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Abstract: We design a mean field and interacting particle interpretation of a class of spatial branching intensity models with spontaneous births arising in multiple-target tracking problems. In contrast to traditional Feynman-Kac type particle models, the transitions of these interacting particle systems depend on the current particle approximation of the total mass process.

In the first part, we analyze the stability properties and the long time behavior of these spatial branching intensity distribution flows. We study the asymptotic behavior of total mass processes and we provide a series of weak Lipschitz type functional contraction inequalities.

In the second part, we study the convergence of the mean field particle approximations of these models. Under some appropriate stability conditions on the exploration transitions, we derive uniform and non asymptotic estimates as well as a sub-gaussian concentration inequality and a functional central limit theorem. The stability analysis and the uniform estimates presented in the present article seem to be the first results of this type for this class of spatial branching models.

Key-words: Spatial branching processes, multi-target tracking problems, mean field and interacting particle systems, Feynman-Kac semigroups, uniform estimates w.r.t. time, functional central limit theorems.

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Particle approximations of a class of branching distribution flows arising in multi-target tracking

Résumé : Nous développons une interprétation particulaire de type champ moyen d'une classe de modèles d'intensités de branchements spatiaux liés à des naissances spontanées que l'on rencontre dans des problèmes de filtrage stochastique multicibles. A la différence des modèles classiques de type Feynman-Kac, les transitions de ces systèmes de particules dépendent des approximations du processus de masse totale.

Dans la première partie de cette étude, nous analysons les propriétés de stabilité et le comportement en temps long des semigroupes d'intensités de branchements spatiaux. Nous étudions le comportement asymptotique du du processus de masse totale, puis nous présentons une série d'inégalités de contraction fonctionnelles.

Dans la seconde partie, nous analysons la convergence des approximations particulaires de type champ moyen de ces modèles. Sous certaines hypothèses de régularité sur les transitions d'exploration, nous développons des estimation non asymptotiques et uniformes par rapport à l'horizon temporel, ainsi que des inégalités de concentration exponentielles sous gaussiennes, et un théorème de la limite centrale fonctionnel. L'analyse de la stabilité ainsi que les estimations d'erreurs uniformes par rapport au temps semblent être les premiers résultats de ce type pour ces classes de modèles de branchements spatio-temporels.

Mots-clés : Processus de branchements, prob—èmes de filtrage multicibles, systèmes de particules de type champ moyen, semigroupes de Feynman-Kac, estimations uniformes par rapport au temps, théorèmes de la limite centrale fonctionnels.

1 Introduction

Multiple-target tracking problems deal with correctly estimating several interacting maneuvering targets simultaneously given a sequence of noisy and partial observations. This rapidly developing subject is one of the most interesting contact points between the theory of spatial branching processes, mean field particle systems and advanced signal processing. The first connections between stochastic branching processes and multi-target tracking problems seem to go back to the article by S. Mori, and al. [15] published in 1986. A more systematic treatment of multi-sensor multi-target filtering using random finite sets theory can be found in the series of articles by R. Malher and his co-authors [11, 12, 13, 14].

With the exception of some very special cases, the first moment statistics of the optimal multi-target filter cannot be represented in a closed form, even on infinite dimensional state-spaces. The central idea behind these multi-target approximation filtering techniques is twofold : First, we need to find an approximate recursive equation for the first moment statistics of a branching evolution model given a partial and noisy observation point process. The so-called probability hypothesis density (known as PHD filter in the literature) is a first moment Poisson type approximation to the optimal multi-target filtering equation [11, 12, 13, 14]. Given these closed equations, we approximate these distributions numerically using particle approximations [9, 10].

Despite many advances in recent years, the theoretical performance of these multi-target particle type filters remain poorly understood. The aim of this work is to study the stability properties of these branching signal processes and more particularly the uniform nature of their mean field particle approximations. It is important to observe that without any observations, the"a priori" signal process is a spatial branching model whose first moment statistics always satisfy a closed recursive equation in the space of bounded positive measures. The present article is centered around these branching signal processes. The analysis of the particle interpretations of PHD filters is more involved mainly because it is built on a nonlinear updating transition of the current intensity distribution. These non linear models are studied in a separate article [7].

The rest of this paper is organized as follows:

In section 1.1, we design a general class of spatial branching models that encapsulates all the aspects of a multi-target motion, including time-varying random number of targets due to new targets appearing and old targets disappearing from the scene; the branching rates and the survival probabilities of each target may also vary depending on its kinetic parameters and its type. We also introduce geometric time clocks for spawning and spontaneous branching schemes. The closed linear evolution equations associated with the intensity distributions of these branching models is presented in section 1.2. Special attention is paid to the analysis of the stability properties and the long time behavior of these distribution flows, including the asymptotic behavior of the total mass process and the convergence to equilibrium of the corresponding sequence of normalized distributions. For time homogeneous models, we exhibit three different types of asymptotic behavior. The analysis of these stability properties is essential in order to guarantee the robustness of the model and to obtain reliable performance of any numerical approximation scheme.

In section 1.3, we design an original mean field particle interpretation model. In contrast to the mean field models developed in [4], these new particle algorithms also interact with the particle approximation of the total mass process. The second main result of the present article is a non asymptotic convergence theorem for these particle models. Under some appropriate stability properties, we provide a rather sharp analysis that allows us to obtain uniform estimates w.r.t. the time parameter, yielding the first results of this type for this class of models.

1.1 Spatial branching models

Let us suppose that at a given time n there are N_n targets $(X_n^i)_{1\leq i\leq N_n}$ taking values in some measurable state space E_n enlarged with a virtual and auxiliary cemetery point c. The state spaces E_n depends on the problem at hand. It may vary with the time parameter and it encapsulates all the characteristics of the targets, including the possibly different types of targets and their kinetic parameters, as well as the complete path of a given target from the origin. As usual, we extend the measures γ_n and the bounded and measurable functions f_n on E_n by setting $\gamma_n(\{c\}) = 0$ and $f_n(c) = 0$.

Each target has a survival probability $e_n(X_n^i) \in [0,1]$. When a target dies, it goes to the cemetery point c. We also use the convention $e_n(c) = 0$ so that a killed target remains in the cemetery point. Survival targets give birth to a random number of random individuals $h_n^i(X_n^i)$, where $h_n^i(x_n)$, $x_n \in E_n$, is a collection of independent and identically distributed random variables with a prescribed mean value $\mathbb{E}\left(h_n^i(x_n)\right) := H_n(x_n)$ dictated by a given collection of bounded potential type functions H_n on E_n. This branching transition is called spawning in the mutli-target tracking literature. Geometric time clocks spawning can be considered by setting

$$h_n^i(X_n^i) = U_n^i(X_n^i) + (1 - U_n^i(X_n^i)\ )\ \overline{h}_n^i(X_n^i)$$

where $U_n^i(x_n)$, $x_n \in E_n$, is a collection of $\{0,1\}$-valued, independent and identically distributed random variables with a prescribed mean value $\mathbb{E}\left(U_n^i(x_n)\right) := \mathbb{P}(U_n^i(x_n) = 1) = p_n(x_n)$ dictated by a given collection of $[0,1]$-valued functions p_n on E_n; and $\overline{h}_n^i(x_n)$, $x_n \in E_n$, is a collection of independent and identically distributed random variables with a prescribed mean value $\mathbb{E}\left(\overline{h}_n^i(x_n)\right) := \overline{H}_n(x_n)$ associated with a given collection of bounded potential type functions $\overline{H}_n$ on E_n. Notice that in this case we have $H_n = p_n + (1-p_n)\overline{H}_n$. In both cases, we set $G_n = e_n H_n$.

After this branching transition, the system consists of a random number $\widehat{N}_n$ of individuals $(\widehat{X}_n^i)_{1\leq i\leq \widehat{N}_n}$. Each of them evolves randomly $\widehat{X}_n^i = x_n \rightsquigarrow X_{n+1}^i$ according to a given elementary Markov transition $M_{n+1}(x_n, dx_{n+1})$ from E_n into E_{n+1}. We use the convention $M_{n+1}(c,c) = 1$, so that any killed target remains in the cemetery state.

At the same time, an independent collection of new targets is added to the current configuration. This additional and spontaneous branching process is often modeled by a spatial Poisson process with a prescribed intensity measure μ_{n+1} on E_{n+1}. This spontaneous branching scheme is used to model the new maneuvering targets entering in the observation scene, such as point targets entering the radar or the sonar screens.

At the end of this transition, we obtain $N_{n+1} = \widehat{N}_n + N'_{n+1}$ targets $(X_{n+1}^i)_{1\leq j\leq N_{n+1}}$, where N'_{n+1} is a Poisson random variable with parameter given by the total mass $\mu_{n+1}(1)$ of the positive measure μ_{n+1}, and $(X_{n+1}^{\widehat{N}_n+i})_{1\leq i\leq N'_{n+1}}$ are independent and identically distributed random variables with common law $\overline{\mu}_{n+1} := \mu_{n+1}/\mu_{n+1}(1)$. As above, spontaneous branching schemes on geometric time clocks are defined by setting $N'_{n+1} = U_n \times N''_{n+1}$, where U_n is a $\{0,1\}$-valued Bernoulli random variable with a given parameter $\mathbb{P}(U_n = 1) = s_n \in [0,1]$, and N''_{n+1} is a Poisson random variable with parameter given by the total mass $\mu_{n+1}(1)/s_n$.

1.2 Intensity distribution flows

At every time n, the first moment of the occupation measure $\mathcal{X}_n := \sum_{i=1}^{N_n} \delta_{X_n^i}$ of the spatial branching signal defined in section 1.1 is given for any bounded measurable function f on $E_n \cup \{c\}$ by the following formula:

$$\gamma_n(f) := \mathbb{E}\left(\mathcal{X}_n(f)\right) \quad \text{with} \quad \mathcal{X}_n(f) := \int f(x)\ \mathcal{X}_n(dx)$$

To simplify the presentation, we suppose that the initial configuration of the targets is a spatial Poisson process with intensity measure μ_0 on the state space E_0.

By construction, the measures γ_n on E_n satisfy the following recursive equation

$$\gamma_{n+1}(dx') = \int \gamma_n(dx)\ Q_{n+1}(x,dx') + \mu_{n+1}(dx') \tag{1.1}$$

with the initial condition $\gamma_0 = \mu_0$, and the integral operator Q_{n+1} from E_n into E_{n+1} defined by

$$Q_{n+1}(x_n, dx_{n+1}) = G_n(x_n)\ M_{n+1}(x_n, dx_{n+1}) \tag{1.2}$$

For null spontaneous branching measures $\mu_n = 0$, we observe that

$$\gamma_{n+1}(dx') = (\gamma_n Q_{n+1})(dx') := \int \gamma_n(dx)\ Q_{n+1}(x, dx') \tag{1.3}$$

In this particular situation, the solution of the equation (1.1) is given by the following Feynman-Kac path integral formulae

$$\gamma_n(f) = \gamma_0(1)\ \mathbb{E}\left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right) \tag{1.4}$$

where X_n stands for a Markov chain taking values in the state spaces E_n with initial distribution $\eta_0 = \gamma_0/\gamma_0(1)$ and Markov transitions M_n (see for instance section 1.4.4.in [4]). Thse measure valued equations also arise in a variety of application domains, including in physics, biology, and rare event analysis; see for instance [4, 9], and references therein.

These measures typically do not admit any closed-form expression. One natural way to solve these equations numerically is to use a mean field particle interpretation of the normalized distributions flow given by

$$\eta_n(dx) := \gamma_n(dx_n)/\gamma_n(1) \quad \text{with} \quad \gamma_n(1) := \int \gamma_n(dx)$$

To avoid unnecessary technical details, we further assume that the potential functions G_n are chosen so that $\sup_{x,y}(G_n(x)/G_n(y)) < \infty$, for any time parameter $n \geq 0$. The forthcoming analysis can be extended to more general models using the techniques developed in section 4.4 in [4]; see also [3]. We denote by $\mathcal{P}(E_n)$ the convex set of all probability measures on the state space E_n.

