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THE REMOTE POINT PROBLEM, SMALL BIAS SPACES, AND EXPANDING GENERATOR SETS

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ABSTRACT. Using ε -bias spaces over \mathbb{F}_2 , we show that the Remote Point Problem (RPP), introduced by Alon et al [APY09], has an NC^2 algorithm (achieving the same parameters as [APY09]). We study a generalization of the Remote Point Problem to groups: we replace \mathbb{F}_2^n by \mathcal{G}^n for an arbitrary fixed group \mathcal{G} . When \mathcal{G} is Abelian we give an NC^2 algorithm for RPP, again using ε -bias spaces. For nonabelian \mathcal{G} , we give a deterministic polynomial-time algorithm for RPP. We also show the connection to construction of expanding generator sets for the group \mathcal{G}^n . All our algorithms for the RPP achieve essentially the same parameters as [APY09].

1. Introduction

Valiant, in his celebrated work [V77] on circuit lower bounds for computing linear transformations $A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ for a field \mathbb{F} , initiated the study of rigid matrices. If explicit rigid matrices of certain parameters can be constructed it would result in superlinear lower bounds for logarithmic depth linear circuits over \mathbb{F} . This problem and the construction of such rigid matrices has remained elusive for over three decades.

Alon, Panigrahy and Yekhanin [APY09] recently proposed a problem that appears to be of intermediate difficulty. Given a subspace L of \mathbb{F}_2^n by its basis and a number $r \in [n]$ as input, the problem is to compute in deterministic polynomial time a point $v \in \mathbb{F}_2^n$ such that $\Delta(u, v) \geq r$ for all $u \in L$, where $\Delta(u, v)$ is the Hamming distance. They call this the *Remote Point Problem*. The point v is said to be r -far from the subspace L .

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Alon et al [APY09] give a nice polynomial time-bounded (in n) algorithm for computing a $v \in \mathbb{F}_2^n$ that is $c \log n$ -far from a given subspace L of dimension $n/2$ and c is a fixed constant. For L such that $\dim(L) = k < n/2$ they give a polynomial-time algorithm for computing a point $v \in \mathbb{F}_2^n$ that is $\frac{cn \log k}{k}$ -far from L .

Results of this paper. In [AS09a] we recently investigated the problem of proving circuit lower bounds in the presence of help functions. Specifically, one of the problems we consider is proving lower bounds for constant-depth Boolean circuits which can take a given set of (arbitrary) help functions $\{h_1, h_2, \dots, h_m\}$ at the input level, where $h_i : \{0, 1\}^n \rightarrow \{0, 1\}$ for each i . Proving explicit lower bounds for this model would allow us to separate EXP from the polynomial-time many-one closure of nonuniform AC^0 . We show that it suffices to find a polynomial-time solution to the Remote Point Problem for parameters $k = 2^{(\log \log n)^c}$ and $r = \frac{n}{2^{(\log \log n)^d}}$ for all constants c and d . Unfortunately, the parameters of the Alon et al algorithm are inadequate for our application.

However, motivated by this connection, in the present paper we carry out a more detailed study of the Remote Point Problem as an algorithmic question. We briefly summarize our results.

1. The first question we address is whether we can give a deterministic parallel (i.e. NC) algorithm for the problem — Alon et al’s algorithm is inherently sequential as it is based on the method of conditional probabilities and pessimistic estimators.

It turns out an element of an ε -bias space for suitably chosen ε is a solution to the Remote Point Problem which gives us an NC algorithm quite easily.

2. Since the RPP for \mathbb{F}_2^n can be solved using small bias spaces, it naturally leads us to address the problem in a more general group-theoretic setting.

In the generalization we study we will replace \mathbb{F}_2 with an arbitrary fixed finite group \mathcal{G} such that $|\mathcal{G}| \geq 2$. Hence we will have the n -fold product group \mathcal{G}^n instead of the vector space \mathbb{F}_2^n .

Given elements $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$ of \mathcal{G}^n , let $\Delta(x, y) = |\{i \mid x_i \neq y_i\}|$. I.e. $\Delta(x, y)$ is the *Hamming distance* between x and y . Furthermore, for $S \subseteq \mathcal{G}^n$, let $\Delta(x, S)$ denote $\min_{y \in S} \Delta(x, y)$.

We now define the *Remote Point Problem (RPP) over a finite group \mathcal{G}* . The input is a subgroup \mathcal{H} of \mathcal{G}^n , where \mathcal{H} is given by a generating set, and a number $r \in [n]$. The problem is to compute in deterministic polynomial (in n) time an element $x \in \mathcal{G}^n$ such that $\Delta(x, \mathcal{H}) > r$. The results we show in this general setting are the following.

- (a) The Remote Point Problem over any *Abelian group* \mathcal{G} has an NC^2 algorithm for $r = O(\frac{n \log k}{k})$ and $k \leq n/2$, where $k = \log_{|\mathcal{G}|} |\mathcal{H}|$.
- (b) Over an arbitrary group \mathcal{G} the Remote point problem has a polynomial-time algorithm for $r = O(\frac{n \log k}{k})$ and $k \leq n/2$, where $k = \log_{|\mathcal{G}|} |\mathcal{H}|$.

