Stochastic geometry modelling of hybrid wireless/optical networks

Christian Farinetto, Sergei Zuyev

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Abstract

We propose a new method for the economic evaluation of hybrid fixed wireless/optical access networks. These networks are cost-effective solutions in countries with a limited infrastructure. The evaluation of the cost function of such networks requires a detailed description of their spatial features that is usually not available and unsuitable for economic analysis. Our model captures the essential spatial characteristics of optical ring networks by representing its components as a family of random object generated by a spatial Poisson process. We give bounds on the cost of feeder rings and the exact cost function of distribution subnetworks as functional of the intensities of Poisson processes.

1 Introduction

Broadband wireless systems enables easy deployment of networks to deliver Mb/s service. Several broadband wireless technologies are now available, including LMDS [8] and digital microwave [12]. These systems operate over limited distance, need line-of-sight access and suffer attenuation from rain. Given these limitations, a critical issue facing wireless operators is how to aggregate traffic from multiple base stations and distribute them. A cost

[*]Department of Statistics and Modelling Science, University of Strathclyde, Glasgow G1 1XH, U.K.
E-mails: [christian|sergei]@stams.strath.ac.uk

effective solution is to build an optical feeder ring network connected to wireless base stations through Passive Optical Networks (PON). This type of PON-fed hybrid optical/wireless network is an attractive alternative to cabled networks when providing services to a low density residential area, and perhaps most significant of all, developing countries [4]. For example, in India, where telephone penetration is approximately 1 telephone per 40 people (4.5 per 1000 people for Internet connections), low-cost solutions will be needed to reach the goal of extending basic telephone service and Internet service. If Internet access and telecommunications are to become common place in developing countries such as in Africa and Asia then this technology has huge potential.

Feeder networks primarily serve two functions: aggregate traffic from distribution networks via access nodes and transport them to the corresponding backbone network via egress nodes; and transfer local traffic between distribution networks on the same feeder network.

A common approach used by field operators to secure their feeder network is to organise it as SDH/SONET rings. These rings are "self healing" since they incorporate autonomous protection mechanisms that detect failures and rapidly reroute traffic away from failed links and nodes onto other routes.

In the area of optical communications, the emergence of Wavelength Division Multiplexing (WDM) is one of the most important technological developments of the last years. WDM is a transmission technique that has the potential to utilise a large region of fiber bandwidth. SDH/SONET protection concepts have also been applied to WDM ring networks and analogous architectures have been devised. We therefore assume for our modelling that the feeder network adopts a ring design and is connected to wireless base stations through access nodes by Passive Optical Networks. Although self healing rings can also be used for the design of PONs (see [10]), we will assume for PONs more commonly to-date used architecture: a double star configuration using passive optical splitters (see Figure 1).

In this paper we study the expected cost of building such an hybrid ring/double star network in an area with a limited infrastructure. Until now a complete description of such an architecture (the locations and connection scheme of all of its components) is too cumbersome and usually not particularly suitable for macroscopic analysis or strategic planning. The statistical information on the characteristics of the network is often given only in the form of aggregate or unreliable data which is not sufficient for economic analysis. Stochastic geometry methods can help to solve this prob-

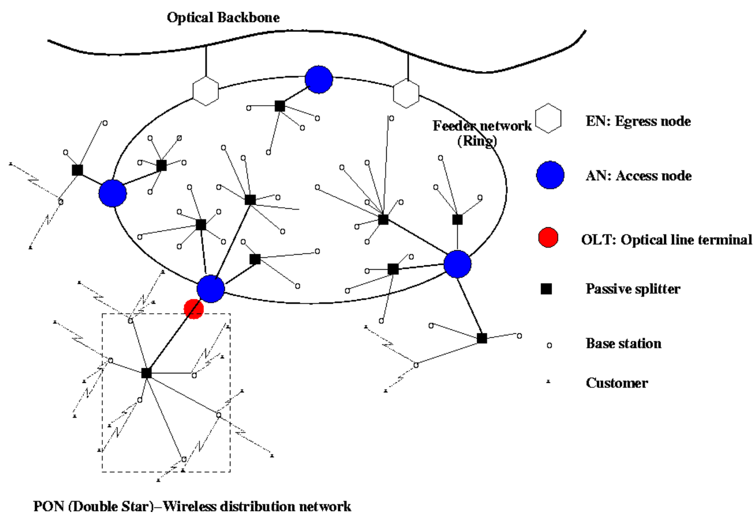

Figure 1: Example of optical access network.

lem of uncertainty by modelling the external environment, the configuration and the behaviour of the communication network. In the framework of our analysis we model optical ring networks and distribution subnetworks by homogeneous Poisson processes and evaluate the cost functions associated with their construction.