To describe these stochastic particle models, it is important to observe that the pair process $(\gamma_n(1), \eta_n) \in (\mathbb{R}_+ \times \mathcal{P}(E_n))$ satisfies an evolution equation of the following form

$$(\gamma_n(1), \eta_n) = \Gamma_n(\gamma_{n-1}(1), \eta_{n-1}) \tag{1.5}$$

We let Γ_n^1 and Γ_n^2 be the first and the second component mappings from $(\mathbb{R}_+ \times \mathcal{P}(E_n))$ into $\mathbb{R}_+$, and from $(\mathbb{R}_+ \times \mathcal{P}(E_n))$ into $\mathcal{P}(E_n)$. The mean field type interacting particle system associated with the equation (1.5) relies on the fact that the one step mappings Γ_{n+1}^2 can be rewritten in the following form

$$\Gamma_{n+1}^2(\gamma_n(1), \eta_n) = \eta_n K_{n+1,(\gamma_n(1),\eta_n)} \tag{1.6}$$

for some collection of Markov kernels $K_{n+1,(m,\eta)}$ indexed by the time parameter n and the set of probability measures η on the space E_n and the mass parameter $m \in \mathbb{R}_+$. We mention that the choice of the Markov transitions $K_{n,(m,\eta)}$ is not unique. In the literature on mean field particle models, $K_{n,(m,\eta)}$ are called McKean transitions. For null spontaneous branching measures $\mu_n = 0$, using (1.3) we can readily prove that

$$\begin{array}{rcl} \gamma_n(1) & = & \eta_{n-1}(G_n)\ \gamma_{n-1}(1) := \Gamma_n^1(\gamma_{n-1}(1), \eta_{n-1}) \\ \eta_n(dx') & = & \displaystyle\int \Psi_{G_{n-1}}(\eta_{n-1})(dx)\ M_n(x, dx') := \Gamma_n^2(\gamma_{n-1}(1), \eta_{n-1})(dx') \end{array}$$

with the Boltzmann-Gibbs measure

$$\Psi_{G_{n-1}}(\eta_{n-1})(dx) := \frac{1}{\eta_{n-1}(G_{n-1})}\ G_{n-1}(x)\ \eta_{n-1}(dx)$$

In this situation, we observe that the second component mapping

$$\Gamma_n^2(\gamma_{n-1}(1), \eta_{n-1}) := \Phi_n(\eta_{n-1})$$

reduces to a mapping Φ_n that does not depend on the total mass process $\gamma_{n-1}(1)$.

1.3 Mean field particle models

The transport formula discussed in (1.6) provides a natural interpretation of the distribution laws η_n as the laws of a non linear Markov chain $\overline{X}_n$ whose elementary transitions $\overline{X}_n \rightsquigarrow \overline{X}_{n+1}$ depends on the distribution $\eta_n = \mathrm{Law}(X_n)$ as well as on the current mass process $\gamma_n(1)$. In contrast to traditional McKean models, the dependency on the mass process induces a dependency of the whole sequence of measures η_p, from the origin $p = 0$ up to the current time $p = n$. For a thorough description of these non linear McKean type models, we refer the reader to [4]. A more detailed presentation of these models is provided in section 4.1 in the present article. In the further developments of the article, we always assume that the mappings

$$\left(m, (x^i)_{1\leq i\leq N}\right) \in \left(\mathbb{R}_+ \times E_n^N\right) \mapsto K_{n+1,\left(m, \frac{1}{N}\sum_{j=1}^N \delta_{x^j}\right)}\left(x^i, A_{n+1}\right)$$

are measurable w.r.t. the product sigma fields on $(\mathbb{R}_+ \times E_n^N)$, for any $n \geq 0$, $N \geq 1$, and $1 \leq i \leq N$, and any measurable subset $A_{n+1} \subset E_{n+1}$. In this situation, the mean field particle interpretation of this non linear measure valued model is an E_n^N-valued Markov chain $\xi_n^{(N)} = \left(\xi_n^{(N,i)}\right)_{1\leq i\leq N}$, with elementary transitions defined as

$$\left\{\begin{array}{rcl} \gamma_{n+1}^N(1) & = & \gamma_n^N(1)\ \eta_n^N(G_n) + \mu_{n+1}(1) \\ \mathbb{P}\left(\xi_{n+1}^{(N)} \in dx \;\middle|\; \mathcal{F}_n^{(N)}\right) & = & \prod_{i=1}^N\ K_{n+1,(\gamma_n^N(1),\eta_n^N)}(\xi_n^{(N,i)}, dx^i) \end{array}\right. \tag{1.7}$$

with the pair of occupation measures $\left(\gamma_n^N, \eta_n^N\right)$ defined below

$$\eta_n^N := \frac{1}{N}\sum_{j=1}^N \delta_{\xi_n^{(N,j)}} \quad \text{and} \quad \gamma_n^N(dx) := \gamma_n^N(1)\ \eta_n^N(dx)$$

In the above displayed formula, $\mathcal{F}_n^N$ stands for the σ-field generated by the random sequence $(\xi_p^{(N)})_{0\leq p\leq n}$, and $dx = dx^1 \times \ldots \times dx^N$ stands for an infinitesimal neighborhood of a point $x = (x^1, \ldots, x^N) \in E_n^N$. The initial system $\xi_0^{(N)}$ consists of N independent and identically distributed random variables with common law η_0. As usual, to simplify the presentation, when there is no possible confusion we suppress the parameter N, so that we write ξ_n and ξ_n^i instead of $\xi_n^{(N)}$ and $\xi_n^{(N,i)}$.

In the above discussion, we have implicitly assumed that the quantities $\mu_n(1)$ are known and it is easy to sample from the probability distribution $\overline{\mu}_n(dx) := \mu_n(dx)/\mu_n(1)$. In practice, we often need to resort to an additional approximation scheme to approximate the quantities $\mu_n(1)$ and $\overline{\mu}_n$. This situation is discussed in section 5. Using an additional level of approximation, we will show that this case essentially reduces to the analysis of the one discussed above.

For the convenience of the reader, we end this introduction with some notation used in the present article. We denote by $\mathcal{B}(E)$, the Banach space of all bounded and measurable functions f on some measurable state space $(E, \mathcal{E})$, equipped with the uniform norm $\|f\|$. We also denote by $\mathrm{Osc}_1(E)$, the convex set of $\mathcal{E}$-measurable functions f with oscillations $\mathrm{osc}(f) \leq 1$.

We let $\mu(f) = \int\ \mu(dx)\ f(x)$, be the Lebesgue integral of a function $f \in \mathcal{B}(E)$, with respect to a measure $\mu \in \mathcal{M}(E)$. We recall that a bounded integral kernel $M(x, dy)$ from a measurable space $(E, \mathcal{E})$ into an auxiliary measurable space $(E', \mathcal{E}')$ is an operator $f \mapsto M(f)$ from $\mathcal{B}(E')$ into $\mathcal{B}(E)$ such that the functions $x \mapsto M(f)(x) := \int_{E'} M(x, dy)f(y)$ are $\mathcal{E}$-measurable and bounded, for any $f \in \mathcal{B}(E')$. In the above displayed formulae, dy stands for an infinitesimal neighborhood of a point y in E'. The kernel M also generates a dual operator $\mu \mapsto \mu M$ from $\mathcal{M}(E)$ into $\mathcal{M}(E')$ defined by $(\mu M)(f) := \mu(M(f))$. A Markov kernel is a positive and bounded integral operator M with $M(1) = 1$. Given a pair of bounded integral operators (M_1, M_2), we let $(M_1 M_2)$ the composition operator defined by $(M_1 M_2)(f) = M_1(M_2(f))$. For time homogenous state spaces, we denote by

$M^k = M^{k-1}M = MM^{k-1}$ the k-th composition of a given bounded integral operator M, with $k \geq 0$, with the convention $M^0 = Id$ the identity operator. We also used the notation

$$M\left([f_1 - M(f_1)]\,[f_2 - M(f_2)]\right)(x) := M\left([f_1 - M(f_1)(x)]\,[f_2 - M(f_2)(x)]\right)(x)$$

for some bounded functions f_1, f_2.

We also denote by $\|\mu\|_{\rm tv} = \sup_{f\in {\rm Osc}_1(E)} |\mu(f)|$, the total variation norm on $\mathcal{M}(E)$. When the bounded integral operator M has a constant mass, that is $M(1)(x) = M(1)(y)$ for any $(x,y) \in E^2$, the operator $\mu \mapsto \mu M$ maps $\mathcal{M}_0(E)$ into $\mathcal{M}_0(E')$. In this situation, we let $\beta(M)$ be the Dobrushin coefficient of a bounded integral operator M defined by the following formula

$$\beta(M) := \sup\ \{{\rm osc}(M(f))\ ;\ \ f \in {\rm Osc}_1(F)\}$$

Given a positive function G on E, we let $\Psi_G\ :\ \eta \in \mathcal{P}(E) \mapsto \Psi_G(\eta) \in \mathcal{P}(E)$ be the Boltzmann-Gibbs transformation defined by

$$\Psi_G(\eta)(dx) := \frac{1}{\eta(G)}\ G(x)\ \eta(dx)$$

We recall that $\Psi_G(\eta)$ can be expressed in terms of a Markov transport equation

$$\eta S_\eta = \Psi_G(\eta) \tag{1.8}$$

for some selection type transition $S_\eta(x, dy)$. For instance, we can take

$$S_\eta(x,dy) := \frac{\epsilon}{\eta(G)}\ \delta_x(dy) + \left(1 - \frac{\epsilon}{\eta(G)}\right)\ \Psi_{(G-\epsilon)}(\eta)(dy) \tag{1.9}$$

for any $\epsilon \geq 0$ s.t. $G(x) \geq \epsilon$. Notice that for $\epsilon = 0$, we have $S_\eta(x,dy) = \Psi_G(\eta)(dy)$. We can also choose

$$S_\eta(x,dy) := \epsilon G(x)\ \delta_x(dy) + (1 - \epsilon G(x))\ \ \Psi_G(\eta)(dy) \tag{1.10}$$

for any $\epsilon \geq 0$ that may depend on the current measure η, and s.t. $\epsilon G(x) \leq 1$. For instance, we can choose $1/\epsilon$ to be the η-essential maximum of the potential function G.

2 Statement of the main results

In the introduction, we have seen that for null spontaneous birth measures, the evolution equation (1.1) coincides with that of a Feynman-Kac model (1.4). In this situation, the distributions γ_n are simply given by the recursive equation

$$\gamma_n = \gamma_{n-1}Q_n \implies \forall 0 \leq p \leq n \qquad \gamma_n = \gamma_p Q_{p,n} \quad \text{with} \quad Q_{p,n} = Q_{p+1}\ldots Q_{n-1}Q_n \tag{2.1}$$

For $p = n$, we use the convention $Q_{n,n} = Id$, the identity operator. In addition, the non linear semigroup of the normalized distribution flow is given by

$$\eta_n(f) = \Phi_{p,n}(\eta_p)(f) := \eta Q_{p,n}(f)/\eta Q_{p,n}(1) = \eta\left(Q_{p,n}(1)P_{p,n}(f)\right)/\eta Q_{p,n}(1) \tag{2.2}$$

with the Markov transition operator $P_{p,n}(x_p, dx_n) = Q_{p,n}(x_p, dx_n)/Q_{p,n}(x_p, E_n)$. The analysis of the mean field particle interpretations of such models have been studied in [4]. Various properties including contraction inequalities, fluctuations, large deviations and concentration properties have been developed for this class of models. In this context, the fluctuations properties as well as $\mathbb{L}_r$-mean error estimates, including uniform estimates w.r.t. the time parameter are often expressed in terms of two central parameters:

$$q_{p,n} = \sup_{x,y} \frac{Q_{p,n}(1)(x)}{Q_{p,n}(1)(y)} \quad \text{and} \quad \beta(P_{p,n}) = \sup_{x,y\in E_p} \|P_{p,n}(x,.) - P_{p,n}(y,.)\|_{\rm tv} \tag{2.3}$$

with the pair of Feynman-Kac semigroups $(P_{p,n}, Q_{p,n})$ introduced in (2.1) and (2.2).

