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On the Conditional Distributions of Spatial Point Processes

Franois Caron* , Pierre Del Moral†, Arnaud Doucet‡ , Michele Pace§

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Abstract: We consider the problem of estimating a latent point process, given the realization of another point process on abstract measurable state spaces. First, we establish an expression of the conditional distribution of a latent Poisson point process given the observation process when the transformation from the latent process to the observed process includes displacement, thinning and augmentation with extra points. We present an original analysis based on a self-contained random measure theoretic approach combined with reversed Markov kernel techniques. This simplifies and complements previous derivations given in [5], [6]. Second, we show how to extend our analysis to the more complicated case where the latent point process is associated to triangular array sequences, yielding what seems to be the first results of this type for this class of spatial point processes.

Key-words: filtering, multitarget tracking, spatial point processes, triangular array sequences probability hypothesis density filter.

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On the Conditional Distributions of Spatial Point Processes

Résumé : Cette étude concerne l'estimation conditionnelle d'un processus ponctuel spatial par rapport à la donnée d'un autre processus spatial sur des espaces mesurables abstraits.

Tout d'abord, nous établissons une formulation analytique et fonctionnelle explicite des lois conditionnelles dans le cadre de modèles de branchements dynamiques liés à des naissances spontanées, des taux de détection variables, des bruits d'observations de type Poisson, et des processus de naissance et morts nonhomogènes. Nous présentons une analyse complète et originale fondée sur l'analyse de variables aléatoires à valeurs mesures et des techniques d'opérateurs de Markov duaux. Cette approche complète et simplifie les études antérieures présentées dans [5] et [6].

Dans la seconde partie de cet article, nous étendons l'analyse à l'étude de processus ponctuels liés à des tableaux triangulaires de variables aléatoires. Ces derniers résultats semblent être les premiers résultats de ce type pour ces classes de processus ponctuels non homogènes.

Mots-clés : filtrage, poursuite multicibles, processus ponctuels spatiaux, tableaux triangulaires de variables aléatoires, probability hypothesis density filter.

1 Introduction

Spatial point processes occur in a wide variety of scientific disciplines including environmetrics, epidemiology and seismology; see [1] and [7] for recent books on the subject. In this paper, we are interested in scenarios where the spatial point process of interest is unobserved and we only have access to another spatial point process which is obtained from the original process through displacement, thinning and augmentation with extra points. Such problems arise in forestry [3], [4] but our motivation for this work stems from target tracking applications [5], [6], [8]. In this context, we want to infer the number of targets and their locations; this number can vary as targets enter and exit the surveillance area. We only have access to measurements from a sensor. Some targets may not be detected by the sensor and additionally this sensor also provides us with a random number of false measurements.

From a mathematical point of view, we are interested in the computation of the conditional distributions of a sequence of random measures with respect to a sequence of noisy and partial observations given by spatial point processes. Recently a few articles have addressed this problem. In a seminal paper [5], R. Malher has proposed an original and elegant multi-object filtering algorithm known as the PHD (Probability Hypothesis Density) filter which relies on a first order moment approximation of the posterior. The mathematical techniques used by R. Mahler are essentially based on random finite sets techniques including set derivatives and probability generating functionals. In a more recent article [6], S.S. Singh, B.N. Vo, A. Baddeley and S. Zuyev have clarified some important technicalities concerning the use of the derivatives of the joint probability generating functionals to characterize conditional distributions. They have proposed a simplified derivation of the PHD filter and have extended this algorithm to include second moment information.

The main contribution of this article is to propose an original analysis based on a self-contained random measure theoretic approach. The rather elementary techniques developed in this paper complement the more traditional random finite sets analysis involving symmetrisation techniques or related to other technicalities associated with the computation of moment generating functions derivatives. It allows us to derive functional versions of the conditional distributions of spatial point processes associated with triangular arrays of random variables.

The rest of this article is organized as follows. In section 2 we first present a static model associated to a pair of signal-observation Poisson point processes. We establish a functional representation of the conditional distribution of a Poisson signal process w.r.t. noisy and partial observations. The proof is elementary and is used to establish the PHD equations [5], [6]. In section 3, we show how this analysis can be extended to spatial point processes associated with triangular arrays of random variables. We conclude by a short discussion in section 4.

We end this introductory section with some standard notation used in the paper.

We denote respectively by $\mathcal{M}(E)$, $\mathcal{P}(E)$, and $\mathcal{B}(E)$, the set of all finite positive measures μ on some measurable space $(E,\mathcal{E})$, the convex subset of all probability measures, and the Banach space of all bounded and measurable functions f equipped with the uniform norm $\|f\|$. We let $\mu(f)=\int\ \mu(dx)\ f(x)$, be the Lebesgue integral, for measurable subsets $A\in\mathcal{E}$, sometimes we slightly abuse notation and we write $\mu(A)$ instead of $\mu(1_A)$; and we set δ_a the Dirac measure at $a\in E$. We denote by $\mu^{\otimes p}$, the p-tensor product of measure $\mu\in\mathcal{M}(E)$ on the product space E^p.

We associate with a bounded positive potential function $G : x\in E\mapsto G(x)\in[0,\infty)$, the Bayes-Boltzmann-Gibbs transformations

$$\Psi_G\ :\ \eta\in\mathcal{M}(E)\mapsto\Psi_G(\eta)\in\mathcal{P}(E)\quad\text{with}\quad\Psi_G(\eta)(dx):=\frac{1}{\eta(G)}\ G(x)\ \eta(dx)$$

In various places in this article, we shall add an auxiliary cemetery or coffin state c to the original state space E. The functions $f\in\mathcal{B}(E)$ are extended to the augmented space $E\cup\{c\}$ by setting $f(c)=0$.

For every sequence of points $x = (x^i)_{i\geq 1}$ in E and every $p \geq 0$, we denote by $m_p(x)$ the occupation measure of the first p coordinates $m_p(x) = \sum_{1\leq i\leq p} \delta_{x^i}$. For $p = 0$, we use the convention $m_0(x) = 0$, the null measure on E.

We recall that a bounded and positive integral operator Q from a measurable space $(E_1, \mathcal{E}_1)$ into an auxiliary measurable space $(E_2, \mathcal{E}_2)$ is an operator $f \mapsto Q(f)$ from $\mathcal{B}(E_2)$ into $\mathcal{B}(E_1)$ such that the functions

$$x \mapsto Q(f)(x) := \int_{E_2} Q(x, dy) f(y)$$

are $\mathcal{E}_1$-measurable and bounded for some measures $Q(x, .) \in \mathcal{M}(E_2)$. These operators also generate a dual operator $\mu \mapsto \mu Q$ from $\mathcal{M}(E_1)$ into $\mathcal{M}(E_2)$ defined by $(\mu Q)(f) := \mu(Q(f))$. A Markov kernel is a positive and bounded integral operator M with $M(1) = 1$.