We consider a Poisson process Π with intensity λ that generates a set of points z_i in the space $\mathbb{R}^2$. The Voronoi cell V_i with nucleus z_j is the set of the points of the plane closer to z_j than to the other points of the process. The edges of the Voronoi cells constructed with respect to Π represent the WDM rings and the cell's nodes represent the access nodes on the ring. The passive splitters and the wireless base stations of the network are represented by the points of two other independent homogeneous Poisson processes Π_1 and Π_0 with intensities λ_1 and λ_0 which are also assumed independent of Π. Every splitter is connected to the closest node of the Voronoi cell it lies in and every base station is connected to its closest splitter. This part of the network is known as the optical *distribution*.

2 Feeder Ring/Single Star costs

If one knows the cost P of a typical ring access network, then the average cost of a network consisting of N ring access networks can be evaluated as the product PN. Our task thus reduces to evaluation of cost of a typical ring. It is well known that a typical Voronoi cell generated by Poisson process can be simulated by adding a point into the origin for every configuration of the process. Then the cell $V_0(\Pi)$ centered in the origin has a distribution of a typical cell, i.e. a cell "uniformly chosen" among all Voronoi cells. The corresponding rigorous definition is that of a *Palm version* of the process. The corresponding Palm probability and expectation are denoted by $\mathbf{P}^0$ and $\mathbf{E}^0$, respectively (see, e. g., [13, p.39]).

For now we will ignore existence of base stations and evaluate the cost of a typical feeder ring plus PONs down to splitters only, so a typical ring/single star architecture.

Denote by $\hat{V}_0(\Pi)$ the nodes of the cell $V_0(\Pi)$ and $l(V_0(\Pi))$ its perimeter; d represent the cost of a length unit in the ring, which may comprise the cost of the fiber, civil engineering cost as well as the installation and maintenance costs. The cost associated with the construction, maintenance and functioning of an access node, splitter and base station are respectively denoted by $c_{\text{acc}}, c_{\text{spl}}$ and c_{bs}.

We define the cost function P_{feed} of the feeder network centered in the origin to be

$$P_{\text{feed}} = d\, l(V_0(\Pi)) + c_{\text{acc}}\, \text{card}(\hat{V}_0(\Pi)).$$

Next, we assume that the cost of connection of a splitter $x_i \in V_0(\Pi)$ to the corresponding closest access node could be represented by a quadratic function of a distance $d(x_i, \hat{V}_0(\Pi))$ between them, so that the cost of the subnetworks between access nodes and passive splitters contained in the ring centered in the origin is

$$P_{\text{rss}} = \sum_{x_i \in \Pi_0 \cap V_0(\Pi)} \left[a_1 d(x_i, \hat{V}_0(\Pi)) + a_2 d(x_i, \hat{V}_0(\Pi))^2 \right] + c_{\text{spl}} \Pi_1(V_0(\Pi)),$$

Applying Campbell's theorem (see, e. g., [13, p.99]) and well-known moment properties of a typical Voronoi cell (see [9]) we find that

$$\mathbf{E}^0\, P_{\text{feed}} = \frac{4d}{\sqrt{\lambda}} + 6c_{\text{acc}}\,,$$

and

$$\mathbf{E}^0\, P_{\mathrm{rss}} = \lambda_1 \int \mathbf{E}^0 \left(a_1 d(x, \hat{V}_0(\Pi)) + a_2 d(x, \hat{V}_0(\Pi))^2 \right) \mathbb{1}_{x \in V_0(\Pi)}\, dx + \frac{c_{\mathrm{spl}}\lambda_1}{\lambda}. \quad (1)$$

The domain of an integral is the whole plane $\mathbb{R}^2$ unless explicitly written otherwise. The expression (1) reduces to

$$\mathbf{E}^0\, P_{\mathrm{rss}} = \lambda_1(a_1 C_1 + a_2 C_2) + \frac{c_{\mathrm{spl}}\lambda_1}{\lambda}\,, \quad (2)$$

where

$$C_\alpha = \int \mathbf{E}^0 \big[d(x, \hat{V}_0(\Pi))^\alpha\, \mathbb{1}_{x \in V_0(\Pi)} \big]\, dx\,. \quad (3)$$

The last expression can also be written as

$$C_\alpha = \int dx \int_0^{+\infty} \alpha t^{\alpha-1} \mathbf{P}^0 \big\{ d(x, \hat{V}_0(\Pi)) > t, x \in V_0(\Pi) \big\}\, dt\,. \quad (4)$$

Unfortunately this integral is not analytically tractable so the only way to deal with it is either to evaluate it from simulations or find analytical expressions bounding it from above and from below. This is the content of the next two sections.