We also consider the pair of parameters $(g_-(n), g_+(n))$ defined below

$$g_-(n) = \inf_{0\leq p<n} \inf_{E_p} G_p \leq \sup_{0\leq p<n} \sup_{E_p} G_p = g_+(n)$$

The first main objective of this article is to extend some of these properties to models with non necessarily null spontaneous birth distributions. We illustrate our estimates in three typical situations

$$1) \quad G = g_{-/+} = 1 \qquad 2) \quad g_+ < 1 \quad \text{and} \quad 3) \quad g_- > 1 \tag{2.4}$$

arising in time homogeneous models

$$(E_n, G_n, M_n, \mu_n, g_-(n), g_+(n)) = (E, G, M, \mu, g_-, g_+) \tag{2.5}$$

Our first main result concerns three different types of long time behavior for these three types of models. This result can basically be stated as follows.

Theorem 2.1 *For time homogeneous models (2.5), the limiting behavior of the flow $(\gamma_n(1), \eta_n)$ in the three typical situations (2.4) is as follows:*

1. *For unit potential functions $G(x) = 1$, $x \in E$, we have*

$$\gamma_n(1) = \gamma_0(1) + \mu(1)\ n \quad \textit{and} \quad \|\eta_n - \eta_\infty\|_{\rm tv} = O\left(\frac{1}{n}\right)$$

as soon as M is chosen so that

$$\sum_{n\geq 0} \sup_{x\in E} \|M^n(x, \textbf{.}) - \eta_\infty\|_{\rm tv} < \infty \quad \textit{for some invariant measure } \eta_\infty = \eta_\infty M. \tag{2.6}$$

2. *When $g_+ < 1$, there exists some finite constant $c < \infty$ such that*

$$\forall f \in \mathcal{B}(E), \qquad |\gamma_n(f) - \gamma_\infty(f)| \vee |\eta_n(f) - \eta_\infty(f)| \leq c\ g_+^n\ \|f\|$$

$$\textit{with the limiting measures } \gamma_\infty(f) := \sum_{n\geq 0} \mu Q^n(f) \textit{ and } \eta_\infty(f) := \gamma_\infty(f)/\gamma_\infty(1) \tag{2.7}$$

3. *Assume that $g_- > 1$ and $M^k(x, \textbf{.}) \geq \epsilon\ M^k(y, \textbf{.})$, for any $x, y \in E$ and some pair of parameters $k \geq 1$ and $\epsilon > 0$. In this situation, the mapping $\Phi = \Phi_{n-1,n}$ introduced in (2.2) has an unique fixed point $\eta_\infty = \Phi(\eta_\infty)$, and we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \gamma_n(1) = \log \eta_\infty(G) \quad \textit{and} \quad \|\eta_n - \eta_\infty\|_{\rm tv} \leq c\ e^{-\lambda n}$$

for some finite constant $c < \infty$ and some $\lambda > 0$.

A more precise statement and a detailed proof of the above theorem can be found in section 3.2.

Our second main result concerns the convergence of the mean field particle approximations presented in (1.7). We provide rather sharp non asymptotic estimates including uniform convergence results w.r.t. the time parameter. Our results can be basically stated as follows.

Theorem 2.2 *For any $n \geq 0$, and any $N \geq 1$, we have $\gamma_n(1)$ and $\gamma_n^N(1) \in I_n$ with the compact interval I_n defined below*

$$I_n := [m_-(n), m_+(n)] \quad \textit{where} \quad m_{-/+}(n) := \sum_{p=0}^{n} \mu_p(1) g_{-/+}(n)^{(n-p)} \tag{2.8}$$

In addition, for any $r \geq 1$, $f \in \mathrm{Osc}_1(E_n)$, and any $N \geq 1$, we have the estimates:

$$\sqrt{N}\ \mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right|^r\right)^{\frac{1}{r}} \leq a_r\ b_n \quad \textit{with} \quad b_n \leq \sum_{p=0}^{n} b_{p,n} \tag{2.9}$$

In the above displayed formulae, $a_r < \infty$ stands for some constants whose values only depend on the parameter r, and $b_{p,n}$ is the collection of constants given by

$$b_{p,n} := 2\ \left(1 \wedge m_{p,n}\right)\ q_{p,n}\left[q_{p,n}\ \beta(P_{p,n}) + \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}}\ \beta(P_{q,n})\right] \tag{2.10}$$

with the pair of parameters

$$m_{p,n} = m_+(p)\|Q_{p,n}(1)\|/\sum_{p<q\leq n} c_{q,n} \quad \textit{and} \quad c_{p,n} := \mu_p Q_{p,n}(1)$$

Furthermore, the particle measures γ_n^N are unbiased, and for the three classes (2.4) of time homogenous models s.t. $M^k(x,\centerdot) \geq \epsilon\ M^k(y,\centerdot)$, for any $x,y \in E$ and some pair of parameters $k \geq 1$ and $\epsilon > 0$, the constant b_n in (2.9) can be chosen so that $\sup_{n\geq 0} b_n < \infty$; in addition, we have the non asymptotic variance estimates for some $d < \infty$, any $n \geq 1$ and for any $N > 1$

$$\mathbb{E}\left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1\right]^2\right) \leq d\ \frac{n+1}{N-1}\ \left(1 + \frac{d}{N-1}\right)^{n-1} \tag{2.11}$$

The non asymptotic estimates stated in the above theorem extend the one presented in [3, 4] for Feynman-Kac type models (1.4) corresponding to null spontaneous birth measures. For such models, the $\mathbb{L}_r$-mean error estimates (2.9) are satisfied with the collection of parameters $b_{p,n} := 2q_{p,n}^2\ \beta(P_{p,n})$, with $p \leq n$. The extra terms in (2.10) are intimately related to the spontaneous birth measures whose effects in the semigroup stability depend on the nature of the potential functions G_n. We refer to theorem 2.1, section 3.2 and section 3.3, for a discussion on three different behaviors in the three cases presented in (2.4).

Our last main result, is a functional central limit theorem. We let W_n^N be the centered random fields the following stochastic perturbation formulae:

$$\eta_n^N = \eta_{n-1}^N K_{n,(\gamma_n^N(1),\eta_{n-1}^N)} + \frac{1}{\sqrt{N}}\ W_n^N\ . \tag{2.12}$$

We also consider the pair of random fields

$$V_n^{\eta,N} := \sqrt{N}[\eta_n^N - \eta_n] \quad \text{and} \quad V_n^{\gamma,N} := \sqrt{N}[\gamma_n^N - \gamma_n]$$

For $n = 0$, we use the convention $W_0^N = V_0^{\eta,N}$.

Theorem 2.3 *The sequence of random fields $(W_n^N)_{n\geq 0}$ converges in law, as N tends to infinity, to the sequence of n independent, Gaussian and centered random fields $(W_n)_{n\geq 0}$ with a covariance function given for any $f,g \in \mathcal{B}(E_n)$, and $n \geq 0$,*

$$\begin{aligned} &\mathbb{E}(W_n(f)W_n(g)) \\ &= \eta_{n-1}K_{n,(\gamma_{n-1}(1),\eta_{n-1})}\left([f - K_{n,(\gamma_{n-1}(1),\eta_{n-1})}(f)][g - K_{n,(\gamma_{n-1}(1),\eta_{n-1})}(g)]\right). \end{aligned} \tag{2.13}$$

In addition, the pair of random fields $V_n^{\gamma,N}$, and $V_n^{\eta,N}$, converge in law, as $N \to \infty$, to a pair of centered Gaussian fields V_n^γ and V_n^η, defined below

$$V_n^\gamma(f) := \sum_{p=0}^{n} \gamma_p(1)\ W_p(Q_{p,n}(f)) \quad \textit{and} \quad V_n^\eta(f) := V_n^\gamma\left(\frac{1}{\gamma_n(1)}(f - \eta_n(f))\right)$$

The details of the proof of theorem 2.2 and theorem 2.3 can be found in section 4.2. The proof of the non asymptotic variance estimate (2.11) is given in section 4.2.1 dedicated to the convergence of the unnormalized particle measures γ_n^N. The $\mathbb{L}_r$-mean error estimates (2.9) and the fluctuation theorem 2.3 are proved in section 4.2.2.

Let us discuss some more or less direct consequences of these theorems:

Firstly, we observe that the mean error estimates stated in the above theorem clearly implies the almost sure convergence results

$$\lim_{N\to\infty} \eta_n^N(f) = \eta_n(f) \quad \text{and} \quad \lim_{N\to\infty} \gamma_n^N(f) = \gamma_n(f)$$

for any bounded function f on E_n. Moreover, with some information on the constants a_r, these $\mathbb{L}_r$-mean error bounds can be used to establish sub-gaussian concentration estimates. For instance, arguing as in the end of section 3 in [2], we deduce from (2.9) the following non asymptotic Gaussian tail estimates:

$$\mathbb{P}\left(\left| \left[\eta_n^N - \eta_n \right](f) \right| \geq \frac{b_n}{\sqrt{N}} + \epsilon \right) \leq \exp\left(-\frac{N\epsilon^2}{2b_n^2} \right) \tag{2.14}$$

In addition, for the three classes (2.4) of time homogenous models s.t. $M^k(x,.) \geq \epsilon\, M^k(y,.)$, for any $x, y \in E$ and some pair of parameters $k \geq 1$ and $\epsilon > 0$, the constant b_n in (2.14) can be replaced by $b := \sup_{n\geq 0} b_n < \infty$. It is also worth mentioning that the above constructions allow us to consider without any further work branching particle models in path spaces. These path space models arise in the analysis of the historical process associated with a branching model as well as the analysis of a filtering problem of the whole signal path given a series of observations. For instance, let us assume that the integral kernel Q_n defined in (1.2) is associated with the elementary transition M_n of a Markov chain of the following form

$$X_n := \left(X'_p\right)_{0\leq p\leq n} \in E_n := \prod_{0\leq p\leq n} E'_p$$

In other words X_n represents the paths from the origin up to the current time of an auxiliary Markov chain X'_n taking values in some measurable state spaces E'_n, with Markov transitions M'_n. Also assume that the potential functions G_n only depend on the terminal state of the path, in the sense that $G_n(X_n) = G'_n(X'_n)$, for some potential function G'_n on E'_n. In filtering problems, these path space models provide a way to estimate the intensity distribution of the path of a given target.

In practice, it is essential to observe that the mean field particle interpretations of these path space models simply consist of keeping track of the whole history of each particle. It can be shown that the resulting particle model can be interpreted as the genealogical tree model associated to a genetic type model (see for instance [4]). In this situation, η_n^N is the occupation measure of a random genealogical tree and particles represent the ancestral lines of the current individuals. For time homogeneous models, under appropriate mixing type conditions on the Markov transitions M'_n (see for instance condition $(M)_k$ in (3.1)), the mean error estimates (2.9) implies that

$$\sqrt{N}\ \mathbb{E}\left(\left| \left[\eta_n^N - \eta_n \right](f) \right|^r \right)^{\frac{1}{r}} \leq a_r\ b\ n$$

for some finite constant $b < \infty$, whose values do not depend on the time parameter.

The distribution flow models and their particle approximations described in the present article suggest avenue of fundamental research problems, including the analysis of the large deviation principles and the convergence of empirical processes. Roughly speaking, the main difficulty with these models comes from the fact that the McKean transitions of the particle models depend on all the occupation measures of the system from the origin, up to the current time. We hope to discuss some of these properties in a future article.

The rest of the article is organized as follows.

In section 3, we analyze the semigroup properties of the total mass process $\gamma_n(1)$ and the normalized distribution flow η_n. This section is mainly concerned with the proof of theorem 2.1. The long time behavior of the total mass process is discussed in section 3.1, while the asymptotic behavior of the normalized distributions is discussed in section 3.2. In section 3.3, we develop a series of Lipschitz type functional inequalities for uniform estimates w.r.t. the time horizon for particle approximation models. In section 4, we present the McKean models associated with the flow $(\gamma_n(1), \eta_n)$ and their mean field particle interpretations. Section 4.2 is concerned with the convergence analysis of these particle models. In section 4.2.1, we discuss the convergence of the unnormalized distributions, including their unbiasedness property and the non asymptotic variance estimates presented in (2.11). The proof of the $\mathbb{L}_r$-mean error estimates (2.9) is presented in section 4.2.2. The proof of the functional central limit theorem 2.3 is a a more or less direct consequence of the decomposition formulae presented in 4.2. The proof of these fluctuations is sketched at the end of this section.