The parallel algorithm stated in part(a) above is based on ε -bias space constructions for finite Abelian groups described in Azar et al [AMN98]. The sequential algorithm stated in part(b) above is a group-theoretic generalization of the Alon et al algorithm for \mathbb{F}_2^n [APY09].

Due to lack of space, some proofs have been omitted. They may be found in the full version which has been published as an ECCC report [AS09b].

2. Preliminaries

Fix a finite group \mathcal{G} such that $|\mathcal{G}| \geq 2$. Given any $x \in \mathcal{G}^n$, let $wt(x)$ denote the number of coordinates i such that $x_i \neq 1$, where 1 is the identity of the group \mathcal{G} . By $B(r)$, we will refer to the set of $x \in \mathcal{G}^n$ such that $wt(x) \leq r$. Given a subset S of \mathcal{G}^n , $B(S, r)$ will denote the set $S \cdot B(r) = \{sx \mid s \in S, x \in B(r)\}$. Clearly, for any $S \subseteq \mathcal{G}^n$ and any $x \in \mathcal{G}^n$, $x \in B(S, r)$ if and only if $\Delta(x, S) \leq r$. We say that x is r -close to S if $x \in B(S, r)$ and r -far from S if $x \notin B(S, r)$.

The *Remote Point Problem (RPP)* over \mathcal{G} is defined to be the following algorithmic problem:

INPUT: A subgroup \mathcal{H} of \mathcal{G}^n (given by its generators) and an $r \in \mathbb{N}$.
 OUTPUT: An $x \in \mathcal{G}^n$ such that $x \notin B(\mathcal{H}, r)$.

Clearly, there are inputs to the above problem where no solution can be found. But the input instances of the kind that we will study will clearly have a solution (in fact, a random point of \mathcal{G}^n will be a solution with high probability).

Given a subgroup \mathcal{H} of \mathcal{G}^n , denote by $\delta(\mathcal{H})$ the quantity $\log_{|\mathcal{G}|} |\mathcal{H}|$. We will call $\delta(\mathcal{H})$ the *dimension of \mathcal{H} in \mathcal{G}^n* .

We say that the RPP over \mathcal{G} has a $(k(n), r(n))$ -algorithm if there is an efficient algorithm that solves the Remote Point Problem when given as input a subgroup \mathcal{H} of \mathcal{G}^n of dimension at most $k(n)$ and an r that is bounded by $r(n)$. (Here, ‘efficient’ can correspond to polynomial time or some smaller complexity class.)

A simple counting argument shows that there is a valid solution to the RPP over \mathcal{G} on inputs (\mathcal{H}, r) where $\delta(\mathcal{H}) + r \leq n(1 - \frac{H(r/n)}{\log |\mathcal{G}|} - \varepsilon)$, for any fixed $\varepsilon > 0$ (where $H(\cdot)$ denotes the binary entropy function). However, the best known deterministic solution to the RPP – from [APY09] – is a polynomial time $(k, \frac{cn \log k}{k})$ -algorithm which works over \mathbb{F}_2^n (i.e, the group \mathcal{G} involved is the additive group of the field \mathbb{F}_2).

2.1. Some Group-Theoretic Algorithms

We introduce basic definitions and review some group-theoretic algorithms. Let $\text{Sym}(\Omega)$ denote the group of all permutations on a finite set Ω of size m . In this section we use G, H etc. to denote *permutation groups on Ω* , which are simply subgroups of $\text{Sym}(\Omega)$.

Let G be a subgroup of $\text{Sym}(\Omega)$. For a subset $\Delta \subseteq \Omega$ denote by $G_{\{\Delta\}}$ the *point-wise stabilizer* of Δ . I.e $G_{\{\Delta\}}$ is the subgroup consisting of exactly those elements of G that fix each element of Δ .

Theorem 2.1 (Schreier-Sims). [Lu93]

- (1) *If a subgroup G of $\text{Sym}(\Omega)$ is given by a generating set as input along with the subset Δ there is a polynomial-time (sequential) algorithm for computing a generator set for $G_{\{\Delta\}}$.*
- (2) *If a subgroup G of $\text{Sym}(\Omega)$ is given by a generating set as input, then there is a polynomial time algorithm for computing $|G|$.*
- (3) *Given as input a permutation $\sigma \in \text{Sym}(\Omega)$ and a generator set for a subgroup G of $\text{Sym}(\Omega)$, we can test in deterministic polynomial time if σ is an element of G .*

We are also interested in a special case of this problem which we now define. A subset $\Gamma \subseteq \Omega$ is an *orbit* of G if $\Gamma = \{\sigma(i) \mid \sigma \in G\}$ for some $i \in \Omega$. Any subgroup G of $\text{Sym}(\Omega)$ partitions Ω into orbits (called G -orbits).

For a constant $b > 0$, a subgroup G of $\text{Sym}(\Omega)$ is defined to be a *b -bounded permutation group* if every G -orbit is of size at most b .

