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IS RAMSEY'S THEOREM ω-AUTOMATIC?

DIETRICH KUSKE

Centre national de la recherche scientifique (CNRS) and
Laboratoire Bordelais de Recherche en Informatique (LaBRI), Bordeaux, France

Abstract. We study the existence of infinite cliques in ω-automatic (hyper-)graphs. It turns out that the situation is much nicer than in general uncountable graphs, but not as nice as for automatic graphs.

More specifically, we show that every uncountable ω-automatic graph contains an uncountable co-context-free clique or anticlique, but not necessarily a context-free (let alone regular) clique or anticlique. We also show that uncountable ω-automatic ternary hypergraphs need not have uncountable cliques or anticliques at all.

Introduction

Every infinite graph has an infinite clique or an infinite anticlique – this is the paradigmatic formulation of Ramsey's theorem [Ram30]. But this theorem is highly non-constructive since there are recursive infinite graphs whose infinite cliques and anticliques are all non-recursive (not even in Σ_2^0, [Joc72], cf. [Gas98, Thm. 4.6]). Recall that a graph is recursive if both its set of nodes and its set of edges can be decided by a Turing machine. Replacing these Turing machines by finite automata, one obtains the more restrictive notion of an *automatic graph*: the set of nodes is a regular set and whether a pair of nodes forms an edge can be decided by a synchronous two-tape automaton (this concept is known since the beginning of automata theory, a systematic study started with [KN95, BG04], see [Rub08] for a recent overview). In this context, the situation is much more favourable: every infinite automatic graph contains an infinite regular clique or an infinite regular anticlique (cf. [Rub08]).

Soon after Ramsey's paper from 1930, authors got interested in a quantitative analysis. For finite graphs, one can ask for the minimal number of nodes that guarantee the existence of a clique or anticlique of some prescribed size. This also makes sense in the infinite: how many nodes are necessary and sufficient to obtain a clique or anticlique of size $\aleph_0$ (Ramsey's theorem tells us: $\aleph_0$) or $\aleph_1$ (here one needs more than $2^{\aleph_0}$ nodes [Sie33, ER56]).

Since automatic graphs contain at most $\aleph_0$ nodes, we need a more general notion for a recursion-theoretic analysis of this situation. For this, we use Blumensath & Grädel's [BG04] ω-automatic graphs: the names of nodes form a regular ω-language and the edge

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relation (on names) as well as the relation "these two names denote the same node" can be decided by a synchronous 2-tape Büchi-automaton. In this paper, we answer the question whether these ω-automatic graphs are more like automatic graphs (i.e., large cliques or anticliques with nice properties exist) or like general graphs (large cliques need not exist).

Our answer to this question is a clear "somewhere in between": We show that every ω-automatic graph of size $2^{\aleph_0}$ contains a clique or anticlique of size $2^{\aleph_0}$ (Theorem 3.1) – this is in contrast to the case of arbitrary graphs where such a subgraph need not exist [Sie33]. But in general, there is no regular clique or anticlique (Theorem 3.13) – this is in contrast with the case of automatic graphs where we always find a large regular clique or anticlique. Finally, we also provide an ω-automatic "ternary hypergraph" of size $2^{\aleph_0}$ without any clique or anticlique of size $\aleph_1$, let alone $2^{\aleph_0}$ (Theorem 3.11).

For Theorem 3.1, we re-use the proof from [BKR08] that was originally constructed to deal with infinity quantifiers in ω-automatic structures. The proof of Theorem 3.13 makes use of the "ultimately equal" relation. This relation was also crucial in the separation of injectively from general ω-automatic structures [HKMN08] as well as in the handling of infinity quantifiers in [KL08] and [BKR08]. In the ternary hypergraph from Theorem 3.11, a 3-set $\{x, y, z\}$ of infinite words with $x <_{\text{lex}} y <_{\text{lex}} z$ forms an undirected hyperedge iff the longest common prefix of x and y is shorter than the longest common prefix of y and z.

¿From Theorem 3.1 (i.e., the existence of large cliques or anticliques in ω-automatic graphs), we derive that any ω-automatic partial order of size $2^{\aleph_0}$ contains an antichain of size $2^{\aleph_0}$ or a copy of the real line.

1. Preliminaries

1.1. Ramsey-theory

For a set V and a natural number $k \geq 1$, let $[V]^k$ denote the set of k-element subsets of V. A (k, ℓ)-*partition* is a pair $G = (V, E_1, \ldots, E_\ell)$ where V is a set and $(E_1, \ldots, E_\ell)$ is a partition of $[V]^k$ into (possibly empty) sets. For $1 \leq i \leq \ell$, a set $W \subseteq V$ is E_i-*homogeneous* if $[W]^k \subseteq E_i$; it is *homogeneous* if it is E_i-homogeneous for some $1 \leq i \leq \ell$. The case $k = \ell = 2$ is special: any $(2,2)$-partition $G = (V, E_1, E_2)$ can be considered as an (undirected loop-free) graph (V, E_1). Homogeneous sets in G are then complete or discrete induced subgraphs of (V, E_1).

Ramsey theory is concerned with the following question: Does every (k, ℓ)-partition $G = (V, E_1, \ldots, E_\ell)$ with $|V| = \kappa$ have a homogeneous set of size λ (where κ and λ are cardinal numbers and $k, \ell \geq 2$ are natural numbers). If this is the case, one writes

$$\kappa \to (\lambda)^k_\ell$$

(a notation due to Erdős and Rado [ER56]). This allows to formulate Ramsey's theorem concisely:

Theorem 1.1 (Ramsey [Ram30])**.** *If $k, \ell \geq 2$, then $\aleph_0 \to (\aleph_0)^k_\ell$.*

In particular, every graph with $\aleph_0$ nodes contains a complete or discrete induced subgraph of the same size. If one wants to find homogeneous sets of size $\aleph_1$, the base set has to be much larger:

Theorem 1.2 (Sierpiński [Sie33])**.** *If $k, \ell \geq 2$, then $2^{\aleph_0} \not\to (\aleph_1)^k_\ell$ and therefore in particular $2^{\aleph_0} \not\to (2^{\aleph_0})^k_\ell$.*

Erdős and Rado [ER56] proved that partitions of size properly larger than $2^{\aleph_0}$ have homogeneous sets of size $\aleph_1$. For more details on infinite Ramsey theory, see [Jec02, Chapter 9].

