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Rustem Takhanov

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A DICHOTOMY THEOREM FOR THE GENERAL MINIMUM COST HOMOMORPHISM PROBLEM

RUSTEM TAKHANOV [1]

[1] Department of Computer and Information Science, Linköping University
E-mail address: g-rusta@ida.liu.se
E-mail address: takhanov@mail.ru

ABSTRACT. In the constraint satisfaction problem (CSP), the aim is to find an assignment of values to a set of variables subject to specified constraints. In the minimum cost homomorphism problem ($MinHom$), one is additionally given weights c_{va} for every variable v and value a, and the aim is to find an assignment f to the variables that minimizes $\sum_v c_{vf(v)}$. Let $MinHom(\Gamma)$ denote the $MinHom$ problem parameterized by the set of predicates allowed for constraints. $MinHom(\Gamma)$ is related to many well-studied combinatorial optimization problems, and concrete applications can be found in, for instance, defence logistics and machine learning. We show that $MinHom(\Gamma)$ can be studied by using algebraic methods similar to those used for CSPs. With the aid of algebraic techniques, we classify the computational complexity of $MinHom(\Gamma)$ for all choices of Γ. Our result settles a general dichotomy conjecture previously resolved only for certain classes of directed graphs, [Gutin, Hell, Rafiey, Yeo, European J. of Combinatorics, 2008].

1. Introduction

Constraint satisfaction problems (CSP) are a natural way of formalizing a large number of computational problems arising in combinatorial optimization, artificial intelligence, and database theory. This problem has the following two equivalent formulations: (1) to find an assignment of values to a given set of variables, subject to constraints on the values that can be assigned simultaneously to specified subsets of variables, and (2) to find a homomorphism between two finite relational structures A and B. Applications of CSPs arise in the propositional logic, database and graph theory, scheduling and many other areas. During the past 30 years, CSP and its subproblems has been intensively studied by computer scientists and mathematicians. Considerable attention has been given to the case where the constraints are restricted to a given finite set of relations Γ, called a constraint language [3, 6, 13, 17]. For example, when Γ is a constraint language over the boolean set $\{0,1\}$ with four ternary predicates $x \vee y \vee z$, $\overline{x} \vee y \vee z$, $\overline{x} \vee \overline{y} \vee z$, $\overline{x} \vee \overline{y} \vee \overline{z}$ we obtain 3-SAT. This direction of research has been mainly concerned with the computational complexity of $CSP(\Gamma)$ as a function of Γ. It has been shown that the complexity of $CSP(\Gamma)$ is highly

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connected with relational clones of universal algebra [13]. For every constraint language Γ, it has been conjectured that $CSP\,(\Gamma)$ is either in P or NP-complete [6].

In the minimum cost homomorphism problem ($MinHom$), we are given variables subject to constraints and, additionally, costs on variable/value pairs. Now, the task is not just to find any satisfying assignment to the variables, but one that minimizes the total cost.

Definition 1.1. Suppose we are given a finite domain set A and a finite constraint language $\Gamma \subseteq \bigcup\limits_{k=1}^{\infty} 2^{A^k}$. Denote by $MinHom\,(\Gamma)$ the following minimization task:

Instance: A first-order formula $\Phi\,(x_1, \ldots, x_n) = \bigwedge\limits_{i=1}^{N} \rho_i\,(y_{i1}, \ldots, y_{in_i}),\ \rho_i \in \Gamma, y_{ij} \in \{x_1, \ldots, x_n\}$, and weights $w_{ia} \in \mathbb{N}, 1 \le i \le n, a \in A$.
Solution: Assignment $f : \{x_1, \ldots, x_n\} \to A$, that satisfies the formula Φ. If there is no such assignment, then indicate it.
Measure: $\sum\limits_{i=1}^{n} w_{if(x_i)}$.

Remark 1.2. Note that when we require weights to be positive we do not lose generality, since $MinHom\,(\Gamma)$ with arbitrary weights can be polynomial-time reduced to $MinHom\,(\Gamma)$ with positive weights by the following trick: we can add s to all weights, where s is some integer. This trick only adds ns to the value of the optimized measure. Hence, we can make all weights negative, and $MinHom\,(\Gamma)$ modified this way is equivalent to maximization but with positive weights only. This remark explains why both names $MinHom$ and $MaxHom$ can be allowed, though we prefer $MinHom$ due to historical reasons.

$MinHom$ was introduced in [11] where it was motivated by a real-world problem in defence logistics. The question for which directed graphs H the problem $MinHom\,(\{H\})$ is polynomial-time solvable was considered in [8, 9, 10, 11, 12]. In this paper, we approach the problem in its most general form by algebraic methods and give a complete algebraic characterization of tractable constraint languages. From this characterization, we obtain a dichotomy for $MinHom$, i.e., if $MinHom\,(\Gamma)$ is not polynomial-time solvable, then it is NP-hard. Of course, this dichotomy implies the dichotomy for directed graphs.

In Section 2, we present some preliminaries together with results connecting the complexity of $MinHom$ with conservative algebras. The main dichotomy theorem is stated in Section 3 and its proof is divided into several parts which can be found in Sections 4-8. The NP-hardness results are collected in Section 4 followed by the building blocks for the tractability result: existence of majority polymorphisms (Section 5) and connections with optimization in perfect graphs (Section 6). Section 7 introduces the concept of *arithmetical deadlocks* which lays the foundation for the final proof in Section 8. Finally, in Section 9 we explain the relation of our results to previous research and present directions for future research.

2. Algebraic structure of tractable constraint languages

Recall that an optimization problem A is called NP-hard if some NP-complete language can be recognized in polynomial time with the aid of an oracle for A. We assume that $P \neq NP$.

Definition 2.1. Suppose we are given a finite set A and a constraint language $\Gamma \subseteq \bigcup\limits_{k=1}^{\infty} 2^{A^k}$. The language Γ is said to be *tractable* if, for every finite subset $\Gamma' \subseteq \Gamma$, $MinHom\,(\Gamma')$ is polynomial-time solvable, and Γ is called *NP-hard* if there is a finite subset $\Gamma' \subseteq \Gamma$ such that $MinHom\,(\Gamma')$ is NP-hard.

First, we will state some standard definitions from universal algebra.