2.1 Lower bound on the cost

A point y is a node of the Voronoi tessellation constructed with respect to Π if and only if it is the center of a circle having three points of Π on its boundary and no point of the process inside. Therefore, if there is at most one point of the process Π in the set $K_{t,x}$ defined by

$$K_{t,x} = \bigcup_{y \in \partial B(x,t)} B(y, \|y\|),$$

then no point of the disk $B(x,t)$ can be a node as the above property does not hold for it. This implies that

$$\begin{aligned} \mathbf{P}^0 \big\{ d(x, \hat{V}_0(\Pi)) > t, x \in V_0(\Pi) \big\} &\geq \mathbf{P}^0 \big\{ \Pi(K_t, x) \leq 1 \big\} \\ &= (1 + \lambda |K_{x,t}|)\, e^{-\lambda |K_{x,t}|}\,. \quad (5) \end{aligned}$$

With polar coordinates (r, φ) centered in the origin and the polar axis passing through x the area is

$$|K_{t,x}| = K_1(t,\rho) = 2(2t^2 + \rho^2)(\pi - \arccos\frac{t}{\rho}) + 6t\sqrt{\rho^2 - t^2} \tag{6}$$

for $t < \rho$ and

$$|K_{t,x}| = K_2(t,\rho) = 2\pi(2t^2 + \rho^2) \tag{7}$$

for $t \geq \rho$.

Using (5) in the expression (4) and switching to polar coordinates (θ, ρ) for x we then derive

$$C_\alpha \geq I_\alpha(\lambda) + J_\alpha(\lambda), \tag{8}$$

where

$$\begin{aligned} I_\alpha(\lambda) &= \int_0^{2\pi} d\theta \int_0^\infty \rho d\rho \int_0^\rho \alpha t^{\alpha-1}(1 + \lambda K_1(t,\rho))e^{-\lambda K_1(t,\rho)}dt\,, \\ J_\alpha(\lambda) &= \int_0^{2\pi} d\theta \int_0^\infty \rho d\rho \int_\rho^\infty \alpha t^{\alpha-1}(1 + \lambda K_2(t,\rho))e^{-\lambda K_2(t,\rho)}dt\,. \end{aligned}$$

In particular, we have

$$C_1 \geq \frac{0.24895}{\lambda^{3/2}}, \quad C_2 \geq \frac{0.06886}{\lambda^2}.$$

If $a_1, a_2 \geq 0$, this gives the following lower bound for the expectation of the cost:

$$\mathbf{E}^0(P_{\text{feed}} + P_{\text{rss}}) \;\geq\; \lambda_1\left[\frac{0.24895a_1}{\lambda^{3/2}} + \frac{0.06886a_2}{\lambda^2} + \frac{c_{\text{spl}}}{\lambda}\right] + 4\frac{d}{\sqrt{\lambda}} + 6c_{\text{acc}}.$$

A better bound can be achieved by numerical integration of $I_\alpha(\cdot)$ and $J_\alpha(\cdot)$ for given values of λ.

2.2 Upper bound on the cost

Similarly to the previous section, we can obtain an upper bound for the cost (2) by estimating from above the Palm probability of the event $\{d(x, \hat{V}_0(\Pi)) > t, x \in V_0(\Pi)\}$ involved in (4).