1.2. ω-languages

Let Γ be a finite alphabet. With Γ^* we denote the set of all finite words over the alphabet Γ. The set of all nonempty finite words is Γ^+. An *ω-word* over Γ is an infinite ω-sequence $x = a_0a_1a_2\cdots$ with $a_i \in \Gamma$, we set $x[i,j) = a_ia_{i+1}\dots a_{j-1}$ for natural numbers $i \leq j$. In the same spirit, $x[i,\omega)$ denotes the ω-word $a_ia_{i+1}\dots$. The set of all ω-words over Γ is denoted by Γ^ω and $\Gamma^\infty = \Gamma^* \cup \Gamma^\omega$. For a set $V \subseteq \Gamma^+$ of finite words let $V^\omega \subseteq \Gamma^\omega$ be the set of all ω-words of the form $v_0v_1v_2\cdots$ with $v_i \in V$. Two infinite words $x, y \in \Gamma^\omega$ are *ultimately equal*, briefly $x \sim_e y$, if there exists $i \in \mathbb{N}$ with $x[i,\omega) = y[i,\omega)$. By $\leq_{\text{lex}}$, we denote the lexicographic order on the set Σ^ω (with some, implicitly assumed linear order on the letters from Σ) and $\leq_{\text{pref}}$ the prefix order on Σ^∞.

For $\Sigma = \{0,1\}$, the support $\text{supp}(x) \subseteq \mathbb{N}$ is the set of positions of the letter 1 in the word $x \in \Sigma^\omega$.

A (nondeterministic) *Büchi-automaton* M is a tuple $M = (Q, \Gamma, \delta, \iota, F)$ where Q is a finite set of states, $\iota \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and $\delta \subseteq Q \times \Gamma \times Q$ is the transition relation. If $\Gamma = \Sigma^n$ for some alphabet Σ, then we speak of an *n-dimensional Büchi-automaton over* Σ. A *run* of M on an ω-word $x = a_0a_1a_2\cdots$ is an ω-word $r = p_0p_1p_2\cdots$ over the set of states Q such that $(p_i, a_i, p_{i+1}) \in \delta$ for all $i \geq 0$. The run r is *successful* if $p_0 = \iota$ and there exists a final state from F that occurs infinitely often in r. The ω-language $L(M) \subseteq \Gamma^\omega$ defined by M is the set of all ω-words that admit a successful run. An ω-language $L \subseteq \Gamma^\omega$ is *regular* if there exists a Büchi-automaton M with $L(M) = L$.

Alternatively, regular ω-languages can be represented algebraically. To this end, one defines *ω-semigroups* to be two-sorted algebras $S = (S_+, S_\omega; \cdot, *, \pi)$ where $\cdot : S_+ \times S_+ \to S_+$ and $* : S_+ \times S_\omega \to S_\omega$ are binary operations and $\pi : (S_+)^\omega \to S_\omega$ is an ω-ary operation such that the following hold:

- $(S_+, \cdot)$ is a semigroup,
- $s * (t * u) = (s \cdot t) * u$,
- $s_0 \cdot \pi((s_i)_{i\geq 1}) = \pi((s_i)_{i\geq 0})$,
- $\pi((s_i^1 \cdot s_i^2 \cdots s_i^{k_i})_{i\geq 0}) = \pi((t_j)_{j\geq 0})$ whenever
$$(t_j)_{j\geq 0} = (s_0^1, s_0^2, \dots, s_0^{k_0}, s_1^1, \dots, s_1^{k_1}, \dots)\ .$$

The ω-semigroup S is *finite* if both, S_+ and S_ω are finite. The free ω-semigroup generated by Γ is
$$\Gamma^\infty = (\Gamma^+, \Gamma^\omega; \cdot, *, \pi)$$
where $u \cdot v$ and $u * x$ are the natural operations of prefixing a word by the finite word u, and $\pi((u_i)_{i\geq 0})$ is the omega-word $u_0u_1u_2\dots$. A homomorphism $h : \Gamma^\infty \to S$ of ω-semigroups maps finite words to elements of S_+ and ω-words to elements of S_ω and commutes with the operations $\cdot$, $*$, and π. The algebraic characterisation of regular ω-languages then reads as follows.

Proposition 1.3. *An ω-language $L \subseteq \Gamma^\omega$ is regular if and only if there exists a finite ω-semigroup S, a set $T \subseteq S_\omega$, and a homomorphism $\eta : \Gamma^\infty \to S$ such that $L = \eta^{-1}(T)$.*

Hence, every Büchi-automaton is "equivalent" to a homomorphism into some finite ω-semigroup together with a distinguished set T (and vice versa).

For ω-words $x_i = a_i^0 a_i^1 a_i^2 \cdots \in \Gamma^\omega$, the *convolution* $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in (\Gamma^n)^\omega$ is defined by

$$(x_1, \ldots, x_n)^\otimes = (a_1^0, \ldots, x_n^0)\,(a_1^1, \ldots, a_n^1)\,(a_1^2, \ldots, a_n^2) \cdots \ .$$

An n-ary relation $R \subseteq (\Gamma^\omega)^n$ is called *ω-automatic* if the ω-language $\{(x_1, \ldots, x_n)^\otimes \mid (x_1, \ldots, x_n) \in R\}$ is regular.

To describe the complexity of ω-languages, we will use language-theoretic terms. Let LANG denote the class of all languages (i.e., sets of finite words over some finite set of symbols) and ωLANG the class of all ω-languages. By REG and ωREG, we denote the regular languages and ω-languages, resp. An ω-language is *context-free* if it can be accepted by a pushdown-automaton with Büchi-acceptance (on states), it is *co-context-free* if its complement is context-free. We denote by ωCF the set of context-free ω-languages and by co-ωCF their complements. An ω-language belongs to LANG* if it is of the form $\bigcup_{1 \le i \le n} U_i V_i^\omega$ with $U_i, V_i \in$ LANG. Then ωREG $\subseteq$ LANG* and ωCF $\subseteq$ LANG* where the sets U_i and V_i are regular and context-free, resp [Sta97]. In between these two classes, we define the class ωerCF of *eventually regular context-free* ω-languages that comprises all sets of the form $\bigcup_{1 \le i \le n} U_i V_i^\omega$ with $U_i \in$ LANG context-free and $V_i \in$ LANG regular. Alternatively, eventually regular context-free ω-languages are the finite unions of ω-languages of the form $C \cdot L$ where C is a context free-language and L a regular ω-language. Let co-ωerCF denote the set of complements of eventually regular context-free ω-languages.

A final, rather peculiar class of ω-languages is $\mathbf{\Lambda}$: it is the class of ω-languages L such that $(\mathbb{R}, \le)$ embeds into $(L, \le_{\text{lex}})$ (the name derives from the notation $\boldsymbol{\lambda}$ for the order type of $(\mathbb{R}, \le)$).