2 Conditional distributions for Poisson processes

2.1 Conditioning formulae for Poisson processes

Assume the unobserved point process is a Poisson point process $\mathcal{X}_1 := \sum_{1\leq i\leq N_0} \delta_{X^i}$ with intensity measure γ on some measurable state space $(E_1, \mathcal{E}_1)$. We set $\eta(dx) := \gamma(dx)/\gamma(1)$. The Poisson point process $\mathcal{X}_1$ is partially observed on some possibly different measurable state space E_2. The observation point process consists of a collection of random observations directly generated by a random number of points of $\mathcal{X}_1$ plus some random observations unrelated to $\mathcal{X}_1$. To describe more precisely this observation point process, we let α be a measurable function from E_1 into $[0, 1]$ and we consider a Markov transition $L(x, dy)$ from E_1 to E_2.

Given a realization of $\mathcal{X}_1$, every random point $X^i = x$ generates with probability $\alpha(x)$ an observation Y^i on E_2 with distribution $L(x, dy)$; otherwise it goes into an auxiliary cemetery or coffin state c. Hence $\alpha(x)$ measures the “detectability” degree of x. In other words, a given point x generates a random observation in $E_{2,c} := E_2 \cup \{c\}$ with distribution

$$L_c(x, dy) := \alpha(x)\ L(x, dy) + (1 - \alpha(x))\ \delta_c(dy). \tag{2.1}$$

The resulting observation point process is the random measure $\sum_{1\leq i\leq N_0} \delta_{Y^i}$ on the augmented state space $E_{2,c}$. In addition to this partial observation point process we also observe an additional, and independent of $\mathcal{X}_1$, Poisson point process $\sum_{1\leq i\leq N'} \delta_{Y'^i}$ with intensity measure ν on E_2; this is known as the clutter noise in multitarget tracking. In other words, the full observation process on $E_{2,c}$ is given by the random measure

$$\mathcal{Y} := \sum_{1\leq i\leq N_0} \delta_{Y^i} + \sum_{1\leq i\leq N'} \delta_{Y'^i}.$$

The coffin state c being unobservable, the “real world” observation point process is the random measure $\mathcal{Y}_1$ on E_2 given by

$$\mathcal{Y}_1 := \sum_{1\leq i\leq N_0} 1_{E_2}(Y^i)\ \delta_{Y^i} + \sum_{1\leq i\leq N'} \delta_{Y'^i} = \mathcal{Y} - \mathcal{Y}_0 = \sum_{1\leq i\leq N_1} \delta_{Y_1^i}$$

where $\mathcal{Y}_0 := N_c\ \delta_c$, with $N_c := \left(\sum_{1\leq i\leq N_0} 1_c(Y^i)\ \right)$ corresponds to the coffin observations associated to the undetected points.

We present here an explicit integral representation of a version of the conditional distribution of $\mathcal{X} := \mathcal{X}_1 + N'\delta_c$ given $\mathcal{Y}_1$.

Proposition 2.1 *A version of the conditional distribution of $\mathcal{Y}$ given $\mathcal{X}_1$ is given for any function $F \in \mathcal{B}(\mathcal{M}(E_{2,c}))$ by*

$$\mathbb{E}\left(F\left(\mathcal{Y}\right)|\mathcal{X}_1\right) = e^{-\nu(1)} \sum_{k\geq 0} \frac{1}{k!} \int F\left(m_k(y') + m_{N_0}(y)\right)\ \nu^{\otimes k}(dy') \prod_{1\leq i\leq N_0} L_c(X^i, dy^i)\ . \tag{2.2}$$

We further assume that $\nu \ll \lambda$ and $L(x, \centerdot) \ll \lambda$, for any $x \in E_1$, for some reference measure $\lambda \in \mathcal{M}(E_2)$, with Radon Nikodym derivatives given by

$$g(x,y) = \frac{dL(x,\centerdot)}{d\lambda}(y) \quad \text{and} \quad h(y) := \frac{d\nu}{d\lambda}(y) \tag{2.3}$$

and such that $h(y) + \gamma(\alpha g(y, \centerdot)) > 0$, for any $y \in E_2$.

In this situation, a version of the conditional distribution of $\mathcal{X}$ defined on $E_{1,c} := E_1 \cup \{c\}$ given the observation point process $\mathcal{Y}_1$ is given for any function $F \in \mathcal{B}(\mathcal{M}(E_{1,c}))$ by

$$\begin{aligned} &\mathbb{E}\left(F\left(\mathcal{X}\right)|\mathcal{Y}_1\right) \\ &= e^{-\gamma(1-\alpha)} \sum_{k\geq 0} \frac{\gamma(1-\alpha)^k}{k!} \int F\left(m_k(x') + m_{N_1}(x)\right)\ \Psi_{(1-\alpha)}(\eta)^{\otimes k}\left(dx'\right) \prod_{i=1}^{N_1} Q\left(Y_1^i, dx^i\right) \end{aligned} \tag{2.4}$$

where Q is a Markov transition from E_2 into $E_{1,c}$ defined by the following formula

$$Q(y,dx) = (1-\beta(y))\ \Psi_{\alpha g(y,\centerdot)}(\eta)(dx) + \beta(y)\ \delta_c(dx) \ \text{with}\ \beta(y) = \frac{h(y)}{h(y)+\gamma(\alpha g(y,\centerdot))}\ . \tag{2.5}$$

Proof:
The proof of the first assertion in Eq. (2.2) is elementary, thus it is skipped. We provide here a proof of the second result given in Eq. (2.4). First, we observe that the random measure

$$\mathcal{Z} := \sum_{1\leq i\leq N_0} \delta_{(X^i,Y^i)} + \sum_{1\leq i\leq N'} \delta_{(c,Y'^i)} := \sum_{1\leq i\leq N} \delta_{(Z_1^i,Z_2^i)} \tag{2.6}$$

is a Poisson point process in $E_c = E_{1,c} \times E_{2,c}$. More precisely, the random variable $N = N_0 + N'$ is a Poisson random variable with parameter $\kappa = \gamma(1) + \nu(1)$, and $(Z_1^i, Z_2^i)_{i\geq 0}$ is a sequence of independent random variables with common distribution

$$\begin{aligned} \Gamma(d(z_1,z_2)) &= \eta'(dz_1)M'(z_1,dz_2) \qquad \text{with} \qquad \kappa\eta' := \gamma(1)\ \eta + \nu(1)\ \delta_c\ , \\ M'(z_1,dz_2) &= 1_{E_1}(z_1)\ L_c(z_1,dz_2) + 1_c(z_1)\ \overline{\nu}(dz_2) \quad \text{with} \quad \overline{\nu}(dz_2) = \nu(dz_2)/\nu(1)\ . \end{aligned}$$

