

# Bounded extremal problems in Hardy spaces for the conjugate Beltrami equation in simply-connected domains

Yannick Fischer, Juliette Leblond, Jonathan R. Partington, Eva Sincich

► **To cite this version:**

Yannick Fischer, Juliette Leblond, Jonathan R. Partington, Eva Sincich. Bounded extremal problems in Hardy spaces for the conjugate Beltrami equation in simply-connected domains. [Research Report] RR-7219, INRIA. 2010, pp.26. <inria-00460820v3>

**HAL Id: inria-00460820**

**<https://hal.inria.fr/inria-00460820v3>**

Submitted on 9 Jun 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

***Bounded extremal problems in Hardy spaces for the  
conjugate Beltrami equation in simply-connected  
domains***

Yannick Fischer — Juliette Leblond — Jonathan R. Partington — Eva Sincich .

**N° 7219**

Mars 2010

Modeling, Optimization, and Control of Dynamic Systems



*R*apport  
de recherche



## Bounded extremal problems in Hardy spaces for the conjugate Beltrami equation in simply-connected domains

Yannick Fischer<sup>\*</sup>, Juliette Leblond<sup>\*</sup>, Jonathan R. Partington<sup>†</sup>, Eva Sincich<sup>\*</sup>.

Theme : Modeling, Optimization, and Control of Dynamic Systems  
Équipe-Projet Apics

Rapport de recherche n° 7219 — Mars 2010 — 23 pages

**Abstract:** Techniques of constrained approximation are used to recover solutions to elliptic partial differential equations from incomplete and corrupted boundary data. The approach involves the use of generalized Hardy spaces of functions whose real and imaginary parts are related by formulae similar to the Cauchy–Riemann equations. A prime motivation for this is the modelling of plasma confinement in a tokamak reactor. Constructive and numerical aspects are also discussed.

**Key-words:** Hardy classes, extremal problems, inverse Dirichlet-Neumann problems

<sup>\*</sup> INRIA Sophia Antipolis Méditerranée, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis, France, Yannick.Fischer@sophia.inria.fr, Juliette.Leblond@sophia.inria.fr

<sup>†</sup> School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K., J.R.Partington@leeds.ac.uk

# Problèmes extrémaux bornés dans des espaces de Hardy pour l'équation de Beltrami conjuguée dans des domaines simplement connexes

**Résumé :** Des techniques d'approximation sous contraintes sont mises en oeuvre pour retrouver les solutions d'une equation aux dérivées partielles elliptique à partir de données incomplètes et bruitées disponibles sur le bord du domaine d'étude. Cette approche nécessite l'utilisation de fonctions dans des espaces de Hardy généralisés dont les parties réelles et imaginaires sont reliées par des formules analogues aux équations de Cauchy-Riemann. Une importante motivation pour ce travail concerne la modélisation du confinement d'un plasma au sein d'un tokamak. Les aspects constructifs et numériques seront par ailleurs examinés.

**Mots-clés :** classes de Hardy, problèmes extrémaux, problèmes inverses Dirichlet-Neumann

# 1 Introduction

## 1.1 Problems, motivation

Let  $\Omega$  be a bounded open domain in  $\mathbb{R}^2$  with  $C^1$  boundary and

$$\sigma \in W^{1,\infty}(\Omega) \quad \text{such that} \quad 0 < c \leq \sigma \leq C \quad \text{a.e. in } \Omega, \tag{1}$$

for two constants  $0 < c < C < +\infty$ . The elliptic equation we look at is

$$\operatorname{div}(\sigma \nabla u) = 0 \text{ in } \Omega. \tag{2}$$

The divergence and gradient operators are classically understood (as derivation with respect to real variables  $x, y$ , in  $\mathbb{R}^2$ ). We are interested in the following inverse problem of Cauchy type: assume that  $u$  and  $\sigma \partial_n u$  (where  $\partial_n u$  denotes the normal derivative of  $u$  on  $\partial\Omega$ ) are given on an open subset  $I \subset \partial\Omega$ . From these data, we want to recover  $u$  on the complementary subset  $J = \partial\Omega \setminus I$ , also assumed to be of non-empty interior. This problem is ill-posed in the sense of Hadamard, for instance in Sobolev spaces  $W^{1-1/p,p}(I)$  [18],  $1 < p < +\infty$ , but also in  $L^p(I)$ , where it still makes sense [9], while the associated (direct) Dirichlet or Neumann problems have a unique (up to normalisation) and stable solution. However, well-posedness is ensured if a constraint is added on  $J$  and the extrapolation issue expressed as a best  $L^p$  approximation problem on  $I$  (in the class of bounded extremal problems (BEP), which will be explained later).

When  $\sigma$  is constant, this problem amounts to recovering or approximating the values of a holomorphic function in a domain of analyticity from part of its boundary values. This problem was extensively studied in Hardy spaces on simply and doubly connected domains (see [3, 7, 8, 20]).

In the present paper, we are interested in Hardy classes of the *conjugate* Beltrami equation, which is

$$\bar{\partial} f = \nu \bar{\partial} \bar{f} \quad \text{a.e. in } \Omega, \tag{CB}$$

where

$$\nu \in W^{1,\infty}(\Omega) \quad \text{is real valued,} \quad \text{and} \quad \|\nu\|_{L^\infty(\Omega)} \leq \kappa < 1, \tag{\kappa}$$

for some  $\kappa \in (0, 1)$ . There,  $\partial$  and  $\bar{\partial}$  respectively denote the derivation operators with respect to complex variables  $z = x + iy$  and  $\bar{z}$  in  $\mathbb{C} \simeq \mathbb{R}^2$  (recall that  $\partial \bar{\partial} = \bar{\partial} \partial = \Delta = \operatorname{div}(\nabla)$ ).

The equation (CB) is related to (2), in fact to a system of second order elliptic equations in divergence form, in the following way [9, Sec. 3.1] which generalizes the well-known link between holomorphic (analytic) and harmonic functions.

Let  $p \in (1, +\infty)$  and  $f \in W^{1,p}(\Omega)$  be a solution of (CB). If  $f = u + iv$  where  $u$  and  $v$  are real-valued, then  $u, v \in W^{1,p}(\Omega)$ ,  $u$  satisfies equation (2) and (its  $\sigma$ -harmonic conjugate function)  $v$  satisfies

$$\operatorname{div} \left( \frac{1}{\sigma} \nabla v \right) = 0 \text{ in } \Omega, \tag{3}$$

where  $\sigma = (1 - \nu)/(1 + \nu)$ . It follows from  $(\kappa)$  that there exist  $c, C > 0$  such that (1) holds.

Conversely, let  $\sigma \in W^{1,\infty}(\Omega)$  satisfying (1) and  $u \in W^{1,p}(\Omega)$  be a real valued solution of (2), [18]. Then, if  $\Omega$  is simply connected, there exists a real-valued function  $v \in W^{1,p}(\Omega)$ , unique up to an additive constant, such that  $f = u + iv \in W^{1,p}(\Omega)$  satisfies (CB), with  $\nu = (1 - \sigma)/(1 + \sigma)$ . Indeed, with this definition of  $\nu$ , (CB) is equivalent to the system of generalized Cauchy–Riemann equations:

$$\begin{cases} \partial_x v = -\sigma \partial_y u, \\ \partial_y v = \sigma \partial_x u, \end{cases} \tag{4}$$

where  $\partial_x$  and  $\partial_y$  stand for the partial derivatives with respect to  $x$  and  $y$ , respectively. System (4) admits a solution because  $\partial_y(-\sigma \partial_y u) = \partial_x(\sigma \partial_x u)$  by (2), and  $\Omega$  is simply connected. Assumption (1) implies that  $\nu$  satisfies  $(\kappa)$  for some  $\kappa \in (0, 1)$ .

Even more general classes of “analytic” functions were studied in [13, 34], that are strongly related to quasi-conformal maps [2]. The above link between (2) and (CB) is the basis of the approaches of [4, 5, 25] and of the work [9] where fundamental properties of associated generalized Hardy classes of solutions (in the disc) are established, and Dirichlet-type issues considered. Here we shall complete some of these results, set up and solve best constrained approximation (“bounded extremal”) problems in these classes, constructively in the Hilbertian framework of  $L^2$ .

An important practical motivation for such issues comes from plasma confinement in tokamaks, for thermonuclear controlled fusion, when trying to recover the boundary of the plasma, or for numerical simulations. Indeed, in poloidal plane sections, from the so-called Grad–Shafranov equation in the vacuum, an annular domain  $\Omega$  (with  $0 \notin \Omega$ ) lying between the plasma and the chamber, the magnetic flux is a solution to the conductivity equation (2) with the conductivity  $\sigma(x, y) = \sigma_*(x, y) = 1/x$  (see Fig.1). However, in the present work, we stick to the situation of (Dini) smooth simply connected domains; the more involved one of annular domains will be analysed in a forthcoming study. Up to conformal mappings, this allows us to handle the issue in the open unit disc (see [9, Sec. 6]), where we consider Hardy classes.

The overview is as follows.

- Notation and definitions: Section 2.
- Properties of generalized Hardy classes, bounded extremal problems (BEP): Section 3.
- Complete families of solutions : Section 4.
- Constructive aspects: Section 5.
- Concluding remarks: Section 6.

## 2 Notation and definitions

We first present the notation that is used in the next section and later on.

### 2.1 General settings

Throughout the paper,  $\mathbb{D}$  refers to the open unit disc and  $\mathbb{T}$  to the unit circle of the complex plane  $\mathbb{C}$ , both centered at 0.

Let  $\mathcal{O}$  be a bounded open set in  $\mathbb{R}^d$ ,  $d = 1$  or  $2$  with a Lipschitz-continuous boundary. The generic point in  $\mathcal{O}$  is denoted by  $x$  in dimension  $d = 1$  and by  $\mathbf{x} = (x, y)$  in the dimension  $d = 2$ . For each  $p$ ,  $1 \leq p < +\infty$ , we introduce the space

$$L^p(\mathcal{O}) = \{\varphi : \mathcal{O} \rightarrow \mathbb{C} \text{ measurable}; \|\varphi\|_{L^p(\mathcal{O})} = \left( \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} |\varphi(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} < +\infty\}.$$

Now, let  $\mathcal{D}(\mathcal{O})$  designate the space of complex-valued  $C^\infty$  functions with compact support in  $\mathcal{O}$ . Its dual  $\mathcal{D}'(\mathcal{O})$  is the usual space of distributions on  $\mathcal{O}$ . If  $\mathbf{k}$  denotes a  $d$ -tuple  $(k_1, \dots, k_d)$  of nonnegative integers, we use the notation  $\partial^{\mathbf{k}}\varphi$  for the partial derivative (in the distribution sense) of a function  $\varphi$  of total order  $|\mathbf{k}| = k_1 + \dots + k_d$  and partial order  $k_i$  with respect to the  $i$ -th variable.

So, for each nonnegative integer  $m$  and each  $p$  with  $1 \leq p < +\infty$ , we consider the space  $W^{m,p}(\mathcal{O})$  of functions such that all their partial derivatives of total order  $\leq m$  belong to  $L^p(\mathcal{O})$ , i.e.,

$$W^{m,p}(\mathcal{O}) = \{\varphi \in \mathcal{D}'(\mathcal{O}); \|\varphi\|_{W^{m,p}(\mathcal{O})} = \left( \sum_{|\mathbf{k}| \leq m} \|\partial^{\mathbf{k}}\varphi\|_{L^p(\mathcal{O})}^p \right)^{1/p} < +\infty\}.$$

Next, the Sobolev spaces of non-integer order can be defined by an intrinsic norm [12]. Indeed, any positive real number  $s$  which is not an integer can be written  $[s] + \alpha$ , where  $[s]$  denotes its integral part

while its fractional part  $\alpha$  satisfies  $0 < \alpha < 1$ . Then, for any  $p$  with  $1 \leq p < +\infty$ , the space  $W^{s,p}(\mathcal{O})$  is the space of distributions  $\varphi$  in  $\mathcal{D}'(\mathcal{O})$  such that

$$\|\varphi\|_{W^{s,p}(\mathcal{O})} = \left( \|\varphi\|_{W^{[s],p}(\mathcal{O})}^p + \sum_{|k|=s} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|\partial^k \varphi(\mathbf{x}) - \partial^k \varphi(\mathbf{x}')|^p}{|\mathbf{x} - \mathbf{x}'|^{\alpha \mathbf{p} + \mathbf{d}}} d\mathbf{x} d\mathbf{x}' \right)^{1/p} < +\infty.$$

Observe that such Sobolev spaces may also be defined in several other ways, for instance by interpolation methods [1, Ch. VII], or using trace operators [30].

The subscript  $\mathbb{R}$  (for instance in  $L_{\mathbb{R}}^p$  or in  $W_{\mathbb{R}}^{s,p}$ ) indicates real-valued function spaces.