3 Semigroup analysis

The purpose of this section is to analyze the semigroup properties of the spatial branching models (1.1). We establish a framework for the analysis of the long time behavior of these models and their mean field interacting particle approximations (1.7). Firstly, we briefly recall some estimate of the quantities $(q_{p,n}, \beta(P_{p,n}))$ in terms of the potential functions G_n and the Markov transitions M_n. Further details on this subject can be found in [4], and in references therein.

We assume here that the following condition is satisfied for some $k \geq 1$, some collection of numbers $\epsilon_p \in (0,1)$

$$(M)_k \quad M_{p,p+k}(x_p, .) \geq \epsilon_p \; M_{p,p+k}(y_p, .) \;\; \text{with} \;\; M_{p,p+k} = M_{p+1} M_{p+2} \ldots M_{p+k} \tag{3.1}$$

for any time parameter p and any pair of states $(x_p, y_p) \in E_p^2$. It is well known that the mixing type condition $(M)_k$ is satisfied for any aperiodic and irreducible Markov chains on finite spaces, as well as for bi-Laplace exponential transitions associated with a bounded drift function and for Gaussian transitions with a mean drift function that is constant outside some compact domain. We introduce the following quantities

$$\delta_{p,n} := \sup \prod_{p \leq q < n} (G_q(x_q)/G_q(y_q)) \quad \text{and} \quad \delta_p^{(k)} := \delta_{p+1,p+k} \tag{3.2}$$

where the supremum is taken over all admissible pair of paths with elementary transitions M_q. Under the above conditions, we have that

$$\beta(P_{p,p+n}) \leq \prod_{l=0}^{\lfloor n/k \rfloor - 1} \left(1 - \epsilon_{p+lk}^2 / \delta_{p+lk}^{(k)}\right) \quad \text{and} \quad q_{p,p+n} \leq \delta_{p,p+k}/\epsilon_p \tag{3.3}$$

For time homogeneous Feynman-Kac models we set $\epsilon := \epsilon_k$ and $\delta_k := \delta_{0,k}$, for any $k \geq 0$. In this notation, the above estimates reduce to

$$q_{p,p+n} \leq \; \delta_k/\epsilon \quad \text{and} \quad \beta(P_{p,p+n}) \leq \left(1 - \epsilon^2/\delta_{k-1}\right)^{\lfloor n/k \rfloor} \tag{3.4}$$

3.1 Description of the models

The next proposition gives a Markov transport formulation of the one step transformations Γ_n introduced in (1.5).

Proposition 3.1 *For any $n \geq 0$, we have the recursive formula*

$$\left\{ \begin{array}{rcl} \gamma_{n+1}(1) & = & \gamma_n(1) \; \eta_n(G_n) + \mu_{n+1}(1) \\ \\ \eta_{n+1} & = & \Psi_{G_n}(\eta_n) M_{n+1,(\gamma_n(1),\eta_n)} \end{array} \right. \tag{3.5}$$

with the collection of Markov transitions $M_{n+1,(m,\eta)}$ *indexed by the parameters* $m \in \mathbb{R}_+$ *and the probability measures* $\eta \in \mathcal{P}(E_n)$ *given below*

$$M_{n+1,(m,\eta)}(x,dy) := \alpha_n(m,\eta)\, M_{n+1}(x,dy) + (1-\alpha_n(m,\eta))\ \overline{\mu}_{n+1}(dy) \tag{3.6}$$

with the collection of $[0,1]$*-parameters* $\alpha_n(m,\eta)$ *defined below*

$$\alpha_n(m,\eta) = \frac{m\eta(G_n)}{m\eta(G_n) + \mu_{n+1}(1)}$$

Proof:
Observe that for any function $f \in \mathcal{B}(E_{n+1})$, we have that

$$\eta_{n+1}(f) \quad = \quad \frac{\gamma_n(G_n M_{n+1}(f)) + \mu_{n+1}(f)}{\gamma_n(G_n) + \mu_{n+1}(1)} = \frac{\gamma_n(1)\ \eta_n(G_n M_{n+1}(f)) + \mu_{n+1}(f)}{\gamma_n(1)\ \eta_n(G_n) + \mu_{n+1}(1)}$$

from which we find that

$$\eta_{n+1} \quad = \quad \alpha_n(\gamma_n(1),\eta_n)\ \Phi_{n+1}(\eta_n) + (1-\alpha_n(\gamma_n(1),\eta_n))\ \overline{\mu}_{n+1}$$

From these observations, we prove (3.5). This ends the proof of the proposition. ■

We let Γ_{n+1} be the mapping from $\mathbb{R}_+ \times \mathcal{P}(E_n)$ into $\mathbb{R}_+ \times \mathcal{P}(E_{n+1})$ given by

$$\Gamma_{n+1}(m,\eta) = \left(\Gamma^1_{n+1}(m,\eta), \Gamma^2_{n+1}(m,\eta)\right) \tag{3.7}$$

with the pair of transformations:

$$\Gamma^1_{n+1}(m,\eta) = m\ \eta(G_n) + \mu_{n+1}(1) \quad \text{and} \quad \Gamma^2_{n+1}(m,\eta) = \Psi_{G_n}(\eta) M_{n+1,(m,\eta)}$$

We also denote by $(\Gamma_{p,n})_{0\leq p\leq n}$ the corresponding semigroup defined by

$$\forall 0 \leq p \leq n \qquad \Gamma_{p,n} = \Gamma_{p+1,n}\Gamma_{p+1} = \Gamma_n\Gamma_{n-1}\ldots\Gamma_{p+1}$$

with the convention $\Gamma_{n,n} = Id$, the identity operator for $p = n$.
The following lemma collects some important properties of distribution flow γ_n.

Lemma 3.2 *For any* $0 \leq p \leq n$, *we have the semigroup decomposition*

$$\gamma_n = \gamma_p Q_{p,n} + \sum_{p<q\leq n} \mu_q Q_{q,n} \quad \textit{and} \quad \gamma_n = \sum_{0\leq p\leq n} \mu_p Q_{p,n} \tag{3.8}$$

In addition, we also have the following formula

$$\gamma_n(1) = \sum_{p=0}^{n} \mu_p(1) \prod_{p\leq q<n} \eta_q(G_q) \tag{3.9}$$

Proof:

The first pair of formulae are easily proved using a simple induction, and recalling that $\gamma_0 = \mu_0$. To prove the last assertion, we use an induction on the parameter $n \geq 0$. The result is obvious for $n = 0$. We also have by (1.1)

$$\gamma_{n+1}(1) = \gamma_n Q_{n+1}(1) + \mu_{n+1}(1) = \gamma_n(G_n) + \mu_{n+1}(1)$$

This implies that

$$\begin{aligned}
\gamma_{n+1}(1) &= \gamma_n(1)\ \eta_n(G_n) + \mu_{n+1}(1) \\
&= \gamma_{n-1}(1)\ \eta_{n-1}(G_{n-1})\ \eta_n(G_n) + \mu_n(1)\ \ \eta_n(G_n) + \mu_{n+1}(1) \\
&= \ldots \\
&= \gamma_0(1) \prod_{p=0}^{n} \eta_p(G_p)\ + \sum_{p=1}^{n+1} \mu_p(1) \prod_{p\leq q\leq n} \eta_q(G_q)
\end{aligned}$$

Recalling that $\gamma_0(dx_0) = \mu_0(dx_0)$, we prove (3.9). This ends the proof of the lemma. ∎

Using lemma 3.2, one proves that the semigroup $\Gamma_{p,n}$ is given by the pair of formulae described below

Proposition 3.3 *For any $0 \leq p \leq n$, we have*

$$\Gamma^1_{p,n}(m,\eta) = m\ \eta Q_{p,n}(1) + \sum_{p<q\leq n} \mu_q Q_{q,n}(1) \tag{3.10}$$

$$\Gamma^2_{p,n}(m,\eta) = \alpha_{p,n}(m,\eta)\ \ \Phi_{p,n}(\eta) + (1-\alpha_{p,n}(m,\eta)) \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}}\ \Phi_{q,n}(\overline{\mu}_q) \tag{3.11}$$

with the collection of parameters $c_{p,n} := \mu_p Q_{p,n}(1)$ and the $[0,1]$-valued parameters $\alpha_{p,n}(m,\eta)$ defined below

$$\alpha_{p,n}(m,\eta) = \frac{m\eta Q_{p,n}(1)}{m\eta Q_{p,n}(1) + \sum_{p<q\leq n} c_{q,n}} \leq \alpha^\star_{p,n}(m) := 1 \wedge \left[m \left\| \frac{Q_{p,n}(1)}{\sum_{p<q\leq n} c_{q,n}} \right\| \right] \tag{3.12}$$

One central question in the theory of branching processes is the long time behavior of the total mass process $\gamma_n(1)$. Notice that $\gamma_n(1) = \mathbb{E}(\mathcal{X}_n(1))$ is the expected size of the n-th generation. For time homogeneous models with null spontaneous branching $\mu_n = 0$, $n \geq 1$, the exponential growth of these quantities are related to the logarithmic Lyapunov exponents of the semigroup $Q_{p,n}$. The prototype of these models is the Galton-Watson branching process. In this context three typical situations may occur: 1) $\gamma_n(1)$ remains constant and equals to the initial mean number of individuals. 2) $\gamma_n(1)$ goes exponentially fast to 0, 3) $\gamma_n(1)$ grows exponentially fast to infinity,

The analysis of spatial branching model with general spontaneous birth measure is a little different. Loosely speaking, in the first situation the total mass process is generally strictly increasing; while in the second situation discussed above the additional mass injected in the system stabilizes the total mass process. Before giving further details, by lemma 3.2 we observe $\gamma_n(1) \in I_n$, for any $n \geq 0$, with the compact interval I_n defined in 2.8.

We end this section with a more precise, analysis of the effect of this additional spontaneous branching in the three typical situations (2.4).

In the further development of this section, we illustrate the stability properties of the normalized distribution flow η_n in these three situations.

1. Firstly, we observe that for unit potential functions, the total mass process $\gamma_n(1)$ grows linearly w.r.t. the time parameter. That is, we have that

$$(\forall x \in E \quad G(x) = 1) \Longrightarrow \gamma_n(1) = m_-(n) = m_+(n) = \gamma_0(1) + \mu(1)\ n \tag{3.13}$$

   Notice that the estimates in (3.12) take the following form

$$\alpha_{p,n}(\gamma_p(1),\eta_p) \leq \alpha^\star_{p,n}(\gamma_p(1)) := 1 \wedge \frac{\gamma_0(1) + \mu(1)\ p}{\mu(1)\ (n-p)} \rightarrow_{(n-p)\to\infty} 0$$

2. When the potential functions are chosen so that $g_+ < 1$, the total mass process $\gamma_n(1)$ is uniformly bounded w.r.t. the time parameter. More precisely, we have that

$$m_{-/+}(n) = g_{-/+}^n \ \gamma_0(1) + \left(1 - g_{-/+}^n\right) \ \frac{\mu(1)}{1 - g_{-/+}}$$

This yields the rather crude estimates

$$\gamma_0(1) \wedge \frac{\mu(1)}{1 - g_-} \leq \gamma_n(1) \leq \gamma_0(1) \vee \frac{\mu(1)}{1 - g_+} \tag{3.14}$$

We end this discussion with an estimate of the parameter $\alpha_{p,n}(m)$ given in (3.12). When the mixing condition $(M)_k$ stated in (3.1) is satisfied for some fixed parameters $\epsilon_p = \epsilon$, using (3.4) we prove that

$$\sum_{p<r\leq n} \frac{\mu Q_{r,n}(1)}{Q_{p,r}(Q_{r,n}(1))} \geq \frac{\epsilon\mu(1)}{\delta_k} \sum_{p<r\leq n} \frac{1}{Q_{p,r}(1)} \geq \frac{\epsilon\mu(1)}{\delta_k} \ \frac{g_+^{-(n-p)} - 1}{1 - g_+}$$

from which we conclude that for any $n > p$ and any $m \in I_p$

$$\begin{aligned}
\alpha^\star_{p,n}(m) &\leq 1 \wedge \left[m \ g_+^{(n-p)} \ \frac{\delta_k \ (1 - g_+)}{\epsilon\mu(1)(1 - g_+^{(n-p)})}\right] \\
&\leq 1 \wedge \left[m \ g_+^{(n-p)} \ \delta_k/(\epsilon\mu(1))\right] \\
&\leq 1 \wedge \left[\left(\gamma_0(1) \vee \frac{\mu(1)}{1 - g_+}\right) \ g_+^{(n-p)} \ \delta_k/(\epsilon\mu(1))\right] \to_{(n-p)\to\infty} 0
\end{aligned} \tag{3.15}$$

3. In the reverse angle, when the potential functions are strictly greater than 1, the total mass process $\gamma_n(1)$ grows exponentially fast w.r.t. the time parameter. We can easily show that

$$g_- > 1 \Longrightarrow \gamma_n(1) \geq m_-(n) = \gamma_0(1) \ g_-^n + \mu(1) \ \frac{g_-^n - 1}{g_- - 1} \tag{3.16}$$

3.2 Asymptotic properties

This section is concerned with the long time behavior of the semigroups $\Gamma_{p,n}$ in the three situations discussed in (3.13), (3.14), and (3.16). Our results are summarized in theorem 2.1. We consider time homogeneous models $(E_n, G_n, M_n, \mu_n) = (E, G, M, \mu)$.

