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Abstract. Tutte proved that every 3-connected graph on more than 4 nodes has a *contractible edge*. Barnette and Grünbaum proved the existence of a *removable edge* in the same setting. We show that the sequence of contractions and the sequence of removals from G to the K_4 can be computed in $O(|V|^2)$ time by extending Barnette and Grünbaum's theorem. As an application, we derive a certificate for the 3-connectedness of graphs that can be easily computed and verified.

1. Introduction

Instead of dealing with contractions or removals in a 3-connected graph $G = (V, E)$ we take the equivalent view of starting with the complete graph on four vertices K_4 and applying their inverse operations until G is constructed. Such a sequence is called a *construction sequence* of G. We will define contractions, removals and their inverse operations in Section 2.

Although existence theorems on contractible and removable edges are used frequently in graph theory [14, 10, 11], we are not aware of any computational results to find the whole construction sequence, except when contractions and removals are allowed to intermix [1]. Moreover, efficient algorithms are unlikely to be derived from the existence proofs as they, e. g., in the case of Barnette and Grünbaum, depend heavily on adding longest paths, which are NP-hard to find. In contrast, we show that it is possible to find a construction sequence for a graph G in time $O(|V|^2)$ for Barnette and Grünbaum's characterization, at the expense of having parallel edges in intermediate graphs. In addition, we show that Barnette and Grünbaum's sequence can be transformed in linear time to Tutte's sequence of contractions and is therefore algorithmically at least as powerful. Both algorithms do not rely on the 3-connectedness test of Hopcroft and Tarjan [6], which runs in linear time but is rather involved.

Blum and Kannan [3] introduced the concept of *certifying algorithms*, which give an easy-to-verify proof of correctness along with their output. While being important for program verification, certifying algorithms provide often new insights into a problem, which

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can lead to new methods. For that reasons they are a major goal for problems on which the fast solutions known are complicated and difficult to implement. Testing a graph on 3-connectedness is such a problem, but surprisingly few work has been devoted to certifying algorithms, although a sophisticated linear-time algorithm without certificates is known for over 35 years [6, 15, 16]. In fact, we are aware of only one certifying algorithm for that problem [1], which runs in quadratic time, but is quite involved. Using construction sequences, we give a simple, alternative solution with running time $O(|V|^2)$ and show that the used certificate is easy to verify in time $O(|E|)$.

We first recapitulate well-known results on the existence of construction sequences in Sections 2.1 and 2.2 and point out how Tutte's sequence can be obtained from Barnette and Grünbaum's sequence in linear time. Sections 2.3 and 3 cover the main idea for the existence result that we use for computing Barnette and Grünbaum's sequence. Section 4 deals with the question how construction sequences are efficiently represented and Section 5 shows how to use construction sequences for a certifying 3-connectedness test.

2. Construction Sequences

Let $G = (V, E)$ be a finite graph with $n := |V|$, $m := |E|$, $V(G) = V$ and $E(G) = E$. A graph is *connected* if there is a path between any two nodes and *disconnected* otherwise. For $k \geq 1$, a graph is *k-connected* if $n > k$ and deleting every $k-1$ nodes leaves a connected graph. A node (a pair of nodes) that leaves a disconnected graph upon deletion is called a *cut vertex* (a *separation pair*). Note that k-connectedness does not depend on parallel edges nor on self-loops. A path leading from node v to node w is denoted by $v \to w$. For a node v in a graph, let $N(v) = \{w \mid vw \in E\}$ denote its set of neighbors and $deg(v)$ its degree. For a graph G, let $\delta(G)$ be the minimum degree of its vertices.

A *subdivision* of a graph replaces each edge by a path of length at least one. Conversely, we want a notation to get back to the graph without subdivided edges. If $deg(v) = 2$, $|N(v)| = 2$ and $v \notin N(v)$ for a graph G, let $smooth_v(G)$ be the graph obtained from G by deleting v followed by adding an edge between its neighbors; we say v is *smoothed*. If one of the conditions is violated, let $smooth_v(G) = G$. Let $smooth(G)$ be the graph obtained by smoothing every node in G. For an edge $e \in E$, let $G \setminus e$ denote the graph obtained from G by deleting e. Let K_n be the complete graph on n nodes.

The following are well-known corollaries of Menger's theorem [8].