1.3. ω-automatic (k, ℓ)-partitions

An *ω-automatic presentation of a (k, ℓ)-partition* $(V, E_1, \ldots, E_\ell)$ is a pair (L, h) consisting of a regular ω-language L and a surjection $h : L \to V$ such that $\{(x_1, x_2, \ldots, x_k) \in L^k \mid \{h(x_1), h(x_2), \ldots, h(x_k)\} \in E_i\}$ for $1 \le i \le k$ and $R_\approx = \{(x_1, x_2) \in L^2 \mid h(x_1) = h(x_2)\}$ are ω-automatic. An ω-automatic presentation is *injective* if h is a bijection. A (k, ℓ)-partition is *(injectively) ω-automatic* if it has an (injective) ω-automatic presentation. From [BKR08], it follows that an uncountable ω-automatic (k, ℓ)-partition has $2^{\aleph_0}$ elements.

This paper is concerned with the question whether every (injective) ω-automatic presentation (L, h) of a (k, ℓ)-partition admits a "simple" set $H \subseteq L$ such that $h(H)$ has λ elements and is homogeneous. More precisely, let $\mathcal{C}$ be a class of ω-languages, $k, \ell \ge 2$ natural numbers, and κ and λ cardinal numbers. Then we write

$$(\kappa, \omega\mathsf{A}) \to (\lambda, \mathcal{C})^k_\ell$$

if the following partition property holds: for every ω-automatic presentation (L, h) of a (k, ℓ)-partition G of size κ, there exists $H \subseteq L$ in $\mathcal{C}$ such that $h(H)$ is homogeneous in G and of size λ.

$$(\kappa, \omega\mathsf{iA}) \to (\lambda, \mathcal{C})^k_\ell$$

is to be understood similarly where we only consider injective ω-automatic presentations.

Remark 1.4. Let $G = (V, E_1, \ldots, E_\ell)$ be some (k, ℓ)-partition with ω-automatic presentation (L, h). Then the partition property above requires that there is a "large" homogeneous set $X \subseteq V$ and an ω-language $H \in \mathcal{C}$ such that $h(H) = X$, in particular, every element of

X has at least one representative in H. Alternatively, one could require that $h^{-1}(X) \subseteq L$ is an ω-language from $\mathcal{C}$. In this paper, we only encounter classes $\mathcal{C}$ of ω-languages such that the following closure property holds: if $H \in \mathcal{C}$ and R is an ω-automatic relation, then also $R(H) = \{y \mid \exists x \in H : (x,y) \in R\} \in \mathcal{C}$. Since $h^{-1}h(H) = R_{\approx}(H)$, all our results also hold for this alternative requirement $h^{-1}(X) \in \mathcal{C}$.

This paper shows

(0) if $k, \ell \geq 2$, then $(\aleph_0, \omega\mathsf{A}) \to (\aleph_0, \omega\mathrm{REG})^k_\ell$, but $(2^{\aleph_0}, \omega\mathsf{A}) \not\to (\aleph_0, \omega\mathrm{REG})^k_\ell$, see Theorem 2.1.
(1) if $\ell \geq 2$, then $(2^{\aleph_0}, \omega\mathsf{A}) \to (2^{\aleph_0}, \text{co-}\omega\mathrm{erCF})^2_\ell$, see Theorem 3.1.
(2) if $k \geq 3$, $\ell \geq 2$, and $\lambda > \aleph_0$, then $(2^{\aleph_0}, \omega\mathsf{iA}) \not\to (\lambda, \omega\mathrm{LANG})^k_\ell$, see Theorem 3.11.
(3) if $k, \ell \geq 2$ and $\lambda > \aleph_0$, then $(2^{\aleph_0}, \omega\mathsf{iA}) \not\to (\lambda, \omega\mathrm{CF})^k_\ell$, see Theorem 3.13.

Here, the first part of (0) is a strengthening of Ramsey's theorem since the infinite homogeneous set is regular. The second part might look surprising since larger (k,ℓ)-partitions should have larger homogeneous sets – but not necessarily regular ones! In contrast to Sierpiński's result, (1) shows that ω-automatic $(2,\ell)$-partitions have a larger degree of homogeneity than arbitrary $(2,\ell)$-partitions. Even more, the complexity of the homogeneous set can be bound in language-theoretic terms (there is always a homogeneous set that is the complement of an eventually regular context-free ω-language). Statement (2) is an analogue of Sierpiński's Theorem 1.2 showing that (injective) ω-automatic (k,ℓ)-partitions are as in-homogeneous as arbitrary (k,ℓ)-partitions provided $k \geq 3$. The complexity bound from (1) is shown to be optimal by (3) proving that one cannot always find context-free homogeneous sets. Hence, despite the existence of large homogeneous sets for $k = 2$, for some ω-automatic presentations, they are bound to have a certain (low) level of complexity that is higher than the regular ω-languages.

2. Countably infinite homogeneous sets

Let $k, \ell \geq 2$ be arbitrary. Then, from Ramsey's theorem, we obtain immediately $(\aleph_0, \omega\mathsf{A}) \to (\aleph_0, \omega\mathrm{LANG})^k_\ell$ and $(2^{\aleph_0}, \omega\mathsf{A}) \to (\aleph_0, \omega\mathrm{LANG})^k_\ell$, i.e., all infinite ω-automatic (k,ℓ)-partitions have homogeneous sets of size $\aleph_0$. In this section, we ask whether such homogeneous sets can always be chosen regular:

Theorem 2.1. *Let $k, \ell \geq 2$. Then*
(a) $(\aleph_0, \omega\mathsf{A}) \to (\aleph_0, \omega\mathrm{REG})^k_\ell$.
(b) $(2^{\aleph_0}, \omega\mathsf{iA}) \to (\aleph_0, \omega\mathrm{REG})^k_\ell$.
(c) $(2^{\aleph_0}, \omega\mathsf{A}) \not\to (\aleph_0, \mathrm{LANG}^*)^k_\ell$, *and therefore in particular* $(2^{\aleph_0}, \omega\mathsf{A}) \not\to (\aleph_0, \omega\mathrm{CF})^k_\ell$ *and* $(2^{\aleph_0}, \omega\mathsf{A}) \not\to (\aleph_0, \omega\mathrm{REG})^k_\ell$.

Proof. Let (L,h) be an ω-automatic presentation of some (k,ℓ)-partition $G = (V, E_1, \ldots, E_\ell)$ with $|V| = \aleph_0$. By [BKR08], there exists $L' \subseteq L$ regular such that (L', h) is an injective ω-automatic presentation of G. From a Büchi-automaton for L', one can compute a finite automaton accepting some language K such that (K, h') is an injective automatic presentation of G [Blu99]. Hence, by [Rub08], there exists a regular set $H' \subseteq K$ such that $h'(H')$ is homogeneous in G and countably infinite. From this set, one obtains a regular ω-language $H \subseteq L' \subseteq L$ with $h(H) = h'(H')$, i.e., $h(H)$ is a homogeneous set of size $\aleph_0$. This proves (a).