1. For unit potential functions $G(x) = 1$, $x \in E$, we have seen in (3.13) that the total mass process is explicitly known, and it is given by $\gamma_n(1) = \gamma_0(1) + \mu(1) \ n$. In this particular situation, the time inhomogeneous Markov transitions $M_{n,(\gamma_{n-1}(1),\eta_{n-1})} := \overline{M}_n$ introduced in (3.5) are given by

$$\overline{M}_n(x, dy) = \left(1 - \frac{\mu(1)}{\gamma_0(1) + n\mu(1)}\right) \ M(x, dy) + \frac{\mu(1)}{\gamma_0(1) + n\mu(1)} \ \overline{\mu}(dy)$$

This shows that $\eta_n = \mathrm{Law}(\overline{X}_n)$ can be interpreted as the distribution of the states $\overline{X}_n$ of a time inhomogeneous Markov chain with transitions $\overline{M}_n$, and initial distribution η_0. If we choose in (1.6) $K_{n+1,(\gamma_n(1),\eta_n)} = \overline{M}_{n+1}$, the N-particle model (1.7) reduces to a series of N independent copies of $\overline{X}_n$. In this situation, the mapping $\Gamma^2_{0,n}$ is explicitly known, and it is given by

$$\Gamma^2_{0,n}(\gamma_0(1), \eta_0) := \frac{\gamma_0(1)}{\gamma_0(1) + n\mu(1)} \ \eta_0 M^n + \frac{n\mu(1)}{\gamma_0(1) + n\mu(1)} \ \frac{1}{n} \sum_{0\leq p<n} \overline{\mu} M^p$$

The above formula shows that for a large time horizon n, the normalized distribution flow η_n is almost equal to $\frac{1}{n}\sum_{0\leq p<n}\overline{\mu}M^p$. Let us assume that the Markov M is chosen so that (2.6) is satisfied for some invariant measure $\eta_\infty = \eta_\infty M$. In this case, for any starting measure γ_0, we have

$$\|\eta_n - \eta_\infty\|_{\rm tv} \leq \frac{\gamma_0(1)}{\gamma_0(1)+n\mu(1)}\ \tau_n + \frac{n\mu(1)}{\gamma_0(1)+n\mu(1)}\quad \frac{1}{n}\sum_{0\leq p<n}\tau_p = O\left(\frac{1}{n}\right)$$

with $\tau_n = \sup_{x\in E}\|M^n(x,.)-\eta_\infty\|_{\rm tv}$. For instance, suppose the mixing condition $(M)_k$ presented in (3.1) is met for some parameters $k\geq 1$, and some $\epsilon > 0$. In this case, the above upper bound is satisfied with $\tau_n = (1-\epsilon)^{\lfloor n/k\rfloor}$.

2. We examine the situation where $g_+ < 1$. In this situation, the pair of measures (2.7) are well defined. Furthermore, for any $f\in\mathcal{B}(E)$ with $\|f\|\leq 1$, we have the estimates

$$\begin{aligned}|\gamma_n(f)-\gamma_\infty(f)| &\leq \gamma_0(1)\ \eta_0 Q^n(1) + \sum_{p\geq n}\mu Q^p(1)\\ &\leq g_+^n\ [\gamma_0(1)+\mu(1)/(1-g_+)] \longrightarrow_{n\to\infty} 0\end{aligned}$$

In addition, using the fact that $\gamma_n(1)\geq\mu(1)$, for any $f\in\mathrm{Osc}_1(E)$ we find that

$$\begin{aligned}|\eta_n(f)-\eta_\infty(f)| &\leq \frac{1}{\gamma_n(1)}\,|\gamma_n[f-\eta_\infty(f)]-\gamma_\infty[f-\eta_\infty(f)]|\\ &\leq g_+^n\ [\gamma_0(1)/\mu(1)+1/(1-g_+)] \longrightarrow_{n\to\infty} 0\end{aligned}$$

3. We examine the situation where $g_- > 1$. We further assume that the mixing condition $(M)_k$ presented in (3.1) is met for some parameters $k\geq 1$, and some $\epsilon>0$. In this situation, it is well known that the mapping $\Phi = \Phi_{n-1,n}$ introduced in (2.2) has a unique fixed point $\eta_\infty = \Phi(\eta_\infty)$, and for any initial distribution η_0, we have

$$\|\Phi_{0,n}(\eta_0)-\eta_\infty\|_{\rm tv} \leq a\ e^{-\lambda\ n} \tag{3.17}$$

with

$$\lambda = -\frac{1}{k}\log\left(1-\epsilon^2/\delta_{0,k-1}\right) \quad\text{and}\quad a = 1/\left(1-\epsilon^2/\delta_{0,k-1}\right)$$

as well as

$$\sup_{\eta\in\mathcal{P}(E)}\left|\frac{1}{n}\log\eta Q^n(1) - \log\eta_\infty(G)\right| \leq b/n \tag{3.18}$$

for some finite constant $b<\infty$. For a more thorough discussion on the stability properties of the semigroup $\Phi_{0,n}$ and the limiting measures η_∞, we refer the reader to [4]. Our next objective is to transfer these stability properties to the one of the flow η_n. Firstly, using (3.18), we readily prove that

$$\lim_{n\to\infty}\frac{1}{n}\log\gamma_n(1) = \log\eta_\infty(G)$$

Next, we simplify the notation, and we set $\alpha_n := \alpha_{0,n}(\gamma_0(1),\eta_0)$ and $c_n := c_{0,n}$. Using (3.11), we find that

$$a^{-1}\ \|\eta_n-\eta_\infty\|_{\rm tv} \leq \alpha_n\ e^{-\lambda n} + (1-\alpha_n)\sum_{0\leq p<n}\frac{c_p}{\sum_{0\leq q<n}c_q}\ e^{-\lambda p}$$

for any $n>1$. Recalling that

$$\mu(1)\ g_-^p \leq c_p = \mu Q^p(1) \leq \mu(1)\ g_+^p$$

we find that

$$\begin{aligned}
\sum_{0\leq p<n} \frac{c_p}{\sum_{1\leq q<n} c_q}\, e^{-\lambda p} &\leq \frac{1}{\left[\sum_{0\leq q<n} c_q\right]^{1/r}} \left[\sum_{0\leq p<n} c_p e^{-\lambda pr}\right]^{1/r} \\
&\leq \frac{1}{\left[\sum_{0\leq q<n} g_-^q\right]^{1/r}} \left[\sum_{0\leq p<n} (e^{-\lambda r} g_+)^p\right]^{1/r} \qquad (3.19)
\end{aligned}$$

for any $r \geq 1$. We conclude that

$$r > \frac{1}{\lambda}\, \log g_+ \Longrightarrow \sum_{0\leq p<n} \frac{c_p}{\sum_{0\leq q<n} c_q}\, e^{-\lambda p} \leq g_-^{-(n-1)/r}/(1-e^{-\lambda r} g_+)^{1/r}$$

and therefore

$$a^{-1}\, \|\eta_n - \eta_\infty\|_{\rm tv} \leq e^{-\lambda n} + g_-^{-(n-1)/r}\ /(1-e^{-\lambda r} g_+)^{1/r} \to_{n\to\infty} 0$$

3.3 Stability and Lipschitz regularity properties

We describe in this section a framework that allows to transfer the regularity properties of the Feynman-Kac semigroups $\Phi_{p,n}$ introduced in (2.2) to the ones of the semigroup $\Gamma_{p,n}$ of the flow $(\gamma_n(1), \eta_n)$. Before proceeding we recall a more or less well known lemma that provides some weak Lipschitz type inequalities for the Feynman-Kac semigroup $\Phi_{p,n}$ in terms of the Dobrushin contraction coefficient associated with the Markov transitions $P_{p,n}$ introduced in (2.2). The details of the proof of this result can be found in [4] or in [5] (see Lemma 4.4. in [5], or proposition 4.3.7 on page 146 in [4]).

Lemma 3.4 ([5]) *For any $0 \leq p \leq n$, any $\eta, \mu \in \mathcal{P}(E_p)$ and any $f \in \mathrm{Osc}_1(E_n)$, we have*

$$|[\Phi_{p,n}(\mu) - \Phi_{p,n}(\eta)]\,(f)| \leq 2\ q_{p,n}^2\ \beta(P_{p,n})\ \ |(\mu-\eta)\mathcal{D}_{p,n,\eta}(f)| \qquad (3.20)$$

with a collection of functions $\mathcal{D}_{p,n,\eta}(f) \in \mathrm{Osc}_1(E_p)$, whose values only depends on the parameters (p, n, η).

Proposition 3.5 *For any $0 \leq p \leq n$, any $\eta, \eta' \in \mathcal{P}(E_p)$ and any $f \in \mathrm{Osc}_1(E_n)$, there exits a a collection of functions $\mathcal{D}_{p,n,\eta'}(f) \in \mathrm{Osc}_1(E_p)$ whose values only depends on the parameters (p, n, η), and such that for any $m \in I_p$ we have*

$$\begin{aligned}
&\left|\left[\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m,\eta')\right](f)\right| \\
&\leq 2\ \alpha^\star_{p,n}\ q_{p,n}\left[q_{p,n}\ \beta(P_{p,n})\ \ |(\eta-\eta')\mathcal{D}_{p,n,\eta'}(f)| + \beta_{p,n}\,|(\eta-\eta')h_{p,n,\eta'}|\right]
\end{aligned} \qquad (3.21)$$

with the collection of functions $h_{p,n,\eta'} = \frac{1}{2q_{p,n}}\ \frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)} \in \mathrm{Osc}_1(E_p)$, and the sequence of parameters $\epsilon_{p,n}$ and $\beta_{p,n}$ defined below

$$\alpha^\star_{p,n} := \alpha^\star_{p,n}(m_+(p)) \quad \textit{and} \quad \beta_{p,n} := \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}}\ \beta(P_{q,n}) \qquad (3.22)$$

Before getting into the details of the proof of proposition 3.5, we illustrate the impact of these weak functional inequalities for time homogeneous models $(E_n, G_n, M_n, \mu_n) = (E, G, M, \mu)$, in the three situations discussed in (3.13), (3.14), and (3.16).