Definition 2.2. Let $\rho \subseteq A^m$ and $f : A^n \to A$. We say that the function (operation) f *preserves* the predicate ρ if, for every $\left(x_1^i, \ldots, x_m^i\right) \in \rho, 1 \leq i \leq n$, we have that $\left(f\left(x_1^1, \ldots, x_1^n\right), \ldots, f\left(x_m^1, \ldots, x_m^n\right)\right) \in \rho$.

For a constraint language Γ, let $Pol\,(\Gamma)$ denote the set of operations preserving all predicates in Γ. Throughout the paper, we let A denote a finite domain and Γ a constraint language over A. We assume the domain A to be finite.

Definition 2.3. A constraint language Γ is called *a relational clone* if it contains every predicate expressible by a first-order formula involving only

a) predicates from $\Gamma \cup \left\{=^A\right\}$;
b) conjunction; and
c) existential quantification.

First-order formulas involving only conjunction and existential quantification are often called *primitive positive (pp) formulas.* For a given constraint language Γ, the set of all predicates that can be described by pp-formulas over Γ is called the *closure* of Γ and is denoted by $\langle\Gamma\rangle$.

For a set of operations F on A, let $Inv\,(F)$ denote the set of predicates preserved under the operations of F. Obviously, $Inv\,(F)$ is a relational clone. The next result is well-known [2, 7].

Theorem 2.4. *For a constraint language Γ over a finite set A, $\langle\Gamma\rangle = Inv\,(Pol\,(\Gamma))$.*

Theorem 2.4 tells us that the Galois closure of a constraint language Γ is equal to the set of all predicates that can be obtained via pp-formulas from the predicates in Γ.

Theorem 2.5. *For any finite constraint language Γ and any finite $\Gamma' \subseteq \langle\Gamma\rangle$, there is a polynomial time reduction from $MinHom\,(\Gamma')$ to $MinHom\,(\Gamma)$.*

Proof. Since any predicate from Γ' can be viewed as a pp-formula with predicates in Γ, an input formula to $MinHom\,(\Gamma')$ can be represented on the form $\Phi\left(x_1, \ldots, x_n\right) = \bigwedge\limits_{i=1}^{N} \exists z_{i1}, \ldots, z_{im_i} \Phi_i\left(y_{i1}, \ldots, y_{in_i}, z_{i1}, \ldots, z_{im_i}\right)$, where $y_{ij} \in \left\{x_1, \ldots, x_n\right\}$ and Φ_i is a first-order formula involving only predicates in Γ, equality, and conjunction. Obviously, this formula is equivalent to $\exists z_{11}, \ldots, z_{Nm_N} \bigwedge\limits_{i=1}^{N} \Phi_i\left(y_{i1}, \ldots, y_{in_i}, z_{i1}, \ldots, z_{im_i}\right)$. $\bigwedge\limits_{i=1}^{N} \Phi_i\left(y_{i1}, \ldots, y_{in_i}, z_{i1}, \ldots, z_{im_i}\right)$ can be considered as an instance of $MinHom\left(\Gamma \cup \left\{=^A\right\}\right)$ with variables $x_1, \ldots, x_n, z_{11}, \ldots, z_{Nm_N}$ where weights w_{ij} will remain the same and for additional variables z_{kl} we define $w_{z_{kl}j} = 0$. By solving $MinHom\left(\Gamma \cup \left\{=^A\right\}\right)$ with the described input, we can find a solution of the initial $MinHom\,(\Gamma')$ problem. It is easy to see that the number of added variables is bounded by a polynomial in n. So this reduction can be carried out in polynomial time. Finally, $MinHom\left(\Gamma \cup \left\{=^A\right\}\right)$ can be reduced polynomially to $MinHom\,(\Gamma)$ because an equality constraint for a pair of variables is equivalent to replacement of all inclusions of the first variable in a formula by the second one. ∎

The previous theorem tells us that the complexity of $MinHom\,(\Gamma)$ is basically determined by $Inv\,(Pol\,(\Gamma))$, i.e., by $Pol\,(\Gamma)$. That is why we will be concerned with the classification of sets of operations F for which $Inv\,(F)$ is a tractable constraint language.

Definition 2.6. An *algebra* is an ordered pair $\mathbb{A} = (A, F)$ such that A is a nonempty set (called a universe) and F is a family of finitary operations on A. An algebra with a finite universe is referred to as a finite algebra.

Definition 2.7. An algebra $\mathbb{A} = (A, F)$ is called *tractable* if $Inv(F)$ is a tractable constraint language and $\mathbb{A}$ is called *NP-hard* if $Inv(F)$ is an NP-hard constraint language.

In the following theorem, we show that we only need to consider a very special type of algebras, so called *conservative* algebras.

Definition 2.8. An algebra $\mathbb{A} = (A, F)$ is called conservative if for every operation $f \in F$ we have that $f\,(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$.

Theorem 2.9. *For any finite constraint language Γ over A and $C \subseteq A$, there is a polynomial time Turing reduction from $MinHom\,(\Gamma \cup \{C\})$ to $MinHom\,(\Gamma)$.*

Proof. Let the first-order formula $\Phi\,(x_1, \ldots, x_n) = \mathop{\wedge}\limits_{i=1}^{M} C\,(y_i) \wedge \mathop{\wedge}\limits_{i=1}^{N} \rho_i\,(z_{i1}, \ldots, z_{in_i})$, where $\rho_i \in \Gamma, y_i, z_{ij} \in \{x_1, \ldots, x_n\}$, and weights $w_{ia}, 1 \le i \le n, a \in A$ be an instance of $MinHom\,(\Gamma \cup \{C\})$. We assume without loss of generality that $y_i \neq y_j$, when $i \neq j$. Let $W = \sum\limits_{i=1}^{n} \sum\limits_{a \in A} w_{ia} + 1$ and define a new formula and weights

$$\Phi'\,(x_1, \ldots, x_n) = \mathop{\wedge}\limits_{i=1}^{N} \rho_i\,(z_{i1}, \ldots, z_{in_i})$$

$$w'_{ia} = \begin{cases} w_{ia} + W, \text{if } a \notin C, \exists j\ x_i = y_j \\ w_{ia}, \text{ otherwise} \end{cases}$$

Then, using an oracle for $MinHom\,(\Gamma)$, we can solve

$$\min_{f \text{ satisfies } \Phi'} \sum_j w'_{jf(x_j)}.$$

Suppose that $\Phi\,(x_1, \ldots, x_n)$ is satisfiable and f is a satisfying assignment. It is easy to see that the part of the measure $\sum\limits_j w'_{jf(x_j)}$ that corresponds to the added values W is equal to 0 and the measure cannot be greater than $W - 1$. If g is any assignment that does not satisfy $\mathop{\wedge}\limits_{i=1}^{M} C\,(y_i)$, then we see that this part of measure cannot be 0, and hence, is greater or equal to W. This means that the minimum in the task is achieved on satisfying assignments of $\Phi\,(x_1, \ldots, x_n)$ and any such assignment minimize the part of the measure that corresponds to the initial weights, i.e., $\sum\limits_i w_{if(x_i)}$.