To prove (b), let (L,h) be an injective ω-automatic presentation of some (k,ℓ)-partition $G = (V, E_1, \ldots, E_\ell)$ of size $2^{\aleph_0}$. Then there exists a regular ω-language $L' \subseteq L$ with

$|L'| = \aleph_0$. Consider the sub-partition $G' = (h(L'), E'_1, \ldots, E'_\ell)$ with $E'_i = E_i \cap [h(L')]^k$. This (k,ℓ)-partition has as ω-automatic presentation the pair (L', h). Then, by (a), there exists $L'' \subseteq L'$ regular and infinite such that $h(L'')$ is homogeneous in G' and therefore in G. Since h is injective, this implies $|h(L')| = |L'| = \aleph_0$.

Finally, we show (c) by a counterexample. Let $L = \{0,1\}^\omega$, $V = L/{\sim_e}$, and $h : L \to V$ the canonical mapping. Furthermore, set $E_1 = [L]^k$. Then $G = (V, E_1, \emptyset, \ldots, \emptyset)$ is a (k,ℓ)-partition with ω-automatic presentation (L, h).

Now let $H = \bigcup_{1 \le i \le n} U_i V_i^\omega \subseteq L$ for some non-empty languages $U_i, V_i \subseteq \{0,1\}^+$ such that $h(H)$ is homogeneous and infinite.

If $|V_i^\omega| = 1$, then $U_i V_i^\omega/{\sim_e}$ is finite. Since $h(H)$ is infinite, there exists $1 \le i \le n$ with $|V_i^\omega| > 1$ implying the existence of words $v, w \in V_i^+$ such that $|v| = |w|$ and $v \neq w$. For $u \in U_i$, the set $u\{v,w\}^\omega \subseteq H$ has $2^{\aleph_0}$ equivalence classes wrt. $\sim_e$. Hence $|h(H)| = 2^{\aleph_0}$. ∎

3. Uncountable homogeneous sets

3.1. A Ramsey theorem for ω-automatic $(2,\ell)$-partitions

The main result of this section is the following theorem that follows immediately from Prop. 3.7 and Lemma 3.5.

Theorem 3.1. *For all $\ell \ge 2$, we have $(2^{\aleph_0}, \omega\mathsf{A}) \to (2^{\aleph_0}, \text{co-}\omega\text{erCF} \cap \mathbf{\Lambda})^2_\ell$.*

3.1.1. *The proof.* The proof of this theorem will construct a language from co-ωerCF that describes a homogeneous set. This language is closely related to the following language

$$N = 1\{0,1\}^\omega \cap \bigcap_{n \ge 0} \{0,1\}^n (0\{0,1\}^n 00 \cup 10^n\{01,10\})\{0,1\}^\omega \ ,$$

i.e., an ω-word x belongs to N iff it starts with 1 and, for every $n \ge 0$, we have $x[n, 2n+3) \in 0\{0,1\}^*00 \cup 10^*01 \cup 10^*10$. We first list some useful properties of this language N:

Lemma 3.2. *The ω-language N is contained in $(1^+0^+)^\omega$, belongs to* co-ωerCF $\cap\, \mathbf{\Lambda}$*, and* $\mathrm{supp}(x) \cap \mathrm{supp}(y)$ *is finite for any $x, y \in N$ distinct.*

Proof. Let $b_i \in \{0,1\}$ for all $i \ge 0$ and suppose the word $x = b_0 b_1 \ldots$ belongs to N. Then $b_0 = 1$, hence the word x contains at least one occurrence of 1. Note that, whenever $b_n = 1$, then $\{b_{2n+1}, b_{2n+2}\} = \{0,1\}$, hence x contains infinitely many occurrences of 1 and therefore infinitely many occurrences of 0, i.e., $N \subseteq (1^+0^+)^\omega$.

Note that the complement of N equals

$$\begin{aligned} & 0\{0,1\}^\omega \cup \bigcup_{n \ge 0} \Big(\{0,1\}^n (0\{0,1\}^n\{01,10,11\} \cup 1\{0,1\}^n\{00,11\})\{0,1\}^\omega \Big) \\ = & \left[0 \cup \bigcup_{n \ge 0} \{0,1\}^n (0\{0,1\}^n\{01,10,11\} \cup 1\{0,1\}^n\{00,11\}) \right] \{0,1\}^\omega \ . \end{aligned}$$

Since the expression in square brackets denotes a context-free language, $\{0,1\}^\omega \setminus N$ is an eventually regular context-free ω-language.

Note that a word $10^{n_0}10^{n_1}10^{n_2}\dots$ belongs to N iff, for all $k \geq 0$, we have $0 \leq n_k - |10^{n_0}10^{n_1}\dots 10^{n_{k-1}}| \leq 1$. Hence, when building a word from N, we have two choices for any n_k, say n_k^0 and n_k^1 with $n_k^0 < n_k^1$. But then $a_0a_1a_2\dots \mapsto 10^{n_0^{a_0}}10^{n_1^{a_1}}10^{n_2^{a_2}}\dots$ defines an order embedding $(\{0,1\}^\omega, \leq_{\text{lex}}) \hookrightarrow (N, \leq_{\text{lex}})$. Since $(\mathbb{R}, \leq) \hookrightarrow (\{0,1\}^\omega, \leq_{\text{lex}})$, we get $N \in \mathbf{\Lambda}$.

Now let $x, y \in N$ with $\text{supp}(x) \cap \text{supp}(y)$ infinite. Then there are arbitrarily long finite words u and v of equal length such that $u1$ and $v1$ are prefixes of x and y, resp. Since $u1$ is a prefix of $x \in N$, it is of the form $u1 = u'10^{|u'|}1$ (if $|u|$ is even) or $u1 = u'10^{|u'|}01$ (if $|u|$ is odd) and analogously for v. Inductively, one obtains $u' = v'$ and therefore $u = v$. Since u and v are arbitrarily long, we showed $x = y$. ∎

Lemma 3.3. *Let $\sim$ and $\approx$ be two equivalence relations on some set L such that any equivalence class $[x]_\sim$ of $\sim$ is countable and $\approx$ has $2^{\aleph_0}$ equivalence classes. Then there are elements $(x_\alpha)_{\alpha<2^{\aleph_0}}$ of L such that $[x_\alpha]_{\sim_e} \cap [x_\beta]_\approx = \emptyset$ for all $\alpha < \beta$.*

Proof. We construct the sequence $(x_\alpha)_{\alpha<2^{\aleph_0}}$ by ordinal induction. So assume we have elements $(x_\alpha)_{\alpha<\kappa}$ for some ordinal $\kappa < 2^{\aleph_0}$ with $[x_\alpha]_\sim \cap [x_\beta]_\approx = \emptyset$ for all $\alpha < \beta < \kappa$.