1. Firstly, we observe that for unit potential functions $G(x) = 1$, $x \in E$, we have

$$\Phi_{p,n}(\eta) = \eta M^{(n-p)}, \quad h_{p,n,\eta'} = 1/2 \quad c_{p,n} = \mu(1) \quad q_{p,n} = 1 \quad \alpha^\star_{p,n} \leq 1$$

Let us assume that $\beta(M^n) \leq ae^{-\lambda n}$, for any $n \geq 0$, and for some finite constants $a < \infty$, and $0 < \lambda < \infty$. In this situation, using (3.21) we prove that

$$\left|\left[\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m,\eta')\right](f)\right| \leq 2ae^{-\lambda(n-p)} \ |(\mu-\eta)\mathcal{D}_{p,n,\eta'}(f)|$$

2. We examine the situation where $g_+ < 1$. When the mixing condition $(M)_k$ stated in (3.1) is satisfied for some fixed parameters $\epsilon_p = \epsilon$, we have seen in (3.15) that

$$\sup_{m \in I_p} \alpha^\star_{p,n}(m) \leq 1 \wedge \left(d \ g_+^{(n-p)}\right) \quad \text{with} \quad d = \left((\gamma_0(1)/\mu(1)) \vee (1-g_+)^{-1}\right) \delta_{0,k}\epsilon^{-1}$$

Furthermore, using the estimates given in (3.3) and (3.4), we also have that

$$q_{p,n} \leq \delta_k/\epsilon \qquad \beta_{p,n} \leq 1 \quad \text{and} \quad \beta(P_{p,n}) \leq a \ e^{-\lambda \ (n-p)} \quad \text{with } (a,\lambda) \text{ given in (3.17)}$$

In this situation, using (3.21) we prove that

$$\begin{aligned}&\left|\left[\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m,\eta')\right](f)\right| \\ &\leq 2 \ \left[1 \wedge \left(d \ g_+^{(n-p)}\right)\right] \ (\delta_k/\epsilon) \left[(\delta_k/\epsilon) \ a \ e^{-\lambda(n-p)} \ |(\mu-\eta)\mathcal{D}_{p,n,\eta'}(f)| + |(\mu-\eta)h_{p,n,\eta'}|\right]\end{aligned}$$

Notice that for $(n-p) \geq \log(d)/\log(1/g_+)$, this yields

$$\begin{aligned}&\left|\left[\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m,\eta')\right](f)\right| \\ &\leq a_0 \ e^{-\lambda_0(n-p)} \ |(\mu-\eta)\mathcal{D}_{p,n,\eta'}(f)| + a_1 \ e^{-\lambda_1(n-p)} \,|(\mu-\eta)h_{p,n,\eta'}|\end{aligned}$$

with

$$a_0 = 2ad(\delta_k/\epsilon)^2 \quad a_1 = 2d(\delta_k/\epsilon) \quad \lambda_0 = \lambda + \log(1/g_+) \quad \text{and} \quad \lambda_1 = \log(1/g_+)$$

3. Finally, we examine the situation where $g_- > 1$. We further assume that the mixing condition $(M)_k$ presented in (3.1) is met for some parameters $k \geq 1$, and some $\epsilon > 0$. In this case, we use the fact that

$$\alpha^\star_{p,n} \leq 1 \quad q_{p,n} \leq \delta_k/\epsilon \quad \text{and} \quad \beta(P_{p,n}) \leq a \ e^{-\lambda(n-p)} \quad \text{with } (a,\lambda) \text{ given in (3.17)}$$

Arguing as in (3.19), we prove that for any $r > \frac{1}{\lambda} \ \log g_+$

$$\beta_{p,n} \leq g_-^{-(n-p-1)/r}/(1-e^{-\lambda r}g_+)^{1/r}$$

from which we conclude that

$$\begin{aligned}&\left|\left[\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m,\eta')\right](f)\right| \\ &\leq a_0 \ e^{-\lambda_0(n-p)} \ |(\mu-\eta)\mathcal{D}_{p,n,\eta'}(f)| + a_1 \ e^{-\lambda_1(n-p)} \,|(\mu-\eta)h_{p,n,\eta'}|\end{aligned}$$

with

$$a_0 = 2a(\delta_k/\epsilon)^2 \quad a_1 = 2g_-^r(\delta_k/\epsilon)/(1-e^{-\lambda r}g_+)^{1/r} \quad \lambda_0 = \lambda \quad \text{and} \quad \lambda_1 = \log(g_-)$$

Now, we come to the proof of proposition 3.5.
Proof of proposition 3.5:
Firstly, we observe that

$$\begin{aligned}
&\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m',\eta') \\
&= \alpha_{p,n}(m,\eta) \; \left[\Phi_{p,n}(\eta) - \sum_{p<q\leq n} \tfrac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \; \Phi_{q,n}(\overline{\mu}_q)\right] \\
&\qquad -\alpha_{p,n}(m',\eta') \; \left[\Phi_{p,n}(\eta') - \sum_{p<q\leq n} \tfrac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \; \Phi_{q,n}(\overline{\mu}_q)\right]
\end{aligned}$$

Using the following decomposition

$$ab - a'b' = a'(b-b') + (a-a')b' + (a-a')(b-b') \tag{3.23}$$

which is valid for any $a, a', b, b' \in \mathbb{R}$, we prove that

$$\begin{aligned}
&\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m',\eta') \\
&= \alpha_{p,n}(m',\eta') \left[\Phi_{p,n}(\eta) - \Phi_{p,n}(\eta')\right] \\
&\quad + \left[\Phi_{p,n}(\eta') - \sum_{p<q\leq n} \tfrac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \; \Phi_{q,n}(\overline{\mu}_q)\right] \left[\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m',\eta')\right] \\
&\qquad + \left[\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m',\eta')\right] \left[\Phi_{p,n}(\eta) - \Phi_{p,n}(\eta')\right]
\end{aligned} \tag{3.24}$$

For $m = m'$, using (3.24) we find that

$$\begin{aligned}
&\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m,\eta') \\
&= \alpha_{p,n}(m,\eta) \; \left[\Phi_{p,n}(\eta) - \Phi_{p,n}(\eta')\right] \\
&\quad + \left[\Phi_{p,n}(\eta') - \sum_{p<q\leq n} \tfrac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \; \Phi_{q,n}(\overline{\mu}_q)\right] \left[\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m,\eta')\right]
\end{aligned}$$

We also notice that

$$\alpha_{p,n}(m,\eta) = \frac{1}{1 + \mu_{p,n}/[m\eta Q_{p,n}(1)]}$$

from which we easily prove that

$$\begin{aligned}
&\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m',\eta') \\
&= \tfrac{\mu_{p,n}}{\mu_{p,n} + m\eta Q_{p,n}(1)} \tfrac{1}{\mu_{p,n} + m'\eta' Q_{p,n}(1)} \left[m\eta Q_{p,n}(1) - m'\eta' Q_{p,n}(1)\right]
\end{aligned}$$

and therefore

$$\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m,\eta') = \left(\alpha_{p,n}(m,\eta')\left(1 - \alpha_{p,n}(m,\eta)\right)\right) \; [\eta - \eta'] \left(\frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)}\right)$$

The proof of $\alpha_{p,n}(m,\eta) \leq \alpha^{\star}_{p,n}(m)$ is elementary. From the above decomposition, we prove the following upper bounds

$$|\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m,\eta')| \leq \alpha^{\star}_{p,n}(m) \; \left|[\eta - \eta'] \left(\frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)}\right)\right|$$

and

$$\left|\left[\Gamma^2_{p,n}(m,\eta)-\Gamma^2_{p,n}(m,\eta')\right](f)\right|$$

$$\leq \alpha^{\star}_{p,n}(m)\left[\left|\left[\Phi_{p,n}(\eta)-\Phi_{p,n}(\eta')\right](f)\right|\right.$$

$$\left.+\left|[\eta-\eta']\left(\frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)}\right)\right|\ \left|\sum_{p<q\leq n}\frac{c_{q,n}}{\sum_{p<r\leq n}c_{r,n}}\ \left[\Phi_{q,n}(\overline{\mu}_q)-\Phi_{q,n}\left(\Phi_{p,q}(\eta')\right)\right](f)\right|\right]$$

This yields

$$\left|\left[\Gamma^2_{p,n}(m,\eta)-\Gamma^2_{p,n}(m,\eta')\right](f)\right|$$

$$\leq \alpha^{\star}_{p,n}(m)\left[\left|\left[\Phi_{p,n}(\eta)-\Phi_{p,n}(\eta')\right](f)\right|+\beta_{p,n}\ \left|[\eta-\eta']\left(\frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)}\right)\right|\right]$$

The last formula comes from the fact that

$$\beta(P_{q,n}):=\sup_{\nu,\nu'\in\mathcal{P}(E_q)}\|\Phi_{q,n}(\nu)-\Phi_{q,n}(\nu')\|_{\rm tv}$$

The proof of this result can be found in [4] (proposition 4.3.1 on page 134). The end of the proof is now a direct consequence of lemma 3.4. This ends the proof of the proposition. ■

4 Mean field particle models

4.1 McKean particle interpretation models

In proposition 3.1, the evolution equation (3.5) of the flow of measures $\eta_n \rightsquigarrow \eta_{n+1}$ is a combination of an updating type transition $\eta_n \rightsquigarrow \Psi_{G_n}(\eta_n)$ and an integral transformation w.r.t. a Markov transition $M_{n+1,(\gamma_n(1),\eta_n)}$ that depends on the current mass process $\gamma_n(1)$ and the current normalized distribution η_n. The operator $M_{n+1,(\gamma_n(1),\eta_n)}$ defined in (3.6) is a mixture of the Markov transition M_{n+1} and the spontaneous birth measure $\overline{\mu}_{n+1}$. We let S_{n,η_n} any Markov transition from E_n into itself satisfying the compatibility condition

$$\Psi_{G_n}(\eta_n)=\eta_n S_{n,\eta_n}$$

The choice of these transitions is not unique. We can choose for instance one of the collection of transitions presented in (1.8) and (1.10). Further examples of McKean acceptance-rejection type transitions can also be found in section 2.5.3 in [4] By construction, we have the recursive formula

$$\eta_{n+1}=\eta_n K_{n+1,(\gamma_n(1),\eta_n)}\quad \text{with}\quad K_{n+1,(\gamma_n(1),\eta_n)}=S_{n,\eta_n}M_{n+1,(\gamma_n(1),\eta_n)} \tag{4.1}$$

with the auxiliary total mass evolution equation

$$\gamma_{n+1}(1)=\gamma_n(1)\ \eta_n(G_n)+\mu_{n+1}(1) \tag{4.2}$$

As we already mention in the introduction, we notice that the normalized distribution flow η_n can be interpreted as the distributions of the states $\overline{X}_n$ of a non linear Markov chain defined by the elementary transitions

$$\mathbb{P}\left(\overline{X}_{n+1}\in dx\mid \overline{X}_n\right)=K_{n,(\gamma_n(1),\eta_n)}\left(\overline{X}_n,dx\right)\quad\text{with}\quad \eta_n=\mathrm{Law}(\overline{X}_n)$$

Next, we define the mean field particle interpretations of the flow $(\gamma_n(1),\eta_n)$ given in (4.1) and (4.2). Firstly, mimicking formula (4.2) we set

$$\gamma^N_{n+1}(1):=\gamma^N_n(1)\ \eta^N_n(G_n)+\mu_{n+1}(1)\quad\text{and}\quad \gamma^N_n(f)=\gamma^N_n(1)\ \times\ \eta^N_n(f)$$

for any $f \in \mathcal{B}(E_n)$, with the initial measure $\gamma_0^N = \gamma_0$. It is important to notice that

$$\gamma_n^N(1) = \gamma_0(1) \prod_{0 \leq q < n} \eta_q^N(G_q) + \sum_{p=1}^{n} \mu_p(1) \prod_{p \leq q < n} \eta_q^N(G_q) \Longrightarrow \gamma_n^N(1) \in I_n$$

The mean field particle interpretation of the nonlinear measure valued model (4.1) is an E_n^N-valued Markov chain ξ_n with elementary transitions defined in (1.7) and (4.1). By construction, the particle evolution is a simple combination of a selection and a mutation genetic type transition

$$\xi_n \rightsquigarrow \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1 \leq i \leq N} \rightsquigarrow \xi_{n+1}$$

During the selection transitions $\xi_n \rightsquigarrow \widehat{\xi}_n$, each particle $\xi_n^i \rightsquigarrow \widehat{\xi}_n^i$ evolves according to the selection type transition $S_{n,\eta_n^N}(\xi_n^i, dx)$. During the mutation stage, each of the selected particles $\widehat{\xi}_n^i \rightsquigarrow \xi_{n+1}^i$ evolves according to the Markov transition

$$M_{n+1,(\gamma_n^N(1),\eta_n^N)}(x,dy) := \alpha_n\left(\gamma_n^N(1),\eta_n^N\right) M_{n+1}(x,dy) + \left(1 - \alpha_n\left(\gamma_n^N(1),\eta_n^N\right)\right)\ \overline{\mu}_{n+1}(dy)$$

4.2 Asymptotic behavior

This section is mainly concerned with the proof of theorem 2.2. In the first section, section 4.2.1, we discuss the unibiasedness property of the particle measures γ_n^N and their convergence properties towards γ_n, as the number of particles N tends to infinity. We mention that the proof of the non asymptotic variance estimates (2.11) is simpler than the one provided in a recent article by the second author with F. Cerou and A. Guyader [3]. Section 4.2.2 is concerned with the convergence and the fluctuations of the occupation measures η_n^N around their limiting measures η_n.