We let  $L : L^p(\mathbb{T}) \rightarrow \mathbb{C}$  be the (bounded) mean value operator on  $\mathbb{T}$  (recall that  $p > 1$ ):

$$L\phi = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) d\theta, \quad \forall \phi \in L^p(\mathbb{T}).$$

## 2.2 Generalized Hardy classes

If  $1 < p < +\infty$  and  $\nu \in W_{\mathbb{R}}^{1,\infty}(\mathbb{D})$  satisfies  $(\kappa)$ , we define the ‘‘generalized’’ Hardy space  $H_{\nu}^p(\mathbb{D}) = H_{\nu}^p$  to consist of those Lebesgue measurable functions  $f$  on  $\mathbb{D}$  such that

$$\operatorname{ess\,sup}_{0 < r < 1} \|f\|_{L^p(\mathbb{T}_r)} = \operatorname{ess\,sup}_{r > 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < +\infty$$

(which therefore belong to  $L^p(\mathbb{D})$ ) and satisfying (CB) in the sense of distributions on  $\mathbb{D}$ , see [9]. Equipped with the norm

$$\|f\|_{H_{\nu}^p(\mathbb{D})} := \operatorname{ess\,sup}_{0 < r < 1} \|f\|_{L^p(\mathbb{T}_r)},$$

$H_{\nu}^p$  is a Banach space. When  $\nu = 0$ ,  $H_0^p = H^p = H^p(\mathbb{D})$  is the classical Hardy space of holomorphic functions on the unit disc (see [21, 22]).

By analogy with classical Hardy spaces on  $\mathbb{C} \setminus \overline{\mathbb{D}}$ , let  $\overline{H_{\nu}^p} = \overline{H_{\nu}^p}(\mathbb{D})$  be the space of functions  $f$  such that

$$\operatorname{ess\,sup}_{r > 1} \|f\|_{L^p(\mathbb{T}_r)} < +\infty,$$

and satisfying (CB) in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ . For each function  $f \in L^p(\mathbb{D})$ , set

$$\check{f}(z) = \overline{f\left(\frac{1}{\bar{z}}\right)}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

A direct computation shows that

$$f \in H_{\nu}^p \iff \check{f} \in \overline{H_{\nu}^p}, \quad (5)$$

for the dilation  $\check{\nu} \in W_{\mathbb{R}}^{1,\infty}(\mathbb{C} \setminus \overline{\mathbb{D}})$  (since  $\nu$  is real-valued,  $\check{\nu}(z) = \nu(1/\bar{z})$ ).

We introduce  $H_{\nu}^{p,0} \subset H_{\nu}^p$  (resp.  $\overline{H_{\nu}^{p,0}} \subset \overline{H_{\nu}^p}$ ), the (closed) subset of functions  $f \in H_{\nu}^p$  (resp.  $\overline{H_{\nu}^p}$ ) subject to the normalization condition  $L(\operatorname{Im}(\operatorname{tr} f)) = 0$ . Let  $\overline{H_{\nu}^{p,0,0}}$  be the subspace of functions  $f \in \overline{H_{\nu}^{p,0}}$  such that

$$\int_0^{2\pi} \operatorname{tr} f(e^{i\theta}) d\theta = 0 = L(\operatorname{tr} f). \quad (6)$$

Finally, let  $\Omega \subset \mathbb{C}$  be a simply connected bounded Dini-smooth domain (its boundary being a Jordan curve with nonsingular Dini-smooth parametrization). Let  $\nu$  satisfying  $(\kappa)$  on  $\Omega$ . We introduce the Hardy classes  $H_{\nu}^p(\Omega)$  as the space of functions  $f$  on  $\Omega$  such that  $f \circ \psi \in H_{\nu \circ \psi}^p(\mathbb{D})$ , for some conformal transformation  $\psi$  from  $\mathbb{D}$  onto  $\Omega$ . A straightforward computation [2, Ch. 1] shows that this class does not depend on the particular choice of  $\psi$  and consists of distributional solutions to (CB).



### 3 Properties of Hardy classes, approximation issues

Let  $1 < p < \infty$  and assume that  $\nu$  satisfies  $(\kappa)$ . We first state and discuss some fundamental properties from [4, 5, 9].

#### 3.1 Background

- Classes  $H_\nu^p$ :

Basic properties of the above classes  $H_\nu^p$  were established in the preliminary work [9], of which we give a brief account below. These  $H_\nu^p$  spaces mainly share properties of the classical Hardy spaces  $H^p$ . From [9, Prop. 4.3.1,4.3.2] we have:

##### Proposition 1

- Any function  $f$  in  $H_\nu^p$  has a non-tangential limit almost everywhere on  $\mathbb{T}$ , which we call the trace of  $f$  and denote by  $\text{tr } f$ ; moreover,  $\text{tr } f \in L^p(\mathbb{T})$  and its  $L^p(\mathbb{T})$  norm is equivalent to the  $H_\nu^p$  norm of  $f$ .
- The space  $\text{tr } H_\nu^p$  is closed in  $L^p(\mathbb{T})$ .
- If  $f \in H_\nu^p$  and  $f \not\equiv 0$ , then its zeros are isolated in  $\mathbb{D}$  and  $\log |\text{tr } f| \in L^1(\mathbb{T})$  (in particular,  $\text{tr } f$  does not vanish on a subset of  $\mathbb{T}$  having positive Lebesgue measure).
- Each  $f \in H_\nu^p$  satisfies the maximum principle (in modulus).
- Let  $f \in H_\nu^{p,0}$ . If  $\text{Re}(\text{tr } f) = 0$  a.e. on  $\mathbb{T}$ , then  $f \equiv 0$ .

Concerning the Dirichlet problem, we have [9, Thm 4.4.1.2]:

**Theorem 1** For all  $\varphi \in L^p_{\mathbb{R}}(\mathbb{T})$ , there exists a unique  $f \in H_\nu^{p,0}$  such that, a.e. on  $\mathbb{T}$ ,  $\text{Re}(\text{tr } f) = \varphi$ . Moreover, there exists  $c_{p,\nu} > 0$  such that:  $\|f\|_{H_\nu^p(\mathbb{D})} \leq c_{p,\nu} \|\varphi\|_{L^p(\mathbb{T})}$ .

This result allows us to define a generalized conjugation operator  $\mathcal{H}_\nu$  from  $L^p(\mathbb{T})$  into itself. Indeed, to each  $\varphi \in L^p_{\mathbb{R}}(\mathbb{T})$ , associate the unique function  $f \in H_\nu^{p,0}$  such that  $\text{Re } \text{tr } f = \varphi$ , and set  $\mathcal{H}_\nu \varphi = \text{Im } \text{tr } f \in L^p(\mathbb{T})$ . Note that  $\mathcal{H}_\nu$  was introduced on  $W^{1/2,2}(\mathbb{T})$  in [5] as the  $\nu$ -Hilbert transform. When  $\nu = 0$ , we observe that  $\mathcal{H}_0 \varphi$  is just the harmonic conjugate of  $\varphi$  normalized to have zero mean on  $\mathbb{T}$ .

Finally, the following density property of traces hold from [9, Thm 4.5.2.1]:

**Theorem 2** Let  $I \subset \mathbb{T}$  be a measurable subset such that  $\mathbb{T} \setminus I$  has positive Lebesgue measure. The restrictions to  $I$  of traces of  $H_\nu^p$ -functions are dense in  $L^p(I)$ .

- Conjugation operator  $\mathcal{H}_\nu$ :

It follows from Theorem 1 that [9, Cor. 4.4.2.1]:

**Corollary 1** The operator  $\mathcal{H}_\nu$  is bounded on  $L^p_{\mathbb{R}}(\mathbb{T})$ .

Further, the following properties hold [4, 5]:

- (i) On  $L^p_{\mathbb{R}}(\mathbb{T})$ ,  $\mathcal{H}_{-\nu} \circ \mathcal{H}_\nu u = \mathcal{H}_\nu \circ \mathcal{H}_{-\nu} u = -u + Lu$ :  $(\mathcal{H}_\nu + L)^{-1} = -\mathcal{H}_{-\nu} + L$ . This follows immediately from the fact that  $iH_\nu^p = H_{-\nu}^p$  ( $f$  is a solution to (CB)  $\Leftrightarrow g = if$  is a solution to  $\bar{\partial}g = -\nu \bar{\partial}g$ ).
- (ii) It is easily checked that, by definition:  $\mathcal{H}_\nu \circ L = L \circ \mathcal{H}_\nu = 0$  on  $L^p_{\mathbb{R}}(\mathbb{T})$ .
- (iii) So far,  $\mathcal{H}_\nu(u)$  was defined only for real-valued functions. It is however natural to set as in [4, 5]:  $\mathcal{H}_\nu(iu) = i\mathcal{H}_{-\nu}(u)$ , thereby extending  $\mathcal{H}_\nu$  to complex-valued functions, as an  $\mathbb{R}$ -linear bounded operator on  $L^p(\mathbb{T})$ .

**Remark 1** The generalized conjugation operator  $\mathcal{H}_\nu$  may be expressed as a compact perturbation of the usual one, thereby extending to  $L^p(\mathbb{T})$  situations with  $1 < p < \infty$  the results established in [5] in  $W^{1/2,2}(\mathbb{T})$ .

- Projection operator  $P_\nu$ :

(i) On  $L^2(\mathbb{T})$  let:

$$P_\nu \operatorname{tr} f = \frac{1}{2}(I + i\mathcal{H}_\nu) \operatorname{tr} f + \frac{1}{2}L(\operatorname{tr} f).$$

This defines a generalized Riesz projection operator  $P_\nu$  from  $L^p(\mathbb{T})$  onto  $\operatorname{tr} H_\nu^p$ , [5, 9]. In particular  $P_\nu$  is a bounded operator on  $L^2(\mathbb{T})$ . Equivalently, for  $f = u + iv$ ,  $u, v \in L^2_{\mathbb{R}}(\mathbb{T})$ ,

$$P_\nu \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u - \mathcal{H}_{-\nu}v \\ \mathcal{H}_\nu u + v \end{pmatrix} + \frac{1}{2} \begin{pmatrix} Lu \\ Lv \end{pmatrix}$$

(ii) Observe that  $P_\nu \circ L = L \circ P_\nu = L$ .

(iii) Let  $u_1, u_2 \in L^2_{\mathbb{R}}(\mathbb{T})$ .

$$P_\nu(u_1 + i\mathcal{H}_\nu u_2 + ic) = P_\nu \begin{pmatrix} u_1 \\ \mathcal{H}_\nu u_2 + c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_1 + u_2 + L(u_1 - u_2) \\ \mathcal{H}_\nu(u_1 + u_2) + 2c \end{pmatrix}.$$

### 3.2 Further properties of Hardy classes

By definition of the Hilbert transform  $\mathcal{H}_\nu$ , we have the following M. Riesz-like result, which is also a generalization of [26, Lem. 3]:

**Corollary 2** *Let  $f = u + iv \in L^p(\mathbb{T})$  ( $u, v \in L^p_{\mathbb{R}}(\mathbb{T})$ ).*

(i)  $f \in \operatorname{tr} H_\nu^{p,0} \Leftrightarrow v = \mathcal{H}_\nu u$ . Further, there exists an absolute constant  $c = c_{p,\nu} > 0$  such that:

$$\forall f = u + iv \in \operatorname{tr} H_\nu^{p,0}, \quad \|f\|_{L^p(\mathbb{T})} \leq c \|u\|_{L^p(\mathbb{T})}.$$

(ii)  $f \in \operatorname{tr} H_\nu^p \Leftrightarrow v = \mathcal{H}_\nu u + Lv$ . If  $I \subset \mathbb{T}$  is a connected subset of  $\mathbb{T}$  such that both  $I$  and  $J = \mathbb{T} \setminus I$  have positive Lebesgue measure, there exists an absolute constant  $c = c_{p,\nu,I} > 0$  such that:

$$\forall f = u + iv \in \operatorname{tr} H_\nu^p, \quad \|f\|_{L^p(\mathbb{T})} \leq c(\|u\|_{L^p(\mathbb{T})} + \|v\|_{L^p(I)}).$$

*Proof.* (i) Use Theorem 1 and Corollary 1.

(ii) Let  $f = u + iv \in \operatorname{tr} H_\nu^p$ . Apply point (i) to  $f - iLv \in \operatorname{tr} H_\nu^{p,0}$ . We then have  $v = \mathcal{H}_\nu u + \alpha$ , for some constant  $\alpha = Lv$ , and there exists  $C = C_{p,\nu} > 0$  such that

$$\|v - \alpha\|_{L^p(\mathbb{T})} \leq C \|u\|_{L^p(\mathbb{T})}.$$

Then

$$\|f - \alpha\|_{L^p(I)} \leq 2C \|u\|_{L^p(\mathbb{T})},$$

whence

$$\|\alpha\|_{L^p(I)} \leq 2C \|u\|_{L^p(\mathbb{T})} + \|f\|_{L^p(I)}.$$

This gives the required bound on  $\alpha$ . □

As another consequence of Theorem 1, we get:

**Corollary 3**  $L^p(\mathbb{T}) = \operatorname{tr} H_\nu^p \oplus \overline{\operatorname{tr} H_\nu^{p,0}}$ , and the decomposition is topological.