Lemma 2.1. (Fan Lemma) *Let v be a node in a graph G that is k-connected with $k \geq 1$ and let A be a set of at least k nodes in G with $v \notin A$. Then there are k internally node-disjoint paths $P_1, \ldots, P_k$ from v to distinct nodes $a_1, \ldots, a_k \in A$ such that for each of these paths $V(P_i) \cap A = a_i$.*

Lemma 2.2. (Expansion Lemma [17]) *Let G be a k-connected graph. Then the graph obtained by adding a new node v joined to at least k nodes in G is still k-connected.*

2.1. Tutte's Characterization and their Inverse

From now on we assume for simplicity that our input graph $G = (V, E)$ is simple although all results can be extended to multigraphs. Generally, contractions cannot always avoid parallel edges in intermediate graphs, e. g., for wheels. That is why we define contractions to preserve graphs to be simple: *Contracting* an edge $e = xy$ in a graph deletes e, identifies nodes x and y and replaces iteratively all 2-cycles by an edge. An edge e is called *contractible* if contracting e results in a 3-connected graph.

A *node splitting* takes a node v of a 3-connected graph, replaces v by two nodes x and y with an edge between them and replaces every former edge uv that was incident to v with either the edge ux, uy or both such that $|N(x)| \geq 3$ and $|N(y)| \geq 3$ in the new graph. Node splitting as defined here is therefore the exact inverse of contracting a contractible edge that has on both endnodes at least 3 neighbors.

Theorem 2.3. (Corollary of Tutte [13]) *The following statements are equivalent:*

$$\begin{aligned} & \textit{A simple graph } G \textit{ is } 3\textit{-connected} \\ \Leftrightarrow\ & \exists \textit{ sequence of contractions from } G \textit{ to } K_4 \textit{ on contractible edges } e = xy \\ & \textit{with } |N(x)| \geq 3 \textit{ and } |N(y)| \geq 3 && (2.1) \\ \Leftrightarrow\ & \exists \textit{ construction sequence from } K_4 \textit{ to } G \textit{ using node splittings} && (2.2) \end{aligned}$$

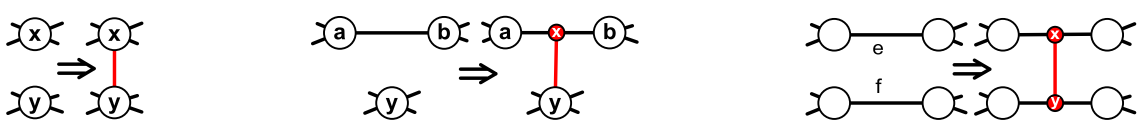
We describe next a straight-forward $O(n^2)$ algorithm to compute (2.1) for a graph G on more than 4 vertices. First, we decrease the number of edges to $O(n)$ in G by applying the algorithm of Nagamochi and Ibaraki [9]. This preserves the 3-connectedness or respectively, the non 3-connectedness of G. Moreover, it is known that the resulting graph contains a vertex v of degree 3. By a result of Halin [5], every node of degree 3 is incident to a contractible edge e. We get e by subsequently contracting each of the three incident edges and testing the resulting graph with the algorithm of Hopcroft and Tarjan [6] for 3-connectedness. Iteration of both subroutines gives us the whole contraction sequence in $O(n^2)$ time. However, the Hopcroft-Tarjan test is difficult to implement and we will give a much simpler algorithm that is capable of computing both characterizations later.

2.2. Barnette and Grünbaum's Characterization and their Inverse

The Barnette and Grünbaum operations (*BG-operations*) consist of the following operations on a 3-connected graph (see Figures 1(a)-1(c)).

(a) add an edge xy (possibly a parallel edge)
(b) subdivide an edge ab by a node x and add the edge xy for a node $y \notin \{a, b\}$
(c) subdivide two distinct, non-parallel edges by nodes x and y, respectively, and add the edge xy

In all three cases, let xy be the edge that was *added* by the BG-operation.



(a) parallel edges allowed (b) y, a, b distinct (c) $e \neq f$, e and f not parallel

Figure 1: The three operations of Barnette and Grünbaum.

Theorem 2.4. (Barnette and Grünbaum [2], Tutte [14]) *A graph G is 3-connected if and only if G can be constructed from the K_4 using BG-operations.*

Theorem 2.4 was proven in this notation by Barnette and Grünbaum [2], but implicitly described in a theorem about *nodal connectivity* by Tutte [14, Theorem 12.65]. If not stated otherwise, every construction sequence uses only BG-operations. Let a BG-operation be *basic*, if it does not create parallel edges and let a construction sequence be *basic*, if it only uses basic BG-operations.