If $\Phi\,(x_1, \ldots, x_n)$ is not satisfiable, then either Φ' is not satisfiable or $\min\limits_{f \text{ satisfies } \Phi'} \sum\limits_j w'_{jf(x_j)} \ge W$. Using an oracle for $MinHom\,(\Gamma)$, we can easily check this.

Consequently, $MinHom\,(\Gamma \cup \{C\})$ is polynomial-time reducible to $MinHom\,(\Gamma)$. ∎

Theorem 2.10. *If* Γ *is a constraint language over* A *that contains all unary relations, then* $\mathbb{A} = (A, Pol\,(\Gamma))$ *is conservative.*

Proof. Let $C = \{x_1, \ldots, x_n\} \subseteq A$. If a function $f : A^n \to A$ preserves the predicate C, then $f\,(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$. ∎

3. Structure of tractable conservative algebras

Let $g : A^k \to A$ be an arbitrary conservative function and $S \subseteq A$. Define the function $g|_S : S^k \to S$, such that $\forall x_1, \ldots, x_k \in S\ g|_S\,(x_1, \ldots, x_k) = g\,(x_1, \ldots, x_k)$, i.e. the restriction of g to the set S. Throughout this paper we will consider a conservative algebra (A, F). For every $B \subseteq A$, let $F|_B = \{f_B | f \in F\}$. We assume that F is closed under superposition and variable change and contains all projections, i.e., it is *a functional clone*, because closing the set F under these operations does not change the set $Inv\,(F)$.

Sometimes we will consider clones as algebras and to describe them we will use the terms (conservativeness, tractability, NP-hardness) defined for algebras. All tractable clones, in case $A = \{0, 1\}$, can be easily found using well-known classification of boolean clones [15].

Theorem 3.1. *The boolean functional clone* H *is tractable if either* $\{x \wedge y, x \vee y\} \subseteq H$ *or* $\{(x \wedge \overline{y}) \vee (\overline{y} \wedge z) \vee (x \wedge z)\} \subseteq H$*, where* $\wedge, \vee$ *denote conjunction and disjunction. Otherwise,* H *is NP-hard.*

Every 2-element subalgebra of a tractable algebra must be tractable, which motivates the following definition.

Definition 3.2. Let F be a conservative functional clone. We say that F satisfies the *necessary local conditions* if and only if for every 2-element subset $B \subseteq A$, either

(1) there exists $f^\wedge, f^\vee \in F$ s.t. $f^\wedge|_B$ and $f^\vee|_B$ are different binary commutative functions; or
(2) there exists $f \in F$ s.t. $f|_B\,(x, x, y) = f|_B\,(y, x, x) = f|_B\,(y, x, y) = y$.

Theorem 3.3. *Suppose* F *is a conservative functional clone. If* F *is tractable, then it satisfies the necessary local conditions. If* F *does not satisfy the necessary local conditions, then it is NP-hard.*

In general, the necessary local conditions are not sufficient for tractability of a conservative clone. Let $M = \{B | B \subseteq A, |B| = 2, F|_B \text{ contains different binary commutative functions}\}$ and $\overline{M} = \{B | B \subseteq A, |B| = 2\} \setminus M$.

Suppose $f \in F$. By $\overset{a}{\underset{b}{\downarrow}} f$ we mean $a \neq b$ and $f\,(a, b) = f\,(b, a) = b$. For example, $\overset{1}{\underset{2}{\downarrow}}\overset{2}{\underset{3}{\downarrow}}\overset{1}{\underset{3}{\downarrow}} f$ means that $f|_{\{1,2,3\}}\,(x, y) = \max\,(x, y)$.

Introduce an undirected graph without loops $T_F = (M^o, P)$ where $M^o = \{(a, b)\,|\,\{a, b\} \in M\}$ and $P = \left\{\langle (a, b), (c, d)\rangle \,|\, (a, b), (c, d) \in M^o, \text{ there is no } f \in F : \overset{a}{\underset{b}{\downarrow}}\overset{c}{\underset{d}{\downarrow}} f\right\}$. The core result of the paper is the following.

Theorem 3.4. *Suppose* F *satisfy the necessary local conditions. If the graph* $T_F = (M^o, P)$ *is bipartite, then* F *is tractable. Otherwise,* F *is NP-hard.*

The proof of this theorem will be given in two steps. Firstly, in the following section, we will prove NP-hardness of F when $T_F = (M^o, P)$ is not bipartite. The final sections will be dedicated to the polynomial-time solvable cases.

4. NP-hard case

In this section, we will prove that if a set of functions F satisfies the necessary local conditions and $T_F = (M^o, P)$ (as defined in the previous section) is not bipartite, then F is NP-hard. Let ${}^{a}_{b}\overline{\times}^{c}_{d}$ and ${}^{a}_{b}\times^{c}_{d}$ denote the predicates $\{a,b\} \times \{c,d\} \setminus \{(b,d)\}$ and $\{(a,d),(b,c)\}$, where $a \neq b, c \neq d$. We need the following lemmas.

Lemma 4.1. *A constraint language that contains* $\left\{ {}^{a_0}_{b_0}\overline{\times}^{a_1}_{b_1}, \ldots, {}^{a_{2k-1}}_{b_{2k-1}}\overline{\times}^{a_{2k}}_{b_{2k}}, {}^{a_{2k}}_{b_{2k}}\overline{\times}^{a_0}_{b_0} \right\}$ *is NP-hard.*

Lemma 4.2. *If* $\langle (a,b),(c,d) \rangle \in P$*, then either* ${}^{a}_{b}\overline{\times}^{c}_{d} \in Inv(F)$*, or* ${}^{a}_{b}\times^{c}_{d} \in Inv(F)$*.*

Proof of NP-hard case of Theorem 3.4. For binary predicates α, β, let $\alpha \circ \beta = \{(x,y) | \exists z : \alpha(x,z) \wedge \beta(z,y)\}$. Obviously, if $\alpha, \beta \in Inv(F)$, then $\alpha \circ \beta \in Inv(F)$, too.