*Proof.*

In view of (5), Proposition 1 also holds for  $\overline{H_\nu^p}$  functions (while  $f \in H_\nu^{p,0} \iff \check{f} \in \overline{H_\nu^{p,0}}$  as well). In particular,  $\operatorname{tr} H_\nu^p + \overline{\operatorname{tr} H_\nu^{p,0}} \subset L^p(\mathbb{T})$ . Note also that  $\operatorname{Re}(\operatorname{tr} \check{f}) = \operatorname{Re}(\operatorname{tr} f)$ ,  $\operatorname{Im}(\operatorname{tr} \check{f}) = -\operatorname{Im}(\operatorname{tr} f)$  a.e. on  $\mathbb{T}$ . Conversely, it follows first from Theorem 1 that  $L^p_{\mathbb{R}}(\mathbb{T}) \subset \operatorname{Re}(\operatorname{tr} H_\nu^{p,0}) + \operatorname{Re}(\operatorname{tr} \overline{H_\nu^{p,0}})$ . Indeed, assume that  $\varphi \in L^p_{\mathbb{R}}(\mathbb{T})$ . In view of Theorem 1, there exists  $f \in H_\nu^{p,0}$  such that, a.e. on  $\mathbb{T}$ :  $\operatorname{Re}(\operatorname{tr} f) = \varphi/2$ . Hence,  $\check{f} \in \overline{H_\nu^{p,0}}$  and a.e. on  $\mathbb{T}$ ,  $\operatorname{Re}(\operatorname{tr} \check{f}) = \varphi/2$ , and  $\operatorname{Re} \operatorname{tr}(f + \check{f}) = \varphi$ ,  $\operatorname{Im} \operatorname{tr}(f + \check{f}) = 0$ .

Next, consider  $\Phi = \varphi + i\psi \in L^p(\mathbb{T})$ , with  $\varphi, \psi \in L^p_{\mathbb{R}}(\mathbb{T})$ . Let  $f$  and  $\check{f}$  as above, and  $g \in H^{p,0}_{-\nu}$  such that  $\operatorname{Re}(\operatorname{tr} g) = \psi/2$  a.e. on  $\mathbb{T}$  ( $\operatorname{Re} \operatorname{tr}(g + \check{g}) = \psi$ ,  $\operatorname{Im} \operatorname{tr}(g + \check{g}) = 0$ ). Then, one can check that  $ig \in H^p_{\nu}$  and  $i\check{g} \in \overline{H^p_{\nu}}$ , while  $\operatorname{Im} i \operatorname{tr}(g + \check{g}) = \psi$  and  $\operatorname{Re} i \operatorname{tr}(g + \check{g}) = 0$  a.e. on  $\mathbb{T}$ . Finally, a.e. on  $\mathbb{T}$ ,

$$\Phi = \operatorname{tr}(f + \check{f}) + i \operatorname{tr}(g + \check{g}) = \operatorname{tr}(f + ig) + c_{f,g} + \operatorname{tr}(\check{f} + i\check{g}) - c_{f,g} \in \operatorname{tr} H^p_{\nu} + \operatorname{tr} \overline{H^{p,0}_{\nu}},$$

with

$$c_{f,g} = \int_{\mathbb{T}} \operatorname{Re} \operatorname{tr} f |d\theta| + i \int_{\mathbb{T}} \operatorname{Re} \operatorname{tr} g |d\theta|.$$

To check that this decomposition is direct, let  $f \in H^p_{\nu}$  and  $g \in \overline{H^{p,0}_{\nu}}$  such that  $\operatorname{tr} f = \operatorname{tr} g$  a.e. on  $\mathbb{T}$ . Hence  $f + \check{g} \in H^p_{\nu}$  and  $\operatorname{Im} \operatorname{tr}(f + \check{g}) = 0$ : from Theorem 1,  $f + \check{g}$  is identically equal in  $\mathbb{D}$  to a real-valued constant:  $f + \check{g} = 2 \operatorname{Re} \operatorname{tr} f = 2 \operatorname{Re} \operatorname{tr} g = c$ . From the normalization assumption (6), we necessarily have that  $\operatorname{Re} \operatorname{tr} f = \operatorname{Re} \operatorname{tr} g = 0$  a.e. on  $\mathbb{T}$ , hence also  $\operatorname{Im} \operatorname{tr} f = \operatorname{Im} \operatorname{tr} g = 0$  a.e. on  $\mathbb{T}$ .  $\square$

We can also derive an explicit conjugation formula on  $L^p_{\mathbb{R}}(\mathbb{T})$ :

**Proposition 2** For  $z \in \mathbb{T}$ ,  $u \in L^p_{\mathbb{R}}(\mathbb{T})$ ,

$$(\mathcal{H}_{\nu}u)(z) - (\mathcal{H}_0u)(z) = -\frac{1}{\pi} \iint_{\mathbb{D}} (1 - \sigma(\zeta)) [\partial_x \log |\zeta - z| \partial_y u(\zeta) - \partial_y \log |\zeta - z| \partial_x u(\zeta)] dx dy.$$

*Proof.* Consider  $f \in H^{p,0}_{-\nu}$  such that  $\operatorname{Re} \operatorname{tr} f = u$  on  $\mathbb{T}$  (Theorem 1). Put  $\operatorname{tr} f = u + iv$ , with  $v = \mathcal{H}_{\nu}u$  (Corollary 2). For  $z \in \mathbb{T}$ , from the Cauchy–Green formula,

$$v(z) = \operatorname{Im} \operatorname{tr} f(z) = \operatorname{Im} \left[ P_+(\operatorname{tr} f)(z) + \frac{1}{2i\pi} \iint_{\mathbb{D}} \frac{\bar{\partial} f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta} \right].$$

But from [22]:

$$P_+(\operatorname{tr} f)(z) = \frac{1}{2}(\operatorname{tr} f + i\mathcal{H}_0(\operatorname{tr} f))(z) + \frac{1}{2}L(\operatorname{tr} f).$$

Hence, since  $f$  satisfies the normalisation condition  $\operatorname{Im} L(f) = 0$ , we get:

$$\operatorname{Im} P_+(\operatorname{tr} f) = \frac{1}{2} \operatorname{Im}(\operatorname{tr} f + i\mathcal{H}_0(\operatorname{tr} f)) = \frac{1}{2}(v + \mathcal{H}_0u)$$

and

$$\frac{1}{2}(v(z) - \mathcal{H}_0u(z)) = \operatorname{Im} \frac{1}{2i\pi} \iint_{\mathbb{D}} \frac{\bar{\partial} f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

Next,  $f$  being a solution to (CB) on  $\mathbb{D}$ ,  $u$  and  $v$  satisfy the conjugation relations (4), from which we get:

$$\bar{\partial} f = (1 - \sigma)\bar{\partial}u.$$

Using now:

$$\frac{1}{\zeta - z} = 2\partial_{\zeta} \log |\zeta - z| = 2i\partial_{\zeta} \arg(\zeta - z),$$

we obtain the equality:

$$\frac{1}{2}(v(z) - \mathcal{H}_0u(z)) = -\frac{2}{\pi} \operatorname{Im} \iint_{\mathbb{D}} (1 - \sigma(\zeta)) \bar{\partial}u(\zeta) \partial \log |\zeta - z| d\zeta \wedge d\bar{\zeta}.$$

$\square$

**Remark 2** Let  $\Omega \subset \mathbb{C}$  be a simply connected bounded Dini-smooth domain and  $\nu$  satisfying  $(\kappa)$  on  $\Omega$ . The present results still hold in the Hardy class  $H^p_{\nu}(\Omega)$ . Note that  $\nu \circ \psi \in W^{1,\infty}(\mathbb{D})$  with  $\|\nu \circ \psi\|_{L^{\infty}(\mathbb{D})} \leq \kappa < 1$ , and that any conformal transformation  $\psi$  from  $\mathbb{D}$  onto  $\Omega$  extends continuously from  $\mathbb{D}$  onto  $\bar{\Omega}$  together with its derivative, in such a way that the latter is never zero [33, Thm 3.5].

### 3.3 Bounded extremal problems (BEP) in the disc

Let  $I \subset \mathbb{T}$  be a connected subset of  $\mathbb{T}$  such that both  $I$  and  $J = \mathbb{T} \setminus I$  have positive Lebesgue measure. For  $M \geq 0$ ,  $\phi \in L^p_{\mathbb{R}}(J)$ , define

$$\mathcal{B}_p = \left\{ g \in \text{tr } H^p_{\nu}; \|\text{Re } g - \phi\|_{L^p(J)} \leq M \right\} \Big|_I \subset L^p(I).$$

**Theorem 3** Fix  $M > 0$ ,  $\phi \in L^p_{\mathbb{R}}(J)$ . For every function  $F_d \in L^p(I)$ , there exists a unique function  $g_* \in \mathcal{B}_p$  such that

$$\|F_d - g_*\|_{L^p(I)} = \min_{g \in \mathcal{B}_p} \|F_d - g\|_{L^p(I)}. \quad (\text{BEP})$$

Moreover, if  $F_d \notin \mathcal{B}_p$ , then  $\|\text{Re } g_* - \phi\|_{L^p(J)} = M$ .

*Proof.* We argue as in [7, 20]. Since  $\mathcal{B}_p$  is clearly convex, to prove the existence and the uniqueness of  $g_*$ , it is enough to check that  $\mathcal{B}_p$  is closed in  $L^p(I)$ , since this Banach space is uniformly convex.

Let  $\varphi_k|_I \in \mathcal{B}_p$ ,  $\varphi_k|_I \rightarrow \varphi_I$  in  $L^p(I)$  as  $k \rightarrow \infty$ . Put  $\varphi_k = u_k + iv_k \in \text{tr } H^p_{\nu}$ . By assumption,  $(u_k)$  is bounded in  $L^p(\mathbb{T})$  and  $(v_k)$  in  $L^p(I)$ . Hence, we get from Corollary 2, (ii), that  $(\varphi_k)$  is bounded in  $L^p(\mathbb{T})$  ( $(v_k)$  is bounded in  $L^p(J)$ ) and thus, up to extracting a subsequence, weakly converges to  $\psi \in L^p(\mathbb{T})$ ; necessarily  $\psi|_I = \varphi_I$ .

Finally,  $\varphi_k \in \text{tr } H^p_{\nu}$ , which is closed thus weakly closed in  $L^p(\mathbb{T})$  (Mazur's theorem, [23]); this implies that  $\psi \in \text{tr } H^p_{\nu}$ . Because  $\text{Re } \varphi_k = u_k$  satisfies the constraint on  $J$ , so does  $\text{Re } \psi$ , whence  $\varphi_I \in \mathcal{B}_p$ .

Let us now prove that, if  $F_d \notin \mathcal{B}_p$ , then  $\|\text{Re } g_* - \phi\|_{L^p(J)} = M$ . Assume for a contradiction that  $\|\text{Re } g_* - \phi\|_{L^p(J)} < M$ . By Theorem 2, there is a function  $h \in \text{tr } H^p_{\nu}$  such that

$$\|F_d - g_* - h\|_{L^p(I)} < \|F_d - g_*\|_{L^p(I)},$$

and by the triangle inequality we have

$$\|F_d - g_* - \lambda h\|_{L^p(I)} < \|F_d - g_*\|_{L^p(I)}$$

for all  $0 < \lambda < 1$ . Now for  $\lambda > 0$  sufficiently small we have  $\|\text{Re}(g_* + \lambda h) - \phi\|_{L^p(J)} \leq M$ , contradicting the optimality of  $g_*$ .  $\square$

**Remark 3** Theorem 3 still holds if the constraint on  $J$  is replaced by  $\|g - \phi\|_{L^p(J)} \leq M$ , for  $\phi \in L^p(J)$ , as in [7, 8], or more generally by a combination:

$$\alpha \|\text{Re } g - \phi_r\|_{L^p(J)} + \beta \|\text{Im } g - \phi_i\|_{L^p(J)} \leq M,$$

for  $\phi_r, \phi_i \in L^p_{\mathbb{R}}(J)$ , and  $\alpha, \beta \geq 0$  with  $\alpha + \beta > 0$ , as in [28].

We now pay particular attention to the Hilbertian case  $p = 2$ . Note that  $\text{tr } H^2_{\nu}$  is a closed subspace of  $L^2(\mathbb{T})$  (see Prop. 1, Cor. 3), and that a (best approximation) Riesz projection operator  $P_{\nu} : L^2(\mathbb{T}) \rightarrow \text{tr } H^2_{\nu}$  has been introduced in Section 3.1 that generalizes the usual one. Using this we can formulate a solution to the extremal problem posed in Theorem 3 (more complicated expressions can be derived in the case  $p \neq 2$ , using the methods of [7, 11, 20]).