Like in Theorem 2.3, we want the inverse of a BG-operation. Let *removing* the edge $e = xy$ of a graph be the operation of deleting e followed by smoothing x and y. An edge $e = xy$ in G is called *removable*, if removing e yields a 3-connected graph. We show that removing a removable edge $e = xy$ with $|N(x)| \geq 3$, $|N(y)| \geq 3$ and $|N(x) \cup N(y)| \geq 5$ is exactly the inverse of a BG-operation.

Theorem 2.5. *The following statements are equivalent:*

$$\begin{aligned} & \text{A simple graph } G \text{ is 3-connected} && (2.3)\\ \Leftrightarrow\ & \exists \text{ sequence of removals from } G \text{ to } K_4 \text{ on removable edges } e = xy \\ & \text{with } |N(x)| \geq 3,\ |N(y)| \geq 3 \text{ and } |N(x) \cup N(y)| \geq 5 && (2.4)\\ \Leftrightarrow\ & \exists \text{ construction sequence from } K_4 \text{ to } G \text{ using BG-operations} && (2.5)\\ \Leftrightarrow\ & \exists \text{ basic construction sequence from } K_4 \text{ to } G \text{ using BG-operations} && (2.6) \end{aligned}$$

Proof. Theorem 2.4 establishes (2.3) $\Leftrightarrow$ (2.5). Moreover, the proof of Theorem 2.4 in [2] implicitly shows that on simple graphs basic operations suffice, thus only the equivalence for (2.4) remains. We first prove (2.6) $\Rightarrow$ (2.4) and then (2.4) $\Rightarrow$ (2.5).

BG-operations operate by definition on 3-connected graphs, this holds in particular for the ones in (2.5). Let G' be the graph obtained by a basic BG-operation in (2.5) that adds the edge $e = xy$. The operation can clearly be undone by removing e in G'. Since BG-operations preserve 3-connectedness with Theorem 2.4, $|N(x)| \geq 3$ and $|N(y)| \geq 3$ hold in G'.

It remains to show that $|N(x) \cup N(y)| \geq 5$ in G'. If $|N(x)| \geq 4$ or $|N(y)| \geq 4$, $|N(x) \cup N(y)| \geq 5$ follows, since x and y are neighbors and no self-loops exist. Thus, let $|N(x)| = |N(y)| = 3$. Having $N(x) \setminus \{y\} \neq N(y) \setminus \{x\}$ yields $|N(x) \cup N(y)| \geq 5$ as well, so let $N(x) \setminus \{y\}$ and $N(y) \setminus \{x\}$ contain the same two nodes a and b. If $|V(G)| > 4$, a or b must be adjacent to a node c that is neither adjacent to x nor y. But then $\{a, b\}$ is a separation pair, contradicting the 3-connectedness of G. On the other hand, $|V(G)| = 4$ is not possible, since that implies the BG-operation to be (a) (since only (b) and (c) create new vertices) and that is no basic operation on the K_4.

We prove (2.4) $\Rightarrow$ (2.5). Let G' be the graph containing a removable edge $e = xy$ that is removed in (2.4). Note that G' can have parallel edges due to previous removals but no self-loops. The removal can be undone by one of the BG-operations. Which one, is dependent on the number i of endnodes of e on which smoothing changed the graph, i. e., the number of endnodes u of e with $|N(u)| = deg(u) = 3$ in G'. If $i = 0$, removing e just deletes e which is inversed by operation (a). For $i = 1$, let x be the node with $|N(x)| = deg(x) = 3$ in G' and f be the edge in which x was smoothed. Then (b) can be applied, because $y \notin f$ (see Figure 8(a)) since otherwise x would have had only 2 neighbors in G', contradicting the assumption $|N(x)| \geq 3$.

If $i = 2$, let f_1 and f_2 be the edges in which x and y were smoothed. Operation (c) can only be applied if f_1 and f_2 are neither identical (see Figure 8(b)) nor parallel. But $f_1 = f_2$ would again contradict $|N(x)| \geq 3$ in G' and f_1 being parallel to f_2 would contradicts

$|N(x) \cup N(y)| \geq 5$ in G, since in that case x and y are only adjacent to each other and the two nodes $f_1 \cap f_2$. ■

We show that Barnette and Grünbaum's characterization is algorithmically at least as powerful as Tutte's by giving a simple linear time transformation. Lemma 2.6 allows us to focus on computing BG-operations only.