Since $T_F = (M^o, P)$ is not bipartite, we can find a shortest odd cycle in it, i.e. a sequence $(a_0,b_0),(a_1,b_1),\ldots,(a_{2k},b_{2k}) \in M^o, k \geq 1$, such that $\langle (a_i,b_i),(a_{i\oplus 1},b_{i\oplus 1}) \rangle \in P$. Here, $i \oplus j$ denotes $i + j (\bmod\ 2k+1)$.

By Lemma 4.2, there is a cyclic sequence $\rho_{0,1}, \rho_{1,2}, \ldots, \rho_{2k,0} \in Inv(F)$ such that $\rho_{i,i\oplus 1}$ is either equal to ${}^{a_i}_{b_i}\overline{\times}^{a_{i\oplus 1}}_{b_{i\oplus 1}}$ or equal to ${}^{a_i}_{b_i}\times^{a_{i\oplus 1}}_{b_{i\oplus 1}}$. Note that all predicates cannot be of the second type: otherwise, we have $\rho_{0,1} \circ \rho_{1,2} \circ \cdots \circ \rho_{2k,0} = {}^{a_0}_{b_0}\times^{a_0}_{b_0}$ which contradicts that $\{a_0, b_0\} \in M$.

If the sequence contains a fragment $\rho_{i,i\oplus 1} = {}^{a_i}_{b_i}\times^{a_{i\oplus 1}}_{b_{i\oplus 1}}$, $\rho_{i\oplus 1,i\oplus 2} = {}^{a_{i\oplus 1}}_{b_{i\oplus 1}}\times^{a_{i\oplus 2}}_{b_{i\oplus 2}}$, $\rho_{i\oplus 2,i\oplus 3} = {}^{a_{i\oplus 2}}_{b_{i\oplus 2}}\overline{\times}^{a_{i\oplus 3}}_{b_{i\oplus 3}}$, then these predicates can be replaced by:

$$\rho_{i,i\oplus 3} \stackrel{\Delta}{=} \rho_{i,i\oplus 1} \circ \rho_{i\oplus 1,i\oplus 2} \circ \rho_{i\oplus 2,i\oplus 3} = {}^{a_i}_{b_i}\times^{a_{i\oplus 1}}_{b_{i\oplus 1}} \circ {}^{a_{i\oplus 1}}_{b_{i\oplus 1}}\times^{a_{i\oplus 2}}_{b_{i\oplus 2}} \circ {}^{a_{i\oplus 2}}_{b_{i\oplus 2}}\overline{\times}^{a_{i\oplus 3}}_{b_{i\oplus 3}} = {}^{a_i}_{b_i}\overline{\times}^{a_{i\oplus 3}}_{b_{i\oplus 3}}$$

Let us replace $\rho_{i,i\oplus 1}$, $\rho_{i\oplus 1,i\oplus 2}$, $\rho_{i\oplus 2,i\oplus 3}$ by $\rho_{i,i\oplus 3}$ in the sequence $\rho_{0,1}, \rho_{1,2}, \ldots, \rho_{2k,0}$. We have $\langle (a_i,b_i),(a_{i\oplus 3},b_{i\oplus 3}) \rangle \in P$, since otherwise the predicate $\rho_{i,i\oplus 3}$ is not preserved. Hence, we can delete two vertices in the cycle $(a_0,b_0),(a_1,b_1),\ldots,(a_{2k},b_{2k}) \in M^o$. This contradicts that this sequence is the shortest among odd sequences. Therefore, such a fragment does not exist.

If the sequence contains a fragment $\rho_{i,i\oplus 1} = {}^{a_i}_{b_i}\overline{\times}^{a_{i\oplus 1}}_{b_{i\oplus 1}}$, $\rho_{i\oplus 1,i\oplus 2} = {}^{a_{i\oplus 1}}_{b_{i\oplus 1}}\times^{a_{i\oplus 2}}_{b_{i\oplus 2}}$, $\rho_{i\oplus 2,i\oplus 3} = {}^{a_{i\oplus 2}}_{b_{i\oplus 2}}\overline{\times}^{a_{i\oplus 3}}_{b_{i\oplus 3}}$, then these predicates can be replaced by:

$$\rho_{i,i\oplus 3} \stackrel{\Delta}{=} \rho_{i,i\oplus 1} \circ \rho_{i\oplus 1,i\oplus 2} \circ \rho_{i\oplus 2,i\oplus 3} = {}^{a_i}_{b_i}\overline{\times}^{a_{i\oplus 1}}_{b_{i\oplus 1}} \circ {}^{a_{i\oplus 1}}_{b_{i\oplus 1}}\times^{a_{i\oplus 2}}_{b_{i\oplus 2}} \circ {}^{a_{i\oplus 2}}_{b_{i\oplus 2}}\overline{\times}^{a_{i\oplus 3}}_{b_{i\oplus 3}} = {}^{a_i}_{b_i}\overline{\times}^{a_{i\oplus 3}}_{b_{i\oplus 3}}$$

As in the previous case, we obtain a contradiction. Consequently, we have an odd sequence ${}^{a_0}_{b_0}\overline{\times}^{a_1}_{b_1}, {}^{a_1}_{b_1}\overline{\times}^{a_2}_{b_2}, \ldots, {}^{a_{2k-1}}_{b_{2k-1}}\overline{\times}^{a_{2k}}_{b_{2k}}, {}^{a_{2k}}_{b_{2k}}\overline{\times}^{a_0}_{b_0} \in Inv(F)$. By Lemma 4.1, this class of predicates is NP-hard. ∎

5. Existence of the majority operation

The necessary local conditions tell that every two-element subalgebra of a tractable algebra contains certain operations. The simplest algebras over a domain A that satisfy these conditions are the following: $F_1 = \{\phi, \psi\}$ where ϕ, ψ are conservative commutative operations such that $\phi(a, b) \neq \psi(a, b)$ for every $a \neq b \in A$, and $F_2 = \{m\}$ where m is a conservative arithmetical operation, i.e. $m(x, x, y) = m(y, x, x) = m(y, x, y) = y$. This leads us to the following definitions.