**Proposition 3** For  $p = 2$ , and if  $F_d \notin \mathcal{B}_2$ , the solution  $g = g_*$  to the extremal problem (BEP) subject to  $\|\text{Re } g - \phi\|_{L^2(J)} \leq M$ ,  $\phi \in L^2_{\mathbb{R}}(J)$ ,  $M > 0$ , is given by

$$P_{\nu} \chi_I g + \gamma P_{\nu} \chi_J \text{Re } g = P_{\nu} \chi_I F_d + \gamma P_{\nu} \chi_J \phi, \quad (7)$$

for the unique  $\gamma > 0$  such that  $\|\text{Re } g - \phi\|_{L^2(J)} = M$ .

*Proof.* The method of proof is variational, as in [3, 28], and indeed more general abstract results can be found in [19, 29]. We therefore give only a brief sketch of the argument.

Let  $\psi \in \text{tr } H_{\nu|_I}^2$  lie in the real tangent space to  $\mathcal{B}_2$  at  $g$ , that is,

$$\text{Re}\langle g - \phi, \psi \rangle_{L^2(J)} = 0.$$

Then optimality implies that

$$\text{Re}\langle g - F_d, \psi \rangle_{L^2(I)} = 0.$$

Thus  $P_\nu \chi_I(g - F_d) = \gamma P_\nu \chi_I(\phi - \text{Re } g)$  for some  $\gamma \in \mathbb{R}$ . Finally, a first-order perturbation argument shows that  $\gamma > 0$ .  $\square$

A discussion of the relations between the (Lagrange-type) parameter  $\gamma$ , the constraint  $M$ , and the error  $e_* = \|F_d - g_*\|_{L^2(I)}$  in the “classical” case  $\nu = 0$  can be found in [3], with further precise information obtained by spectral methods in [6]. All we require in the present situation is the easy observation that  $e_*$  is a strictly decreasing function of  $M$ , and as  $M \rightarrow \infty$  we have  $e_* \rightarrow 0$ .

## 4 Families of solutions and density results

Let  $\sigma(x, y) = \sigma_*(x, y) = 1/x$  be the particular conductivity involved in plasma equations, and the associated dilation coefficient  $\nu = \nu_* = (1 - \sigma_*)/(1 + \sigma_*)$ :

$$\nu_*(z, \bar{z}) = \frac{z + \bar{z} - 2}{z + \bar{z} + 2}.$$

We consider now equations (2) and (3) in the disc  $\Omega = \mathbb{D}_0$ :

$$\mathbb{D}_0 = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + y^2 < R^2\}, \quad \text{where } 0 < R < |x_0|,$$

with  $\mathbb{T}_0$  its boundary (a circle). Let  $\mathcal{R}$  be a rectangle containing  $\mathbb{D}_0$ :  $\mathcal{R} = [a, b] \times [-c, c]$ ,  $b > a > 0$  and  $c > 0$ , with (Lipschitz-continuous) boundary  $\partial\mathcal{R}$ . Its edges are denoted by  $\Gamma_a = \{a\} \times [-c, c]$ ,  $\Gamma_b = \{b\} \times [-c, c]$ ,  $\Gamma_c = [a, b] \times \{c\}$  and  $\Gamma_{-c} = [a, b] \times \{-c\}$ .

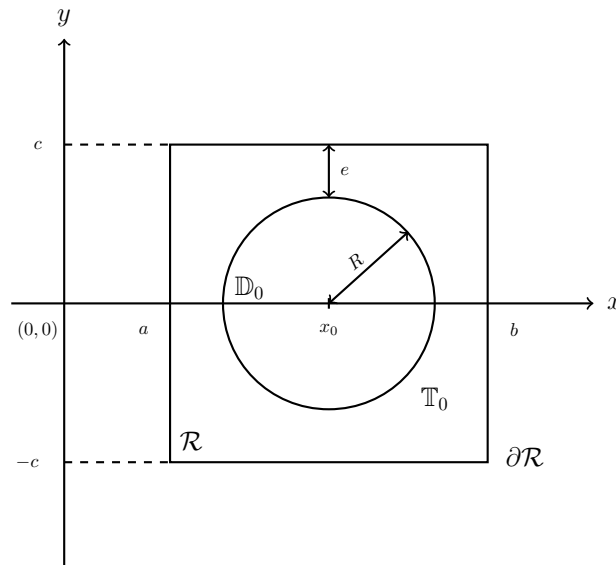


Fig.1 Geometrical setting in a poloidal section

Note that both  $\sigma = \sigma_*$  and  $\nu = \nu_*$  are smooth on  $\mathcal{R}$  and that  $\mathbb{D}_0$  is defined in order to have compact closure in  $\mathcal{R}$ .

#### 4.1 Particular solutions of Bessel-exponential type

We write  $u(x, y) = A(x)B(y)$  for a solution to (2) with separated variables and  $v(x, y) = C(x)D(y)$  for its  $\sigma$ -harmonic conjugate function, a solution to (3) in  $\mathcal{R}$ . In view of the Cauchy–Riemann equations (4), we have

$$u_x = xv_y \quad \text{or} \quad A'B = xCD'$$

and

$$u_y = -xv_x \quad \text{or} \quad AB' = -xC'D.$$

The 2nd order equation for  $v$  gives

$$C''D + CD'' = -\frac{1}{x}C'D, \quad \text{or} \quad \frac{C''}{C} + \frac{1}{x}\frac{C'}{C} = -\frac{D''}{D} = -\lambda^2 \in \mathbb{R}, \quad \text{say,}$$

giving

$$D(y) = \exp(\lambda y), \quad \text{and} \quad C'' + \frac{1}{x}C' + \lambda^2 C = 0.$$

(i) Assume  $\lambda \neq 0$ ,  $\lambda \in \mathbb{R}$ .

There is a fundamental solution in terms of Bessel functions, namely,

$$C(x) = J_0(\lambda x) \quad \text{or} \quad C(x) = Y_0(\lambda x).$$

Our main references for the theory of Bessel functions are [35, Ch. III] and [36, Ch. XVII]. Also, we may take  $B = D$  and then (since  $J'_0 = -J_1$  and  $Y'_0 = Y_1$ ) we have, correspondingly,  $A(x) = -xC'(x)/\lambda$ , i.e.,

$$A(x) = xJ_1(\lambda x) \quad \text{or} \quad A(x) = xY_1(\lambda x).$$

In fact, we shall not require the functions  $Y_n$ .

(ii) Assume  $\lambda \neq 0$ ,  $\lambda \notin \mathbb{R}$ .

Modified Bessel functions (Bessel functions of imaginary argument), namely

$$C(x) = I_0(\lambda x) \quad \text{or} \quad C(x) = K_0(\lambda x),$$

satisfy the equation

$$\frac{C''}{C} + \frac{1}{x}\frac{C'}{C} = \lambda^2,$$

in conjunction with  $D(y) = \cos(\lambda(y + y_0))$ , say. Taking  $B(y) = \sin(\lambda(y + y_0))$  again, and noting that  $I'_0 = I_1$  and  $K'_0 = -K_1$  [35], we have the corresponding solutions:

$$A(x) = -xI_1(\lambda x) \quad \text{or} \quad A(x) = xK_1(\lambda x).$$

(iii) Finally we look at the case  $\lambda = 0$ .

The dependence in  $x$  and  $y$  of each functions allows us to write

$$\frac{A'}{xC} = \frac{D'}{B} = \rho \quad \text{and} \quad \frac{A}{xC'} = -\frac{D}{B'} = -\mu$$

and we have the following identities:

$$D'' = \lambda^2 D = \rho B' = \frac{\rho}{\mu} D \quad \text{so that} \quad \lambda^2 = \frac{\rho}{\mu}$$

so that  $\lambda^2 = \frac{\rho}{\mu}$ . Then  $\lambda = 0$  implies that either  $\rho = 0$  and  $\mu \neq 0$ , or  $\mu = \infty$ .

If  $\rho = 0$  and  $\mu \neq 0$ , we have

$$D(y) = d, B(y) = \frac{d}{\mu}y + b, A(x) = a \quad \text{and} \quad C(x) = -\frac{a}{\mu} \ln(x) + c.$$

In the other case,  $\mu = \infty$ ,

$$B(y) = b, D(y) = \rho by + d, C(x) = c \quad \text{and} \quad A(x) = \frac{\rho c}{2}x^2 + a.$$

Thus, for every  $N \in \mathbb{N}^*$ , and for all sequences of real-valued coefficients  $(\lambda_n), (\mu_n), (\alpha_n), (\beta_n), (\gamma_n), (\delta_n)$ ,  $n = 1, \dots, N$ , and every  $a_0, b_0, c_0 \in \mathbb{R}$ , the following is a legitimate solution  $u$  to (2):

$$\begin{aligned} u(x, y) &= \sum_{n=1}^N x J_1(\lambda_n x) [\alpha_n e^{\lambda_n y} + \beta_n e^{-\lambda_n y}] \\ &\quad + \sum_{n=1}^N x [\gamma_n I_1(\mu_n x) + \delta_n K_1(\mu_n x)] \sin(\mu_n(y + c)) + a_0 x^2 + b_0 y + c_0 \\ &= u_\lambda(x, y) + u_\mu(x, y) + u_0(x, y), \quad \text{say.} \end{aligned} \quad (8)$$

The analogous expression for  $\mathcal{H}_{\nu_*} u$  involves Bessel functions of order zero, as above, whence if  $u$  is given by (8), then  $\mathcal{H}_{\nu_*} u = v - Lv$  with the solution  $v$  to (3) and (4) given by:

$$\begin{aligned} v(x, y) &= \sum_{n=1}^N J_0(\lambda_n x) [\alpha_n e^{\lambda_n y} - \beta_n e^{-\lambda_n y}] \\ &\quad + \sum_{n=1}^N [-\gamma_n I_0(\mu_n x) + \delta_n K_0(\mu_n x)] \cos(\mu_n(y + c)) - b_0 \ln x + 2a_0 y \\ &= v_\lambda(x, y) + v_\mu(x, y) + v_0(x, y). \end{aligned} \quad (9)$$

It is convenient to choose  $\lambda_n$  to be the positive roots of  $J_0(\lambda_n b) = 0$ , in view of the fact that with this choice the functions  $(\sqrt{x} J_0(\lambda_n x))_{n \geq 1}$  are a complete orthogonal system in  $L^2((0, b); x dx)$ . However, many other choices are possible.

An appropriate choice of  $\mu_n$  is  $\frac{n\pi}{2c}$ , so that the second expression vanishes on the lines  $y = \pm c$ .

We then write  $\mathcal{B}$  for the space of solutions  $u$  given by (8) with such  $(\lambda_n), (\mu_n)$ ,  $n = 1, \dots, N$ , and free parameters  $(\alpha_n), (\beta_n), (\gamma_n), (\delta_n)$ ,  $a_0, b_0, c_0 \in \mathbb{R}$ . Meanwhile, we introduce  $\mathcal{F}$  as the corresponding (through (4)) space of solutions  $v$  given by (9).

Then, given  $L^2$  data on the boundary of the rectangle  $\mathcal{R} = [a, b] \times [-c, c]$ , we may approximate it arbitrarily closely on the lines  $y = \pm c$  by taking a finite sum and choosing  $(\alpha_n)$  and  $(\beta_n)$  suitably. Then we may choose the coefficients  $(\gamma_n)$  and  $(\delta_n)$  to obtain approximation on the lines  $x = a$ ,  $x = b$ .

**Remark 4** Note that the Wronskian  $I_1' K_1 - I_1 K_1'$  does not vanish, which implies that  $I_1/K_1$  is monotonic, and so the vectors  $(I_1(\mu_n a), K_1(\mu_n a))$  and  $(I_1(\mu_n b), K_1(\mu_n b))$  in  $\mathbb{R}^2$  are linearly independent.

## 4.2 Density properties of $\mathcal{B}$ in $\mathcal{R}$

**Proposition 4** The family  $\mathcal{B}$  is  $W_{\mathbb{R}}^{1,2}(\mathcal{R})$  dense in

$$\mathcal{S}_{\mathcal{R}} = \left\{ u \in W_{\mathbb{R}}^{1,2}(\mathcal{R}); \operatorname{div} \left( \frac{1}{x} \nabla u \right) = 0 \right\}.$$

The proof relies on the following result that we first establish:

**Lemma 1**  $\mathcal{B}|_{\partial \mathcal{R}}$  is  $W_{\mathbb{R}}^{1,2}$  dense in  $W_{\mathbb{R}}^{1,2}(\partial \mathcal{R})$ .

*Proof of Lemma 1.* Recall that  $\mathcal{B}_{|\partial\mathcal{R}}$  is the set of solutions of the conductivity equation

$$\operatorname{div}\left(\frac{1}{x}\nabla u\right) = 0,$$

obtained by separating the variables on the rectangle  $\mathcal{R} = [a, b] \times [-c, c]$ , restricted to  $\partial\mathcal{R}$ , in  $W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$ .

The proof proceeds in three stages:

- we consider first the horizontal sides  $\Gamma_c$  and  $\Gamma_{-c}$  (Step 1),
- then the vertical sides  $\Gamma_a$  and  $\Gamma_b$  (Step 2),
- then we prove density for the complete boundary  $\partial\mathcal{R}$  of the rectangle  $\mathcal{R}$  (Step 3).