Using the easily checked reversal formula

$$\eta'(dz_1)M'(z_1,dz_2) = \left(\eta'M'\right)(dz_2)\ M'_{\eta'}(z_2,dz_1)$$

where

$$M'_{\eta'}(z_2,dz_1) := 1_c(z_2)\ \Psi_{(1-\alpha)}(\eta)(dz_1) + 1_{E_2}(z_2)\ Q(z_2,dz_1)$$

we conclude that for any function $F \in \mathcal{B}(\mathcal{M}(E_{1,c}))$

$$\mathbb{E}\left(F(\mathcal{Z}_1)\,|\mathcal{Z}_2\right) = \int F\left(m_N(z_1)\right) \prod_{i=1}^{N} M'_{\eta'}\left(Z_2^i, dz_1^i\right).$$

where $\mathcal{Z}_j$ stands for the j-th marginal of $\mathcal{Z}$, with $j \in \{1,2\}$. The end of the proof is now a direct consequence of the fact that $(\mathcal{Z}_1,\mathcal{Z}_2) = (\mathcal{X},\mathcal{Y})$, $\mathbb{E}\left(F\left(\mathcal{X}\right)|\mathcal{Y}_1\right) = \mathbb{E}\left(\mathbb{E}\left(F(\mathcal{X})\,|\mathcal{Y}\right)|\mathcal{Y}_1\right)$ and

$$\mathbb{E}\left(F(\mathcal{Y})\mid \mathcal{Y}_1\right) = e^{-\gamma(1-\alpha)} \sum_{q\geq 0} \frac{\gamma(1-\alpha)^q}{q!}\ F\left(q\delta_c + \mathcal{Y}_1\right)$$

for any function $F \in \mathcal{B}(\mathcal{M}(E_{2,c}))$ as N_c is Poisson$(\gamma(1-\alpha))$ distributed. This ends the proof of the proposition. ∎

The conditional expectation of the random point processes $\mathcal{X}$ and $\mathcal{X}_1$ given the point process $\mathcal{Y}_1$ can be easily computed using Eq. (2.4). Recall that $f(c) = 0$ by convention.

Corollary 2.2 *For any function $f \in \mathcal{B}(E_{1,c})$ we have the almost sure integral representation formula*

$$\begin{aligned}\mathbb{E}\left(\mathcal{X}(f) \mid \mathcal{Y}_1\right) &= \mathbb{E}\left(\mathcal{X}_1(f) \mid \mathcal{Y}_1\right) \\ &= e^{-\gamma(1-\alpha)} \sum_{k\geq 0} \frac{\gamma(1-\alpha)^k}{k!} \left(k\ \Psi_{(1-\alpha)}(\eta)\left(f\right) + \int \mathcal{Y}_1(dy) Q\left(f\right)(y)\right) \\ &= \gamma((1-\alpha)f) + \int \mathcal{Y}_1(dy)\left(1-\beta(y)\right)\ \Psi_{\alpha g(y,\boldsymbol{.})}(\eta)(f)\ . \end{aligned} \tag{2.7}$$

In particular, the conditional expectation of the number of points N_0 given the observations is given by

$$\mathbb{E}\left(N_0 | \mathcal{Y}_1\right) = \mathbb{E}\left(\mathcal{X}_1(1) \mid \mathcal{Y}_1\right) = \gamma(1-\alpha) + \mathcal{Y}_1\left(1-\beta\right). \tag{2.8}$$

2.2 Spatial filtering models and probability hypothesis density equations

We show here how the results obtained in proposition 2.1 and corollary 2.2 allows us to establish directly the PHD filter equations [5], [6].

Let $E_{1,n}$ and $E_{2,n}$ be a sequence of measurable state spaces indexed by $n \in \mathbb{N}$. In what follows the parameter n is interpreted as a discrete time index. We consider a collection of measures $\mu_n \in \mathcal{M}(E_{1,n})$ and a collection of positive operators R_{n+1} from $E_{1,n}$ into $E_{1,n+1}$.

We then define recursively a sequence of random measures $\mathcal{X}_{1,n}$ and $\mathcal{Y}_{1,n}$ on $E_{1,n}$ and $E_{2,n}$ as follows. The initial measure $\mathcal{X}_{1,0}$ is a Poisson point process with intensity measure $\gamma_0 = \mu_0$ on $E_{1,0}$. Given a realization of $\mathcal{X}_{1,0}$, the corresponding observation process $\mathcal{Y}_{1,0}$ on $E_{2,0}$ is defined as in Section 2.1 with a detection function α_0 on $E_{1,0}$, a clutter intensity measure ν_0, and some Markov transitions $L_{c,0}$ and L_0 defined as in (2.1) and satisfying (2.3) for some reference measure λ_0 and some functions h_0 and g_0. From corollary 2.2, we have for any function $f \in \mathcal{B}(E_{1,0})$

$$\begin{aligned}\widehat{\gamma}_0(f) :=& \mathbb{E}\left(\mathcal{X}_{1,0}(f) \mid \mathcal{Y}_{1,0}\right) \\ &= \gamma_0((1-\alpha_0)f) + \int \mathcal{Y}_{1,0}(dy)\left(1-\beta_0(y)\right)\ \Psi_{\alpha_0 g_0(y,\boldsymbol{.})}(\gamma_0)(f)\end{aligned}$$

with a function β_0 defined as in (2.5) by substituting (α_0, h_0, g_0) to (α, h, g). Given a realization of the pair random sequences $(\mathcal{X}_{1,p}, \mathcal{Y}_{1,p})$, with $0 \leq p \leq n$, the pair of random measures $(\mathcal{X}_{1,n+1}, \mathcal{Y}_{1,n+1})$ is *defined* as follows. We set $\mathcal{X}_{1,n+1}$ to be a Poisson point process with intensity measure γ_{n+1} defined by the following recursions