Definition 5.1. Suppose a set of operations H over D is conservative and $B \subseteq \{\{x, y\} | x, y \in D, x \neq y\}$. A pair of binary operations $\phi, \psi \in H$ is called a *tournament pair* on B, if $\forall \{x, y\} \in B$ $\phi(x, y) = \phi(y, x), \psi(x, y) = \psi(y, x), \phi(x, y) \neq \psi(x, y)$ and for arbitrary $\{x, y\} \in \overline{B}$, $\phi(x, y) = x, \psi(x, y) = x$. An operation $m \in H$ is called *arithmetical* on B, if $\forall \{x, y\} \in B$ $m(x, x, y) = m(y, x, x) = m(y, x, y) = y$.

Definition 5.2. An operation $\mu : A^3 \to A$, satisfying the equality

$$\mu(x, y, y) = \mu(y, x, y) = \mu(y, y, x) = y$$

is called majority operation.

Theorem 5.3. *If F satisfies the necessary local conditions and $T_F = (M^o, P)$ is bipartite, then F contains a tournament pair on M.*

Proof. Let M_1, M_2 denote a partitioning of the bipartite graph $T_F = (M^o, P)$. Then, for every $(a, b), (c, d) \in M_1$, there is a function $\phi \in F : \underset{b}{\overset{a}{\downarrow}}\underset{d}{\overset{c}{\downarrow}}\phi$. Let us prove by induction that for every $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \in M_1$, there is a $\phi : \underset{b_1}{\overset{a_1}{\downarrow}}\underset{b_2}{\overset{a_2}{\downarrow}}\ldots\underset{b_n}{\overset{a_n}{\downarrow}}\phi$.

The base of induction $n = 2$ is obvious. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_{n+1}, b_{n+1}) \in M_1$ be given. By the induction hypothesis, there are $\phi_1, \phi_2, \phi_3 \in F$: $\underset{b_2}{\overset{a_2}{\downarrow}}\ldots\underset{b_n}{\overset{a_n}{\downarrow}}\underset{b_{n+1}}{\overset{a_{n+1}}{\downarrow}}\phi_1, \underset{b_1}{\overset{a_1}{\downarrow}}\underset{b_3}{\overset{a_3}{\downarrow}}\ldots\underset{b_n}{\overset{a_n}{\downarrow}}\underset{b_{n+1}}{\overset{a_{n+1}}{\downarrow}}\phi_2, \underset{b_1}{\overset{a_1}{\downarrow}}\underset{b_2}{\overset{a_2}{\downarrow}}\ldots\underset{b_n}{\overset{a_n}{\downarrow}}\phi_3$. Then, it is easy to see that $\underset{b_1}{\overset{a_1}{\downarrow}}\ldots\underset{b_n}{\overset{a_n}{\downarrow}}\underset{b_{n+1}}{\overset{a_{n+1}}{\downarrow}}\phi_3(\phi_1(x, y), \phi_2(x, y))$ which completes the induction proof.

The analogous statement can be proved for M_2. Moreover, $M_2 = \{(x, y) | (y, x) \in M_1\}$. So it follows from the proof that there are binary operations $\phi', \psi' \in F$, such that $\forall (x, y) \in M_1$: $\underset{y}{\overset{x}{\downarrow}}\phi'$ and $\forall (x, y) \in M_2$: $\underset{y}{\overset{x}{\downarrow}}\psi'$. Thus, the operations $\phi(x, y) = \phi'(x, \phi'(y, x))$ and $\psi(x, y) = \psi'(x, \psi'(y, x))$ satisfy the conditions of theorem. ∎

The proof of the following theorem uses the ideas from [3].

Theorem 5.4. *If F satisfies the necessary local conditions and $\overline{M} \neq \emptyset$, then F contains an arithmetical operation on $\overline{M}$.*

Theorem 5.5. *If F satisfies the necessary local conditions and $T_F = (M^o, P)$ is bipartite, then F contains a majority operation μ.*

Proof. If $\overline{M} \neq \emptyset$, then by Theorem 5.4, F contains a function $m : A^3 \to A$ that is arithmetical on $\overline{M}$. Then the function $\mu^1(x, y, z) = m(x, m(x, y, z), z)$ satisfies the conditions $\forall \{x, y\} \in \overline{M}$ $\mu^1(x, y, y) = \mu^1(y, x, y) = \mu^1(y, y, x) = y$. It is clear that, in the case where $M = \emptyset$, we can take μ^1 as majority μ.

If $M \neq \emptyset$, then by Theorem 5.3, there is a tournament pair $\phi, \psi : A^2 \to A$ on M. Then, the function $\mu^2(x,y,z) = \phi(\phi(\psi(x,y), \psi(y,z)), \psi(x,z))$ satisfies conditions $\forall \{x,y\} \in M\ \mu^2(x,y,y) = \mu^2(y,x,y) = \mu^2(y,y,x) = y$, and $\forall \{x,y,z\} \in \overline{M}\ \mu^2(x,y,z) = x$. If $\overline{M} = \emptyset$, then we can take μ^2 as the majority μ.

Finally, if $M, \overline{M} \neq \emptyset$, then $\mu(x,y,z) = \mu^1\left(\mu^2(x,y,z), \mu^2(y,z,x), \mu^2(z,x,y)\right)$. ∎

6. Consistency and microstructure graphs

Every predicate in $Inv(F)$, when F contains a majority operation, is equal to the join of its binary projections [1]. To prove Theorem 3.4, it is consequently sufficient to prove polynomial-time solvability of $MinHom(\Gamma)$ where $\Gamma = \left\{\rho | \rho \subseteq A^2, \rho \in Inv(F)\right\}$, i.e. the $MinHom$ problem restricted to binary constraint languages.

