Lemma 3.4.** *For any constant  $c > 0$  and large enough  $n \in \mathbb{N}$ , the following holds. Let  $S$  be any multiset of elements of  $\mathcal{G}^n$  such that  $\lambda(C(\mathcal{G}^n, S)) < \frac{1}{n^{20c}}$ . Then, for  $m \leq n^{10c}$  and any collection  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$  of subgroups such that  $\delta(\mathcal{H}_i) \leq 2n/3$  for each  $i$ , there is some  $s \in S$  such that  $s \notin \bigcup_i \mathcal{H}_i$ .*

*Proof.* The proof follows easily from Lemma 3.2. Given any  $i \in [m]$ , we know, from Lemma 3.2, that  $|S \cap \mathcal{H}_i| < \frac{|S|}{n^{10c}}$  (where the elements of the multisets are counted with repetitions). Hence,  $|S \cap \bigcup_i \mathcal{H}_i| \leq \sum_i |S \cap \mathcal{H}_i| < \frac{m|S|}{n^{10c}} \leq |S|$ . Therefore, there must be some  $s \in S$  such that  $s \notin \bigcup_i \mathcal{H}_i$ .  $\blacksquare$



Therefore, to find a point  $x$  that is  $c \log n$ -far from the subspace  $\mathcal{H}$ , it suffices to construct an  $S$  such that  $C(\mathcal{G}^n, S)$  is a sufficiently good expander, find the covering subgroups  $\mathcal{H}_i$  ( $i \in [m]$ ), and then to find an  $s \in S$  that does not lie in any of the  $\mathcal{H}_i$ . We follow the above approach to give an efficient parallel algorithm for the RPP in the case that  $\mathcal{G}$  is an Abelian group. For arbitrary groups, we show that the method of [APY09] yields a polynomial time algorithm.

#### 4. Remote Point Problem for Abelian Groups

Fix an Abelian group  $\mathcal{G}$ . Recall that a *character*  $\chi$  of  $\mathcal{G}^n$  is a homomorphism from  $\mathcal{G}^n$  to  $\mathbb{C}_1^*$ , the multiplicative subgroup of the complex numbers of absolute value 1. For  $\varepsilon > 0$ , a distribution  $\mu$  over  $\mathcal{G}^n$  is said to be  $\varepsilon$ -biased if, given any non-trivial character  $\chi$  of  $\mathcal{G}^n$ ,  $|\mathbf{E}_{x \sim \mu}[\chi(x)]| \leq \varepsilon$ .

A multiset  $S$  consisting of elements from  $\mathcal{G}^n$  is said to be an  $\varepsilon$ -biased space in  $\mathcal{G}^n$  if the uniform distribution over  $S$  is an  $\varepsilon$ -biased distribution.

It can be checked that a multiset consisting of  $(\frac{n}{\varepsilon})^{O(1)}$  independent, uniformly random elements from  $\mathcal{G}^n$  form an  $\varepsilon$ -biased space with high probability. Explicit  $\varepsilon$ -biased spaces were constructed for the group  $\mathbb{F}_2^n$  by Naor and Naor in [NN93]; further constructions were given by Alon et al. in [AGHP92]. Explicit constructions of  $\varepsilon$ -biased spaces in  $\mathbb{Z}_d^n$  were given by Azar et al. in [AMN98]. We observe that this last construction yields a construction for all Abelian groups  $\mathcal{G}^n$ , when  $\mathcal{G}$  is of constant size. We first state the result of [AMN98] in a form that we will find suitable.

**Theorem 4.1.** *For any fixed  $d$ , there is an  $\text{NC}^2$  algorithm that does the following. On input  $n$  and  $\varepsilon > 0$  (both in unary), the algorithm produces a symmetric multiset  $S \subseteq \mathbb{Z}_d^n$  of size  $O((\frac{n}{\varepsilon})^2)$  such that  $S$  is an  $\varepsilon$ -biased space in  $\mathbb{Z}_d^n$ .*

*Proof.* It is easy to see that the  $\varepsilon$ -biased space construction in [AMN98] can be implemented in deterministic logspace (and hence in  $\text{NC}^2$ ). If the space  $S$  obtained is not symmetric, we can consider the multiset that is the disjoint union of  $S$  and  $S^{-1}$ , which is also easily seen to be  $\varepsilon$ -biased. ■

**Remark 4.2.** We note that the definition of small bias spaces in [AMN98] differs somewhat from our own definition above. But it is easy to see that an  $\varepsilon$ -bias space in  $\mathbb{Z}_d^n$  in the sense of [AMN98] is a  $(d\varepsilon)$ -bias space according to our definition above.