$$\begin{aligned}\gamma_{n+1} :=& \widehat{\gamma}_n R_{n+1} + \mu_{n+1} \\ \widehat{\gamma}_n(f) :=& \gamma_n((1-\alpha_n)f) + \int \mathcal{Y}_{1,n}(dy)\left(1-\beta_n(y)\right)\ \Psi_{\alpha_n g_n(y,\boldsymbol{.})}(\gamma_n)(f)\end{aligned}$$

for any function $f \in \mathcal{B}(E_{1,n})$. In the context of spatial branching processes, μ_n stands for the intensity measure of a spontaneous birth model while R_n represents the first moment transport kernel associated with a spatial branching type mechanism. For example, assume that each random point/target $X_n^i = x$ at time n dies with probability $\rho(x)$ or survives and evolves according to a Markov kernel K_{n+1} from $E_{1,n}$ into $E_{1,n+1}$ then R_{n+1} corresponds to

$$R_{n+1}\left(x, dx'\right) = \left(1-\rho(x)\right) K_{n+1}\left(x, dx'\right).$$

It is also possible to modify R_{n+1} to include some spawning points [5], [6], [8]. In addition, given a realization of $\mathcal{X}_{1,n+1}$, the corresponding observation process $\mathcal{Y}_{1,n+1}$ is defined as in Section 2.1 with a detection function α_{n+1} on $E_{1,n+1}$, a clutter intensity measure ν_{n+1}, and some Markov transitions $L_{c,(n+1)}$ and L_{n+1} defined as in (2.1) and satisfying (2.3) for some reference measure λ_{n+1} and some functions h_{n+1} and g_{n+1}. We let N'_n be the number of coffin type virtual states associated with clutter observations at time n.

The following elementary corollary proves that the PHD filter propagates the first moment of the multi-target posterior distribution of the filtering model defined above. This is a direct consequence of proposition 2.1 and corollary 2.2.

Corollary 2.3 *A version of the conditional distribution of* $\mathcal{X}_n := \mathcal{X}_{1,n} + N'_n \delta_c$ *given the filtration* $\mathcal{F}_n^Y = \sigma(\mathcal{Y}_{1,p},\ 0 \leq p \leq n)$ *generated by the observation point processes* $\mathcal{Y}_{1,p} := \sum_{1 \leq i \leq N_{1,p}} \delta_{Y^i_{1,p}}$*, from the origin* $p = 0$ *up to the current time* $p = n$*, is given for any function* $F_n \in \mathcal{B}(E_{1,n} \cup \{c\})$ *by the following formula*

$$\mathbb{E}\left(F_n\left(\mathcal{X}_n\right) \middle| \mathcal{F}_n^Y\right) = e^{-\gamma_n(1-\alpha_n)} \sum_{k \geq 0} \frac{\gamma_n(1-\alpha_n)^k}{k!}$$
$$\int F_n\left(m_k(x') + m_{N_{1,n}}(x)\right)\ \Psi_{(1-\alpha_n)}(\gamma_n)^{\otimes k}(dx') \prod_{i=1}^{N_{1,n}} Q_n\left(Y^i_{1,n}, dx^i\right)$$

with the Markov transitions

$$Q_n(y, dx) = (1 - \beta_n(y))\ \Psi_{\alpha_n g_n(y,\bullet)}(\gamma_n)(dx) + \beta_n(y)\ \delta_c(dx)\ .$$

In particular, the random measures γ_n *and* $\widehat{\gamma}_n$ *defined below coincide with the first moment of the random measures* $\mathcal{X}_1^n$ *given the sigma-fields* $\mathcal{F}_{n-1}^Y$ *and* $\mathcal{F}_n^Y$*; that is, for any function* $f \in \mathcal{B}(E_{1,n})$*, we have*

$$\gamma_n(f) := \mathbb{E}\left(\mathcal{X}_{1,n}(f) \mid \mathcal{F}_{n-1}^Y\right) \quad \text{and} \quad \widehat{\gamma}_n(f) := \mathbb{E}\left(\mathcal{X}_{1,n}(f) \mid \mathcal{F}_n^Y\right).$$

3 Conditional distributions for general processes

3.1 Statement of Results

This section is concerned with conditioning principles for spatial point processes associated with triangular arrays of random variables. These models are defined as random measures $\mathcal{Z}$ on $E_c = E_{1,c} \times E_{2,c}$ with the following form

$$\mathcal{Z} := \sum_{1 \leq i \leq N} \delta_{(X^{(N,i)}, Y^{(N,i)})}\ . \tag{3.1}$$

Here N stands for an $\mathbb{N}$-valued random variable with some distribution $\pi \in \mathcal{P}(\mathbb{N})$. Given a realization $N = k$, $(X^{(k,i)}, Y^{(k,i)})_{1 \leq i \leq k}$ is a sequence of independent random variables with distributions on E_c given by

$$\mu^{(k,i)}(dx)\ K^{(k,i)}(x, dy). \tag{3.2}$$

We will need the following notation

$$\mathcal{X}_0 := \sum_{1 \leq i \leq N} 1_c\left(X^{(N,i)}\right)\delta_c,\ \mathcal{X}_1 := \sum_{1 \leq i \leq N} 1_{E_1}\left(X^{(N,i)}\right)\delta_{X^{(N,i)}},\ \mathcal{X} := \mathcal{X}_0 + \mathcal{X}_1\ , \tag{3.3}$$

$$\mathcal{Y}_0 := \sum_{1 \leq i \leq N} 1_c\left(Y^{(N,i)}\right)\delta_c,\ \mathcal{Y}_1 := \sum_{1 \leq i \leq N} 1_{E_2}\left(Y^{(N,i)}\right)\delta_{Y^{(N,i)}},\ \mathcal{Y} := \mathcal{Y}_0 + \mathcal{Y}_1\ . \tag{3.4}$$

We also write

$$\mathcal{Y}_1 := \sum_{1 \leq i \leq N_1} \delta_{Y_1^{(N,i)}} \quad \text{where} \quad N_1 := \mathrm{Card}\left\{1 \leq i \leq N\ :\ Y^{(N,i)} \in E_2\right\}.$$