In [MC87], McKenzie and Cook studied the parallel complexity of *Abelian* permutation group problems. Specifically, they gave an NC^3 algorithm for testing membership in an Abelian permutation group given by a generator set and for computing the order of an Abelian permutation group. When restricted to b -bounded Abelian permutation groups, the algorithms of [MC87] for these problems are actually NC^2 algorithms. We formally state their result and derive a consequence.

Theorem 2.2 ([MC87]). *There is an NC^2 algorithm for membership testing in a b -bounded Abelian permutation group G given by a generator set.*

We now consider problems over \mathcal{G}^n , for a fixed finite group \mathcal{G} . We know from basic group theory that every group \mathcal{G} is a permutation group acting on itself. I.e. every \mathcal{G} can be seen as a subgroup of $\text{Sym}(\mathcal{G})$, where \mathcal{G} acts on itself by left (or right) multiplication. Therefore, \mathcal{G}^n can be easily seen as a permutation group on the set $\Omega = \mathcal{G} \times [n]$ and hence, \mathcal{G}^n can be considered a subgroup of $\text{Sym}(\Omega)$. Furthermore, notice that each subset $\mathcal{G} \times \{i\}$ is an orbit of this group \mathcal{G}^n . Hence, \mathcal{G}^n is a b -bounded permutation group contained in $\text{Sym}(\Omega)$, where $b = |\mathcal{G}|$. Finally, if \mathcal{G} is an Abelian group, then so is this subgroup of $\text{Sym}(\Omega)$. We have the following lemma as an easy consequence of Theorem 2.2.

Lemma 2.3. *Let \mathcal{G} be Abelian. There is an NC^2 algorithm that takes as input a generator set for some subgroup \mathcal{H} of \mathcal{G}^n and an $x \in \mathcal{G}^n$, and accepts iff $x \in \mathcal{H}$.*

Given any $y = (y_1, y_2, \dots, y_i) \in \mathcal{G}^i$ with $1 \leq i \leq n$ and any $S \subseteq \mathcal{G}^n$, let S_y denote the set $\{x \in S \mid x_j = y_j \text{ for } 1 \leq j \leq i\}$.

Lemma 2.4. *Let \mathcal{G} be any fixed finite group. There is a polynomial time algorithm that takes as input a subgroup \mathcal{H} of \mathcal{G}^n , where \mathcal{H} is given by generators, and a $y \in \mathcal{G}^i$ with $1 \leq i \leq n$, and computes $|\mathcal{H}_y|$.*

Proof. Let $\mathcal{K} = \{(x_1, x_2, \dots, x_n) \in \mathcal{H} \mid x_1 = x_2 = \dots = x_n = 1\}$, where 1 denotes the identity element of \mathcal{G} . Clearly, \mathcal{K} is a subgroup of \mathcal{H} . The set \mathcal{H}_y , if nonempty, is simply a coset of \mathcal{K} and thus, we have $|\mathcal{H}_y| = |\mathcal{K}|$. To check if \mathcal{H}_y is nonempty, we consider the map $\pi_i : \mathcal{G}^n \rightarrow \mathcal{G}^i$ that projects its input onto its first i coordinates; note that \mathcal{H}_y is nonempty iff the subgroup $\pi_i(\mathcal{H})$ contains y , which can be checked in polynomial time by point (3) of Theorem 2.1 (here, we are identifying \mathcal{G}^n with a subgroup of $\text{Sym}(\mathcal{G} \times [n])$ as above). If $y \notin \pi_i(\mathcal{H})$, the algorithm outputs 0. Otherwise, we have $|\mathcal{H}_y| = |\mathcal{K}|$ and it suffices to compute $|\mathcal{K}|$. But \mathcal{K} is simply the point-wise stabilizer of the set $\mathcal{G} \times [i]$ in \mathcal{H} , and hence $|\mathcal{K}|$ can be computed in polynomial time by points (1) and (2) of Theorem 2.1. ■

3. Expanding Cayley Graphs and the Remote Point Problem

Fix a group \mathcal{G} such that $|\mathcal{G}| \geq 2$, and consider an instance of the RPP over \mathcal{G} . The main idea that we develop in this section is that if we have a (symmetric) expanding generator set S for the group \mathcal{G}^n with appropriate expansion parameters then for a subgroup \mathcal{H} of \mathcal{G}^n such that $\delta(\mathcal{H}) \leq k$ some element of S will be r -far from H , for suitable k and r .

We review some definitions related to expander graphs (e.g. see the survey of Hoory, Linial, and Wigderson [HLW06]). An undirected multigraph $G = (V, E)$ is an (n, d, α) -graph for $n, d \in \mathbb{N}$ and $\alpha > 0$ if $|V| = n$, the degree of each vertex is d , and the second largest value $\lambda(G)$ from among the absolute values of eigenvalues of $A(G)$ – the adjacency matrix of the graph G – is bounded by αd .

A *random walk* of length $t \in \mathbb{N}$ on an (n, d, α) -graph $G = (V, E)$ is the output of the following random process: a vertex $v_0 \in V$ of picked uniformly at random, and for $0 \leq i < t$, if v_i has been picked, then v_{i+1} is obtained by selecting a neighbour v_{i+1} uniformly at random (i.e a random edge out of v_i is picked, and v_{i+1} is chosen to be the other endpoint of the edge); the output of the process is (v_0, v_1, \dots, v_t) . We now state an important result regarding random walks on expanders (see [HLW06, Theorem 3.6] for details).