Suppose $\bigcup_{\alpha<\kappa}[x_\alpha]_\sim \cap [x]_\approx \neq \emptyset$ for all $x \in L$. For $x, y \in L$ with $x \not\approx y$, we have $(\bigcup_{\alpha<\kappa}[x_\alpha]_\sim \cap [x]_\approx) \cap (\bigcup_{\alpha<\kappa}[x_\alpha]_\sim \cap [y]_\approx) \subseteq [x]_\approx \cap [y]_\approx = \emptyset$. Since $\bigcup_{\alpha<\kappa}[x_\alpha]_\sim$ has $\kappa \cdot \aleph_0 \leq \max(\kappa, \aleph_0) < 2^{\aleph_0}$ elements, we obtain $|L| < 2^{\aleph_0}$, contradicting $|L| \geq |L/{\approx}| = 2^{\aleph_0}$. Hence there exists an element $x_\kappa \in L$ with $[x_\alpha]_\sim \cap [x_\kappa]_\approx = \emptyset$ for all $\alpha < \kappa$. ∎

Definition 3.4. Let u, v, and w be nonempty words with $|v| = |w|$ and $v \neq w$. Define an ω-semigroup homomorphism $h : \{0,1\}^\infty \to \Sigma^\infty$ by $h(0) = v$ and $h(1) = w$ and set

$$H_{u,v,w} = u \cdot h(N)$$

where N is the set from Lemma 3.2.

Lemma 3.5. *Let u, v, and w be as in the previous definition. Then $H_{u,v,w} \in \text{co-}\omega\text{erCF} \cap \mathbf{\Lambda}$.*

Proof. Assume $v <_{\text{lex}} w$. Then the mapping $\chi : \{0,1\}^\omega \to \Sigma^\omega : x \mapsto uh(x)$ (where h is the homomorphism from the above definition) embeds $(N, \leq_{\text{lex}})$ (and hence $(\mathbb{R}, \leq)$) into $(H_{u,v,w}, \leq_{\text{lex}})$. If $w <_{\text{lex}} v$, then $(\mathbb{R}, \leq) \cong (\mathbb{R}, \geq) \hookrightarrow (N, \geq_{\text{lex}}) \hookrightarrow (H_{\alpha,\beta,\gamma}, \leq_{\text{lex}})$. This proves that $H_{u,v,w}$ belongs to $\mathbf{\Lambda}$.

Since $v \neq w$, the mapping χ is injective. Hence

$$\Sigma^\omega \setminus H_{\alpha,\beta,\gamma} = \Sigma^\omega \setminus \chi(N) = \Sigma^\omega \setminus \chi(\{0,1\}^\omega) \cup \chi(\{0,1\}^\omega \setminus N)\ .$$

Since χ can be realized by a generalized sequential machine with Büchi-acceptance, $\chi(\{0,1\}^\omega)$ is regular and $\chi(\{0,1\}^\omega \setminus N)$ (as the image of an eventually regular context-free ω-language) is eventually regular context-free. Hence $\Sigma^\omega \setminus H_{u,v,w}$ is eventually regular context-free. ∎

Proposition 3.6. *Let $G = (L, E_0, E_1, \dots, E_\ell)$ be some $(2, 1+\ell)$-partition with injective ω-automatic presentation (L, id) such that $\{(x,y) \mid \{x,y\} \in E_0\} \cup \{(x,x) \mid x \in L\}$ is an equivalence relation on L (denoted $\approx$) with $2^{\aleph_0}$ equivalence classes. Then there exist nonempty words u, v, and w with v and w distinct, but of the same length, such that $H_{u,v,w}$ is i-homogeneous for some $1 \leq i \leq \ell$.*

Proof. There are finite ω-semigroups S and T and homomorphisms $\gamma : \Sigma^\infty \to S$ and $\delta : (\Sigma \times \Sigma)^\infty \to T$ such that

(a) $x \in L$, $y \in \Sigma^\omega$, and $\gamma(x) = \gamma(y)$ imply $y \in L$ and
(b) $x, x', y, y' \in L$, $\{h(x), h(x')\} \in E_i$, and $\delta(x, x') = \delta(y, y')$ imply $\{h(y), h(y')\} \in E_i$ (for all $0 \leq i \leq \ell$).

By Lemma 3.3, there are words $(x_\alpha)_{\alpha<2^{\aleph_0}}$ in L such that $[x_\alpha]_{\sim_e} \cap [x_\beta]_{\approx} = \emptyset$ for all $\alpha < \beta$.

In the following, we only need the words $x_0, x_1, \dots, x_C$ with $C = |S| \cdot |T|$. Then [BKR08, Sections 3.1-3.3][1] first constructs two ω-words y_1 and y_2 and an infinite sequence $1 \le g_1 < g_2 < \dots$ of natural numbers such that in particular $y_1[g_1, g_2) <_{\text{lex}} y_2[g_1, g_2)$. Set $u = y_2[0, g_1)$, $v = y_1[g_1, g_2)$, and $w = y_2[g_1, g_2)$. In the following, let $h : \{0,1\}^\infty \to \Sigma^\infty$ be the homomorphism from Def. 3.4 and set $\chi(x) = uh(x)$ for $x \in \{0,1\}^*$. As in [BKR08], one can then show that all the words from $H_{u,v,w}$ belong to the ω-language L. In the following, set $x_{\circ\bullet} = \chi((01)^\omega)$ and $x_{\bullet\circ} = \chi((10)^\omega)$. Then obvious alterations in the proofs by Bárány et al. show:

(1) [BKR08, Lemma 3.4][2] If $x, y \in \{0,1\}^\omega$ with $\operatorname{supp}(x) \setminus \operatorname{supp}(y)$ and $\operatorname{supp}(y) \setminus \operatorname{supp}(x)$ infinite, then
$$\{\delta(\chi(x), \chi(y)), \delta(\chi(y), \chi(x))\} = \{\delta(x_{\bullet\circ}, x_{\circ\bullet}), \delta(x_{\circ\bullet}, x_{\bullet\circ})\} .$$
(2) [BKR08, Lemma 3.5] $x_{\bullet\circ} \not\approx x_{\circ\bullet}$.

There exists $0 \le i \le \ell$ with $\{x_{\bullet\circ}, x_{\circ\bullet}\} \in E_i$. Then (2) implies $i > 0$.