The coffin state c being an isolated state, there is no loss of generality to assume that the distributions $\mu^{(k,i)}$ and the Markov transitions $K^{(k,i)}$ have the following form

$$\mu^{(k,i)} := a^{(k,i)}\ \eta^{(k,i)} + (1 - a^{(k,i)})\ \delta_c \quad \text{with} \quad a^{(k,i)} \in [0,1] \tag{3.5}$$

$$K^{(k,i)}(x,dy) := 1_{E_1}(x)\ L_c^{(k,i)}(x,dy) + 1_c(x)\ \overline{\nu}^{(k,i)}(dy) \tag{3.6}$$

for some pair of measures $\eta^{(k,i)} \in \mathcal{P}(E_1)$, $\overline{\nu}^{(k,i)} \in \mathcal{P}(E_2)$, and some Markov transition $L_c^{(k,i)}$ from E_1 into $E_{2,c}$ given by

$$L_c^{(k,i)}(x,dy) := \alpha^{(k,i)}(x)\ g^{(k,i)}(x,y)\ \lambda^{(k,i)}(dy) + (1 - \alpha^{(k,i)}(x))\ \delta_c(dy) \tag{3.7}$$

$$\overline{\nu}^{(k,i)}(dy) := \overline{h}^{(k,i)}(y)\ \lambda^{(k,i)}(dy) \tag{3.8}$$

with some reference measures $\lambda^{(k,i)} \in \mathcal{M}(E_2)$, some non negative functions $\overline{h}^{(k,i)}$, $g^{(k,i)}$, and some parameters $\alpha^{(k,i)}(x) \in [0,1]$.

In contrast to the situation discussed in Section 2.1, the main difficulty in the mathematical analysis of these models comes from the fact that the random variables are not identically distributed. The non-homogeneous version of the conditional transitions discussed in (2.5) are the Markov transitions $\left(Q^{(k,i)}\right)_{1\leq i\leq k}$ from E_2 into $E_{1,c}$ defined by

$$Q^{(k,i)}(y,dx) := \left(1 - \overline{\beta}^{(k,i)}(y)\right) \Psi_{\alpha^{(k,i)} g^{(k,i)}(y,\,.\,)} \left(\eta^{(k,i)}\right)(dx) + \overline{\beta}^{(k,i)}(y)\ \delta_c(dx) \tag{3.9}$$

with the $[0,1]$-valued parameters

$$\overline{\beta}^{(k,i)}(y) = \frac{(1 - a^{(k,i)})\overline{h}^{(k,i)}(y)}{a^{(k,i)}\eta^{(k,i)}\left(\alpha^{(k,i)} g^{(k,i)}(y,\,.\,)\right) + (1 - a^{(k,i)})\overline{h}^{(k,i)}(y)}\ . \tag{3.10}$$

Let $\mathcal{G}_k$ denote the set of permutations of $\{1,...,k\}$. Using this notation, the main result of the article is the following theorem.

Theorem 3.1 *The conditional distribution of $\mathcal{X}$ given the random measure $\mathcal{Y}_1$ is given for any $F \in \mathcal{B}(\mathcal{M}(E_{1,c}))$ by the following almost sure formula*

$$\begin{aligned}
&\mathbb{E}\left(F(\mathcal{X})\,|\mathcal{Y}_1\right)\\
&= \tfrac{1}{c(N_1)} \textstyle\sum_{k\geq N_1} \pi(k)\ b(N_1,k) \tfrac{1}{k!}\\
&\textstyle\sum_{\sigma\in\mathcal{G}_k} \displaystyle\int F\left(m_k(x)\right) \left[\prod_{i=1}^{N_1} Q^{(k,\sigma(i))}\left(Y_1^{(N,i)}, dx^i\right)\right] \left[\prod_{i=N_1+1}^{k} \Psi_{1-\alpha^{(k,\sigma(i))}}\left(\eta^{(k,\sigma(i))}\right)(dx^i)\right]
\end{aligned} \tag{3.11}$$

where $c(N_1)$ is a normalizing constant and

$$b(N_1,k) := \sum_{\substack{I\subset\{1,\ldots,k\}\\ |I|=N_1}} \left[\prod_{i\in I^c} b_0^{(k,i)}\right] \left[\prod_{i\in I} b_1^{(k,i)}\right] \ \text{and}\ \ b_0^{(k,i)} := a^{(k,i)}\eta^{(k,i)}\left(1 - \alpha^{(k,i)}\right) = 1 - b_1^{(k,i)}\ . \tag{3.12}$$

From this theorem, we obtain directly the following corollary. Recall that $f(c) = 0$.

Corollary 3.2 *For any function $f \in \mathcal{B}(E_{1,c})$ we have the almost sure integral representation formula*

$$\mathbb{E}\left(\mathcal{X}(f)\,|\mathcal{Y}_1\right) = \mathbb{E}\left(\mathcal{X}_1(f)\,|\mathcal{Y}_1\right) = \frac{1}{c(N_1)} \sum_{k \geq N_1} \pi(k)\; b(N_1,k) \frac{1}{k!} \sum_{\sigma \in \mathcal{G}_k} \tag{3.13}$$
$$\left[\sum_{i=1}^{N_1} Q^{(k,\sigma(i))}(f) \left(Y_1^{(N,i)}\right) + \sum_{i=N_1+1}^{k} \Psi_{1-\alpha^{(k,\sigma(i))}} \left(\eta^{(k,\sigma(i))}\right)(f) \right].$$

Thus, we obtain

$$\mathbb{E}\left(\mathcal{X}_1(1)\,|\mathcal{Y}_1\right) = \frac{1}{c(N_1)} \sum_{k \geq N_1} \pi(k)\; b(N_1,k) \left((k - N_1) + \left(1 - \sum_{i=1}^{N_1} \frac{1}{k!} \sum_{\sigma \in \mathcal{G}_k} \overline{\beta}^{(k,\sigma(i))} \left(Y_1^{(N,i)}\right) \right) \right).$$

This corollary can be used to obtain a generalized version of the PHD filter in the spirit of Section 2.2.