**Step 1:** density of  $\mathcal{B}_{|\Gamma_c}$  in  $W_{\mathbb{R}}^{1,2}(a, b)$ .

Let  $f \in W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$  so that  $f(\cdot, c) \in W_{\mathbb{R}}^{1,2}(a, b)$  denotes its restriction to  $\Gamma_c$ . Write  $(\lambda_n)_{n \geq 1}$  for the strictly increasing sequence of roots [35, Ch. XV] of the equation  $J_0(\lambda_n b) = 0$  which are all positive. Now we define a function  $\psi$  on  $\mathcal{R}$  by  $\psi(x, y) = \frac{f'(x, c)}{c\sqrt{x}} y$ . Its restriction to  $\Gamma_c$  still belongs to  $L_{\mathbb{R}}^2(a, b)$  and, by [14, Thm. 2],  $\{x^{1/2}J_0(\lambda_n x)\}$  forms a complete system in  $L_{\mathbb{R}}^2(a, b)$ . It follows that for every  $\epsilon > 0$ , there exist a sufficiently large integer  $N$  and scalars  $(a_{n,c})_{n=1,\dots,N}$  such that

$$\left\| \psi(t, c) - \sum_{n=1}^N a_{n,c} \sqrt{t} J_0(\lambda_n t) \right\|_{L^2(a,b)} \leq \frac{\epsilon}{b-a} \sqrt{\frac{2}{b}},$$

Then classical properties of the integral and the Cauchy–Schwarz inequality lead to

$$\frac{1}{\sqrt{b-x}} \left| \int_x^b (f'(t, c) - \sum_{n=1}^N a_{n,c} t J_0(\lambda_n t)) dt \right| \leq \epsilon \sqrt{\frac{2}{b-a}}, \quad x \in (a, b). \quad (10)$$

Using the identity  $J_{\nu}(x) = x^{-\nu-1} \frac{d}{dx}(x^{\nu+1} J_{\nu+1}(x))$  from [35, p. 66] and calculating the integral in (10) allow us to write

$$(\xi_c(x, c) - f(x, c))^2 \leq 2 \left( \frac{b-x}{b-a} \right) \epsilon^2, \quad (11)$$

where  $\xi_c \in \mathcal{B}$  denotes the function defined on  $\mathcal{R}$  by

$$\xi_c(x, y) = \sum_{n=1}^N \frac{a_{n,c}}{\lambda_n} x J_1(\lambda_n x) \left( \frac{e^{\lambda_n(y+c)} - e^{-\lambda_n(y+c)}}{e^{2\lambda_n c} - e^{-2\lambda_n c}} \right) - \left( \frac{K_{N,c} - f(b, c)}{2c} \right) (y+c), \quad (12)$$

and

$$K_{N,c} = \sum_{n=1}^N \frac{a_{n,c}}{\lambda_n} b J_1(\lambda_n b).$$

Integrating (11) between  $a$  and  $b$  we have that  $\|f(x, c) - \xi_c(x, c)\|_{L^2(a,b)} \leq \epsilon$ , which implies the density of  $\mathcal{B}_{|\Gamma_c}$  in  $W_{\mathbb{R}}^{1,2}(a, b)$ .

A similar calculation can be made on the side  $\Gamma_{-c}$  and we denote by  $\xi_{-c}$  the corresponding function. Since  $\xi_c$  vanishes on  $\Gamma_{-c}$  and that  $\xi_{-c}$  vanishes on  $\Gamma_c$ , we conclude that if  $f \in W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$ , then  $(\xi_c + \xi_{-c})$  approximates  $f$  on  $\Gamma_c \cup \Gamma_{-c}$  in  $W_{\mathbb{R}}^{1,2}(a, b)$  norm.

The same principle will be used for the vertical sides  $\Gamma_a$  and  $\Gamma_b$ : construct an approximation on both, such that its contribution to the other side is arbitrarily small (or zero) in  $W_{\mathbb{R}}^{1,2}$  norm.



**Step 2:** density of  $\mathcal{B}_{|\Gamma_b}$  in  $W_{\mathbb{R}}^{1,2}(-c, c)$ .

Let  $f \in W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$  so that  $f(b, \cdot) \in W_{\mathbb{R}}^{1,2}(-c, c)$  denotes its restriction to  $\Gamma_b$ . Hence  $f'(b, \cdot)$  can be expanded as a Fourier series using the family  $\left\{1, \cos\left(\frac{n\pi}{2c}(y+c)\right)\right\}_{n \in \mathbb{N}^*}$ , which is complete in  $L^2_{\mathbb{R}}(-c, c)$ . Hence, for every  $\epsilon > 0$ , there exist an integer  $N$  and scalars  $(a_{n,b})_{n=1, \dots, N}$  such that

$$\left\| f'(b, t) - \left( a_{0,b} + \sum_{n=1}^N a_{n,b} \cos\left(\frac{n\pi}{2c}(t+c)\right) \right) \right\|_{L^2(-c,c)} \leq \frac{\epsilon}{\sqrt{c}}.$$

Let us denote by  $\mathcal{F}_N(b, \cdot)$  the above partial sum of order  $N$  of the Fourier series expansion of  $f'(b, \cdot)$ . As in Step 1, we have that

$$\frac{1}{\sqrt{y+c}} \left| \int_{-c}^y (f'(b, t) - \mathcal{F}_N(b, t)) dt \right| \leq \sqrt{2}\epsilon, \quad y \in (-c, c). \tag{13}$$

Writing  $\mu_n = \frac{n\pi}{2c}$ , (13) can be calculated to give

$$\left| f(b, y) - f(b, -c) - \left( a_{0,b}(y+c) + \sum_{n=1}^N \frac{a_{n,b}}{\mu_n} \sin(\mu_n(y+c)) \right) \right| \leq \sqrt{2(y+c)}\epsilon.$$

That is,

$$(f(b, y) - \xi_b(b, y))^2 \leq 2(y+c)\epsilon^2, \tag{14}$$

where  $\xi_b \in \mathcal{B}$  denotes the function defined on  $\mathcal{R}$  by

$$\begin{aligned} \xi_b(x, y) &= \sum_{n=1}^{n=N} x [\gamma_{n,b} I_1(\mu_n x) + \delta_{n,b} K_1(\mu_n x)] \sin(\mu_n(y+c)) \\ &+ \left( \frac{x^2 - a^2}{b^2 - a^2} \right) f(b, -c) + \frac{f(b, c) - f(b, -c)}{2c} (y+c), \end{aligned}$$

and

$$\begin{cases} \gamma_{n,b} = -\frac{a_{n,b}}{b\mu_n} \left( \frac{K_1(\mu_n a)}{K_1(\mu_n b)I_1(\mu_n a) - K_1(\mu_n a)I_1(\mu_n b)} \right), \\ \delta_{n,b} = \frac{a_{n,b}}{b\mu_n} \left( \frac{I_1(\mu_n a)}{K_1(\mu_n b)I_1(\mu_n a) - K_1(\mu_n a)I_1(\mu_n b)} \right). \end{cases}$$

By Remark 4, the expression of  $\xi_b$  is valid insofar as  $K_1(\mu_n b)I_1(\mu_n a) - K_1(\mu_n a)I_1(\mu_n b) \neq 0$ . Finally, integrating (14) between  $-c$  and  $c$ , we have  $\|f(b, y) - \xi_b(b, y)\|_{L^2(-c,c)} \leq \epsilon$ , which gives the density of  $\mathcal{B}_{|\Gamma_b}$  in  $W_{\mathbb{R}}^{1,2}(-c, c)$ .

As in the previous section, a similar result holds on the side  $\Gamma_a$  with a corresponding function  $\xi_a$ . The functions  $\xi_a$  and  $\xi_b$  are constructed so that if  $f$  is arbitrarily small on  $\Gamma_c$  and  $\Gamma_{-c}$  then  $\xi_a$  is arbitrarily small on  $\Gamma_b$  and so does  $\xi_b$  on  $\Gamma_a$ . We conclude that if  $f \in W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$  and is small enough, then  $(\xi_a + \xi_b)$  is an approximation to  $f$  on  $\Gamma_a \cup \Gamma_b$  in  $W_{\mathbb{R}}^{1,2}(-c, c)$  norm.

**Step 3:** density of  $\mathcal{B}_{|\partial\mathcal{R}}$  in  $W^{1,2}(\partial\mathcal{R})$ .

Let  $f \in W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$ . From Step 1, there exist  $\xi_c$  and  $\xi_{-c}$  belonging to  $\mathcal{B}$  such that on  $\Gamma_c \cup \Gamma_{-c}$ , we have  $(f - \xi_c - \xi_{-c})$  arbitrarily small in  $W_{\mathbb{R}}^{1,2}(a, b)$  norm. Since  $(f - \xi_c - \xi_{-c})$  still belongs to  $W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$ , there

exist, from Step 2,  $\xi_a \in \mathcal{B}$  and  $\xi_b \in \mathcal{B}$  which approximate  $(f - \xi_c - \xi_{-c})$  in  $W_{\mathbb{R}}^{1,2}(-c, c)$  norm on the sides  $\Gamma_a$  and  $\Gamma_b$  respectively. Writing  $I = \{a, b, c, -c\}$ , we now show that

$$\|f - (\xi_c + \xi_{-c} + \xi_a + \xi_b)\|_{W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})}^2 = \sum_{i \in I} \|f - (\xi_c + \xi_{-c} + \xi_a + \xi_b)\|_{W_{\mathbb{R}}^{1,2}(\Gamma_i)}^2$$

can be made arbitrarily small, in other words  $(\xi_c + \xi_{-c} + \xi_a + \xi_b) \in \mathcal{B}$  gives a good approximation to the function  $f$  in  $W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$  norm. For convenience, we only give the details for  $\Gamma_{-c}$ . We let  $\mathcal{H}$  denote the function defined on  $\mathcal{R}$  such that  $\mathcal{H} = (f - \xi_c - \xi_{-c})$ . Let  $A_{-c} = \|f - (\xi_c + \xi_{-c} + \xi_a + \xi_b)\|_{W_{\mathbb{R}}^{1,2}(\Gamma_{-c})}$ . Then

$$\begin{aligned} A_{-c} &= \|f(x, -c) - \xi_{-c}(x, -c) - \left(\frac{x^2 - b^2}{a^2 - b^2}\right) \mathcal{H}(a, -c) - \left(\frac{x^2 - a^2}{b^2 - a^2}\right) \mathcal{H}(b, -c)\|_{W_{\mathbb{R}}^{1,2}(a,b)} \\ &\leq \|f(x, -c) - \xi_{-c}(x, -c)\|_{W_{\mathbb{R}}^{1,2}(a,b)} + \|\mathcal{H}(a, -c)\|_{W_{\mathbb{R}}^{1,2}(a,b)} + \|\mathcal{H}(b, -c)\|_{W_{\mathbb{R}}^{1,2}(a,b)}, \end{aligned}$$

Since  $\xi_{c|\Gamma_{-c}} = 0$  and  $\xi_{-c}$  is an approximation to  $f$  on  $\Gamma_{-c}$  in  $W_{\mathbb{R}}^{1,2}(a, b)$  norm, thus a fortiori at the corner points  $(a, -c)$  and  $(b, -c)$  also, all the terms in the last inequality can be made arbitrarily small.

Finally,  $\|f - (\xi_c + \xi_{-c} + \xi_a + \xi_b)\|_{W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})}$  can be made arbitrarily small. This ends the proof of Lemma 1.  $\square$

*Proof of Proposition 4.*

Recall first that  $\mathcal{R}$  is a simply-connected plane domain with Lipschitz boundary [24, p. 6].