**Remark 4.3.** In a recent paper, Meka and Zuckerman [MZ09] observe, as we do below, that the construction of [AMN98] gives small bias spaces for any arbitrary Abelian group  $\mathcal{G}$ . Nevertheless, we present our own proof of this fact, since the small bias spaces that follow from our proof are of *smaller* size. Specifically, our proof shows how to explicitly construct sample spaces of size  $O(\frac{n^2}{\varepsilon^2})$ , whereas the relevant result in [MZ09] only produces small bias spaces of size  $O((\frac{n}{\varepsilon})^b)$ , where  $b$  is some constant that depends on  $\mathcal{G}$  (and can be as large as  $\Omega(\log |\mathcal{G}|)$ ).

**Lemma 4.4.** *For any fixed group  $\mathcal{G}$ , there is an  $\text{NC}^2$  algorithm which, on input  $n$  and  $\varepsilon > 0$  in unary, produces a symmetric multiset  $S \subseteq \mathcal{G}^n$  of size  $O((\frac{n}{\varepsilon})^2)$  such that  $S$  is an  $\varepsilon$ -biased space in  $\mathcal{G}^n$ .*

*Proof.* By the Fundamental Theorem of finite Abelian groups,  $\mathcal{G} \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ , for positive integers  $d_1, d_2, \dots, d_k$  such that  $d_1 \mid d_2 \mid \cdots \mid d_k$ . Let  $\mathcal{G}_0$  denote  $\mathbb{Z}_{d_k}^k$ . Note that for any  $s, t \in \mathbb{N}$ ,  $\mathbb{Z}_s \cong \mathbb{Z}_{st}/\mathbb{Z}_t$ . Hence, we see that that  $\mathcal{G} \cong \mathcal{G}_0/\mathcal{H}$ , where  $\mathcal{H}$  is the subgroup  $\mathbb{Z}_{e_1} \oplus \mathbb{Z}_{e_2} \oplus \cdots \oplus \mathbb{Z}_{e_k}$ , and  $e_i = d_k/d_i$  for each  $i \in [k]$ . Therefore,  $\mathcal{G}^n \cong \mathcal{G}_0^n/\mathcal{H}^n$ . Let  $\pi : \mathcal{G}_0^n \rightarrow \mathcal{G}^n$  be the natural onto homomorphism with kernel  $\mathcal{H}^n$ . Note that  $\pi$  is just the projection map and can easily be computed in  $\text{NC}^2$ .

Since  $\mathcal{G}_0^n \cong \mathbb{Z}_{d_k}^{nk}$ , by Theorem 4.1, there is an  $\text{NC}^2$  algorithm that constructs a symmetric multiset  $S_0 \subseteq \mathcal{G}_0^n$  of size  $O((\frac{kn}{\varepsilon})^2)$  such that  $S_0$  is an  $\varepsilon$ -biased space in  $\mathcal{G}_0^n$ . We claim that the multiset  $S = \pi(S_0)$  is a symmetric  $\varepsilon$ -biased space in  $\mathcal{G}^n$ . To see this, consider any non-trivial character  $\chi$  of  $\mathcal{G}^n$ ; note that  $\chi_0 = \chi \circ \pi$  is a non-trivial character of  $\mathcal{G}_0^n$ . We have

$$\left| \mathbf{E}_{x \sim S} [\chi(x)] \right| = \left| \mathbf{E}_{x_0 \sim S_0} [\chi(\pi(x_0))] \right| = \left| \mathbf{E}_{x_0 \sim S_0} [\chi_0(x_0)] \right| \leq \varepsilon$$

where the first equality follows from the definition of  $S$ , and the last inequality follows from the fact that  $S_0$  is an  $\varepsilon$ -biased space in  $\mathcal{G}_0^n$ . Since  $\chi$  was an arbitrary non-trivial character of  $\mathcal{G}^n$ , we have proved that  $S$  is indeed an  $\varepsilon$ -biased space in  $\mathcal{G}^n$ . It is easy to see that  $S$  is symmetric. Finally, note that  $S$  can be computed in  $\text{NC}^2$ . This completes the proof.  $\blacksquare$

Finally, we mention a well-known connection between small bias spaces in  $\mathcal{G}^n$  and Cayley graphs over  $\mathcal{G}^n$  (e.g. see Alon and Roichman [AR94]).