Lemma 3.1. *Let $G = (V, E)$ be an (n, d, α) -graph and $B \subseteq V$ with $|B| \leq \beta n$. Then, the probability that a random walk (v_0, v_1, \dots, v_t) is entirely contained inside B (i.e, $v_i \in B$ for each i) is bounded by $(\beta + \alpha)^t$.*

Let \mathcal{H} be a group and S a *symmetric* multiset of elements from \mathcal{H} . I.e. there is a bijection of multisets $\varphi : S \rightarrow S$ such that $\varphi(s) = s^{-1}$ for each $s \in S$. We define the Cayley graph $C(\mathcal{H}, S)$ to be the (multi)graph G with vertex set \mathcal{H} and edges of the form (x, xs) for each $x \in \mathcal{H}$ and each $s \in S$; since S is symmetric, we consider $C(\mathcal{H}, S)$ to be an undirected graph by identifying the edges (x, xs) and $(xs, (xs)\varphi(s))$, for each x and s .

We now show a lemma that will help relate generators of expanding Cayley graphs on \mathcal{G}^n and the RPP over \mathcal{G} . In what follows, let S be a symmetric multiset of elements from \mathcal{G}^n ; let G denote the Cayley graph $C(\mathcal{G}^n, S)$; and let N, D denote $|\mathcal{G}|^n$ and $|S|$ (counted with repetitions) respectively.

Lemma 3.2. *Assume S as above is such that G is an (N, D, α) -graph, where $\alpha \leq \frac{1}{n^d}$, for some fixed $d > 0$. Then, given any subgroup \mathcal{H} of \mathcal{G}^n such that $\delta(\mathcal{H}) \leq 2n/3$, we have $\frac{|S \cap \mathcal{H}|}{|S|} \leq \frac{1}{n^{d/2}}$ for large enough n (where the elements of $S \cap \mathcal{H}$ are counted with repetitions).*

Proof. Let $S' = S \cap \mathcal{H}$ and let $\eta = |S'|/|S|$. We want an upper bound on η . Consider a random walk (x_0, x_1, \dots, x_t) of length t on the graph G (the exact value of t will be fixed later). Let \mathcal{B} denote the following event: there is a $y \in \mathcal{G}^n$ such that all the vertices x_0, x_1, \dots, x_t are all contained in the coset $y\mathcal{H}$ of \mathcal{H} . Let p denote the probability that \mathcal{B} occurs.

We will first lower bound p . At each step of the random walk, a random $s_i \in S$ is chosen and x_{i+1} is set to $x_i s_i$. If these s_i all happen to belong to S' , then the cosets $x_i \mathcal{H}$ and $x_{i+1} \mathcal{H}$ are the same for all i and hence, the event \mathcal{B} does occur. Hence, $p \geq \eta^t$.

We now upper bound p . Fix any coset $y\mathcal{H}$ of the subgroup \mathcal{H} . Since the dimension of \mathcal{H} in \mathcal{G}^n is bounded by $2n/3$, we have $|y\mathcal{H}| = |\mathcal{H}| \leq |\mathcal{G}|^{2n/3} \leq 2^{-n/3} |\mathcal{G}^n|$. That is, the coset $y\mathcal{H}$ is a very small subset of \mathcal{G}^n . Applying Lemma 3.1, we see that the probability that the random walk (x_0, x_1, \dots, x_t) is completely contained inside this coset is bounded by $(2^{-n/3} + n^{-d})^t \leq \frac{2^t}{n^{dt}}$, for large enough n . As the total number of cosets of \mathcal{H} is bounded by $|\mathcal{G}|^n$, an application of the union bound tells us that p is upper bounded by $|\mathcal{G}|^n \frac{2^t}{n^{dt}} \leq \frac{|\mathcal{G}|^{n+t}}{n^{dt}}$. Setting $t = \frac{2n}{d \log_{|\mathcal{G}|} n - 2}$ we see that p is at most $\frac{1}{n^{dt/2}}$.

Putting the upper and lower bounds together, we see that $\eta^t \leq \frac{1}{n^{dt/2}}$ and hence, $\eta \leq \frac{1}{n^{d/2}}$. This completes the proof. \blacksquare

We follow the structure of the algorithm for the RPP over \mathbb{F}_2 in [APY09]. We first describe their $(n/2, c \log n)$ -algorithm for the RPP, followed by our own algorithm. We then describe how they extend this algorithm to a $(k, \frac{cn \log k}{k})$ -algorithm for any $k \leq n/2$; the same procedure works for our algorithm also.

The $(n/2, c \log n)$ -algorithm proceeds as follows. On an input instance consisting of a subgroup V (which is a subspace of \mathbb{F}_2^n) of dimension at most $n/2$ and an $r \leq c \log n$,

- (1) The algorithm first computes a collection of $m = n^{O(c)}$ subspaces V_1, V_2, \dots, V_m , each of dimension at most $2n/3$ such that $B(V, c \log n) \subseteq \bigcup_{i=1}^m V_i$.
- (2) The algorithm then finds an $x \in \mathbb{F}_2^n$ such that $x \notin \bigcup_i V_i$. (This is done using a method similar to the method of pessimistic estimators introduced by Raghavan [Rag88].)