Let $x, y \in N$ be distinct. Then $\operatorname{supp}(x) \cap \operatorname{supp}(y)$ is finite by Lemma 3.2. Since, on the other hand, $\operatorname{supp}(x)$ and $\operatorname{supp}(y)$ are both infinite, the two differences $\operatorname{supp}(x) \setminus \operatorname{supp}(y)$ and $\operatorname{supp}(y) \setminus \operatorname{supp}(x)$ are infinite. Hence we obtain $\delta(\chi(x), \chi(y)) \in \{\delta(x_{\bullet\circ}, x_{\circ\bullet}), \delta(x_{\circ\bullet}, x_{\bullet\circ})\}$ from (1). Hence (b) implies $\{\chi(x), \chi(y)\} \in E_i$, i.e., $H_{u,v,w}$ is E_i-homogeneous.

Since $H_{u,v,w} \in$ co-ωerCF $\cap\, \mathbf{\Lambda}$ by Lemma 3.5, the result follows. ∎

Proposition 3.7. *Let $G = (V, E'_1, \dots, E'_\ell)$ be some $(2, \ell)$-partition with automatic presentation (L, h). Then there exist $u, v, w \in \Sigma^+$ with v and w distinct of equal length such that $h(H_{u,v,w})$ is homogeneous and of size $2^{\aleph_0}$.*

Proof. To apply Prop. 3.6, consider the following $(2, 1+\ell)$-partition $G = (L, E_0, \dots, E_\ell)$:

- The underlying set is the ω-language L,
- E_0 comprises all sets $\{x, y\}$ with $h(x) = h(y)$ and $x \neq y$, and
- E_i (for $1 \le i \le \ell$) comprises all sets $\{x, y\}$ with $\{h(x), h(y)\} \in E'_i$.

Then (L, id) is an injective ω-automatic presentation of the $(2, 1+\ell)$-partition G. By Prop. 3.6, there exists $1 \le i \le \ell$ and words u, v and w such that $H_{u,v,w}$ is i-homogeneous in G. Since $(E_0, \dots, E_\ell)$ is a partition of $[L]^2$, we have $\{x, y\} \notin E_0$ (and therefore $h(x) \neq h(y)$) for all $x, y \in H_{u,v,w}$ distinct. Hence h is injective on $H_{u,v,w}$. Furthermore $[H_{u,v,w}]^2 \subseteq E_i$ implies $[h(H_{u,v,w})]^2 \subseteq E'_i$. Hence $h(H_{u,v,w})$ is an i-homogeneous set in G' of size $2^{\aleph_0}$. ∎

This finishes the proof of Theorem 3.1.

3.1.2. *Effectiveness.* Note that the proof above is non-constructive at several points: Lemma 3.3 is not constructive and the proof proper uses Ramsey's theorem [BKR08, page 390] and makes a Ramseyan factorisation coarser [BKR08, begin of section 3.2]. We now show that nevertheless the words u, v, and w can be computed. By Prop. 3.7, it suffices to decide for a given triple (u, v, w) whether $h(H_{u,v,w})$ is i-homogeneous for some fixed $1 \le i \le \ell$.

To be more precise, let $(V, E_1, \dots, E_\ell)$ be some $(2, \ell)$-partition with ω-automatic presentation (L, h). Furthermore, let $u, v, w \in \Sigma^+$ with $v \neq w$ of the same length and write H

[1] The authors of [BKR08] require $[x_i]_{\sim_e} \cap [x_j]_{\approx} = \emptyset$ for all $0 \le i, j \le C$ distinct, but they use it only for $i < j$. Hence we can apply their result here.

[2] The authors of [BKR08] only require one of the two differences to be infinite, but the proof uses that they both are infinite.

for $H_{u,v,w}$. We have to decide whether $H \subseteq L$ and $H \otimes H \subseteq L_i \cup L_=$. Note that $H \subseteq L$ iff $L \cap \Sigma^\omega \setminus H = \emptyset$. But $\Sigma^\omega \setminus H$ is context-free, so the intersection is context-free. Hence the emptiness of the intersection can be decided.

Towards a decision of the second requirement, note that

$$(\Sigma \times \Sigma)^\omega \setminus (H \otimes H) = (\Sigma^\omega \setminus H \otimes \Sigma^\omega) \cup (\Sigma^\omega \cup \Sigma^\omega \setminus H)$$

is the union of two context-free ω-languages and therefore context-free itself. Since $L_i \cup L_=$ is regular, the intersection $(L_i \cup L_=) \cap (\Sigma \times \Sigma)^\omega \setminus (H \otimes H)$ is context-free implying that its emptiness is decidable. But this emptiness is equivalent to $H \otimes H \subseteq L_1 \cup L_=$.

3.1.3. *ω-automatic partial orders.* ¿From Theorem 3.1, we now derive a necessary condition for a partial order of size $2^{\aleph_0}$ to be ω-automatic. A partial order $(V, \sqsubseteq)$ is ω-automatic iff there exists a regular ω-language L and a surjection $h : L \to V$ such that the relations $R_= = \{(x, y) \in L^2 \mid h(x) = h(y)\}$ and $R_\sqsubseteq = \{(x, y) \in L^2 \mid h(x) \sqsubseteq h(y)\}$ are ω-automatic.

Corollary 3.8 ([BKR08][3])**.** *If $(V, \sqsubseteq)$ is an ω-automatic partial order with $|V| \geq \aleph_1$, then $(\mathbb{R}, \leq)$ or an antichain of size $2^{\aleph_0}$ embeds into $(V, \sqsubseteq)$.*

Proof. Let $(V, \sqsubseteq)$ be a partial order, $L \subseteq \Sigma^\omega$ a regular ω-language and $h : L \to V$ a surjection such that $R_=$ and $R_\sqsubseteq$ are ω-automatic. Define an injective ω-automatic $(2, 4)$-partition $G = (L, E_0, E_1, E_2, E_3)$:

- E_0 comprises all pairs $\{x, y\} \in [L]^2$ with $h(x) = h(y)$,
- E_1 comprises all pairs $\{x, y\} \in [L]^2$ with $h(x) \sqsubset h(y)$ and $x <_{\text{lex}} y$,
- E_2 comprises all pairs $\{x, y\} \in [L]^2$ with $h(x) \sqsupset h(y)$ and $x <_{\text{lex}} y$, and
- $E_3 = [L]^2 \setminus (E_0 \cup E_1 \cup E_2)$ comprises all pairs $\{x, y\} \in [L]^2$ such that $h(x)$ and $h(y)$ are incomparable.