**Lemma 4.5.** *Given any symmetric multiset  $S \subseteq \mathcal{G}^n$ , the Cayley graph  $C(\mathcal{G}^n, S)$  is an  $(|\mathcal{G}^n|, |S|, \alpha)$ -graph iff  $S$  is an  $\alpha$ -biased space.*

Lemmas 4.5 and 4.4 have the following easy consequence:

**Lemma 4.6.** *For any Abelian group  $\mathcal{G}$ , there is an  $\text{NC}^2$  algorithm which, on unary inputs  $n$  and  $\alpha > 0$ , produces a symmetric multiset  $S \subseteq \mathcal{G}^n$  of size  $O((\frac{n}{\alpha})^2)$  such that  $C(\mathcal{G}^n, S)$  is a  $(|\mathcal{G}^n|, |S|, \alpha)$ -graph.*

Putting the above statement together with the results of Section 3, we have the following.

**Theorem 4.7.** *For any constant  $c > 0$ , the RPP over  $\mathcal{G}$  has an  $\text{NC}^2$   $(n/2, c \log n)$ -algorithm.*

*Proof.* Let  $\mathcal{H}$  denote the input subgroup. By Lemma 3.3, there is a logspace (and hence  $\text{NC}^2$ ) algorithm that computes a collection of  $m = n^{O(c)}$  many subgroups  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$  such that  $B(\mathcal{H}, c \log n) \subseteq \bigcup_{i=1}^m \mathcal{H}_i$  and  $\delta(\mathcal{H}_i) \leq 2n/3$  for each  $i \in [m]$ . Now, fix any multiset  $S \subseteq \mathcal{G}^n$  such that the Cayley graph  $C(\mathcal{G}^n, S)$  is a  $(|\mathcal{G}^n|, |S|, \alpha)$ -graph, where  $\alpha = \frac{1}{2n^{20c}}$ ; by Lemma 4.6, such an  $S$  can be constructed in  $\text{NC}^2$ . It follows from Lemma 3.4 that there is some  $s \in S$  such that  $s \notin \bigcup_{i=1}^m \mathcal{H}_i$ . Finally, by Lemma 2.3, there is an  $\text{NC}^2$  algorithm to test if each  $s \in S$  belongs to  $\mathcal{H}_i$ , for any  $i \in [m]$ . Hence, we can find out (in parallel) exactly which  $s \in S$  do not belong to any of the  $\mathcal{H}_i$  and output one of them. The output element  $s$  is surely  $c \log n$ -far from  $\mathcal{H}$ .  $\blacksquare$

Let  $\mathcal{G}$  be Abelian. We observe that a method of [APY09], coupled with Theorem 4.7, yields an efficient  $(k, \frac{cn \log k}{k})$ -algorithm for any constant  $c > 0$ , and  $k \leq n/2$ .

**Theorem 4.8.** *Let  $c > 0$  be any constant. If  $\mathcal{G}$  is an Abelian group, then the RPP over  $\mathcal{G}$  has an  $\text{NC}^2$   $(k, \frac{cn \log k}{k})$ -algorithm for any  $k \leq n/2$ .*

*Proof.* Given as input a subgroup  $\mathcal{H}$  such that  $\delta(\mathcal{H}) = k \leq n/2$ , the algorithm partitions  $[n]$  as  $[n] = \bigcup_{i=1}^m T_i$ , where  $2k \leq |T_i| < 4k$  for each  $i$ ; note that  $m \geq n/4k$ . Let  $\mathcal{H}_i$  denote the subgroup obtained when  $\mathcal{H}$  is projected onto the coordinates in  $T_i$ . Since  $\delta(\mathcal{H}_i) \leq k \leq |T_i|/2$ , we can, by Theorem 4.7, efficiently find a point  $x_i \in \mathcal{G}^{|T_i|}$  that is at least  $4c \log k$ -far from  $\mathcal{H}_i$ . Putting these  $x_i$  together in the natural way, we obtain an  $x \in \mathcal{G}^n$  that is  $\frac{cn \log k}{k}$ -far from the subgroup  $\mathcal{H}$ .

Since  $\mathcal{G}$  is Abelian, using the algorithm of Theorem 4.7, the  $x_i$  can all be computed in parallel in  $\text{NC}^2$ . Hence, the entire procedure can be performed in  $\text{NC}^2$ .  $\blacksquare$

## 5. RPP over General Groups

Let  $\mathcal{G}$  denote some fixed finite group. We can generalize the polynomial-time algorithm of [APY09], described for  $\mathbb{F}_2$ , to compute a point  $x \in \mathcal{G}^n$  that is  $c \log n$ -far from a given input subgroup  $\mathcal{H}$  such that  $\delta(\mathcal{H}) \leq n/2$ . We only state this result below and refer the interested reader to the full version [AS09b] for details.