Our algorithm will proceed exactly as the above algorithm in the first step. The second step of our algorithm will be different (assuming that the group \mathcal{G} is Abelian). We first state Step 1 of the algorithm of [APY09] in greater generality:

Lemma 3.3. *Let \mathcal{G} be any fixed finite group with $|\mathcal{G}| \geq 2$. For any constant $c > 0$ and large enough n , the following holds. Given any subgroup \mathcal{H} of \mathcal{G}^n such that $\delta(\mathcal{H}) \leq \frac{n}{2}$, there is a collection of $m \leq n^{10c}$ subgroups $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$ such that $B(\mathcal{H}, c \log n) \subseteq \bigcup_{i=1}^m \mathcal{H}_i$, and*

$\delta(\mathcal{H}_i) \leq 2n/3$ for each i . Moreover, there is a logspace algorithm that, when given as input \mathcal{H} as a set of generators, produces generators for the subgroups $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$.

Proof. The proof follows exactly as in [APY09]. We reproduce it here for completeness and to analyze the complexity of the procedure.

Let 1 denote the identity element of \mathcal{G} . For each $S \subseteq [n]$, let $\mathcal{G}^n(S)$ denote the subgroup of \mathcal{G}^n consisting of those x such that $x_i = 1$ for each $i \notin S$. Note that $\delta(\mathcal{G}^n(S)) = |S|$. Also note that for each $S \subseteq [n]$, the group $\mathcal{G}^n(S)$ is a normal subgroup; in particular, this implies that the set $\mathcal{K} \cdot \mathcal{G}^n(S)$ is a subgroup of \mathcal{G}^n whenever \mathcal{K} is a subgroup of \mathcal{G}^n .

Partition the set $[n]$ into $\ell \leq 10c \log n$ sets of size at most $\lceil \frac{n}{10c \log n} \rceil$ each – we will call these sets S_1, S_2, \dots, S_ℓ . For each $A \subseteq [\ell]$ of size $\lceil c \log n \rceil$, let \mathcal{K}_A denote the subgroup $\mathcal{G}^n(\bigcup_{i \in A} S_i)$. Note that the number of such subgroups is at most $2^\ell \leq n^{10c}$. Also, for each A as above, $\delta(\mathcal{K}_A) = |\bigcup_{i \in A} S_i| \leq \left(\frac{n}{10c \log n} + 1 \right) (c \log n + 1) < \frac{n}{9}$, for large enough n .

Consider any $x \in B(c \log n)$ (i.e, an element x of \mathcal{G}^n s.t $wt(x) \leq c \log n$). We know that $x \in \mathcal{G}^n(S)$ for some S of size at most $c \log n$. Hence, it can be seen that $x \in \mathcal{G}^n(\bigcup_{i \in A} S_i)$ for some A of size $\lceil c \log n \rceil$; this shows that $B(c \log n) \subseteq \bigcup_A \mathcal{K}_A$. Therefore, we see that $B(\mathcal{H}, c \log n) = \mathcal{H}B(c \log n) \subseteq \bigcup_A \mathcal{H}\mathcal{K}_A$.

For each $A \subseteq [\ell]$ of size $\lceil c \log n \rceil$, let \mathcal{H}_A denote the subgroup $\mathcal{H}\mathcal{K}_A$ (note that this is indeed a subgroup, since \mathcal{K}_A is a normal subgroup). Moreover, the cardinality of this subgroup is bounded by $|\mathcal{H}| \cdot |\mathcal{K}_A| \leq |\mathcal{G}|^{n/2} |\mathcal{G}|^{n/9} < |\mathcal{G}|^{2n/3}$; hence, $\delta(\mathcal{H}_A) \leq 2n/3$. Thus, the collection of subgroups $\{\mathcal{H}_A\}_A$ satisfies all the properties mentioned in the statement of the lemma. That a set of generators for this subgroup can be computed in deterministic logspace – for some suitable choice of S_1, S_2, \dots, S_ℓ – is a routine check from the definition of the subgroups $\{\mathcal{K}_A\}_A$. This completes the proof of the lemma. \blacksquare

Using Lemma 3.3, we are able to efficiently “cover” $B(\mathcal{H}, c \log n)$ for any small subgroup \mathcal{H} of \mathcal{G}^n by a union of small subgroups. Therefore, to find a point that is $c \log n$ -far from \mathcal{H} , it suffices to find a point $x \in \mathcal{G}^n$ not contained in any of the covering subgroups. To do this, we note that if S is a multiset containing elements from \mathcal{G}^n such that $C(\mathcal{G}^n, S)$ is a Cayley graph with good expansion, then S must contain such an element. This is formally stated below.