Let  $f \in \mathcal{S}_{\mathcal{R}}$ . Since  $\text{tr } f \in W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})$  [24, Thm 1.5.1.3], there exist  $(\phi_n)_{n \in \mathbb{N}} \in W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})$  such that

$$\|\text{tr } f - \phi_n\|_{W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})} \rightarrow 0,$$

and by Lemma 1, we also have that for each  $n \in \mathbb{N}$  there exists  $b^{(n)} \in \mathcal{B}$  such that  $\|\phi_n - b^{(n)}\|_{W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})}$  can be made arbitrarily small. Since  $W_{\mathbb{R}}^{1,2}(\partial\mathcal{R}) \subset W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})$  (see [30]), and the injection is bounded (indeed, even compact) by [12, Thm 2.2] (see also [1, Par. 1.24]), we have the existence of a constant  $c > 0$  such that

$$\|\phi_n - b^{(n)}\|_{W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})} \leq c \|\phi_n - b^{(n)}\|_{W_{\mathbb{R}}^{1,2}(\partial\mathcal{R})} \rightarrow 0,$$

and, by the triangle inequality,

$$\|\text{tr } f - b^{(n)}\|_{W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})} \leq \|\text{tr } f - \phi_n\|_{W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})} + \|\phi_n - b^{(n)}\|_{W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})} \rightarrow 0.$$

Finally, by [12, Thm 4.4], there is a continuous lifting of the trace from  $W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})$  to  $W_{\mathbb{R}}^{1,2}(\mathcal{R})$ . Since  $f$  belongs to  $\mathcal{S}_{\mathcal{R}}$  this lifting is unique, so that, for some  $c' > 0$ ,

$$\|f - b^{(n)}\|_{W_{\mathbb{R}}^{1,2}(\mathcal{R})} \leq c' \|\text{tr } f - b^{(n)}\|_{W_{\mathbb{R}}^{1/2,2}(\partial\mathcal{R})} \rightarrow 0,$$

and thus  $\mathcal{B}$  is dense in  $\mathcal{S}_{\mathcal{R}}$ .  $\square$

### 4.3 Density properties of $\mathcal{B}$ in $\mathbb{D}_0$

We show below that  $L_{\mathbb{R}}^2(\mathbb{T}_0)$  functions may be approximated by functions belonging to  $\mathcal{B}_{|\mathbb{T}_0}$  that are solutions to (2). For the particular dilation coefficient  $\nu_*$ , this is a constructive view of the fact that  $L_{\mathbb{R}}^2(\mathbb{T}_0) = \text{Re tr } H_{\nu_*}^2(\mathbb{D}_0)$  (see Theorem 1) and will be used in order to solve bounded extremal problems. Recall that  $H_{\nu_*}^2(\mathbb{D}_0)$  is defined accordingly to Section 2 and Remark 2, using for instance  $\psi(z) = Rz + x_0$  as a conformal map from  $\mathbb{D}$  onto  $\mathbb{D}_0$ .

**Proposition 5** *The restriction  $\mathcal{B}_{|\mathbb{T}_0}$  to  $\mathbb{T}_0$  of  $\mathcal{B}$  is  $L^2$  dense in  $L_{\mathbb{R}}^2(\mathbb{T}_0) = \text{Re tr } H_{\nu_*}^2(\mathbb{D}_0)$ .*

*Proof.* Let  $\psi \in L^2_{\mathbb{R}}(\mathbb{T}_0)$  and  $\varepsilon > 0$ . There exists  $\phi \in W^{1/2,2}_{\mathbb{R}}(\mathbb{T}_0)$  such that

$$\|\psi - \phi\|_{L^2(\mathbb{T}_0)} \leq \varepsilon.$$

However, Dirichlet-type results from [18] are to the effect that there exists a unique solution  $u \in W^{1,2}_{\mathbb{R}}(\mathbb{D}_0)$  such that  $u|_{\mathbb{T}_0} = \phi$ . Then, because of the  $W^{1,2}(\mathcal{R})$  density of  $\mathcal{B}$  in  $\mathcal{S}_{\mathcal{R}}$ , see Proposition 4, approximation results of Browder [16, 17] ensure that there exists  $b \in \mathcal{B}$  such that

$$\|u - b\|_{W^{1,2}(\mathbb{D}_0)} \leq \varepsilon,$$

whence also  $\|u - b\|_{L^2(\mathbb{T}_0)} \leq c\varepsilon$ , for some  $c > 0$ , from [15, Ch. 9] (we are taking the restrictions of  $b$  to the disc or the circle as appropriate). Finally, we get that

$$\|\psi - b\|_{L^2(\mathbb{T}_0)} \leq (1 + c)\varepsilon.$$

This implies that  $\mathcal{B}|_{\mathbb{T}_0}$  is dense in  $L^2_{\mathbb{R}}(\mathbb{T}_0)$ . Further, the result that

$$L^2_{\mathbb{R}}(\mathbb{T}_0) = \operatorname{Re} \operatorname{tr} H^2_{\nu^*}(\mathbb{D}_0) \tag{15}$$

follows from Theorem 1. □

**Remark 5** *Results of Lax [27] and Malgrange [31], assert similar Runge-like density properties, namely that real solutions of the partial differential equation on a disc can be approximated locally uniformly by solutions that extend to a large simply-connected domain.*

We then obtain:

**Corollary 4** (i)  $\mathcal{B}^{\nu^*} = \{b + i\mathcal{H}_{\nu^*}b + ic, b \in \mathcal{B}, c \in \mathbb{R}\}$  is  $L^2(\mathbb{T}_0)$  dense in  $\operatorname{tr} H^2_{\nu^*}(\mathbb{D}_0)$ .  
 (ii)  $\mathcal{B} + i\mathcal{H}_{\nu^*}\mathcal{B} + i\mathbb{R} = \{b_1 + i\mathcal{H}_{\nu^*}b_2 + ic, b_1, b_2 \in \mathcal{B}, c \in \mathbb{R}\}$  is dense in  $L^2(\mathbb{T}_0)$ .

*Proof.* (i) Let  $f \in H^2_{\nu^*}(\mathbb{D}_0)$  and  $\varepsilon > 0$ . So  $u = \operatorname{Re}(\operatorname{tr} f) \in L^2_{\mathbb{R}}(\mathbb{T}_0)$ , and Proposition 5 ensures that there exists  $b \in \mathcal{B}$  such that

$$\|u - b\|_{L^2_{\mathbb{R}}(\mathbb{T}_0)} < \varepsilon.$$

Now, as  $\mathcal{H}_{\nu^*}$  is continuous from  $L^2_{\mathbb{R}}(\mathbb{T}_0)$  to  $L^2_{\mathbb{R}}(\mathbb{T}_0)$  ([9, Cor. 4.4.2.1]), we have

$$\|\mathcal{H}_{\nu^*}(u) - \mathcal{H}_{\nu^*}(b)\|_{L^2_{\mathbb{R}}(\mathbb{T}_0)} = \|\mathcal{H}_{\nu^*}(u - b)\|_{L^2_{\mathbb{R}}(\mathbb{T}_0)} < \varepsilon.$$

Finally, using Corollary 2, we get that  $\mathcal{H}_{\nu^*}(u) = v + c = \operatorname{Im}(\operatorname{tr} f) + c$ , for some  $c \in \mathbb{R}$  and point (i) follows ( $\operatorname{tr} H^2_{\nu^*}(\mathbb{D}_0) = L^2_{\mathbb{R}}(\mathbb{T}_0) + i\mathcal{H}_{\nu^*}(L^2_{\mathbb{R}}(\mathbb{T}_0)) + i\mathbb{R}$ ).

Point (ii) may be obtained from the decomposition  $L^2(\mathbb{T}_0) = L^2_{\mathbb{R}}(\mathbb{T}_0) + iL^2_{\mathbb{R}}(\mathbb{T}_0)$ , using that  $\mathcal{H}_{\nu^*}\mathcal{B} + \mathbb{R}$  is dense in  $L^2_{\mathbb{R}}(\mathbb{T}_0)$ , from point (i). □

**Remark 6** *From point (i) of the above corollary, for  $K \subset \mathbb{T}_0$ , such that  $\mathbb{T}_0 \setminus K$  has positive Lebesgue measure, we get that  $\mathcal{B}^{\nu^*}|_K$  is  $L^2(K)$  dense in  $[\operatorname{tr} H^2_{\nu^*}(\mathbb{D}_0)]|_K$ , whence in  $L^2(K)$ , in view of Theorem 2.*

## 5 Constructive aspects

In order to compute solutions to (BEP) in  $\mathbb{D}_0$  using Proposition 3 (see also Remark 2), we first need to find approximations in  $\mathcal{B}$  and  $\mathcal{H}_{\nu^*}\mathcal{B}$  to given data  $(u, v)$  on the circle  $\mathbb{T}_0$  with centre  $(x_0, 0)$  and radius  $R$ , contained in the rectangle  $\mathcal{R}$ . From now on, we denote by  $e > 0$  the minimal distance between  $\mathbb{T}_0$  and  $\mathcal{R}$  (see Fig.1).

In this section, we take advantage of the properties established for  $\mathcal{B}$  in Section 4. Indeed, recall that given  $b \in \mathcal{B}$ , with a set of coefficients on the family of Bessel-exponential functions (8) that generate  $\mathcal{B}$ , then  $\mathcal{H}_{\nu^*}b$  has the same coefficients on the family (9) of Bessel-exponentials that generate  $\mathcal{H}_{\nu^*}\mathcal{B}$  and  $\mathcal{F}$ .

## 5.1 Approximation algorithms

In order to determine numerically  $P_{\nu_*}$  for given boundary data  $u$  and  $v$  in  $L^2_{\mathbb{R}}(\mathbb{T}_0)$ , it is sufficient to compute an  $L^2$ -approximation of  $\mathcal{H}_{\nu_*}u$  and  $\mathcal{H}_{-\nu_*}v$ , according to the formulas in Section 3.1. To proceed, recall first that  $\mathcal{B}$  is the space of solutions to (2) with separated variables and  $\sigma = \sigma_*$ , while  $\mathcal{F}$  is the corresponding space of solutions to (3), (4), see the expressions (8), (9).

That  $\mathcal{F}$  is dense in  $L^2_{\mathbb{R}}(\mathbb{T}_0)$  follows from Corollary 4. The issue now consists in writing the expansions of  $u$  on  $\mathcal{B}$  and of  $v$  on  $\mathcal{F}$  and then in computing both  $\mathcal{H}_{\nu_*}u$  and  $\mathcal{H}_{-\nu_*}v$  by the following rule (see (8), (9)):

$$\mathcal{H}_{\nu_*}\mathcal{B} = \mathcal{F} - L(\mathcal{F}) \quad \text{and} \quad \mathcal{H}_{-\nu_*}\mathcal{F} = -\mathcal{B} - L(-\mathcal{B}).$$

Here the averaging operator  $L$  is considered on  $\mathbb{T}_0$ :

$$\forall f \in L^2_{\mathbb{R}}(\mathbb{T}_0), \quad Lf = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + R \cos \theta, R \sin \theta) d\theta.$$

We remark that there is no connection between the coefficients resulting from the decomposition of  $u$  in  $\mathcal{B}$  and those of  $v$  in  $\mathcal{F}$  except when  $u + iv \in \text{tr } H^2_{\nu_*}(\mathbb{D}_0)$ . Of course, only a finite number  $N$  of functions in  $\mathcal{B}$  and  $\mathcal{F}$  will be used for computations. More particularly, the same number  $n$  of functions of each kind is involved, and  $N = 4n + 2$ .

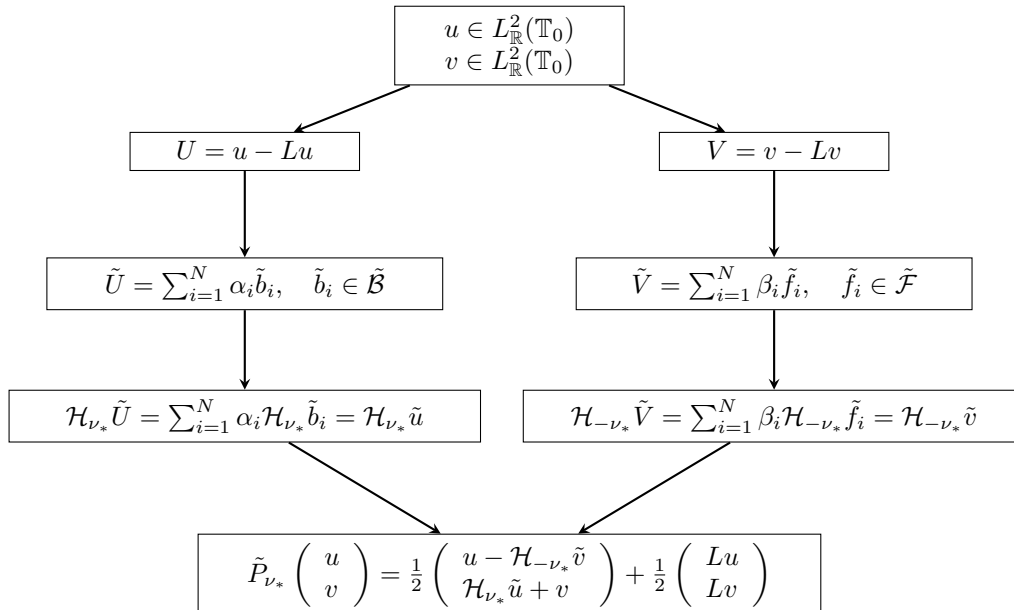
If  $(b_n)_{n=1, \dots, N}$  are the functions of one of these finite systems, which are complete in  $L^2_{\mathbb{R}}(\mathbb{T}_0)$  (as  $N \rightarrow \infty$ , Proposition 5), then an optimal  $L^2(\mathbb{T}_0)$  approximation of a function  $\psi \in L^2(\mathbb{T}_0)$  in the span of  $(b_n)_{n=1, \dots, N}$  is obtained by solving in  $(k_m)_m$  the *normal equations*:

$$\left\langle \sum_{m=1}^N k_m b_m - \psi, b_n \right\rangle_{L^2(\mathbb{T}_0)} = 0, \quad \text{or} \quad \sum_{m=1}^N k_m \langle b_m, b_n \rangle_{L^2(\mathbb{T}_0)} = \langle \psi, b_n \rangle_{L^2(\mathbb{T}_0)}, \quad n = 1, \dots, N.$$