Definition 6.1. Suppose we are given a constraint language Γ over A. Denote by $2 - MinHom(\Gamma)$ the following minimization problem:
Instance: A finite set of variables $X = \{x_1, \ldots, x_n\}$, a constraints pair (U, B) where $U = \langle \rho_i \rangle_{1 \le i \le n}$, $B = \langle \rho_{kl} \rangle_{1 \le k \neq l \le n}$, $\rho_i, \rho_{kl} \in \Gamma$, and weights $w_{ia}, 1 \le i \le n, a \in A$.
Solution: Assignment $f : \{x_1, \ldots, x_n\} \to A$, such that $\forall i\ f(x_i) \in \rho_i$ and $\forall k \neq l\ \ (f(x_k), f(x_l)) \in \rho_{kl}$.
Measure: $\sum\limits_{i=1}^{n} w_{if(x_i)}$.

We suppose everywhere that $\rho_{kl} = \rho^t_{lk}$ (where $\rho^t = \{(b,a) \,|\, (a,b) \in \rho\}$). If $\rho_{kl} \neq \rho^t_{lk}$, then we can always define $\forall k \neq l\ \rho_{kl} := \rho_{kl} \cap \rho^t_{lk}$, which does not change the set $\{(a,b) \,|\, (a,b) \in \rho_{kl}, (b,a) \in \rho_{lk}\}$. For a binary predicate ρ, define projections $\mathrm{Pr}_1\, \rho = \{a | (a,b) \in \rho\}$ and $\mathrm{Pr}_2\, \rho = \{b | (a,b) \in \rho\}$.

Definition 6.2. An instance of $2 - MinHom(\Gamma)$ with constraints pair $U = \langle \rho_i \rangle_{1 \le i \le n}$, $B = \langle \rho_{kl} \rangle_{1 \le k \neq l \le n}$ is called *arc-consistent* if $\forall i \neq j : \mathrm{Pr}_1\, \rho_{ij} = \rho_i, \mathrm{Pr}_2\, \rho_{ij} = \rho_j$ and is called *path-consistent* if for each different $i, j, k : \rho_{ik} \subseteq \rho_{ij} \circ \rho_{jk}$.

Obviously, by applying operations of the type $\rho_i := \rho_i \cap \mathrm{Pr}_1\, \rho_{ij}$, $\rho_j := \rho_j \cap \mathrm{Pr}_2\, \rho_{ij}$, $\rho_{ij} := \rho_{ij} \cap (\rho_i \times A)$, $\rho_{ij} := \rho_{ij} \cap (A \times \rho_j)$, $\rho_{ik} := \rho_{ik} \cap (\rho_{ij} \circ \rho_{jk})$, we can always make an instance arc-consistent and path-consistent in polynomial time. It is clear that under this transformations the set of feasible solutions does not change.

Definition 6.3. The *microstructure graph* [14] of an instance of $2 - MinHom(\Gamma)$ with constraints pair $U = \langle \rho_i \rangle_{1 \le i \le n}$, $B = \langle \rho_{kl} \rangle_{1 \le k \neq l \le n}$ is the graph $M_{U,B} = (V, E)$, where $V = \{(i,a) \,| 1 \le i \le n, a \in \rho_i\}$ and $E = \{\langle (i,a), (j,b) \rangle \,| i \neq j, (a,b) \in \rho_{ij}\}$.

Theorem 6.4. *Let $I = (X, U, B, w)$ be a satisfiable instance of $2 - MinHom(\Gamma)$. Then there is a one-to-one correspondence between maximal-size cliques of $M_{U,B}$ and satisfying assignments of I.*

Proof. The microstructure graph of an instance with constraints pair $U = \langle \rho_i \rangle_{1 \le i \le n}$, $B = \langle \rho_{kl} \rangle_{1 \le k \neq l \le n}$ is, obviously, n-partite, since $V = \bigcup\limits_{i=1}^{n} \{i\} \times \rho_i$ and pairs $(i,a), (i,b), a \neq b$ are not connected. Therefore, the cardinality of a maximal clique of $M_{U,B} = (V, E)$ is not greater than n.

If the cardinality of a maximal clique $S \subseteq V$ is n, then, for every i, $|S \cap (\{i\} \times \rho_i)| = 1$. Then, denoting the only element of $S \cap (\{i\} \times \rho_i)$ by v_i, we see that the assignment $f(x_i) = v_i$ satisfies all constraints. The opposite is also true, i.e., if the constraints $\langle \rho_i \rangle_{1 \le i \le n}, \langle \rho_{kl} \rangle_{1 \le k \ne l \le n}$ can be satisfied by some assignment f, then $\{(i, f(x_i)) \,|1 \le i \le n\}$ is a clique of cardinality n. ■

Hence, $2 - MinHom(\Gamma)$ can be reduced to finding a maximal-size clique $S \subseteq V$ of a microstructure graph that minimizes the following value:

$$\sum_{(i,a) \in S} w_{ia}.$$

Definition 6.5. Let $MMClique$ (Minimal weight among maximal-size cliques) denote the following minimization problem:
Instance: A graph $G = (V, E)$ and weights $w_i \in \mathbb{N}, i \in V$.
Solution: A maximal-size clique $K \subseteq V$ of G.
Measure: $\sum_{v \in K} w_v$.

The following theorem connects perfect microstructure graphs and the complexity of $MinHom$.

Theorem 6.6. *Suppose we are given a class of conservative functions F containing a majority operation. If the microstructure graph is perfect for arbitrary arc-consistent and path-consistent instances of $2 - MinHom(Inv(F))$, then F is tractable.*

Definition 6.7. A cycle C_{2k+1}, $k \ge 2$, is called *an odd hole* and its complement graph *an odd antihole.*

In Section 8 we will use the following conjecture of Berge, which was proved in [4].

Theorem 6.8. *A graph is perfect if and only if it does not contain an induced subgraph isomorphic to an odd hole or antihole.*

We say that a graph is *of the type $S_{2k+1}, k \ge 2$* if it is isomorphic to the graph with vertex set $\{0, 1, \dots, 2k\}$, where vertices $i \pmod{2k+1}$, $i+1 \pmod{2k+1}$ are not connected and vertices $i \pmod{2k+1}$, $i+2 \pmod{2k+1}$ are connected. Other pairs can be connected arbitrarily. Obviously, every odd hole or antihole is of one of the types $S_{2k+1}, k \ge 2$.

7. Arithmetical deadlocks

The key idea for the proof of the polynomial case of Theorem 3.4 is to show that path- and arc-consistent instances of $2 - MinHom(Inv(F))$ have a perfect microstructure graph. We will prove this by showing that the microstructure graph forbids certain types of subgraphs. The exact formulation of the result can be found below in Theorem 8.1. This theorem uses the nonexistence of structures called *arithmetical deadlocks* which are introduced in this section.