Lemma 3.4. *For any constant $c > 0$ and large enough $n \in \mathbb{N}$, the following holds. Let S be any multiset of elements of \mathcal{G}^n such that $\lambda(C(\mathcal{G}^n, S)) < \frac{1}{n^{20c}}$. Then, for $m \leq n^{10c}$ and any collection $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$ of subgroups such that $\delta(\mathcal{H}_i) \leq 2n/3$ for each i , there is some $s \in S$ such that $s \notin \bigcup_i \mathcal{H}_i$.*

Proof. The proof follows easily from Lemma 3.2. Given any $i \in [m]$, we know, from Lemma 3.2, that $|S \cap \mathcal{H}_i| < \frac{|S|}{n^{10c}}$ (where the elements of the multisets are counted with repetitions). Hence, $|S \cap \bigcup_i \mathcal{H}_i| \leq \sum_i |S \cap \mathcal{H}_i| < \frac{m|S|}{n^{10c}} \leq |S|$. Therefore, there must be some $s \in S$ such that $s \notin \bigcup_i \mathcal{H}_i$. \blacksquare

Therefore, to find a point x that is $c \log n$ -far from the subspace \mathcal{H} , it suffices to construct an S such that $C(\mathcal{G}^n, S)$ is a sufficiently good expander, find the covering subgroups \mathcal{H}_i ($i \in [m]$), and then to find an $s \in S$ that does not lie in any of the \mathcal{H}_i . We follow the above approach to give an efficient parallel algorithm for the RPP in the case that \mathcal{G} is an Abelian group. For arbitrary groups, we show that the method of [APY09] yields a polynomial time algorithm.

4. Remote Point Problem for Abelian Groups

Fix an Abelian group \mathcal{G} . Recall that a *character* χ of \mathcal{G}^n is a homomorphism from \mathcal{G}^n to \mathbb{C}_1^* , the multiplicative subgroup of the complex numbers of absolute value 1. For $\varepsilon > 0$, a distribution μ over \mathcal{G}^n is said to be ε -biased if, given any non-trivial character χ of \mathcal{G}^n , $|\mathbf{E}_{x \sim \mu}[\chi(x)]| \leq \varepsilon$.

A multiset S consisting of elements from \mathcal{G}^n is said to be an ε -biased space in \mathcal{G}^n if the uniform distribution over S is an ε -biased distribution.

It can be checked that a multiset consisting of $(\frac{n}{\varepsilon})^{O(1)}$ independent, uniformly random elements from \mathcal{G}^n form an ε -biased space with high probability. Explicit ε -biased spaces were constructed for the group \mathbb{F}_2^n by Naor and Naor in [NN93]; further constructions were given by Alon et al. in [AGHP92]. Explicit constructions of ε -biased spaces in \mathbb{Z}_d^n were given by Azar et al. in [AMN98]. We observe that this last construction yields a construction for all Abelian groups \mathcal{G}^n , when \mathcal{G} is of constant size. We first state the result of [AMN98] in a form that we will find suitable.

Theorem 4.1. *For any fixed d , there is an NC^2 algorithm that does the following. On input n and $\varepsilon > 0$ (both in unary), the algorithm produces a symmetric multiset $S \subseteq \mathbb{Z}_d^n$ of size $O((\frac{n}{\varepsilon})^2)$ such that S is an ε -biased space in \mathbb{Z}_d^n .*

Proof. It is easy to see that the ε -biased space construction in [AMN98] can be implemented in deterministic logspace (and hence in NC^2). If the space S obtained is not symmetric, we can consider the multiset that is the disjoint union of S and S^{-1} , which is also easily seen to be ε -biased. ■

Remark 4.2. We note that the definition of small bias spaces in [AMN98] differs somewhat from our own definition above. But it is easy to see that an ε -bias space in \mathbb{Z}_d^n in the sense of [AMN98] is a $(d\varepsilon)$ -bias space according to our definition above.

Remark 4.3. In a recent paper, Meka and Zuckerman [MZ09] observe, as we do below, that the construction of [AMN98] gives small bias spaces for any arbitrary Abelian group \mathcal{G} . Nevertheless, we present our own proof of this fact, since the small bias spaces that follow from our proof are of *smaller* size. Specifically, our proof shows how to explicitly construct sample spaces of size $O(\frac{n^2}{\varepsilon^2})$, whereas the relevant result in [MZ09] only produces small bias spaces of size $O((\frac{n}{\varepsilon})^b)$, where b is some constant that depends on \mathcal{G} (and can be as large as $\Omega(\log |\mathcal{G}|)$).

Lemma 4.4. *For any fixed group \mathcal{G} , there is an NC^2 algorithm which, on input n and $\varepsilon > 0$ in unary, produces a symmetric multiset $S \subseteq \mathcal{G}^n$ of size $O((\frac{n}{\varepsilon})^2)$ such that S is an ε -biased space in \mathcal{G}^n .*

Proof. By the Fundamental Theorem of finite Abelian groups, $\mathcal{G} \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k}$, for positive integers d_1, d_2, \dots, d_k such that $d_1 \mid d_2 \mid \cdots \mid d_k$. Let \mathcal{G}_0 denote $\mathbb{Z}_{d_k}^k$. Note that for any $s, t \in \mathbb{N}$, $\mathbb{Z}_s \cong \mathbb{Z}_{st}/\mathbb{Z}_t$. Hence, we see that $\mathcal{G} \cong \mathcal{G}_0/\mathcal{H}$, where \mathcal{H} is the subgroup $\mathbb{Z}_{e_1} \oplus \mathbb{Z}_{e_2} \oplus \cdots \oplus \mathbb{Z}_{e_k}$, and $e_i = d_k/d_i$ for each $i \in [k]$. Therefore, $\mathcal{G}^n \cong \mathcal{G}_0^n/\mathcal{H}^n$. Let $\pi : \mathcal{G}_0^n \rightarrow \mathcal{G}^n$ be the natural onto homomorphism with kernel \mathcal{H}^n . Note that π is just the projection map and can easily be computed in NC^2 .