From $|L| \geq |V| > \aleph_0$, we obtain $|L| = 2^{\aleph_0}$. Hence, by Prop. 3.6, there exists $H \subseteq L$ 1-, 2- or 3-homogeneous with $(\mathbb{R}, \leq) \hookrightarrow (H, \leq_{\text{lex}})$. Since $[H]^2 \subseteq E_1 \cup E_2 \cup E_3$ and since G is a partition of L, the mapping h acts injectively on H. If $[H]^2 \subseteq E_1$ (the case $[H]^2 \subseteq E_2$ is symmetrical) then $(\mathbb{R}, \leq) \hookrightarrow (H, \leq_{\text{lex}}) \cong (h(H), \sqsubseteq)$. If $[H]^2 \subseteq E_3$, then $h(H)$ is an antichain of size $2^{\aleph_0}$. ∎

A linear order $(L, \sqsubseteq)$ is *scattered* if $(\mathbb{Q}, \leq)$ cannot be embedded into $(L, \sqsubseteq)$. Automatic partial orders are defined similarly to ω-automatic partial orders with the help of finite automata instead of Büchi-automata.

Corollary 3.9 ([BKR08][3])**.** *Any scattered ω-automatic linear order $(V, \sqsubseteq)$ is countable. Hence,*

- *a scattered linear order is ω-automatic if and only if it is automatic, and*
- *an ordinal α is ω-automatic if and only if $\alpha < \omega^\omega$.*

Proof. If $(V, \sqsubseteq)$ is not countable, then it embeds $(\mathbb{R}, \leq)$ by the previous corollary and therefore in particular $(\mathbb{Q}, \leq)$. The remaining two claims follow immediately from [BKR08] ("countable ω-automatic structures are automatic") and [Del04] ("an ordinal is automatic iff it is properly smaller than ω^ω"), resp. ∎

[3] As pointed out by two referees, the paragraph before Sect. 4.1 in [BKR08] already hints at this result, although in a rather implicit way.

Contrast Theorem 3.1 with Theorem 1.2: any uncountable ω-automatic (k,ℓ)-partition contains an uncountable homogeneous set of size $2^{\aleph_0}$. But we were able to prove this for $k=2$, only. One would also wish the homogeneous set to be regular and not just from co-ωerCF. We now prove that these two shortcomings are unavoidable: Theorem 3.1 does not hold for $k=3$ nor is there always an ω-regular homogeneous set. These negative results hold even for injective presentations.

3.2. A Sierpiński theorem for ω-automatic (k,ℓ)-partitions with $k\geq 3$

We first concentrate on the question whether some form of Theorem 3.1 holds for $k\geq 3$. The following lemma gives the central counterexample for $k=3$ and $\ell=2$, the below theorem then derives the general result.

Lemma 3.10. $(2^{\aleph_0},\omega\mathsf{iA})\not\to(\aleph_1,\omega\mathrm{LANG})^3_2$.

Proof. Let $\Sigma=\{0,1\}$, $V=L=\{0,1\}^\omega$. Furthermore, for $H\subseteq L$, we write $\bigwedge H\in\Sigma^\infty$ for the longest common prefix of all ω-words in H, $\bigwedge\{x,y\}$ is also written $x\wedge y$. Then let E_1 consist of all 3-sets $\{x,y,z\}\in[L]^3$ with $x<_{\mathrm{lex}}y<_{\mathrm{lex}}z$ and $x\wedge y<_{\mathrm{pref}}y\wedge z$; E_2 is the complement of E_1. This finishes the construction of the $(3,2)$-partition (V,E_1,E_2) of size $2^{\aleph_0}$ with injective ω-automatic presentation (L,id).

Note that 1^*0^ω is a countable E_1-homogeneous set and that 0^*1^ω is a countable E_2-homogeneous set. But there is no uncountable homogeneous set: First suppose $H\subseteq L$ is infinite and $x\wedge y<_{\mathrm{pref}}y\wedge z$ for all $x<_{\mathrm{lex}}y<_{\mathrm{lex}}z$ from H. Let $u\in\Sigma^*$ such that $H\cap u0\Sigma^\omega$ and $H\cap u1\Sigma^\omega$ are both nonempty and let $x,y\in H\cap u0\Sigma^\omega$ with $x\leq_{\mathrm{lex}}y$ and $z\in H\cap u1\Sigma^\omega$. Then $x\wedge y>_{\mathrm{pref}}u=y\wedge z$ and therefore $x=y$ (for otherwise, we would have $x<_{\mathrm{lex}}y<_{\mathrm{lex}}z$ in H with $x\wedge y>_{\mathrm{pref}}y\wedge z$). Hence we showed $|H\cap u0\Sigma^\omega|=1$. Let $u_0=\bigwedge H$ and $H_1=H\cap u_0 1\Sigma^\omega$. Since $H\cap u_0 0\Sigma^\omega$ is finite, the set H_1 is infinite. We proceed by induction: $u_n=\bigwedge H_n$ and $H_{n+1}=H_n\cap u_n 1\Sigma^\omega$ satisfying $|H_n\cap u_n 0\Sigma^\omega|=1$. Then $u_0<_{\mathrm{pref}}u_0 1\leq_{\mathrm{pref}}u_1<_{\mathrm{pref}}u_1 1\leq_{\mathrm{pref}}u_2\cdots$ with

$$H=\bigcup_{n\geq 0}(H\cap u_n 0\Sigma^\omega)\cup\bigcap_{n\geq 0}(H\cap u_n 1\Sigma^\omega)\ .$$

Then any of the sets $H\cap u_n 0\Sigma^\omega=H_n\cap u_n 0\Sigma^\omega$ and $\bigcap(H\cap u_n 1\Sigma^\omega)$ is a singleton, proving that H is countable. Thus, there cannot be an uncountable E_1-homogeneous set.

So let $H\subseteq L$ be infinite with $x\wedge y\geq_{\mathrm{pref}}y\wedge z$ for all $x<_{\mathrm{lex}}y<_{\mathrm{lex}}z$. Since we have only two letters, we get $x\wedge y>_{\mathrm{pref}}y\wedge z$ for all $x<_{\mathrm{lex}}y<_{\mathrm{lex}}z$ which allows to argue symmetrically to the above. Thus, indeed, there is no uncountable homogeneous set in L. ■

Theorem 3.11. *For all $k\geq 3$, $\ell\geq 2$, and $\lambda>\aleph_0$, we have $(2^{\aleph_0},\omega\mathsf{iA})\not\to(\lambda,\omega\mathrm{LANG})^k_\ell$.*

Proof. Let G be the $(3,2)$-partition from Lemma 3.10 that does not have homogeneous sets of size λ and let (L,id) be an injective ω-automatic presentation of $G=(V,E_1,E_2)$ (in particular, $V=L$).