4.2.1 Unnormalized distributions

We start this section with a simple unbiasedness property.

Proposition 4.1 *For any $0 \leq p \leq n$, and any $f \in \mathcal{B}(E_n)$, we have*

$$\mathbb{E}\left(\gamma_{n+1}^N(f) \,\middle|\, \mathcal{F}_p^{(N)}\right) = \gamma_p^N Q_{p,n+1}(f) + \sum_{p < q \leq n+1} \mu_q Q_{q,n+1}(f) \tag{4.3}$$

In particular, we have the unbiasedness property: $\mathbb{E}\left(\gamma_n^N(f)\right) = \gamma_n(f)$.

Proof:
By construction of the particle model, for any $f \in \mathcal{B}(E_n)$ we have

$$\mathbb{E}\left(\eta_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) = \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)}(f) = \Gamma_{n+1}^2\left(\gamma_n^N(1),\eta_n^N\right)(f)$$

with the second component Γ_{n+1}^2 of the one step transformation Γ_{n+1} introduced in 3.7. Using the fact that

$$\Gamma_{n+1}^2\left(\gamma_n^N(1),\eta_n^N\right)(f) = \frac{\gamma_n^N(1)\ \eta_n^N(Q_{n+1}(f)) + \mu_{n+1}(f)}{\gamma_n^N(1)\ \eta_n^N(Q_{n+1}(1)) + \mu_{n+1}(1)} = \frac{\gamma_n^N(Q_{n+1}(f)) + \mu_{n+1}(f)}{\gamma_n^N(Q_{n+1}(1)) + \mu_{n+1}(1)}$$

and

$$\gamma_{n+1}^N(1) = \gamma_n^N(1)\ \eta_n^N(G_n) + \mu_{n+1}(1) = \gamma_n^N(Q_{n+1}(1)) + \mu_{n+1}(1)$$

we prove that

$$\begin{aligned}\mathbb{E}\left(\gamma_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) &= \mathbb{E}\left(\gamma_{n+1}^N(1)\ \eta_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) = \gamma_{n+1}^N(1)\ \mathbb{E}\left(\eta_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) \\ &= \gamma_n^N(Q_{n+1}(f)) + \mu_{n+1}(f)\end{aligned}$$

This also implies that

$$\begin{aligned}\mathbb{E}\left(\gamma^N_{n+1}(f)\,\middle|\,\mathcal{F}^{(N)}_{n-1}\right) &= \mathbb{E}\left(\gamma^N_n(Q_{n+1}(f))\,\middle|\,\mathcal{F}^{(N)}_{n-1}\right)+\mu_{n+1}(f)\\ &= \gamma^N_{n-1}(Q_nQ_{n+1}(f))+\mu_n(Q_{n+1}(f))+\mu_{n+1}(f)\end{aligned}$$

Iterating the argument one proves (4.3). The end of the proof is now clear. ■

The next theorem provides a key martingale decomposition and a rather crude non asymptotic variance estimate.

Theorem 4.2 *For any $n\geq 0$ and any function $f\in\mathcal{B}(E_n)$, we have the decomposition*

$$\sqrt{N}\ \left[\gamma^N_n-\gamma_n\right](f) = \sum_{p=0}^n \gamma^N_p(1)\ W^N_p(Q_{p,n}(f)) \tag{4.4}$$

In addition, if the mixing condition $(M)_k$ presented in (3.1) is met for some parameters $k\geq 1$, and some $\epsilon>0$, then we have for any $N>1$ and any $n\geq 1$

$$\mathbb{E}\left(\left[\frac{\gamma^N_n(1)}{\gamma_n(1)}-1\right]^2\right)\leq\frac{n+1}{N-1}\,\frac{\delta_k^2}{\epsilon^2}\left(1+\frac{\delta_k^2}{\epsilon^2(N-1)}\right)^{n-1} \tag{4.5}$$

Before presenting the proof of this theorem, it is convenient to make a couple of comments. On the one hand, we observe that the unbiasedness property follows directly from the decomposition (4.4). On the other hand, using Kintchine's inequality, for any $r\geq 1$, $p\geq 1$, and any $f\in\mathrm{Osc}_1(E_n)$ we have the almost sure estimates

$$\sqrt{N}\ \mathbb{E}\left(\left|W^N_p(f)\right|^r\ \middle|\mathcal{F}^{(N)}_{p-1}\right)^{\frac{1}{r}}\leq a_r$$

A detailed proof of these estimates can be found in [4], see also lemma 7.2 in [1] for a simpler proof by induction on the parameter N. From this elementary observation, and recalling that $\gamma^N_n(1)\in I_n$ for any $n\geq 0$, we find that

$$\sqrt{N}\ \mathbb{E}\left(\left|\left[\gamma^N_n-\gamma_n\right](f)\right|^r\right)^{\frac{1}{r}}\leq a_r\ b_n$$

for some finite constant b_n whose values only depend on the time parameter.

Now, we present the proof of theorem 4.2.

Proof of theorem 4.2:

We use the decomposition:

$$\gamma^N_{n+1}(f)-\gamma_{n+1}(f)=\left[\gamma^N_{n+1}(f)-\mathbb{E}\left(\gamma^N_{n+1}(f)\,\middle|\,\mathcal{F}^{(N)}_n\right)\right]+\left[\mathbb{E}\left(\gamma^N_{n+1}(f)\,\middle|\,\mathcal{F}^{(N)}_n\right)-\gamma_{n+1}(f)\right]$$

By (4.3), we find that

$$\gamma^N_{n+1}(f)-\mathbb{E}\left(\gamma^N_{n+1}(f)\,\middle|\,\mathcal{F}^{(N)}_n\right)=\gamma^N_{n+1}(f)-\left[\gamma^N_n(Q_{n+1}(f))+\mu_{n+1}(f)\right]$$

Since we have

$$\begin{aligned}\gamma^N_n(Q_{n+1}(1))+\mu_{n+1}(1) &= \gamma^N_n(G_n)+\mu_{n+1}(1)\\ &= \gamma^N_n(1)\ \eta^N_n(G_n)+\mu_{n+1}(1)=\gamma^N_{n+1}(1)\end{aligned}$$

this implies that

$$\begin{aligned}\gamma^N_{n+1}(f)-\left[\gamma^N_n(Q_{n+1}(f))+\mu_{n+1}(f)\right] &= \gamma^N_{n+1}(1)\ \left[\eta^N_{n+1}(f)-\frac{\left[\gamma^N_n(Q_{n+1}(f))+\mu_{n+1}(f)\right]}{\left[\gamma^N_n(Q_{n+1}(1))+\mu_{n+1}(1)\right]}\right]\\ &= \gamma^N_{n+1}(1)\ \left[\eta^N_{n+1}(f)-\eta^N_nK_{n+1,(\gamma^N_n(1),\eta^N_n)}(f)\right]\end{aligned}$$

and therefore

$$\gamma_{n+1}^N(f) - \mathbb{E}\left(\gamma_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) = \gamma_{n+1}^N(1) \;\; \left[\eta_{n+1}^N(f) - \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)}(f)\right]$$

Finally, we observe that

$$\mathbb{E}\left(\gamma_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) - \gamma_{n+1}(f) = \gamma_n^N(Q_{n+1}(f)) - \gamma_n(Q_{n+1}(f))$$

from which we find the recursive formula

$$\left[\gamma_{n+1}^N - \gamma_{n+1}\right](f) = \gamma_{n+1}^N(1) \;\; \left[\eta_{n+1}^N - \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)}\right](f) + \left[\gamma_n^N - \gamma_n\right](Q_{n+1}(f))$$

The end of the proof of (4.4) is now obtained by a simple induction on the parameter n.

Now, we come to the proof of (4.5). Using the fact that

$$\begin{aligned}\mathbb{E}\left(\gamma_p^N(1)W_p^N(f^{(1)})\;\gamma_q^N(1)W_q^N(f^{(2)})\right) &= \mathbb{E}\left(\gamma_p^N(1)\gamma_q^N(1)W_p^N(f^{(1)})\;\mathbb{E}\left(W_q^N(f^{(2)}) \mid \mathcal{F}_{q-1}^N\right)\right)\\ &= 0\end{aligned}$$

for any $0 \leq p < q \leq n$, and any $f^{(1)} \in \mathcal{B}(E_p)$, and $f^{(2)} \in \mathcal{B}(E_q)$, we prove that

$$N\;\mathbb{E}\left(\left[\gamma_n^N(1) - \gamma_n(1)\right]^2\right) = \sum_{p=0}^n \mathbb{E}\left(\gamma_p^N(1)^2\;\mathbb{E}\left(W_p^N(Q_{p,n}(1))^2|\mathcal{F}_{p-1}^N\right)\right)$$

Notice that

$$\frac{1}{\gamma_n(1)^2} = \frac{1}{\gamma_p(1)^2}\;\frac{1}{\eta_p(Q_{p,n}(1))^2}\;\left(\frac{\gamma_p(Q_{p,n}(1))}{\gamma_n(1)}\right)^2 \leq \alpha_{p,n}^\star(\gamma_p(1))^2\frac{1}{\gamma_p(1)^2}\;\frac{1}{\eta_p(Q_{p,n}(1))^2} \tag{4.6}$$

The r.h.s. estimate comes from the fact that

$$\frac{\gamma_p(Q_{p,n}(1))}{\gamma_n(1)} = \frac{\gamma_p(1)\;\eta_p(Q_{p,n}(1))}{\gamma_p(1)\;\eta_p(Q_{p,n}(1)) + \sum_{p<q\leq n}\mu_q Q_{q,n}(1)} = \alpha_{p,n}\left(\gamma_p(1),\eta_p\right) \leq \alpha_{p,n}^\star(\gamma_p(1))$$

Using the above decompositions, we readily prove that

$$N\;\mathbb{E}\left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1\right]^2\right) \leq \sum_{p=0}^n \alpha_{p,n}^\star(\gamma_p(1))^2\;\mathbb{E}\left(\left(\frac{\gamma_p^N(1)}{\gamma_p(1)}\right)^2\;\mathbb{E}\left(W_p^N(\overline{Q}_{p,n}(1))^2|\mathcal{F}_{p-1}^N\right)\right)$$

with

$$\overline{Q}_{p,n}(1) = \overline{Q}_{p,n}(1)/\eta_p(Q_{p,n}(1)) \leq q_{p,n}$$

We set

$$U_n^N := \mathbb{E}\left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1\right]^2\right) \quad \text{then we find that} \quad N\;U_n^N \leq a_n + \sum_{p=0}^n b_{p,n}\;U_p^N$$

with the parameters

$$a_n := \sum_{p=0}^n \left(q_{p,n}\alpha_{p,n}^\star(\gamma_p(1))\right)^2 \quad \text{and} \quad b_{p,n} := \left(q_{p,n}\alpha_{p,n}^\star(\gamma_p(1))\right)^2$$

Using the fact that $b_{n,n} \leq 1$, we prove the following recursive equation

$$U_n^N \leq a_n^N + \sum_{0\leq p<n} b_{p,n}^N\;U_p^N \quad \text{with} \quad a_n^N := \frac{a_n}{N-1} \quad \text{and} \quad b_{p,n}^N := \frac{b_{p,n}}{N-1}$$

Using an elementary proof by induction on the time horizon n, we prove the following formula:

$$U_n^N \leq \left[\sum_{p=1}^n a_p^N \sum_{e\in\langle p,n\rangle} b^N(e)\right] + \left[\sum_{e\in\langle 0,n\rangle} b^N(e)\right]\ U_0^N$$

In the above display, $\langle p,n\rangle$ stands for the set of all integer valued paths $e=(e(l))_{0\leq l\leq k}$ of a given length k from p to n

$$e_0=p<e_1<\ldots<e_{k-1}<e_k=n \quad\text{and}\quad b^N(e)=\prod_{1\leq l\leq k} b^N_{e(l-1),e(l)}$$