Lemma 2.6. *Every construction sequence using BG-operations can be transformed in linear time to Tutte's sequence* (2.1) *of contractions.*

Proof. We transform every BG-operation in reverse order of the construction sequence to 0, 1 or 2 contractions each. Operation (a) yields no contraction while operation (b) yields the contraction of exactly one part of the subdivided edge (either xa or xb in Figure 1). For an operation (c), let $e = ab$ and $f = vw$ be the edges that are subdivided with x and y. Both edges share at most one node; let w. l. o. g. $a = v$ be that node if it exists. We create one contraction for each of the edges xb and yw in arbitrary order. In all cases, contractions inverse BG-operations except for the added edge xy, which is left over. But additional edges do not harm the 3-connectedness of the graph nor subsequent contractions. Thus, we have found a contraction sequence to the K_4 unless the first contraction in the case of an operation (c) yields at some point a graph H that is not 3-connected. But H can be obtained from the graph that results from contracting the second edge by applying one operation (b) and therefore is 3-connected. ■

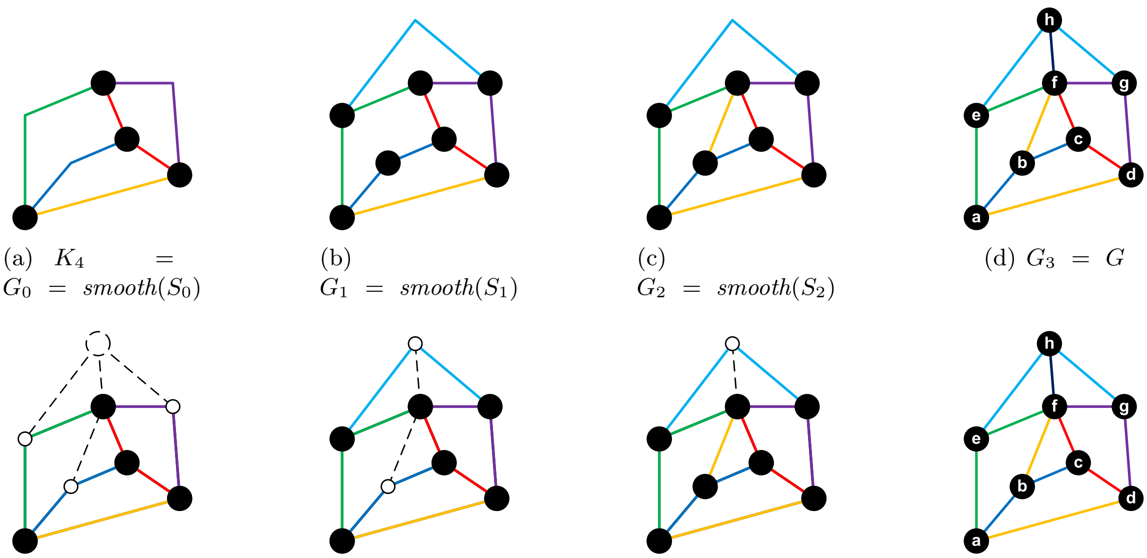
2.3. Identifying Intermediate Graphs with Subdivisions in G

Let $K_4 = G_0, G_1, \ldots, G_z = G$ be the 3-connected graphs obtained in a construction sequence Q to a simple 3-connected graph G using the basic BG-operations $C_0, \ldots, C_{z-1}$. We can reverse Q by starting with G and removing the added edges of BG-operations in reverse order. Suppose we would delete the added edge of every C_i instead of removing it and treat emerging paths containing interior nodes of degree 2 as (topological) edges in G_i (see Figure 2). Then iteratively paths are deleted instead of edges being removed and we obtain the sequence of subdivisions $G = S_z, \ldots, S_0$ in G with S_0 being a subdivision of the K_4. This leads to the following observation.

Lemma 2.7 (Observation)**.** *Let Q be a construction sequence from a graph G_0 to G using BG-operations. Then G contains a subdivision of G_0 that is specified by Q.*

In particular, Observation 2.7 yields with Theorem 2.4 that every 3-connected graph contains a subdivision of the K_4 (Theorem of J. Isbell [2]). Each graph G_i in our construction sequence can be identified with the unique subdivision S_i contained in G. Conversely, $G_i = smooth(S_i)$ for all $0 \leq i \leq z$, since smoothing a graph is exactly the inverse operation of subdividing a graph without nodes of degree two. The nodes x in S_i with $deg(x) \geq 3$ are called *real* nodes, because they correspond to nodes in G_i. Real nodes have at least 3 neighbors in G_i, because G_i is 3-connected.