For a set $X\in[L]^k$, let $X_1<_{\mathrm{lex}}X_2<_{\mathrm{lex}}X_3$ be the three lexicographically least elements of X. Then set $G'=(V,E'_1,E'_2,\dots,E'_\ell)$ with

$$\begin{aligned}E'_1&=\{X\in[V]^k\mid\{X_1,X_2,X_3\}\in E_1\},\\E'_2&=\{X\in[V]^k\mid\{X_1,X_2,X_3\}\in E_2\},\ \text{and}\\E'_i&=\emptyset\ \text{for}\ 3\leq i\leq\ell\ .\end{aligned}$$

Then (L, id) is an injective ω-automatic presentation of G'. Now suppose $H' \subseteq L$ is homogeneous in G' and of size λ. Then there exists $H \subseteq H'$ of size λ such that for any words $x_1 <_{\mathrm{lex}} x_2 <_{\mathrm{lex}} x_3$ from H, there exists $X \subseteq H'$ with $X_i = x_i$ for $1 \leq i \leq 3$ (if necessary, throw away some lexicographically largest elements of H'). Hence H is homogeneous in G, contradicting Lemma 3.10. ∎

3.3. Complexity of homogeneous sets in ω-automatic $(2, \ell)$-partitions

Having shown that $k = 2$ is a central assumption in Theorem 3.1, we now turn to the question whether homogeneous sets of lower complexity can be found.

Construction. Let $V = L$ denote the regular ω-language $(1^+0^+)^\omega$. Furthermore, $E_1 \subseteq [L]^2$ comprises all 2-sets $\{x, y\} \subseteq L$ such that $\mathrm{supp}(x) \cap \mathrm{supp}(y)$ is finite or $x \sim_e y$. The set E_2 is the complement of E_1 in $[L]^2$. This completes the construction of the $(2, 2)$-partition $G = (L, E_1, E_2)$. Note that (L, id_L) is an injective ω-automatic presentation of G.

By Theorem 3.1, G has an E_1- or an E_2-homogeneous set of size $2^{\aleph_0}$. We convince ourselves that G has large homogeneous sets of both types. By Lemma 3.2, there is an ω-language $N \subseteq (1^+0^+)^\omega$ of size $2^{\aleph_0}$ such that the supports of any two words from N have finite intersection. Hence $[N]^2 \subseteq E_1$ and N has size $2^{\aleph_0}$. But there is also an E_2-homogeneous set L_2 of size $2^{\aleph_0}$: Note that the words from N are mutually non-$\sim_e$-equivalent and let L_2 denote the set of all words $1a_11a_21a_3\dots$ for $a_1a_2a_3 \dots \in N$. Then for any $x, y \in L_2$ distinct, we have $2\mathbb{N} \subseteq \mathrm{supp}(x) \cap \mathrm{supp}(y)$ and $x \not\sim_e y$, i.e., $\{x, y\} \in E_2$.

Lemma 3.12. *Let $H \in \mathrm{LANG}^*$ have size $\lambda > \aleph_0$. Then H is not homogeneous in G.*

Proof. By definition of LANG^*, there are languages $U_i, V_i \in \mathrm{LANG}$ with $H = \bigcup_{1 \leq i \leq n} U_i V_i^\omega$.

Since H is infinite, there are $1 \leq i \leq n$ and $x, y \in U_i V_i^\omega$ distinct with $x \sim_e y$ and therefore $\{x, y\} \in E_1$.

Since $|H| > \aleph_0$, there is $1 \leq i \leq n$ with $|U_i V_i^\omega| > \aleph_0$; we set $U = U_i$ and $V = V_i$. From $|U| \leq \aleph_0$, we obtain $|V^\omega| > \aleph_0$. Hence there are $v_1, v_2 \in V^+$ distinct with $|v_1| = |v_2|$. Since $uv_1^\omega \in H$ and each element of H contains infinitely many occurrences of 1, the word v_1 belongs to $\{0,1\}^*10^*$. Let $u \in U$ be arbitrary (such a word exists since $UV^\omega \neq \emptyset$) and consider the ω-words $x' = u(v_1v_2)^\omega$ and $y' = u(v_1v_1)^\omega$ from $UV^\omega \subseteq H$. Then $x' \not\sim_e y'$ since $v_1 \neq v_2$ and $|v_1| = |v_2|$. At the same time, $\mathrm{supp}(x') \cap \mathrm{supp}(y')$ is infinite since v_1 contains an occurrence of 1. Hence $\{x', y'\} \in E_2$.

Thus, we found ω-words $x, y, x', y' \in H$ with $\{x, y\} \in E_1$ and $\{x', y'\} \notin E_1$ proving that H is not homogeneous. ∎

Thus, we found a $(2, 2)$-partition $G = (V, E_1, E_2)$ with $2^{\aleph_0}$ elements and an injective ω-automatic presentation (L, h) such that

(1) G has sets L_1 and L_2 in co-ωerCF of size $2^{\aleph_0}$ with $[L_i]^2 \subseteq E_i$ for $1 \leq i \leq 2$.
(2) There is no ω-language $H \in \mathrm{LANG}^*$ with $H \subseteq L$ such that $h(H)$ is homogeneous of size $2^{\aleph_0}$.

Since all context-free ω-languages belong to LANG^*, the following theorem follows the same way that Lemma 3.10 implied Theorem 3.11.

Theorem 3.13. *For all $k, \ell \geq 2$ and $\lambda > \aleph_0$, we have $(2^{\aleph_0}, \omega\mathsf{iA}) \not\to (\lambda, \omega\mathrm{CF})^k_\ell$ and $(2^{\aleph_0}, \omega\mathsf{iA}) \not\to (\lambda, \omega\mathrm{REG})^k_\ell$.*

This result can be understood as another Sierpiński theorem for ω-automatic (k, ℓ)-partitions. This time, it holds for all $k \geq 2$ (not only for $k \geq 3$ as Theorem 3.11). The price to be paid for this is the restriction of homogeneous sets to "simple" ones. In particular the non-existence f regular homogeneous sets provides a Sierpiński theorem in the spirit of automatic structures.

Open questions

Our positive result Theorem 3.1 guarantees the existence of some clique or anticlique of size $2^{\aleph_0}$ (and such a clique or anticlique can even be constructed). But the following situation is conceivable: the ω-automatic graph contains large cliques without containing large cliques that can be described by a language from co-ωerCF. In particular, it is not clear whether the existence of a large clique is decidable.

A related question concerns Ramsey quantifiers. Rubin [Rub08] has shown that the set of nodes of an automatic graph whose neighbors contain an infinite anticlique is regular (his result is much more general, but this formulation suffices for our purpose). It is not clear whether this also holds for ω-automatic graphs. A positive answer to this second question (assuming that it is effective) would entail an affirmative answer to the decidability question above.

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