Note that functions in the family  $\mathcal{B}$  (or  $\mathcal{F}$ ) are mutually non-orthogonal on  $\mathbb{T}_0$ . As always in this case, the stiffness matrix  $(\langle b_m, b_n \rangle_{L^2(\mathbb{T}_0)})$ ,  $m, n = 1, \dots, N$ , appearing in the resolution of the normal equations is ill-conditioned. A first approach to decreasing the conditioning number and thereby to improve the accuracy of the approximation is to average the basis functions after subtracting their mean value. Hence, we consider the families:

$$\tilde{\mathcal{B}} = \left\{ \frac{b - L(b)}{\|b - L(b)\|_{L^2(\mathbb{T}_0)}}; b \in \mathcal{B} \right\} \quad \text{and} \quad \tilde{\mathcal{F}} = \left\{ \frac{f - L(f)}{\|f - L(f)\|_{L^2(\mathbb{T}_0)}}; f \in \mathcal{F} \right\}$$

and approximations are made for  $u - Lu$  and  $v - Lv$ . Then a representation of the algorithm is:



where  $\tilde{u}, \tilde{v}, \tilde{U}, \tilde{V}, \mathcal{H}_{\nu_*} \tilde{u}, \mathcal{H}_{-\nu_*} \tilde{v}$  represent the approximations of order  $N$  in  $\mathcal{B}$  and  $\mathcal{F}$  of the corresponding functions. As a consequence, the algorithm leads us to define the operator  $\tilde{P}_{\nu_*}$  which is a  $L^2(\mathbb{T}_0)$  approximation of  $P_{\nu_*}$ . For given data  $u$  and  $v$ , the error between  $\tilde{P}_{\nu_*}(u, v)$  and  $P_{\nu_*}(u, v)$  only stems from those between  $\mathcal{H}_{\nu_*} \tilde{u}$  and  $\mathcal{H}_{\nu_*} u$  on the one hand, and between  $\mathcal{H}_{-\nu_*} \tilde{v}$  and  $\mathcal{H}_{-\nu_*} v$  on the other hand.

## 5.2 Numerical computations

In this section we present several numerical results obtained with the software Matlab (R2008b). All the simulations are computed with the specific choice of parameters  $(x_0, R, e) = (5, 2, 1)$  and with a discretization of  $\mathbb{T}_0$  by 62 points ( $p = 1, \dots, 62$ ). Furthermore we specify that the integrations are performed with the *quad* method which is a recursive adaptive Simpson quadrature method (see [32]). The first simulations consist in approximating a function  $u \in L^2_{\mathbb{R}}(\mathbb{T}_0)$ , or more exactly  $u - Lu$ , with  $N = 4n+2$  functions of the family  $\mathcal{B}$ . In the first case (Fig. 2, 3)  $u$  is defined on  $\mathbb{T}_0$  by  $(x, y) \mapsto u(x, y) = xy$  and in the second case (Fig. 4, 5) by  $(x, y) \mapsto u(x, y) = x \ln(x + y)$ . Recall that on  $\mathbb{T}_0$ ,  $x = x_0 + R \cos \theta$  and  $y = R \sin \theta$ . In both situations, we show the behaviour with respect to  $n \in \{1, \dots, 6\}$  of the error between  $U = u - Lu$  and its approximant  $\tilde{U}$ :  $\sum_p (U - \tilde{U})^2 / \sum_p U^2$  (Fig. 3, 5).

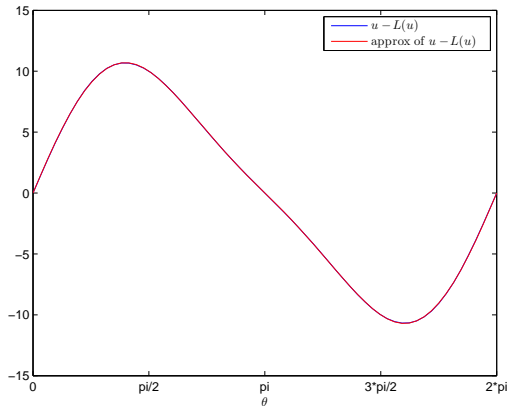


Fig.2 Approximation of  $u = xy$   
for  $n = 3$

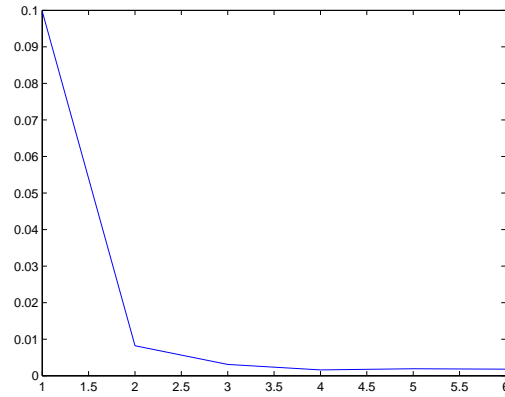


Fig.3 Error between  $U$  and  $\tilde{U}$   
when  $u = xy$

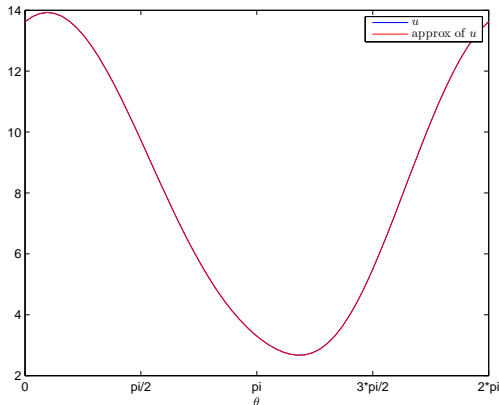


Fig.4 Approximation of  $u = x \ln(x + y)$   
for  $n = 3$

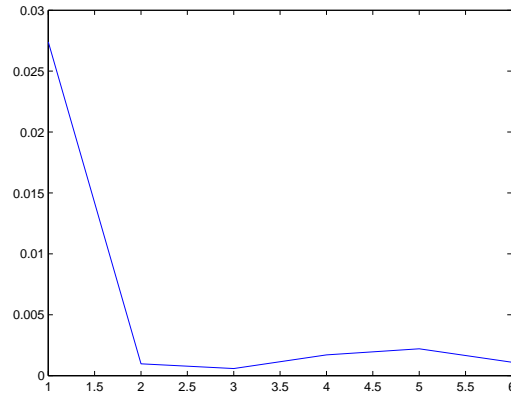


Fig.5 Error between  $U$  and  $\tilde{U}$   
when  $u = x \ln(x + y)$

In both cases (Fig. 2, 4), the fact that  $n = 3$  implies that the approximation is composed of  $N = 4n + 2 = 14$  different elementary functions of the system  $\mathcal{B}$ . The graphs of the associated errors (Fig. 3, 5) show that the approximation keeps a high accuracy as the order increases. We just mention that it is of the

same quality when computed with the system  $\tilde{\mathcal{F}}$ .

From these approximations of functions belonging to  $L^2_{\mathbb{R}}(\mathbb{T}_0)$ , we now compute the operator  $\tilde{P}_{\nu_*}$  in order to verify that it matches the projection from  $L^2(\mathbb{T}_0)$  onto  $\text{tr } H^2_{\nu_*}(\mathbb{D}_0)$ . Let  $f$  be the function defined on  $\mathbb{T}_0$  as

$$f : (x, y) \mapsto u(x, y) + iv(x, y) = x^2 + i(2y + 7).$$

It is an easy computation to see that  $f \in \text{tr } H^2_{\nu_*}(\mathbb{D}_0)$ . Then  $P_{\nu_*}(f) = f$ . Related computations are plotted in Fig. 6, 7:

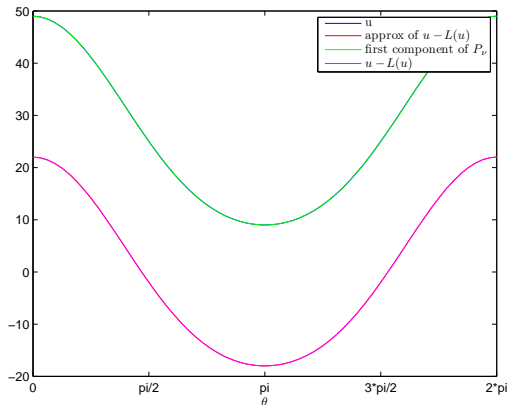


Fig.6 Approximation of  $u = x^2$  and  $P_{\nu_*}(u)$  for  $n = 1$

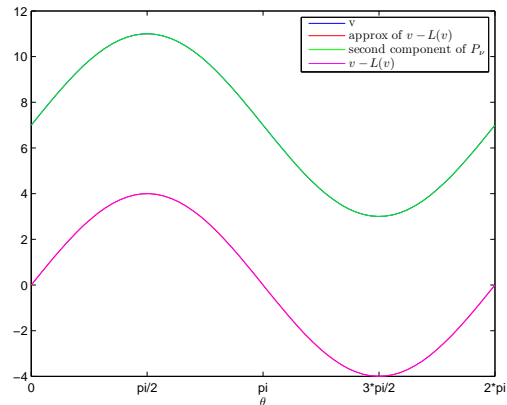


Fig.7 Approximation of  $v = 2y + 7$  and  $P_{\nu_*}(v)$  for  $n = 1$

The error between  $U$  and  $\tilde{U}$  is approximately equal to  $10^{-15}$ , and between  $V$  and  $\tilde{V}$ , to  $10^{-7}$ . A single elementary function is needed ( $n = 1$ ) in  $\tilde{\mathcal{B}}$  and  $\tilde{\mathcal{F}}$  ( $u$  and  $v$  already belong to  $\mathcal{B}$  and  $\mathcal{F}$ , respectively). Then the relation  $\tilde{P}_{\nu_*}(f) \simeq f = P_{\nu_*}(f)$  is verified.

We now test the accuracy of  $\tilde{P}_{\nu_*}$  on functions which still belong to  $\text{tr } H^2_{\nu_*}(\mathbb{D}_0)$  but no longer to the families  $\mathcal{B}$  and  $\mathcal{F}$ . Indeed, it is possible to find homogeneous polynomials  $u$  and  $v$  satisfying (4). With  $\text{deg } u = k$  and  $\text{deg } v = k - 1$  (for  $k \geq 2$ ) the unique solution is to take a constant multiple of the pair:

$$u_k(x, y) = x^2 y^{k-2} - \frac{(k-2)(k-3)}{2.4} x^4 y^{k-4} + \frac{(k-2)(k-3)(k-4)(k-5)}{2.4.4.6} x^6 y^{k-6} - \dots$$

and

$$v_{k-1}(x, y) = \frac{2}{(k-1)} y^{k-1} - \frac{k-2}{2} x^2 y^{k-3} + \frac{(k-2)(k-3)(k-4)}{2.4.4} x^4 y^{k-5} - \dots$$

where each sum has finitely-many non-zero terms. We denote by  $\mathcal{G}$  the corresponding family of polynomials  $f_{k,k-1} = u_k + iv_{k-1}$ . The first test (Fig. 8-11) is computed with the function  $f_{3,2}$  defined on  $\mathbb{T}_0$  by:

$$f_{3,2} : (x, y) \mapsto u_3(x, y) + iv_2(x, y) = x^2 y + i\left(y^2 - \frac{x^2}{2}\right).$$