Note that in non-basic construction sequences $smooth(S_i)$ can have parallel edges, although S_i is always simple. We define the *links* of each S_i to be the unique paths in S_i with only their endnodes being real. The links of S_i partition $E(S_i)$ because S_i is 2-connected, has therefore minimum degree two and is not a cycle. Let two links be *parallel* if they share the same endnodes.



(a) $K_4 = G_0 = smooth(S_0)$ (b) $G_1 = smooth(S_1)$ (c) $G_2 = smooth(S_2)$ (d) $G_3 = G$

(e) S_0 (f) S_1 (g) S_2 (h) $S_3 = G$

Figure 2: The graphs $G_0, \ldots, G_z$ and $S_0, \ldots, S_z$ of a construction sequence of G. On graphs S_i, the dashed edges and nodes are in G but not in S_i and nodes depicted in black are *real* nodes. For example, the path $C_0 = e \rightarrow h \rightarrow g$ is a *BG-path* for S_0, yielding S_1. The *links* of S_1 are the paths C_0, $a \rightarrow b \rightarrow c$ and the single edges ae, ef, fc, cd, da, fg, gd.

Definition 2.8. A *BG-path for S_i* is a path $P = x \rightarrow y$ in G with the following properties:

(1) $S_i \cap P = \{x, y\}$
(2) x and y are not both contained in a link of S_i except as endnodes
(3) x and y are not inner nodes of links of S_i that are parallel

It is easy to see that every BG-path for S_i corresponds to a BG-operation on G_i and vice versa. We will exploit this duality in the next section.

In general, construction sequences are not bound to start with the K_4. Titov and Kelmans [12, 7] extended Theorem 2.4 by proving the existence of a construction sequence even when starting with arbitrary 3-connected graphs G_0 instead of the K_4, as long as a subdivision of G_0 is contained in G. This is a generalization, since every 3-connected graph contains a subdivision of the K_4 by Observation 2.7.

Theorem 2.9. [7, 12] *Let G_0 be a 3-connected graph. Then a simple graph G is 3-connected and contains a subdivision of G_0 if and only if G can be constructed from G_0 using basic BG-operations.*

3. Prescribing Subdivisions

Both Theorems 2.4 and 2.9 choose a very special subdivision of the K_4 (resp. G_0) on which the construction sequence starts, in fact one having the maximum number of edges in G. The construction sequence is then obtained by adding longest BG-paths. Unfortunately, computing these depends heavily on solving the longest paths problem, which is known to be NP-hard even for 3-connected graphs [4].

This gives rise to the question whether Theorems 2.4 and 2.9 can be strengthened to start at a *prescribed* subdivision $H \subseteq G$ of G_0 instead of an arbitrary one. Note that this is equivalent to the constraint $S_0 = H$. Such a result would provide an efficient computational approach to construction sequences, since it allows us to search the neighborhood of H for BG-paths, yielding a new prescribed subdivision of a 3-connected graph.

However, when restricted to basic operations it is not possible to prescribe H, as the minimal counterexample in Figure 3 shows: Consider the graph G consisting of a $K_4 = H$ depicted in black with an additional node connected to three nodes of the K_4. Then every BG-path for H will create a parallel link, although G is simple. But what if we drop the condition that construction sequences have to be basic? The following theorem shows that at this expense we can indeed start a construction sequence from any prescribed subdivision.

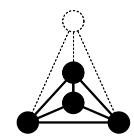


Figure 3: Every possible BG-operation adds a parallel edge.

Theorem 3.1. *Let G be a 3-connected graph and $H \subset G$ with H being a subdivision of a 3-connected graph. Then there is a BG-path for H in G. Moreover, every link of H of length at least 2 contains an inner node on which a BG-path for H starts.*

Proof. We distinguish two cases.

- $H \neq \mathit{smooth}(H)$.
  Then links of length at least 2 exist in H and we pick an arbitrary one of them, say T. Let x be an inner node of T, and let Q be the set of paths in G from x to a node in $V(H) \setminus V(T)$ avoiding the endnodes of T (see Figure 5). By the 3-connectedness of G, the set Q cannot be empty and every path in Q fulfills Definition 2.8.2. There is at least one path $P = x \to y$ in Q with y being not contained in a parallel link of T, because otherwise the endnodes of T would form a separation pair. Let x' be the last node in P that is in T or in a parallel link of T and let y' be the first node after x' that is in $V(H)$. Then $x' \to y'$ has properties 2.8.1 and 2.8.3 and is a BG-path for H.
- $H = \mathit{smooth}(H)$.
  Then H consists only of real nodes and since $H \neq G$, there is a node in $V(G) \setminus V(H)$ or an edge in $E(G) \setminus E(H)$. At first, assume that there is a node $x \in V(G) \setminus V(H)$. Then, by the 2-connectedness of G and Fan Lemma 2.1 we can find a path $P = y_1 \to x \to y_2$ with no other nodes in H than y_1 and y_2. For P the properties 2.8.1-2.8.3 hold, because no link in H can have inner nodes. Let now $V(G) = V(H)$ and e an edge in $E(G) \setminus E(H)$. Then e must be a BG-path for H, since both endnodes are real.