Definition 7.1. Suppose H is a conservative set of functions over D, $m \in H$ is an arithmetical operation on $B \subseteq \{\{x, y\} \,| x, y \in D, x \ne y\}$ and the pair $\phi, \psi \in H$ is a tournament pair on $\overline{B}$. An instance of $2 - MinHom(Inv(H))$ with constraints pair $U = \langle \rho_i \rangle_{1 \le i \le n}$, $B = \langle \rho_{kl} \rangle_{1 \le k \ne l \le n}$ is called *an odd arithmetical deadlock* if there is a subset $\{i_0, \dots, i_{k-1}\} \subseteq \{1, \dots, n\}, k \ge 3$ of odd cardinality and $\{x_0, y_0\}, \dots, \{x_{k-1}, y_{k-1}\} \in B$, such that for $0 \le s \le k-1$: $\rho_{i_s, i_{s\oplus 1}} \cap \{x_s, y_s\} \times \{x_{s\oplus 1}, y_{s\oplus 1}\} = \begin{smallmatrix} x_s & & x_{s\oplus 1} \\ & \times & \\ y_s & & y_{s\oplus 1} \end{smallmatrix}$, where $i \oplus j$ denotes $i + j (\mathrm{mod}\ k)$. The subset $\{i_0, \ldots, i_{k-1}\}$ is called *a deadlock subset.*

Theorem 7.2. *Suppose H is a conservative set of functions over D, $m \in H$ is an arithmetical operation on $B \subseteq \{\{x,y\} \,|x, y \in D, x \ne y\}$ and the pair $\phi, \psi \in H$ is a tournament pair on $\overline{B}$. If an instance of $2 - MinHom\,(Inv\,(H))$ is arc-consistent and path-consistent, then it cannot be an odd arithmetical deadlock.*

8. Final step in a proof of polynomial case

Theorem 8.1. *Suppose that F satisfies the necessary local conditions and that the graph $T_F = (M^o, P)$ is bipartite. Then for every path- and arc-consistent instance of $2 - MinHom\,(Inv\,(F))$, its microstructure graph forbids subgraphs of the type $S_{2p+1}, p \ge 2$.*

Proof. Suppose to the contrary that we have a path- and arc-consistent instance $I = (X, U, B, w)$ of $2 - MinHom\,(Inv\,(F))$ with constraints pair $U = \langle \rho_i \rangle_{1 \le i \le n}$, $B = \langle \rho_{kl} \rangle_{1 \le k \ne l \le n}$ and its microstructure graph has a subgraph of the type $S_{2p+1}, p \ge 2$. For convenience, let us introduce $\rho_{ii} = \{(a,a)\,| a \in \rho_i\}$. Then, there is a set of pairs $\{(i_0, b_0), (i_1, b_1), \ldots, (i_{2p}, b_{2p})\}$, such that for $0 \le l \le 2p$: $(b_l, b_{l\oplus 1}) \notin \rho_{i_l i_{l\oplus 1}}$ and $(b_l, b_{l\oplus 2}) \in \rho_{i_l i_{l\oplus 2}}$, where $i \oplus j$ denotes $i + j (\mathrm{mod}\ 2p+1)$.

From $(b_l, b_{l\oplus 2}) \in \rho_{i_l i_{l\oplus 2}}$ and the path-consistency condition $\rho_{i_l i_{l\oplus 2}} \subseteq \rho_{i_l i_{l\oplus 1}} \circ \rho_{i_{l\oplus 1} i_{l\oplus 2}}$, we see that there is $a_{l\oplus 1}$, such that $(b_l, a_{l\oplus 1}) \in \rho_{i_l i_{l\oplus 1}}$ and $(a_{l\oplus 1}, b_{l\oplus 2}) \in \rho_{i_{l\oplus 1} i_{l\oplus 2}}$.

Consider the predicate $\rho'_{l,l\oplus 1} = \rho_{i_l i_{l\oplus 1}} \cap \{a_l, b_l\} \times \{a_{l\oplus 1}, b_{l\oplus 1}\} \in Inv\,(F)$. Obviously, $\rho'_{l,l\oplus 1}$ equals to either $\begin{smallmatrix} a_l & & a_{l\oplus 1} \\ & \overline{\times} & \\ b_l & & b_{l\oplus 1} \end{smallmatrix}$ or $\begin{smallmatrix} a_l & & a_{l\oplus 1} \\ & \times & \\ b_l & & b_{l\oplus 1} \end{smallmatrix}$.

Let us show that if $\{a_l, b_l\} \in \overline{M}$, then $\{a_{l\oplus 1}, b_{l\oplus 1}\} \in \overline{M}$, too. Assume to the contrary that $\{a_{l\oplus 1}, b_{l\oplus 1}\} \in M$. Then, by Theorem 5.3, there is a $\phi \in F : \begin{smallmatrix} a_{l\oplus 1} \\ \downarrow \\ b_{l\oplus 1} \end{smallmatrix}\ \phi$, where $\phi|_{\{a_l, b_l\}}$ is a projection on the first coordinate. In this case, ϕ preserves neither $\begin{smallmatrix} a_l & & a_{l\oplus 1} \\ & \overline{\times} & \\ b_l & & b_{l\oplus 1} \end{smallmatrix}$ nor $\begin{smallmatrix} a_l & & a_{l\oplus 1} \\ & \times & \\ b_l & & b_{l\oplus 1} \end{smallmatrix}$, because

$$\begin{pmatrix} b_l \\ b_{l\oplus 1} \end{pmatrix} = \begin{pmatrix} \phi\,(b_l, a_l) \\ \phi\,(a_{l\oplus 1}, b_{l\oplus 1}) \end{pmatrix}.$$

Hence, we need to consider two cases only: 1) $\forall l\ \{a_l, b_l\} \in M$ and 2) $\forall l\ \{a_l, b_l\} \in \overline{M}$. In the first case, we have $\langle (a_l, b_l), (a_{l\oplus 1}, b_{l\oplus 1}) \rangle \in P$, i.e., there is an odd cycle in T_F which contradicts that T_F is bipartite.