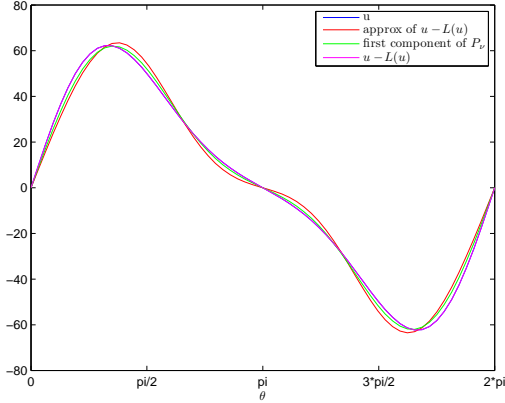


Fig.8 Approximation of  $u_3$  and  $P_{\nu_*}(u_3)$  for  $n = 1$

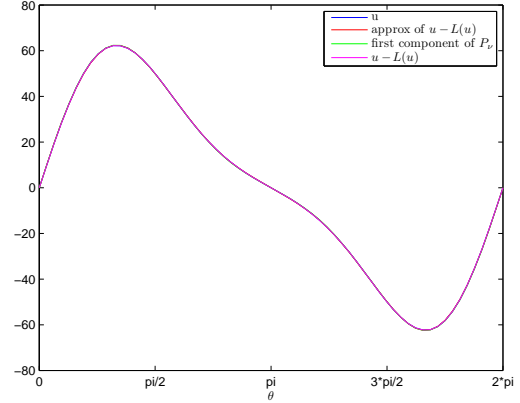


Fig.9 Approximation of  $u_3$  and  $P_{\nu_*}(u_3)$  for  $n = 3$

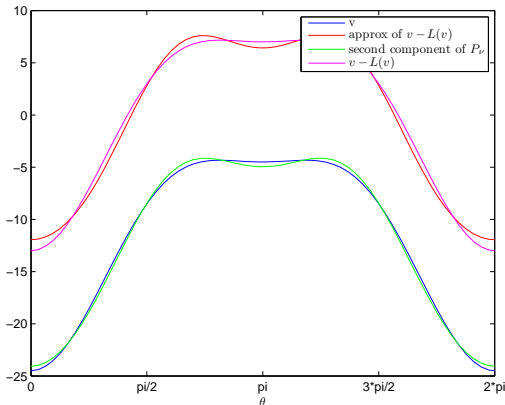


Fig.10 Approximation of  $v_2$  and  $P_{\nu_*}(v_2)$  for  $n = 1$

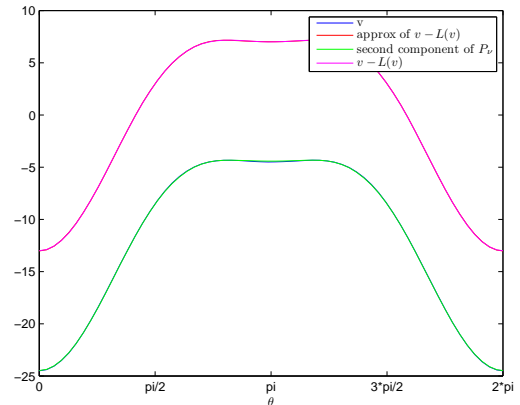


Fig.11 Approximation of  $v_2$  and  $P_{\nu_*}(v_2)$  for  $n = 3$

The error between  $U$  and  $\tilde{U}$  is approximately equal to  $10^{-2}$ , and between  $V$  and  $\tilde{V}$ , to  $10^{-3}$  when  $n = 3$ . On this example (Fig. 9, 11), it can be noticed that the operator  $P_{\nu_*}$  is well approximated with a limited number of functions in the families  $\tilde{\mathcal{B}}$  and  $\tilde{\mathcal{F}}$ .

Moreover this remark still holds when the degree of the polynomials  $u$  and  $v$  increases. The following simulation (Fig. 12-15) is obtained with the function  $f_{5,4} \in \mathcal{G}$  defined on  $\mathbb{T}_0$  by:

$$f_{5,4} : (x, y) \mapsto u_5(x, y) + iv_4(x, y) = x^2y^3 - \frac{3}{4}x^4y + i\left(\frac{y^4}{2} - \frac{3}{2}x^2y^2 + \frac{3}{16}x^4\right).$$

This time the error between  $U$  and  $\tilde{U}$  is approximately equal to  $10^{-2}$ , and between  $V$  and  $\tilde{V}$ , to  $10^{-3}$  when  $n = 5$ . Even if the error increases compared with the last simulation, the operator  $P_{\nu_*}$  remains well approximated as can be observed in Fig. 13, 15.

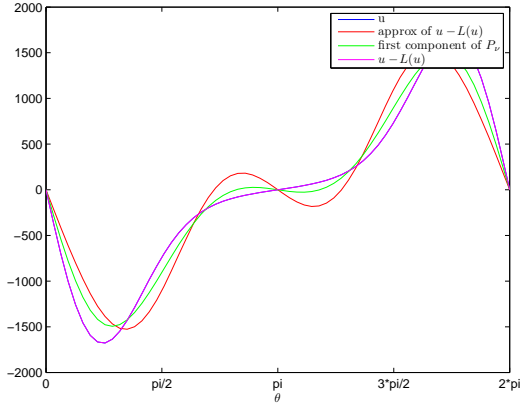


Fig.12 Approximation of  $u_5$  and  $P_{\nu_*}(u_5)$  for  $n = 1$

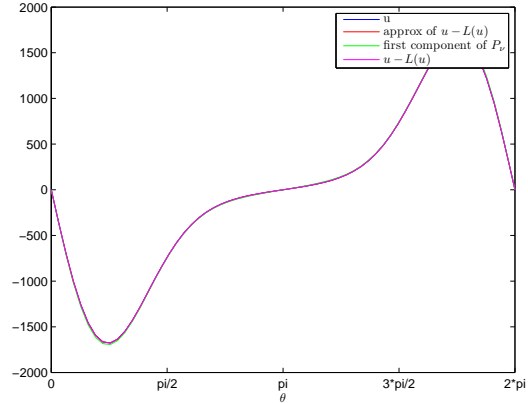


Fig.13 Approximation of  $u_5$  and  $P_{\nu_*}(u_5)$  for  $n = 5$

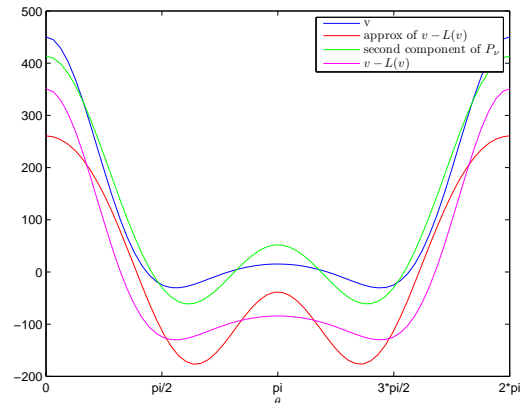


Fig.14 Approximation of  $v_4$  and  $P_{\nu_*}(v_4)$  for  $n = 1$

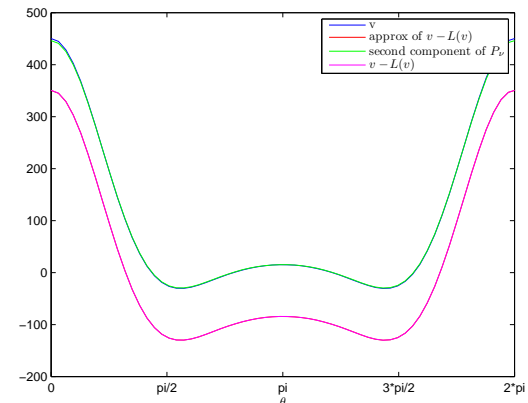


Fig.15 Approximation of  $v_4$  and  $P_{\nu_*}(v_4)$  for  $n = 5$

## 6 Conclusion

Although preliminary numerical results are encouraging, a deep study of the constructive aspects of the presented approximation issues is still to be undertaken. The use of alternative basis functions may be considered (for example, the family  $\mathcal{G}$  of polynomials that satisfy the Cauchy–Riemann equations, although it is not clear whether it is dense).

Concerning the link with plasma modelling in tokamaks, Hardy spaces  $H^p_\nu$  of annular domains are already under study. As in the classical case, they can be expressed as direct sums of Hardy classes of related discs. It would also be of interest to solve such approximation problems in the  $L^\infty$  norm (in this case the classical version of the problem was analysed in [8]); finally, one may also hope to consider “mixed”  $L^2$ – $L^\infty$  problems as in [10].

**Acknowledgments.** Two of the authors (JL and JRP) are grateful to the EPSRC for financial support under joint project EP/F020341/1.

## References

- [1] R. Adams, *Sobolev spaces*, Academic Press, 1975.



- 
- [2] L. Ahlfors, *Lectures on quasiconformal mappings*, Wadsworth and Brooks/Cole Advanced Books and Software, Monterey, CA, 1987.
- [3] D. Alpay, L. Baratchart and J. Leblond, Some extremal problems linked with identification from partial frequency data, *10th conference in Analysis and optimization of systems, Sophia Antipolis, 1992*, in J. L. Lions, R. F. Curtain, A. Bensoussan, Lecture notes in Control and Information Sc., 185, Springer Verlag, 563–573, 1993.
- [4] K. Astala and L. Päiväranta, Calderón’s inverse conductivity problem in the plane, *Ann. of Math.* (2) 163, no. 1, 265–299, 2006.
- [5] K. Astala and L. Päiväranta, A boundary integral equation for Calderón’s inverse conductivity problem, *Proc. 7th Int. Conf. on Harmonic Analysis and PDEs., Madrid (Spain), 2004, Collect. Math.*, Vol. Extra, 127–139, 2006.
- [6] L. Baratchart, J. Grimm, J. Leblond and J. R. Partington, Asymptotic estimates for interpolation and constrained approximation in  $H^2$  by diagonalization of Toeplitz operators. *Integral Equations Operator Theory* 45 (2003), no. 3, 269–299.
- [7] L. Baratchart and J. Leblond, Hardy approximation to  $L^p$  functions on subsets of the circle with  $1 \leq p < \infty$ , *Constr. Approx.* 14, no. 1, 41–56, 1998.
- [8] L. Baratchart, J. Leblond and J. R. Partington, Hardy approximation to  $L^\infty$  functions on subsets of the circle, *Constr. Approx.* 12, no. 3, 423–435, 1996.
- [9] L. Baratchart, J. Leblond, S. Rigat and E. Russ, Hardy spaces of the conjugate Beltrami equation, *Journal of Functional Analysis* 259, 384–427, 2010.
- [10] L. Baratchart, J. Leblond and F. Seyfert, Constrained extremal problems in the Hardy space  $H^2$  and Carleman’s formulas, <http://hal.archives-ouvertes.fr/inria-00429898/fr/>, Submitted for publication.
- [11] L. Baratchart and F. Seyfert, An  $L^p$  analog to AAK theory for  $p \geq 2$ , *J. Funct. Anal.*, 191, no. 1, 52–122, 2002.
- [12] C. Bernardi, M. Dauge and Y. Maday. Polynomials in the Sobolev world (version 2). Preprint 07-14, IRMAR Rennes, 2007.
- [13] L. Bers and L. Nirenberg, On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications, *Convegno internazionale sulle equazioni derivate e parziali*, Cremonese, Roma, 111–138, 1954.
- [14] R. P. Boas and H. Pollard. Complete sets of Bessel and Legendre functions. *Ann. of Math.* (2) 48, 366–384, 1947.
- [15] H. Brézis, *Analyse fonctionnelle, Théorie et applications*, Masson, 1999.
- [16] F. E. Browder, Approximation in uniform norm by solutions of elliptic differential equations, *Bull. Amer. Math. Soc.* 67, 400–404, 1961.
- [17] F. E. Browder, Functional analysis and partial differential equations II, *Math. Ann.* 145, 81–226, 1961/1962.
- [18] S. Campanato, Elliptic systems in divergence form, Interior regularity, Quaderni, Scuola Normale Superiore Pisa, 1980.
- [19] I. Chalendar and J. R. Partington, Constrained approximation and invariant subspaces, *J. Math. Anal. Appl.* 280, no. 1, 176–187, 2003.

- 
- [20] I. Chalendar, J. R. Partington and M. Smith, Approximation in reflexive Banach spaces and applications to the invariant subspace problem, *Proc. Amer. Math. Soc.* 132, no. 4, 1133–1142, 2004.
- [21] P. L. Duren, *Theory of  $H^p$  spaces*, Pure and Applied Mathematics, Vol. 38 Academic Press, New York–London, 1970.
- [22] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Math. 96, Academic Press, 1981.
- [23] I. Ekeland and R. Temam. *Convex analysis and variational problems*. SIAM, 1999.
- [24] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman, 1985.
- [25] T. Iwaniec and G. Martin, What’s new for the Beltrami equation?, in *National Research Symposium on Geometric Analysis and Applications*, Proc. of the CMA, 39, 2000.
- [26] B. Jacob, J. Leblond, J.-P. Marmorat and J.R. Partington, A constrained approximation problem arising in parameter identification, *Linear Algebra and its Applications*, 351-352 (2002), 487–500.
- [27] P.D. Lax, A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations. *Comm. Pure Appl. Math.* 9, 747–766, 1956.
- [28] J. Leblond, J.-P. Marmorat and J.R. Partington, Analytic approximation with real constraints, with applications to inverse diffusion problems, *J. Inverse Ill-Posed Probl.* 16, no. 1, 89–105, 2008.
- [29] J. Leblond and J.R. Partington, Constrained approximation and interpolation in Hilbert function spaces, *J. Math. Anal. Appl.* 234-2, 500–513, 1999.
- [30] J.L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications*, volume 1. Dunod, 1968.
- [31] B. Malgrange, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. *Ann. Inst. Fourier, Grenoble* 6, 271–355, 1955–1956.
- [32] MATLAB user’s guide, <http://www.mathworks.com>, 2009.
- [33] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer Verlag, 1992.
- [34] I. N. Vekua, *Generalized Analytic Functions*, Addison-Wesley Publ. Co., Inc., Reading, Mass, 1962.
- [35] G.N. Watson, *A treatise on the theory of Bessel functions*. Reprint of the second (1944) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.
- [36] E. T. Whittaker and G. N. Watson, *A course of modern analysis*. Reprint of the fourth (1927) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996.



---

Centre de recherche INRIA Sophia Antipolis – Méditerranée  
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex  
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier  
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq  
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique  
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex  
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex  
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex  
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex

---

Éditeur  
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)  
<http://www.inria.fr>  
ISSN 0249-6399