■

In Theorem 3.1, non-basic operations can only occur in the case $H = \mathit{smooth}(H)$ when a path through a node of $V(G) \setminus V(H)$ is chosen. Although we cannot avoid that, it is possible to obtain a basic construction by augmenting the BG-operations with a fourth operation (d).

(d) connect a new node to three distinct nodes

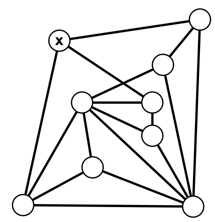


Figure 4: A 3-connected graph having a node x of degree 3 with no incident edge being removable.

Operation (d) preserves 3-connectedness with Lemma 2.2 and is basic, because each new edge ends on the new node. Whenever we encounter a node in $V(G) \setminus V(H)$ in Theorem 3.1, we know by the Fan Lemma 2.1 and the 3-connectedness of G that there are three internally node-disjoint paths to real nodes in H with all inner nodes being in $V(G) \setminus V(H)$. Adding these paths to H is called an *expand* operation and corresponds to operation (d) in the smoothed graph. This gives the following result.

Theorem 3.2. *Let G be a simple graph and let H be a subdivision of a 3-connected graph. Then*

$$
\begin{aligned}
& G \textit{ is 3-connected and } H \subseteq G \\
\Leftrightarrow\ & \delta(G) \geq 3 \textit{ and } \exists \textit{ construction sequence from } H \textit{ to } G \textit{ using BG-paths} && (3.1) \\
\Leftrightarrow\ & \delta(G) \geq 3 \textit{ and } \exists \textit{ basic construction sequence from } H \textit{ to } G \textit{ using BG-paths} \\
& \textit{and the expand operation} && (3.2)
\end{aligned}
$$

Proof. Let G be 3-connected and $H \subseteq G$. Then $\delta(G) \geq 3$ holds and if $H = G$, the desired construction sequences are empty and exist. If $H \subset G$, we can apply Theorem 3.1 iteratively with or without the additional expand operation and the construction sequences exist as well. For the sufficiency part, both construction sequences imply $H \subseteq G$, since only paths are added to construct G. Additionally, G must be 3-connected, as adding BG-paths to each S_i preserves S_{i+1} to be a subdivision of a 3-connected graph with Theorem 2.4, and $\delta(G) \geq 3$ ensures that the last subdivision G of a 3-connected graph is 3-connected itself. ∎

4. Representations

A straight-forward algorithm to compute Barnette and Grünbaum's construction sequence of a 3-connected graph is to search iteratively for removable edges. But in contrast to the algorithm in Section 2.1 that computes contractible edges, this approach only leads to an $O(n^3)$ algorithm. The reason for the additional factor of n is that not all nodes with degree 3 must have an incident removable edge (see Figure 4 for a counterexample on 9 nodes) and we have to try every edge in the worst case. Computing BG-paths instead of BG-operations allows us to obtain better running times, but first we need to know how exactly construction sequences can be represented.

An obvious representation of a construction sequence Q would be to store the graph $G_0 = \textit{smooth}(H)$ and in addition every BG-operation, which gives the sequence $G_0, \ldots, G_z = G$. Unfortunately, the graphs G_i are not necessarily subgraphs of G_{i+1}, so we have to take care of relabeled edges when specifying each operation.

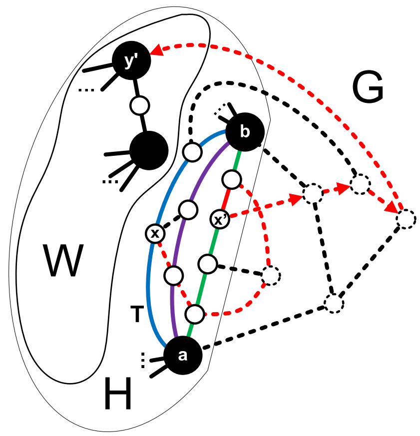


Figure 5: The case $H \neq smooth(H)$. Dashed edges are in $E(G) \setminus E(H)$, arrows depict the BG-path $x' \to y'$.