Note that the Poisson spatial point processes discussed in Section 2 correspond to the following set of parameters

$$\pi(k) = e^{-\kappa}\; \kappa^k/k! \quad \text{and} \quad \kappa a^{(k,i)} := \gamma(1) \quad \text{with} \quad \kappa := \gamma(1) + \nu(1),$$
$$\left(\eta^{(k,i)}, L_c^{(k,i)}, \overline{\nu}^{(k,i)}, \alpha^{(k,i)}, g^{(k,i)}, \overline{h}^{(k,i)}\right) = \left(\eta, L_c, \overline{\nu}, \alpha, g, h/\nu(1)\right).$$

Proposition 3.1 is proven in the following sections. This is achieved by first establishing an expression for $\mathbb{E}\left(F_1(\mathcal{X}) \mid \mathcal{Y}\right)$ where $F_1 \in \mathcal{B}(\mathcal{M}(E_{1,c}))$, then for $\mathbb{E}\left(F_2(\mathcal{Y}) \mid \mathcal{Y}_1\right)$ where $F_2 \in \mathcal{B}(\mathcal{M}(E_{2,c}))$, and, finally, by using the identity $\mathbb{E}\left(F_1(\mathcal{X})\,|\mathcal{Y}_1\right) = \mathbb{E}\left(\mathbb{E}\left(F_1(\mathcal{X})\,|\mathcal{Y}\right)|\mathcal{Y}_1\right)$.

3.2 Conditioning principles for $\mathcal{X}$ given $\mathcal{Y}$

We consider here the random measures $\mathcal{X}$ and $\mathcal{Y}$ defined in Eq. (3.3) and (3.4). The main objective of this section is to establish an expression for $\mathbb{E}\left(F(\mathcal{X}) \mid \mathcal{Y}\right)$ where $F \in \mathcal{B}(\mathcal{M}(E_{1,c}))$.

For a realization $N = k$, we have $\left(X^{(k,i)}, Y^{(k,i)}\right)_{1 \leq i \leq k}$ distributed according to the measure $P^{(k)}$ which satisfies

$$P^{(k)}\left(dz\right) = \mu^{(k)}(dx)\; K^{(k)}(x, dy) \tag{3.14}$$

where $\mu^{(k)}(dx) \in \mathcal{P}(E_{1,c}^k)$ and $K^{(k)}(x, \cdot) \in \mathcal{P}(E_{2,c}^k)$ are given by

$$\mu^{(k)}(dx) = \prod_{1 \leq l \leq k} \mu^{(k,l)}(dx^l) \quad \text{and} \quad K^{(k)}(x, dy) = \prod_{1 \leq l \leq k} K^{(k,l)}(x^l, dy^l). \tag{3.15}$$

In Eq. (3.14), dz, dx, and dy stand for infinitesimal neighborhoods of the points $z = (x^i, y^i)_{1 \leq i \leq k} \in E_c^k$, $x = (x^i)_{1 \leq i \leq k} \in E_{1,c}^k$ and $y = (y^i)_{1 \leq i \leq k} \in E_{2,c}^k$.

Lemma 3.3 *For any function $f \in \mathcal{B}(E_{1,c}^k)$ and any $k \geq 1$, we set $\mu_f^{(k)}(dx) := f(x)\; \mu^{(k)}(dx)$. The integral operator*

$$K_{\mu^{(k)}}^{(k)}(f)(y) := \frac{d\mu_f^{(k)} K^{(k)}}{d\mu^{(k)} K^{(k)}}(y) \tag{3.16}$$

is Markov transition from $E_{1,c}^k$ into $E_{2,c}^k$ satisfying the following time reversal decomposition formula

$$\mu^{(k)}(dx)\; K^{(k)}(x, dy) = \left(\mu^{(k)} K^{(k)}\right)(dy)\; K_{\mu^{(k)}}^{(k)}(y, dx).$$

Proof: We have

$$\mu_f^{(k)}(dx) := f(x)\ \mu^{(k)}(dx) \ll \mu^{(k)}(dx) \Rightarrow \mu_f^{(k)} K^{(k)} \ll \mu^{(k)} K^{(k)}$$

for any $f \in \mathcal{B}(E_{1,c}^k)$. The last implication follows from

$$\begin{aligned}\mu^{(k)} K^{(k)}(A) = 0 \Rightarrow K^{(k)}(1_A) = 0 \quad & \mu^{(k)} - \text{almost everywhere} \\ \Rightarrow K^{(k)}(1_A) = 0 \quad & \mu_f^{(k)} - \text{almost everywhere} \Rightarrow \mu_f^{(k)} K^{(k)}(A) = 0\end{aligned}$$

Using this property, we define the following operator from $\mathcal{B}(E_{1,c}^k)$ into $\mathcal{B}(E_{2,c}^k)$:

$$\forall f \in \mathcal{B}(E_{1,c}^k),\ \forall y \in E_2^k \qquad K_{\mu^{(k)}}^{(k)}(f)(y) := \frac{d\mu_f^{(k)} K^{(k)}}{d\mu^{(k)} K^{(k)}}(y)$$

Notice that $K_{\mu^{(k)}}^{(k)}(1)(y) = 1$, and for any pair of functions $f, g \in \mathcal{B}(E_{1,c}^k)$, we have

$$\mu_{f+g}^{(k)} = \mu_f^{(k)} + \mu_g^{(k)} \Rightarrow K_{\mu^{(k)}}^{(k)}(f+g)(y) = K_{\mu^{(k)}}^{(k)}(f)(y) + K_{\mu^{(k)}}^{(k)}(g)(y)$$

Using the fact that $\lim_{n\to\infty} \mu^{(k)}(A_n) = 0$ for every decreasing sequence of subsets $A_n \in E_1^k$ s.t. $\lim_{n\to\infty} A_n = \emptyset$, we prove that $\lim_{n\to\infty} K_{\mu^{(k)}}^{(k)}(1_{A_n})(y) = 0$, $\mu^{(k)} K^{(k)}$-a.e. This implies that $A \in \mathcal{E}_{1,c}^k \mapsto K_{\mu^{(k)}}^{(k)}(1_A)(y)$ is a well-defined probability measure $K_{\mu^{(k)}}^{(k)}(y, dx)$ on the set $(E_{1,c}^k, \mathcal{E}_{1,c}^k)$, and we have the following $\mu^{(k)} K^{(k)}$-a.e. Lebesgue integral representation

$$K_{\mu^{(k)}}^{(k)}(f)(y) = \int K_{\mu^{(k)}}^{(k)}(y, dx) f(x)$$

This ends the proof of the lemma. ■

We are now able to present the following lemma.