Let P_1, P_2 be two points of the Poisson process Π. Denote by $D(0, P_1, P_2)$ the *open* circumdisk over the origin O, P_1 and P_2, and by $Q(O, P_1, P_2)$ its centre. Then Q is a node of $V_0(\Pi)$ if and only if it does not contain points of Π. Consider thus the set

$$W_0(\Pi) = \{(P_1, P_2) : \ P_1, P_2 \in \Pi, \ P_1 \neq P_2 \text{ and } \Pi(D(O, P_1, P_2)) = 0\},$$

and the set $D_t = \{(z_1, z_2) \in \mathbb{R}^4 : \ \|Q(O, z_1, z_2) - x\| > t\}$. It now follows that:

$$\begin{aligned}
&\mathbf{P}^0\big\{d(x, \hat{V}_0(\Pi) > t, x \in V_0(\Pi)\big\} \\
&= \mathbf{P}^0\Big\{\bigcap_{(P_1,P_2)\in W_0(\Pi)} d(x, Q(O, P_1, P_2)) > t, \ x \in V_0(\Pi)\Big\} \\
&< \mathbf{P}^0\Big\{\bigcup_{P_1 \neq P_2 \in \Pi} d(x, Q(O, P_1, P_2)) > t, \ \Pi(D(O, P_1, P_2)) = 0, \\
&\qquad\qquad \Pi(B(x, \|x\|)) = 0\Big\} \\
&\leq \mathbf{E}^0 \sum_{P_1 \neq P_2 \in \Pi} \mathbb{I}\big\{d(x, Q(O, P_1, P_2)) > t, \ \Pi(D(O, P_1, P_2)) = 0, \\
&\qquad\qquad\qquad \Pi(B(x, \|x\|)) = 0\big\} \\
&= \lambda^2 \iint \mathbf{P}^{0,P_1,P_2}\big\{d(x, Q(O, P_1, P_2)) > t, \\
&\qquad\qquad\qquad \Pi(D(O, P_1, P_2) \cup B(x, \|x\|)) = 0\big\}\, dP_1 dP_2 \\
&= \lambda^2 \iint_{D_t} e^{-\lambda|D(O,P_1,P_2)\cup B(x,\|x\|)|} dP_1 dP_2\,, \qquad (9)
\end{aligned}$$

where $\mathbf{P}^{0,P_1,P_2}$ is the three-fold Palm distribution – (see n-fold Palm theorem in [13]). To evaluate this expression we take a special parametrisation of P_1 and P_2. Namely, let, as before, (θ, ρ) be the polar coordinates of x. Let η be the angle between radius-vectors of x and $Q(O, P_1, P_2)$ measured counterclockwise, and μ be the angle between OQ and the bisector of the angle $2\nu = (QP_1, QP_2)$. Finally, denote by R the radius of $D(O, P_1, P_2)$. If $P_i = (x_i, y_i)$, then in these coordinates we have

$$\begin{aligned}
x_1 &= R\cos(\theta + \eta) + R\cos(\theta + \eta + \mu + \nu) \\
y_1 &= R\sin(\theta + \eta) + R\sin(\theta + \eta + \mu + \nu) \\
x_2 &= R\cos(\theta + \eta) + R\cos(\theta + \eta + \mu - \nu) \\
y_2 &= R\sin(\theta + \eta) + R\sin(\theta + \eta + \mu - \nu)
\end{aligned}$$

and the set D_t writes as

$$\sqrt{R^2+\rho^2-2R\rho\cos\eta} > t\,.$$

We also have the following conditions on the angles:

$$0<\eta<2\pi,\ 0<\nu<\pi,\ 0<\mu<\pi,\ 0<\pi-\mu-\nu\,.$$

It can be easily checked that for the surface element dP_1dP_2 one has the following expression:

$$dP_1dP_2 = 2R^3|\sin 2\nu + 2\sin\nu\cos\mu|\,dR\,d\rho\,d\mu\,d\nu\,.$$

This allows us to express (9) as

$$\frac{3\pi}{2}\lambda^2\int_0^{2\pi}d\eta\int_{D_t}R^3e^{-\lambda|D(O,P_1,P_2)\cup B(x,\|x\|)|}\,dRd\eta, \tag{10}$$

The union of the discs can also then be expressed via the same variables, but the resulting quadruple integral only allows for a numeric evaluation. We bound the intersection of the area of two discs and obtain using (4) and (10) that

$$C_\alpha \le 3\alpha\pi^2\Phi(\alpha,\lambda)\,,$$

with

$$\Phi(\alpha,\lambda)=\lambda^2\int_0^\infty dt\iiint\limits_{D_t}\rho t^{\alpha-1}R^3e^{-\frac{\lambda\pi}{2}(\rho^2+R^2)}d\rho d\eta dR,$$

leading, finally, to

$$C_1\le\frac{21.2131}{\lambda^{3/2}},\quad C_2\le\frac{48.5576}{\lambda^2}\,.$$

This yields

$$\mathbf{E}^0(P_{\text{feed}}+P_{\text{rss}})1\ \le\ \lambda_1\left[\frac{21.2131a_1}{\lambda^{3/2}}+\frac{48.5576a_2}{\lambda^2}+\frac{c_{spl}}{\lambda}\right]+\frac{4d}{\sqrt{\lambda}}+6c_{\text{acc}}.$$

Smaller bounds can be obtained by applying expression (10) directly in (4) and using numerical integration for any given value of λ.