We have also used the convention $b^N(\emptyset)=\prod_\emptyset=1$ and $\langle n,n\rangle=\{\emptyset\}$, for $p=n$. Recalling that $\gamma_0^N=\gamma_0$, we conclude that

$$U_n^N \leq \sum_{p=1}^n a_p^N \sum_{e\in\langle p,n\rangle} b^N(e)$$

We further assume that the mixing condition $(M)_k$ presented in (3.1) is met for some parameters $k\geq 1$, and some $\epsilon>0$. In this case, we use the fact that

$$\alpha^\star_{p,n}\leq 1 \quad\text{and}\quad q_{p,n}\leq \delta_k/\epsilon$$

to prove that

$$\sup_{0\leq p\leq n} a_p^N \leq (n+1)\ (\delta_k/\epsilon)^2/(N-1) \quad\text{and}\quad \sup_{0\leq p\leq n} b^N_{p,n}\leq (\delta_k/\epsilon)^2/(N-1)$$

Using these rather crude estimates, we find that

$$U_n^N \leq a_n^N + \sum_{0<p<n} a_p^N \sum_{l=1}^{(n-p)} \left(\begin{array}{c} n-p-1\\ l-1\end{array}\right)\left(\frac{\delta_k^2}{\epsilon^2(N-1)}\right)^l$$

and therefore

$$\begin{aligned} U_n^N &\leq \frac{(n+1)}{(N-1)}\ \frac{\delta_k^2}{\epsilon^2}\left(1+\frac{\delta_k^2}{\epsilon^2(N-1)}\sum_{0<p<n}\left(1+\left(\frac{\delta_k^2}{\epsilon^2(N-1)}\right)\right)^{n-p-1}\right)\\ &= \frac{(n+1)}{(N-1)}\ \frac{\delta_k^2}{\epsilon^2}\left(1+\frac{\delta_k^2}{\epsilon^2(N-1)}\right)^{n-1}\end{aligned}$$

This ends the proof of the theorem. ■

4.2.2 Normalized distributions

This section is mainly concerned with the proof of the $\mathbb{L}_r$-mean error estimates stated in (2.9). We use the decomposition

$$\begin{aligned}\left(\gamma_n^N(1),\eta_n^N\right)-(\gamma_n(1),\eta_n) &= \left[\Gamma_{0,n}\left(\gamma_0^N(1),\eta_0^N\right)-\Gamma_{0,n}\left(\gamma_0(1),\eta_0\right)\right]\\ &\quad+\sum_{p=1}^n\left[\Gamma_{p,n}\left(\gamma_p^N(1),\eta_p^N\right)-\Gamma_{p-1,n}\left(\gamma_{p-1}^N(1),\eta_{p-1}^N\right)\right] \qquad (4.7)\end{aligned}$$

to prove that

$$\begin{aligned}&\eta_n^N-\eta_n\\ &=\left[\Gamma^2_{0,n}\left(\gamma_0^N(1),\eta_0^N\right)-\Gamma^2_{0,n}\left(\gamma_0(1),\eta_0\right)\right]+\textstyle\sum_{p=1}^n\left[\Gamma^2_{p,n}\left(\gamma_p^N(1),\eta_p^N\right)-\Gamma^2_{p-1,n}\left(\gamma_{p-1}^N(1),\eta_{p-1}^N\right)\right]\end{aligned}$$

Using the fact that

$$\Gamma_{p-1,n}(m,\eta) = \Gamma_{p,n}\left(\Gamma_p(m,\eta)\right) \Rightarrow \Gamma^2_{p-1,n}(m,\eta) = \Gamma^2_{p,n}\left(\Gamma_p(m,\eta)\right)$$

we readily check that

$$\begin{aligned}\Gamma_p\left(\gamma^N_{p-1}(1),\eta^N_{p-1}\right) &= \left(\gamma^N_{p-1}(1)\eta^N_{p-1}(G_{p-1})+\mu_p(1),\Psi_{G_{p-1}}\left(\eta^N_{p-1}\right)M_{p,(\gamma^N_{p-1}(1),\eta^N_{p-1})}\right)\\ &= \left(\gamma^N_p(1),\eta^N_{p-1}K_{p,(\gamma^N_{p-1}(1),\eta^N_{p-1})}\right)\end{aligned}$$

Since we have $\gamma^N_0(1)=\mu_0(1)=\gamma_0(1)$, one concludes that

$$\begin{aligned}\eta^N_n-\eta_n &= \left[\Gamma^2_{0,n}\left(\gamma_0(1),\eta^N_0\right)-\Gamma^2_{0,n}\left(\gamma_0(1),\eta_0\right)\right]\\ &\quad+\sum_{p=1}^n\left[\Gamma^2_{p,n}\left(\gamma^N_p(1),\eta^N_p\right)-\Gamma^2_{p,n}\left(\gamma^N_p(1),\eta^N_{p-1}K_{p,(\gamma^N_{p-1}(1),\eta^N_{p-1})}\right)\right]\end{aligned}$$

Using the fact that $\gamma^N_p(1)\in I_p$, for any $p\geq 0$, the end of the proof is a direct consequence of lemma 3.5 and Kintchine inequality. The proof of the uniform convergence estimates stated in the end of theorem 2.2 are a more or less direct consequence of the functional inequalities derived at the end of section 3.3. The end of the proof of the theorem 2.2 is now completed.

We end this section with the fluctuations properties of the N-particle approximation measures γ^N_n and η^N_n around their limiting values. Using the same line of arguments as those we use in the proof of the functional central limit theorem, theorem 3.3 in [6], we can prove that the sequence $(W^N_n)_{n\geq 0}$ defined in (2.12) converges in law, as N tends to infinity, to the sequence of n independent, Gaussian and centered random fields $(W_n)_{n\geq 0}$ with a covariance function given in (2.13). Using the decompositions (4.4) and

$$\eta^N_n(f)-\eta_n(f)=\frac{\gamma_n(1)}{\gamma^N_n(1)}\left([\gamma^N_n-\gamma_n]\left(\frac{1}{\gamma_n(1)}(f-\eta_n(f))\right)\right)$$

by the continuous mapping theorem, we deduce the functional central limit theorem 2.3.

5 Particle approximations of spontaneous birth measures

We assume that the spontaneous birth measures μ_n are chosen so that $\mu_n\ll\lambda_n$, for some reference probability measures λ_n and the Radon Nikodim derivatives $H_n=d\mu_n/d\lambda_n$ are bounded. For any $n\geq 0$, we let $\lambda^{N'}_n := \frac{1}{N'}\sum_{i=1}^{N'}\delta_{\zeta^i_n}$ be the empirical measure associated with N' independent an identically distributed random variables $\left(\zeta^i_n\right)_{1\leq i\leq N}$ with common distribution λ_n. We also denote by $\mu^{N'}_n$ the particle spontaneous birth measures defined below

$$\forall n\geq 0\qquad \mu^{N'}_n(dx_n):=H_n(x_n)\ \lambda^{N'}_n(dx_n)$$

In this notation, the initial distribution η_0 and the initial mass γ_0 are approximated by the weighted occupation measure $\eta^{N'}_0:=\Psi_{H_0}(\lambda^{N'}_0)$ and $\gamma^{N'}_0(1):=\lambda^{N'}_0(H_0)$.

We let $\widetilde{\gamma}^{N'}_n$ and $\widetilde{\eta}^{N'}_n$ the random measures defined as γ_n and η_n by replacing in (1.1) the measures μ_n by the random measures $\mu^{N'}_n$, for any $n\geq 0$; that is, we have that

$$\widetilde{\gamma}^{N'}_n=\widetilde{\gamma}^{N'}_{n-1}Q_n+\mu^{N'}_n\quad\text{and}\quad\widetilde{\eta}^{N'}_n(f_n)=\widetilde{\gamma}^{N'}_n(f_n)/\widetilde{\gamma}^{N'}_n(1)$$

for any $f_n\in\mathcal{B}(E_n)$. By construction, using the same arguments as the ones we used in the proof of (3.8) we have

$$\widetilde{\gamma}^{N'}_n=\sum_{0\leq p\leq n}\mu^{N'}_pQ_{p,n}$$

This yields for any $f \in \mathcal{B}(E_n)$ the decomposition

$$\left[\widetilde{\gamma}_n^{N'} - \gamma_n\right](f) = \sum_{0\leq p\leq n} \left[\mu_p^{N'} - \mu_p\right] Q_{p,n}(f) = \sum_{0\leq p\leq n} \left[\lambda_p^{N'} - \lambda_p\right] (H_p\ Q_{p,n}(f))$$

Several estimates can be derived from these formulae, including $\mathbb{L}_p$-mean error bounds, functional central limit theorems, empirical process convergence, as well as sharp exponential concentration inequalities. For instance, we have the unbiasedness property

$$\mathbb{E}\left(\widetilde{\gamma}_n^{N'}(f)\right) = \gamma_n(f)$$

and the variance estimate

$$N\ \mathbb{E}\left(\left[\widetilde{\gamma}_n^{N'}(f) - \gamma_n(f)\right]^2\right) = \sum_{0\leq p\leq n} \lambda_p \left[(H_pQ_{p,n}(f) - \lambda_p(H_pQ_{p,n}(f))\right]^2\Big)$$

Using the same arguments as the ones we used in (4.6), we prove the following rather crude upper bound

$$\begin{aligned} N\ \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f)\right]^2\right) &\leq \sum_{0\leq p\leq n} \alpha^\star_{p,n}(\gamma_p(1))^2 \frac{1}{\gamma_p(1)^2}\ \frac{\mu_p\left(H_pQ_{p,n}(f)^2\right)}{\eta_p(Q_{p,n}(1))^2} \\ &\leq \sum_{0\leq p\leq n} \alpha^\star_{p,n}(\gamma_p(1))^2 \frac{1}{\gamma_p(1)^2}\ \|H_p\|\ \mu_p(1)\ q_{p,n}^2 \end{aligned}$$

We illustrate these variance estimates for time homogeneous models $(E_n, G_n, H_n, M_n, \mu_n) = (E, G, H, M, \mu)$, in the three situations discussed in (3.13), (3.14), and (3.16). We further assume that the mixing condition $(M)_k$ presented in (3.1) is met for some parameters $k \geq 1$, and some $\epsilon > 0$. In this case, we use the fact that $q_{p,n} \leq \delta_k/\epsilon$, to prove that

$$N\ \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f)\right]^2\right) \leq c \sum_{0\leq p\leq n} \left[\alpha^\star_{p,n}(\gamma_p(1))/\gamma_p(1)\right]^2$$

with some constant $c := \left(\|H\|\ \mu(1)\ (\delta_k/\epsilon)^2\right)$.

1. Firstly, we observe that for unit potential functions $G(x) = 1$, $x \in E$, we have $\gamma_p(1) = \gamma_0(1) + \mu(1)\ p$. Recalling that $\alpha^\star_{p,n}(\gamma_p(1)) \leq 1$, we prove the uniform estimates

$$N\ \sup_{n\geq 0} \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f)\right]^2\right) \leq c \sum_{p\geq 0} (\gamma_0(1) + \mu(1)\ p)^{-2}$$

2. For $g_+ < 1$, when the mixing condition $(M)_k$ stated in (3.1) is satisfied, we have seen in (3.15) that

$$\alpha^\star_{p,n}(\gamma_p(1)) \leq 1 \wedge \left(d_1\ g_+^{(n-p)}\right) \quad \text{and} \quad \inf_n \gamma_n(1) \geq d_2$$

for some finite constants $d_1 < \infty$ and $d_2 > 0$. From previous calculations, we prove the following uniform variance estimates

$$N\ \sup_{n\geq 0} \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f)\right]^2\right) \leq (c/d_2^2) \sum_{p\geq 0} \left[1 \wedge \left(d_1^2\ g_+^{2p}\right)\right]$$

3. Finally, when $g_- > 1$ we have seen in (3.16) that $\gamma_n(1) \geq d\ g_-^n$ for any $n \geq n_0$, for some finite constant $d < \infty$ and some $n_0 \geq 1$.

$$N\ \sup_{n\geq 0} \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f)\right]^2\right) \leq c\left(\sum_{0\leq p\leq n_0} \gamma_p(1)^{-2} + d \sum_{n\geq n_0} g_-^{-2n}\right)$$

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