Lemma 3.4 *For any functions $F_i \in \mathcal{B}(\mathcal{M}(E_{i,c}))$, with $i = 1, 2$, we have the almost sure formulae:*

$$\mathbb{E}\left(F_1(\mathcal{X})\ \mid \mathcal{Y}\right) = \int F_1\left(m_N(x)\right)\ \frac{1}{N!} \sum_{\sigma \in \mathcal{G}_N} K_{\mu^{(N)}}^{(N)}\left(Y^{(N,\sigma)}, dx\right) \tag{3.17}$$

and

$$\mathbb{E}\left(F_2(\mathcal{Y})\ \mid \mathcal{X}\right) = \int F_2\left(m_N(y)\right)\ \frac{1}{N!} \sum_{\sigma \in \mathcal{G}_N} K^{(N)}\left(X^{(N,\sigma)}, dy\right) \tag{3.18}$$

where $X^{(N,\sigma)}$ resp. $Y^{(N,\sigma)}$ stands for the random point $\left(X^{(N,\sigma(j))}\right)_{1\leq j\leq N}$ resp. $\left(Y^{(N,\sigma(j))}\right)_{1\leq j\leq N}$.

Proof: The additional symmetric operators used in Eq. (3.17)-(3.18) ensure that the resulting formulae are measurable with respect to the conditional random measures $\mathcal{Y}$, and resp. $\mathcal{X}$. From this observation, the proof of (3.18) is immediate, and the proof of (3.17) is a direct consequence of the definition of the reversed Markov transition $K_{\mu^{(N)}}^{(N)}$. ■

3.3 Conditioning principles for $\mathcal{Y}_1$ given $\mathcal{Y}$

We consider here the random measures $\mathcal{Y}_1$ and $\mathcal{Y}$ defined in Eq. (3.4). The main objective of this section is to establish an expression for $\mathbb{E}\left(F(\mathcal{Y})\ \mid \mathcal{Y}_1\right)$ where $F \in \mathcal{B}(\mathcal{M}(E_{2,c}))$.

For a given realization $N = k$, we have $\left(X^{(k,i)}, Y^{(k,i)}\right)$ distributed according to

$$\begin{aligned}
&\mu^{(k,i)}(du)K^{(k,i)}(u,dv) \\
&= a^{(k,i)}\ \eta^{(k,i)}(du)L_c^{(k,i)}(u,dv) + (1-a^{(k,i)})\ \delta_c(du)\ \overline{\nu}^{(k,i)}(dv) \\
&= \left[a^{(k,i)}\ \eta^{(k,i)}(du)\alpha^{(k,i)}(u)\ g^{(k,i)}(u,v) + (1-a^{(k,i)})\ \delta_c(du)\ \overline{h}^{(k,i)}(v)\right]\ \lambda^{(k,i)}(dv) \\
&\qquad\qquad\qquad\qquad + a^{(k,i)}\ \eta^{(k,i)}(du)(1-\alpha^{(k,i)}(u))\ \delta_c(dv).
\end{aligned} \tag{3.19}$$

Therefore we have $Y^{(k,i)}$ distributed according to

$$\begin{aligned}
\rho^{(k,i)}(dv) &= \mu^{(k,i)}K^{(k,i)}(dv) \\
&= \left[a^{(k,i)}\ \eta^{(k,i)}\left(\alpha^{(k,i)}g^{(k,i)}(\,.\,,v)\right) + (1-a^{(k,i)})\ \overline{h}^{(k,i)}(v)\right]\ \lambda^{(k,i)}(dv) \\
&\qquad\qquad + a^{(k,i)}\ \eta^{(k,i)}\left(1-\alpha^{(k,i)}\right)\ \delta_c(dv) \\
&= b_0^{(k,i)}\ \rho_0^{(k,i)}(dv) + b_1^{(k,i)}\ \rho_1^{(k,i)}(dv)
\end{aligned} \tag{3.20}$$

with $b_0^{(k,i)}, b_1^{(k,i)}$ given in Eq. (3.12) and the measures $\rho_0^{(k,i)} :=\ \delta_c$ and

$$\begin{aligned}
\rho_1^{(k,i)}(dv) :=& \frac{\left[a^{(k,i)}\ \eta^{(k,i)}\left(\alpha^{(k,i)}g^{(k,i)}(\,.\,,v)\right) + (1-a^{(k,i)})\ \overline{h}^{(k,i)}(v)\right]\ \lambda^{(k,i)}(dv)}{a^{(k,i)}\ \eta^{(k,i)}\left(\alpha^{(k,i)}\right) + \left(1-a^{(k,i)}\right)} \\
&= \left(1-c^{(k,i)}\right)\Psi_{\alpha^{(k,i)}}\left(\eta^{(k,i)}\right)L^{(k,i)}(dv) + c^{(k,i)}\ \overline{\nu}^{(k,i)}(dv)\ .
\end{aligned}$$

where

$$c^{(k,i)} = \frac{1-a^{(k,i)}}{a^{(k,i)}\ \eta^{(k,i)}\left(\alpha^{(k,i)}\right) + \left(1-a^{(k,i)}\right)}$$

The point process $\mathcal{Y}$ associated to $Y^{(N)} := \left(Y^{(N,i)}\right)_{1\le i\le N}$ can thus be rewritten as

$$\mathcal{Y} := \sum_{i=1}^{N}\left(1-\epsilon^{(N,i)}\right)\ \delta_{Y_0^{(N,i)}} + \sum_{i=1}^{N}\epsilon^{(N,i)}\ \delta_{Y_1^{(N,i)}}$$

where, for a given realization $N = k$, $\epsilon^{(k)} := \left(\epsilon^{(k,i)}\right)_{1\le i\le k}$ a sequence of independent $\{0,1\}$-valued random variables with distributions defined by

$$\forall j \in \{0,1\} \qquad \mathbb{P}\left(\epsilon^{(k,i)} = j\right) = b_j^{(k,i)}$$

and $\left(Y_j^{(k,i)}\right)_{1\le i\le k}$ are independent random variables with distributions $\rho_j^{(k,i)}$. We clearly have $\mathcal{Y}_j = \sum_{i\in I_j^{(N)}}\delta_{Y_j^{(N,i)}}$ where

$$I_1^{(N)} := \left\{i \in \{1,\dots,N\}\ :\ \epsilon_{(N,i)} = 1\right\} \quad \text{and} \quad I_0^{(N)} := \{1,\dots,N\} - I_1^{(N)}\ .$$

We now establish the conditional distribution of $\mathcal{Y}$ given $\mathcal{Y}_1$. To describe precisely this result and to simplify the presentation, it is convenient to introduce the following notation. For every subset $I \subset \{1,\dots,k\}$, $y = (y_i)_{i\in I}$, and any $j = 0,1$, we write

$$b_j^{(k,I)} := \prod_{i\in I} b_j^{(k,i)} \quad \text{and} \quad m_I(y) = \sum_{i\in I}\delta_{y^i} \quad \text{and} \quad \rho_j^{(k,I)}(dy) = \prod_{i\in I}\rho_j^{(k,i)}(dy^i)$$

Proposition 3.5 *For any function $F \in \mathcal{B}(\mathcal{M}(E_{2,c}))$, we have the almost sure formulae:*

$$\mathbb{E}\left(F(\mathcal{Y}) \mid \mathcal{Y}_1\right) = \frac{1}{c(N_1)} \sum_{k \geq N_1} \pi(k) \left(\sum_{|I|=N_1} b_1^{(k,I)} \; b_0^{(k,I^c)} \right) F\left((k - N_1)\,\delta_c + \mathcal{Y}_1\right) \tag{3.21}$$

where the second sum in the r.h.s. is taken over all finite sets $I \subset \{1, \ldots, k\}$ with cardinality $|I| = N_1$ and $b(N_1, k)$ and $c(N_1)$ are given in theorem 3.1.