3 Double star/wireless costs

It is possible to extend evaluation of the cost to the architecture described in Introduction that consists of feeder rings with attached *double-star* passive optical networks (PONs) to their access nodes. The connection principle of base stations is the same as for connection of splitters to the nodes: points of Π_0 (stations) are connected to the closest point of process Π_1 (splitters).

The cost of the subnetwork comprising the second star distribution is thus $P_{\text{dst}} + P_{\text{bs}}$.

Here P_{dst} is the cost of connections from base stations to splitters, P_{bs} is the total cost of base stations attached to the ring. The cost of connection between base station and its closest splitter may have quite a general polynomial form of the distance and still give a computable expression, so we take P_{dst} to be

$$P_{\text{dst}} = \sum_{y_i \in \Pi_1 \cap V_0(\Pi)} \sum_{x_j \in V_{y_i}(\Pi_1) \cap \Pi_0} \sum_l b_l \|y_i - x_j\|^{\gamma_l}.$$

Finally,

$$P_{\text{bs}} = \sum_{y_i \in \Pi_1 \cap V_0(\Pi)} \sum_{x_j \in V_{y_i}(\Pi_1) \cap \Pi_0} c_{\text{bs}}.$$

The set of possible positions of base stations to be attached to an access node on the ring is the so-called iterated Voronoi cell studied in [14]. Using these results,

$$\mathbf{E}^0 P_{\text{bs}} = \frac{\lambda_0 c_{\text{bs}}}{\lambda}.$$

Details of the calculation of the Palm expectation of P_{dst} can be found in [1]. It follows from Neveu's exchange formula (see [6]) that

$$\mathbf{E}^0 P_{\text{dst}} = \frac{\lambda_1}{\lambda} \sum_l b_l \, \mathbf{E}^0 \sum_{x_j \in \Pi_0 \cap V_0(\Pi_1)} \|x_j\|^{\gamma_l}.$$

One can then show that

$$\mathbf{E}^0 \sum_{x_j \in \Pi_0 \cap V_0(\Pi_1)} \|x_j\|^{\gamma_l} = \frac{\lambda_0}{\lambda_1 (\lambda_1 \pi)^{\frac{\gamma_l}{2}}} \Gamma\left(\frac{\gamma_l}{2} + 1\right)$$

leading to

$$\mathbf{E}^0 P_{\text{dst}} = \frac{\lambda_0}{\lambda} \sum_l \frac{b_l}{(\lambda_1 \pi)^{\frac{\gamma_l}{2}}} \Gamma\left(\frac{\gamma_l}{2} + 1\right).$$

Thus we have found all the components of the cost of the double-star architecture attached to a typical ring which we now illustrate on example.

The splitting loss of passive splitters limits the maximal splitting ratio. The number of base stations is not exceeding 64 per splitter in general (in practice 32-48) therefore it is reasonable to take $\lambda_2 < 64\lambda_1$.

The graph below shows the bounds on the expected cost of fiber connection for the following parameters: $\lambda = 1$ rg/sq. mile, $\lambda_1 = \frac{\lambda_0}{32}$, $l = 1$, $a_1 = b_1 = \$10000$/mile,
$a_2 = 0$, $d = \$25000$/mile.

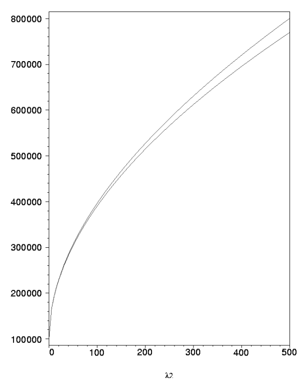

Figure 2: Bounds on a typical WDM feeder ring/PON cost as a function of intensity of base stations.

4 Conclusion and further work

We proposed a macroscopic random network model taking into account essential spatial characteristics of access networks through a small number of parameters. We evaluated the cost of the access network having a ring/(double) star topology. One direction to extend this work is to consider the relation of network components and bandwidth demand in order to give more accurate cost functions. Another related problem is statistical identification of the parameters from the real data which is not covered here.

Acknowledgements

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