Whenever an edge e is subdivided as part of an operation (b) or (c), we specify it by its index in G_i followed by assigning new indices for the new degree-two node and one of the two new separated edge parts in G_{i+1}. The other edge part keeps the index of e.

Similarly, on operations (a) and (b), real endnodes of the added edge are specified by their indices in G_i. We assign a new index for the added edge in G_{i+1}, too. Finally, we have to impose the constraint that G_z is not just isomorphic but identical to G, meaning that nodes and edges of G_z and G are labeled by exactly the same indices, since otherwise we would have to solve the graph isomorphism problem to check that Q really constructs G.

On the other hand, the identification of G_i with a subgraph in G allows us to represent Q without indexing issues: We just store $S_0 \subset G$ and the BG-paths $C_0, \ldots, C_{z-1}$. Hence, we can represent each construction sequence Q of G in the following two ways.

- *Edge representation*: Represent Q by G_0 and a sequence of BG-operations, along with specifying new and old indices for each operation, such that G_z and G are labeled the same.
- *Path representation*: Represent Q by S_0 and BG-paths $C_0, \ldots, C_{z-1}$.

Both representations refer to the same sequence of graphs $G_0, \ldots, G_z$ and are of size $\theta(m)$, assuming the uniform cost model. The next lemma states that it does not matter which of the two representations we compute.

Lemma 4.1. *The edge and path representations of a construction sequence Q can be transformed into each other in $O(m)$ time. Moreover, the representation computed is a unique representation of Q.*

Proof. Omitted. ∎

5. Certifying and Testing 3-Connectedness in $O(n^2)$

We use construction sequences in the path representation as a certificate for the 3-connectedness of graphs. This leads to a new, certifying method for testing graphs on being

3-connected. The total running time of this method is $O(n^2)$, however this is dominated by the time needed for finding the construction sequence and every improvement made there will automatically result in a faster 3-connectedness test. The input graph is a multigraph and does not have to be biconnected nor connected. We follow the steps:

- Apply preprocessing of Nagamochi and Ibaraki to the graph and get G in $O(n+m)$ (This improves the total running time by decreasing the number of edges to $O(n)$.)
- Try to compute a K_4-subdivision S_0 in G and prescribe it in $O(n)$
  - Failure: Return a separation pair
- Try to compute a construction sequence from S_0 to G in $O(n^2)$
  - Success: Return the construction sequence
  - Failure: Return a separation pair

The preprocessing step preserves the graph to be 3-connected or to be not 3-connected. We first describe how to find a K_4-subdivision by one Depth First Search (DFS), which as a byproduct eliminates self-loops and parallel edges and sorts out graphs that are not connected or have nodes with degree at most 2. Let a (resp. b) be the node in the DFS-tree T that is visited first (resp. second). If G is 3-connected, then a and b have exactly one child, otherwise they form a separation pair. We choose two arbitrary neighbors c and d of a that are different from b (see Figure 6). W.l.o.g., let d be visited later by the DFS than c. Let $i \neq b$ the least common ancestor of c and d in T. As $d \neq i$ must hold, let j be the child of i that is contained in the path $i \to d$ in T.

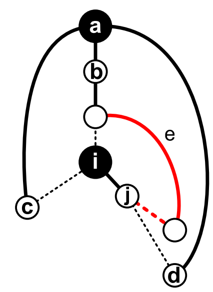


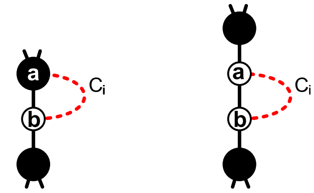
Figure 6: Finding a K_4-subdivision. Dashed edges can be (empty) paths, arcs depict backedges.

If G is 3-connected, we can find a backedge e that starts on a node z in the subtree rooted at j and ends on an inner node z' of $a \to i$ in time $O(n)$. If e does not exist, a and i form a separation pair, otherwise we have found a K_4-subdivision with real nodes a, i, z and z'. The paths connecting this real nodes in T together with the three visited backedges constitute the 6 paths of the K_4-subdivision.

Once the K_4-subdivision S_0 is found, we follow the lines of Theorem 3.1 and try to construct the path representation $C_0, \ldots, C_{z-1}$. If favored, this can be transformed to an edge representation in $O(m)$ later. We assign an index for every link and store it on each of the inner nodes of that link. Moreover, we maintain pointers for each link to its endnodes.