Since $\mathcal{G}_0^n \cong \mathbb{Z}_{d_k}^{nk}$, by Theorem 4.1, there is an NC^2 algorithm that constructs a symmetric multiset $S_0 \subseteq \mathcal{G}_0^n$ of size $O((\frac{kn}{\varepsilon})^2)$ such that S_0 is an ε -biased space in \mathcal{G}_0^n . We claim that the multiset $S = \pi(S_0)$ is a symmetric ε -biased space in \mathcal{G}^n . To see this, consider any non-trivial character χ of \mathcal{G}^n ; note that $\chi_0 = \chi \circ \pi$ is a non-trivial character of \mathcal{G}_0^n . We have

$$\left| \mathbf{E}_{x \sim S} [\chi(x)] \right| = \left| \mathbf{E}_{x_0 \sim S_0} [\chi(\pi(x_0))] \right| = \left| \mathbf{E}_{x_0 \sim S_0} [\chi_0(x_0)] \right| \leq \varepsilon$$

where the first equality follows from the definition of S , and the last inequality follows from the fact that S_0 is an ε -biased space in \mathcal{G}_0^n . Since χ was an arbitrary non-trivial character of \mathcal{G}^n , we have proved that S is indeed an ε -biased space in \mathcal{G}^n . It is easy to see that S is symmetric. Finally, note that S can be computed in NC^2 . This completes the proof. \blacksquare

Finally, we mention a well-known connection between small bias spaces in \mathcal{G}^n and Cayley graphs over \mathcal{G}^n (e.g. see Alon and Roichman [AR94]).

Lemma 4.5. *Given any symmetric multiset $S \subseteq \mathcal{G}^n$, the Cayley graph $C(\mathcal{G}^n, S)$ is an $(|\mathcal{G}|^n, |S|, \alpha)$ -graph iff S is an α -biased space.*

Lemmas 4.5 and 4.4 have the following easy consequence:

Lemma 4.6. *For any Abelian group \mathcal{G} , there is an NC^2 algorithm which, on unary inputs n and $\alpha > 0$, produces a symmetric multiset $S \subseteq \mathcal{G}^n$ of size $O((\frac{n}{\alpha})^2)$ such that $C(\mathcal{G}^n, S)$ is a $(|\mathcal{G}|^n, |S|, \alpha)$ -graph.*

Putting the above statement together with the results of Section 3, we have the following.

Theorem 4.7. *For any constant $c > 0$, the RPP over \mathcal{G} has an NC^2 $(n/2, c \log n)$ -algorithm.*

Proof. Let \mathcal{H} denote the input subgroup. By Lemma 3.3, there is a logspace (and hence NC^2) algorithm that computes a collection of $m = n^{O(c)}$ many subgroups $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$ such that $B(\mathcal{H}, c \log n) \subseteq \bigcup_{i=1}^m \mathcal{H}_i$ and $\delta(\mathcal{H}_i) \leq 2n/3$ for each $i \in [m]$. Now, fix any multiset $S \subseteq \mathcal{G}^n$ such that the Cayley graph $C(\mathcal{G}^n, S)$ is a $(|\mathcal{G}|^n, |S|, \alpha)$ -graph, where $\alpha = \frac{1}{2n^{20c}}$; by Lemma 4.6, such an S can be constructed in NC^2 . It follows from Lemma 3.4 that there is some $s \in S$ such that $s \notin \bigcup_{i=1}^m \mathcal{H}_i$. Finally, by Lemma 2.3, there is an NC^2 algorithm to test if each $s \in S$ belongs to \mathcal{H}_i , for any $i \in [m]$. Hence, we can find out (in parallel) exactly which $s \in S$ do not belong to any of the \mathcal{H}_i and output one of them. The output element s is surely $c \log n$ -far from \mathcal{H} . \blacksquare

Let \mathcal{G} be Abelian. We observe that a method of [APY09], coupled with Theorem 4.7, yields an efficient $(k, \frac{cn \log k}{k})$ -algorithm for any constant $c > 0$, and $k \leq n/2$.