**Theorem 5.1.** *For any constant  $c > 0$ , the RPP over  $\mathcal{G}$  has a polynomial time  $(n/2, c \log n)$ -algorithm.*

Analogous to Theorem 4.8, we have the following solution to RPP for general groups.

**Theorem 5.2.** *Let  $c > 0$  be any constant. For any  $\mathcal{G}$ , the RPP over  $\mathcal{G}$  has a polynomial time  $(k, \frac{cn \log k}{k})$ -algorithm for any  $k \leq n/2$ .*

*Proof.* The construction is exactly the same as in the proof of Theorem 4.8. The only difference is that we will apply the algorithm of Theorem 5.1. In this case, the  $x_i$  can all be found in deterministic polynomial time. Hence, the entire procedure gives us a polynomial-time algorithm.  $\blacksquare$

## 6. Limitations of expanding sets

In the previous sections, we have shown how generators for expanding Cayley graphs on  $\mathcal{G}^n$ , where  $\mathcal{G}$  is a fixed finite group, can help solve the RPP over  $\mathcal{G}$ . In particular, we have the following easy consequence of Lemmas 3.3 and 3.4.

**Corollary 6.1.** *For any constant  $c > 0$ , large enough  $n$ , and any symmetric multiset  $S \subseteq \mathcal{G}^n$  such that  $\lambda(C(\mathcal{G}^n, S)) < \frac{1}{n^{20c}}$ , the following holds. If  $\mathcal{H}$  is any subgroup of  $\mathcal{G}^n$  such that  $\delta(\mathcal{H}) \leq n/2$ , there is some  $s \in S$  such that  $s \notin B(\mathcal{H}, c \log n)$ .*

It makes sense to ask if the parameters in Corollary 6.1 are far from optimal. Is it true that any polynomial-sized symmetric multiset  $S \subseteq \mathcal{G}^n$  with good enough expansion properties is  $\omega(\log n)$ -far from every subgroup of dimension at most  $n/2$ ? We can show that this is not true. Formally, we can prove:

**Theorem 6.2.** *For any constant  $c > 0$  and large enough  $n$ , there is a symmetric multiset  $S \subseteq \mathbb{F}_2^n$  such that  $\lambda(C(\mathbb{F}_2^n, S)) \leq \frac{1}{n^c}$  but there is a subspace  $L$  of dimension  $n/2$  such that  $S \subseteq B(L, 20c \log n)$ .*

It is well known that for any family of  $d$ -regular multigraphs  $G$   $\lambda(G) = \Omega(1/\sqrt{d})$  (see e.g. [HLW06, Theorem 5.3]). As a consequence of this lower bound it follows for any fixed group  $\mathcal{G}$  and any multiset  $S \subseteq \mathcal{G}^n$  that  $\lambda(C(\mathcal{G}, S)) = \Omega(1/\sqrt{|S|})$ . Hence, the above theorem tells us that just the expansion properties of  $C(\mathbb{F}_2^n, S)$  for any poly( $n$ )-sized  $S$  are not sufficient to guarantee  $\omega(\log n)$ -distance from every subspace of dimension  $n/2$ . The proof of the above statement can be found in the full version [AS09b].

## 7. Discussion

For the remote point problem over an Abelian group  $\mathcal{G}$ , we have shown how expanding generating sets for Cayley graphs of  $\mathcal{G}^n$  can be used to obtain deterministic NC<sup>2</sup> algorithms. A natural question is whether we can obtain a similar algorithm for non-Abelian  $\mathcal{G}$ . Note that Lemma 3.4 holds in the non-Abelian setting too. Hence, in order to obtain an NC<sup>2</sup>-algorithm for the RPP over arbitrary non-Abelian  $\mathcal{G}$  along the lines of our algorithm for Abelian groups, we need to be able to check (in NC<sup>2</sup>) for membership in  $\mathcal{G}^n$ , and we need to be able to construct small multisets  $S$  of  $\mathcal{G}^n$  such that  $C(\mathcal{G}^n, S)$  has sufficiently good expansion properties. Luks' work [Lu86] yields an NC<sup>4</sup> test for membership in  $\mathcal{G}^n$  for arbitrary  $\mathcal{G}$ . Building on that, there is also an NC<sup>2</sup> membership test for  $\mathcal{G}^n$  [AKV05]. However, we are unable to compute a (good enough) expanding generator set for the group  $\mathcal{G}^n$  in deterministic NC or even in deterministic polynomial time.

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