In case $H \neq \mathit{smooth}(H)$ of Theorem 3.1 we pick an arbitrary node x of degree two. Let $T = a \to b$ be the link that contains x and let W be the set of nodes $V(H) \setminus V(T)$ minus all nodes in parallel links of T (see Figure 5). We compute the path $P = x \to y'$ by temporarily deleting a and b and performing a DFS on x that stops on the first node $y' \in W$. We can check whether a node lies in a parallel link of T in constant time by comparing the endnodes of its containing link with a and b. Thus, the subpath $x' \to y'$ with x' being the last node contained in T or in a parallel link of T is a BG-path and can be found efficiently. The links and their indices can be updated in $O(n)$.

Similarly, in case $H = \mathit{smooth}(H)$ we delete temporarily all edges in $E(H)$ and start a DFS on a node $x \in V(H)$ that has an incident edge in the remaining graph. The traversal is stopped on the first node $y \in V(H) \setminus \{x\}$. The path $x \to y$ is then the desired BG-path



(a) Either a or b has degree 2.

(b) Both, a and b, have degree 2.

Figure 8: Cases where 2.8.2 fails when $a \in N(b)$.

and we conclude that for 3-connected graphs the construction sequence can be found in time $O(n^2)$.

Otherwise, G is not 3-connected and no construction sequence can exist with Theorem 3.2. In that case a DFS starting at node x fails to find a new BG-path for some subdivision $H \subset G$. If $H \neq \mathit{smooth}(H)$, the endnodes of the link that contains x must form a separation pair. Otherwise, $H = \mathit{smooth}(H)$ and x must be a cut vertex. Thus, if G is not 3-connected, the algorithm returns always a separation pair or cut vertex.

If G is simple, the construction sequence can be transformed to the basic construction sequence (3.2) with the following Lemma.

Lemma 5.1. *For simple graphs G, the construction sequences* (3.1) *and* (3.2) *can be transformed into each other in $O(m)$.*

Proof. Omitted. ∎

Theorem 5.2. *The construction sequences* (3.1) *and* (3.2) *can be computed in $O(n^2)$ and establish a certifying* 3*-connectedness test with the same running time.*

5.1. Verifying the Construction Sequence

It is essential for a certificate that it can be easily validated. We could do this by transforming the path representation to the edge representation using Lemma 4.1 and checking the validity of the BG-operations by comparing indices, but there is a more direct way. First, it can be checked in linear time that all BG-paths $C_0, \ldots, C_{z-1}$ are paths in G and that these paths partition $E(G) \setminus E(S_0)$. We try to remove the BG-paths $C_{z-1}, \ldots, C_0$ from G in that order (i. e., we delete the paths followed by smoothing its endnodes). If the certificate is valid, this is well defined as all removed BG-paths are then edges. On the other hand we can detect longer BG-paths $|C_i| \geq 2$ before their removal, in which case the certificate is not valid, since then the inner nodes of C_i are not attached to BG-paths C_j, $j > i$.

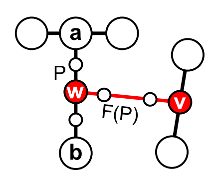


Figure 7: No expand operation can be formed.

We verify that every removed $C_i = ab$ corresponds to a BG-operation by using Definition 2.8 of BG-paths, and start with checking that a and b lie in our current subgraph for condition 2.8.1.

Conditions 2.8.2 and 2.8.3 can now be checked in constant time: Consider the situation immediately after the deletion of ab, but before smoothing a and b. Then all links in our subgraph are single edges, except possibly the ones containing a and b as inner nodes.

Therefore, 2.8.2 is not met for C_i if a is a neighbor of b and at least one of the nodes a and b has degree two (see Figures 8 for possible configurations). Condition 2.8.3 is not met if $N(a) = N(b)$ and both a and b have degree two. Both conditions can be easily checked in constant time. Note that encountering proper BG-paths $C_{z-1}, \dots, C_i$ does not necessarily imply that the current subgraph is 3-connected, since false BG-paths C_j, $j < i$, can exist.

It remains to validate that the graph after removing all BG-paths is the K_4. This can done in constant time by checking it on being simple and having exactly 4 nodes of degree three.

Theorem 5.3. *The construction sequences* (2.4)-(2.6) *and* (3.1)-(3.2) *can be checked on validity in time linearly dependent on their length.*

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