Now, consider the case $\forall l\ \{a_l, b_l\} \in \overline{M}$. By Theorem 5.4, there is a function $m \in F$, arithmetical on $\overline{M}$. If $\rho'_{l,l\oplus 1} = \begin{smallmatrix} a_l & & a_{l\oplus 1} \\ & \overline{\times} & \\ b_l & & b_{l\oplus 1} \end{smallmatrix}$, then we have that

$$\begin{pmatrix} b_l \\ b_{l\oplus 1} \end{pmatrix} = \begin{pmatrix} m\,(a_l, a_l, b_l) \\ m\,(b_{l\oplus 1}, a_{l\oplus 1}, a_{l\oplus 1}) \end{pmatrix} \in \rho'_{l,l\oplus 1}$$

and $\rho'_{l,l\oplus 1} = \begin{smallmatrix} a_l & & a_{l\oplus 1} \\ & \times & \\ b_l & & b_{l\oplus 1} \end{smallmatrix}$.

Consider the set $\{i_0, i_1, \ldots, i_{2p}\}$. Suppose first that all $i_0, i_1, \ldots, i_{2p}$ are distinct. Then, Theorems 5.3 and 5.4 show us that we have an arithmetical operation $m \in F$ on $\overline{M}$ and a tournament pair $\phi, \psi \in F$ on M. It is easy to see that an instance of $2 - MinHom\,(Inv\,(F))$ with constraints pair $U = \langle \rho_i \rangle_{1 \le i \le n}, B = \langle \rho_{kl} \rangle_{1 \le k \ne l \le n}$ is an odd arithmetical deadlock where $\{i_0, i_1, \ldots, i_{2p}\}$ is a deadlock set. This contradicts that I is arc- and path-consistent.

The case when the elements $i_0, i_1, \ldots, i_{2p}$ are not distinct can be reduced to the previous case by the following trick: introduce a new set of variables $X' = \{(i_0, 0), (i_1, 1), \ldots, (i_{2p}, 2p)\}$ and $\rho_{(i_s,s)} = \rho_{i_s}$, where $0 \leq s \leq 2p$. If $i_m \neq i_n$, then $\rho_{(i_m,m),(i_n,n)} = \rho_{i_m,i_n}$, else $\rho_{(i_m,m),(i_n,n)} = \{(a,a) | a \in \rho_{i_m}\}$. It is easy to see that an instance with constraints pair $U = \{\rho_i\}_{i \in X'}, B = \{\rho_{kl}\}_{k \neq l \in X'}$ satisfy the conditions of Theorem 7.2 and is an odd arithmetical deadlock, where the set $\{(i_0, 0), (i_1, 1), \ldots, (i_{2p}, 2p)\}$ is a deadlock set. Therefore, we have a contradiction. ∎

Proof of polynomial case of Theorem 3.4. The conditions of Theorem 3.4 coincides with the conditions of Theorem 8.1 so the microstructure graph of an arc- and path-consistent instance forbids subgraphs of the type $S_{2p+1}, p \geq 2$. By Theorem 6.8, it is perfect and, by Theorem 6.6, we see that the class F is tractable. ∎

Theorems 3.3 and 3.4 give the required dichotomy for conservative algebras, which implies the dichotomy for conservative constraint languages. By Theorem 2.9, we have the following general dichotomy.

Theorem 8.2. *If $MinHom(\Gamma)$ is not tractable then it is NP-hard.*

9. Related work and open problems

$MinHom$ can be viewed as a problem that fits the VCSP (Valued CSP) framework by [5]. By a valued predicate of arity m over a domain D, we mean a function $p : D^m \to \mathbb{N} \cup \{\infty\}$. Informally, if Γ is a finite set of valued predicates over a finite domain D, then an instance of $VCSP(\Gamma)$ is a set of variables together with specified subsets of variables restricted by valued predicates from Γ. Any assignment to variables can be considered a solution and the measure of this solution is the sum of the values that the valued predicates take under the assignments of the specified subsets of variables. The problem is to minimize this measure. It is widely believed that a dichotomy conjecture holds for $VCSP(\Gamma)$, too.

Our dichotomy result for $MinHom$ encourages us to consider generalizations that belong to this framework.

1. Suppose we are given a constraint language Γ and a finite set of unary functions $F \subseteq \{f : D \to \mathbb{N}\}$. Let $MinHom_F(\Gamma)$ denote a minimization problem which is defined completely analogously to $MinHom(\Gamma)$ except that we are restricted to minimizing functionals of the following form: $\sum_{i=1}^{n} \sum_{f \in F} w_{if} f(x_i)$. We believe that the complexity of $MinHom_F(\Gamma)$ is determined by Γ and a certain loopless digraph $G_F = (D, \{(x,y) : \exists f \in F \; f(x) > f(y)\})$. This conjecture holds when Γ is conservative and every two vertices of G_F have an arc (of any direction) between them. Of course, a complete classification of the complexity of this problem is an open question.

2. Suppose we have a finite valued constraint language Γ, i.e. a set of valued predicates over some finite domain set. If Γ contains all unary valued predicates, we call $VCSP(\Gamma)$ a conservative $VCSP$. This name is motivated by the fact that in this case the multimorphisms (which is a generalization of polymorphisms for valued constraint languages [5]) of Γ must consist of conservative functions. Since there is a well-known dichotomy for conservative CSPs [3], we suspect that there is a dichotomy for conservative $VCSPs$.

3. $MinHom$ has (just as CSP) a homomorphism formulation. If we restrict ourselves to relational structures given by digraphs, we arrive at the following problem which we call

digraph $MinHom$: given digraphs S, H and weights $w_{ij}, i \in S, j \in H$, find a homomorphism $h : S \to H$ that minimizes the sum $\sum_{s \in S} w_{sh(s)}$. Suppose we have sets of digraphs $\mathbb{G}_1, \mathbb{G}_2$. Then, $MinHom(\mathbb{G}_1, \mathbb{G}_2)$ denotes the digraph $MinHom$ problem when the first digraph is from $\mathbb{G}_1$ and the second is from $\mathbb{G}_2$. In this case, $MinHom(All, \{H\})$ coincides with $MinHom(\{H\})$ which is characterized in this paper. Another characterization based on digraph theory was announced during the preparation of the camera-ready version of this paper [16]. We believe that this approach could be fruitful for characterizing the complexity of $MinHom(\mathbb{G}, \mathbb{G})$: for example, is there a dichotomy for $MinHom(\mathbb{G}, \mathbb{G})$?

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