Proof:

Using the decomposition

$$\mathbb{E}\left(F(\mathcal{Y})\right) = \sum_{k \geq 0} \pi(k) \sum_I b_1^{(k,I)} \; b_0^{(k,I^c)} \int F\left(m_I(y) + m_{I^c}(y')\right) \; \rho_1^{(k,I)}(dy) \; \rho_0^{(k,I^c)}(dy')$$

it is easily checked that

$$\mathbb{E}\left(F(\mathcal{Y}) \Big| \; \left(N, I_1^{(N)}\right) = (k, I)\right) = \int F\left(m_I(y) + m_{I^c}(y')\right) \; \rho_1^{(k,I)}(dy) \; \rho_0^{(k,I^c)}(dy')$$

and

$$\begin{aligned} \mathbb{E}\left(F(\mathcal{Y}) \Big| \; \mathcal{Y}_1, \left(N, I_1^{(N)}\right) = (k, I)\right) &= \int F\left(\mathcal{Y}_1 + m_{I^c}(y')\right) \; \rho_0^{(k,I^c)}(dy') \\ &= F\left(\mathcal{Y}_1 + (k - N_1)\,\delta_c\right). \end{aligned}$$

Using the fact that

$$\begin{aligned} N_1 = p \Rightarrow \mathbb{E}\left(G\left(N, I_1^{(N)}\right) \mid \mathcal{Y}_1\right) &= \mathbb{E}\left(G\left(N, I_1^{(N)}\right) \; ; \; \left|I_1^{(N)}\right| = p\right) / \mathbb{P}\left(\left|I_1^{(N)}\right| = p\right) \\ &= \frac{1}{c(p)} \sum_{k \geq p} \pi(k) \sum_{|I|=p} b_1^{(k,I)} \; b_0^{(k,I^c)} \; G(k, I) \end{aligned}$$

for any measurable function G on $\cup_{n \geq 0} \left(\{n\} \times \Pi_n\right)$, with Π_n the set of finite subsets of $\{1, \ldots, n\}$, we prove Eq. (3.21). This ends the proof of the proposition. ■

3.4 Proof of Theorem 3.1

We are now in a position to prove theorem 3.1. By a direct application of lemma 3.4, for any functions $F \in \mathcal{B}(\mathcal{M}(E_{1,c}))$ we have the almost sure formulae:

$$\mathbb{E}\left(F(\mathcal{X}) \mid \mathcal{Y}\right) = \int F\left(m_N(x)\right) \frac{1}{N!} \sum_{\sigma \in \mathcal{G}_N} K_{\mu^{(N)}}^{(N)}\left(Y^{(N,\sigma)}, dx\right)$$

where $K_{\mu^{(N)}}^{(N)}$ is defined in (3.16) in lemma 3.3. Given Eq. (3.16) and Eq. (3.15)-(3.14), it is straightforward to check that for any realization $N = k$

$$K_{\mu^{(k)}}^{(k)}(y, dx) = \prod_{i=1}^{k} K_{\mu^{(k,i)}}^{(k,i)}(y^i, dx^i).$$

Moreover, by dividing Eq. (3.19) by Eq. (3.20), we obtain directly

$$\begin{aligned} K_{\mu^{(k,i)}}^{(k,i)}(v, du) = 1_{E_2}(v) &\left[\left(1 - \overline{\beta}^{(k,i)}(v)\right) \Psi_{\alpha^{(k,i)} g^{(k,i)}(v,\cdot)}\left(\eta^{(k,i)}\right)(du) + \overline{\beta}^{(k,i)}(v) \; \delta_c(du)\right] \\ &+ 1_c(v) \Psi_{\left(1 - \alpha^{(k,i)}\right)}\left(\eta^{(k,i)}\right)(du) \end{aligned}$$

with the $[0,1]$-valued parameters

$$\overline{\beta}^{(k,i)}(v) = \frac{(1-a^{(k,i)})\overline{h}^{(k,i)}(v)}{a^{(k,i)}\eta^{(k,i)}\left(\alpha^{(k,i)}g^{(k,i)}(\,.\,,v)\right) + (1-a^{(k,i)})\overline{h}^{(k,i)}(v)}.$$

So we can conclude that

$$\begin{aligned}\mathbb{E}\left(F(\mathcal{X})\,|\mathcal{Y}\,\right) &= \frac{1}{N!}\sum_{\sigma\in\mathcal{G}_N}\int F\left(m_N(x)\right)\prod_{1\leq i\leq N} K^{(N,i)}_{\mu^{(N,i)}}\left(Y^{(N,\sigma(i))}, dx^i\right)\\ &= \frac{1}{N!}\sum_{\sigma\in\mathcal{G}_N}\int F\left(m_N(x)\right)\prod_{1\leq i\leq N} K^{(N,\sigma(i))}_{\mu^{(N,\sigma(i))}}\left(Y^{(N,i)}, dx^i\right). \qquad (3.22)\end{aligned}$$

The final result given in Eq. (3.11) follows directly from Eq. (3.22), proposition 3.5 and the fact that $\mathbb{E}\left(F(\mathcal{X})\,|\mathcal{Y}_1\,\right) = \mathbb{E}\left(\mathbb{E}\left(F(\mathcal{X})\,|\mathcal{Y}\,\right)|\mathcal{Y}_1\,\right)$.

4 Discussion

We have proposed elementary techniques to establish the conditional distributions of spatial point processes. This has allowed us to re-establish the equations of the PHD filter [5] without relying on probability generating functionals and their derivatives. We have then shown how this analysis can be extended to more general spatial point processes which might be of interest in some applications.

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