Theorem 4.8. *Let $c > 0$ be any constant. If \mathcal{G} is an Abelian group, then the RPP over \mathcal{G} has an NC^2 $(k, \frac{cn \log k}{k})$ -algorithm for any $k \leq n/2$.*

Proof. Given as input a subgroup \mathcal{H} such that $\delta(\mathcal{H}) = k \leq n/2$, the algorithm partitions $[n]$ as $[n] = \bigcup_{i=1}^m T_i$, where $2k \leq |T_i| < 4k$ for each i ; note that $m \geq n/4k$. Let \mathcal{H}_i denote the subgroup obtained when \mathcal{H} is projected onto the coordinates in T_i . Since $\delta(\mathcal{H}_i) \leq k \leq |T_i|/2$, we can, by Theorem 4.7, efficiently find a point $x_i \in \mathcal{G}^{|T_i|}$ that is at least $4c \log k$ -far from \mathcal{H}_i . Putting these x_i together in the natural way, we obtain an $x \in \mathcal{G}^n$ that is $\frac{cn \log k}{k}$ -far from the subgroup \mathcal{H} .

Since \mathcal{G} is Abelian, using the algorithm of Theorem 4.7, the x_i can all be computed in parallel in NC^2 . Hence, the entire procedure can be performed in NC^2 . ■

5. RPP over General Groups

Let \mathcal{G} denote some fixed finite group. We can generalize the polynomial-time algorithm of [APY09], described for \mathbb{F}_2 , to compute a point $x \in \mathcal{G}^n$ that is $c \log n$ -far from a given input subgroup \mathcal{H} such that $\delta(\mathcal{H}) \leq n/2$. We only state this result below and refer the interested reader to the full version [AS09b] for details.

Theorem 5.1. *For any constant $c > 0$, the RPP over \mathcal{G} has a polynomial time $(n/2, c \log n)$ -algorithm.*

Analogous to Theorem 4.8, we have the following solution to RPP for general groups.

Theorem 5.2. *Let $c > 0$ be any constant. For any \mathcal{G} , the RPP over \mathcal{G} has a polynomial time $(k, \frac{cn \log k}{k})$ -algorithm for any $k \leq n/2$.*

Proof. The construction is exactly the same as in the proof of Theorem 4.8. The only difference is that we will apply the algorithm of Theorem 5.1. In this case, the x_i can all be found in deterministic polynomial time. Hence, the entire procedure gives us a polynomial-time algorithm. ■

6. Limitations of expanding sets

In the previous sections, we have shown how generators for expanding Cayley graphs on \mathcal{G}^n , where \mathcal{G} is a fixed finite group, can help solve the RPP over \mathcal{G} . In particular, we have the following easy consequence of Lemmas 3.3 and 3.4.

Corollary 6.1. *For any constant $c > 0$, large enough n , and any symmetric multiset $S \subseteq \mathcal{G}^n$ such that $\lambda(C(\mathcal{G}^n, S)) < \frac{1}{n^{20c}}$, the following holds. If \mathcal{H} is any subgroup of \mathcal{G}^n such that $\delta(\mathcal{H}) \leq n/2$, there is some $s \in S$ such that $s \notin B(\mathcal{H}, c \log n)$.*

It makes sense to ask if the parameters in Corollary 6.1 are far from optimal. Is it true that any polynomial-sized symmetric multiset $S \subseteq \mathcal{G}^n$ with good enough expansion properties is $\omega(\log n)$ -far from every subgroup of dimension at most $n/2$? We can show that this is not true. Formally, we can prove:

Theorem 6.2. *For any constant $c > 0$ and large enough n , there is a symmetric multiset $S \subseteq \mathbb{F}_2^n$ such that $\lambda(C(\mathbb{F}_2^n, S)) \leq \frac{1}{n^c}$ but there is a subspace L of dimension $n/2$ such that $S \subseteq B(L, 20c \log n)$.*

It is well known that for any family of d -regular multigraphs G $\lambda(G) = \Omega(1/\sqrt{d})$ (see e.g. [HLW06, Theorem 5.3]). As a consequence of this lower bound it follows for any fixed group \mathcal{G} and any multiset $S \subseteq \mathcal{G}^n$ that $\lambda(C(\mathcal{G}, S)) = \Omega(1/\sqrt{|S|})$. Hence, the above theorem tells us that just the expansion properties of $C(\mathbb{F}_2^n, S)$ for any poly(n)-sized S are not sufficient to guarantee $\omega(\log n)$ -distance from every subspace of dimension $n/2$. The proof of the above statement can be found in the full version [AS09b].

7. Discussion

For the remote point problem over an Abelian group \mathcal{G} , we have shown how expanding generating sets for Cayley graphs of \mathcal{G}^n can be used to obtain deterministic NC² algorithms. A natural question is whether we can obtain a similar algorithm for non-Abelian \mathcal{G} . Note that Lemma 3.4 holds in the non-Abelian setting too. Hence, in order to obtain an NC²-algorithm for the RPP over arbitrary non-Abelian \mathcal{G} along the lines of our algorithm for Abelian groups, we need to be able to check (in NC²) for membership in \mathcal{G}^n , and we need to be able to construct small multisets S of \mathcal{G}^n such that $C(\mathcal{G}^n, S)$ has sufficiently good expansion properties. Luks' work [Lu86] yields an NC⁴ test for membership in \mathcal{G}^n for arbitrary \mathcal{G} . Building on that, there is also an NC² membership test for \mathcal{G}^n [AKV05]. However, we are unable to compute a (good enough) expanding generator set for the group \mathcal{G}^n in deterministic NC or even in deterministic polynomial time.

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