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# Weighted Coloring on $P_4$ -sparse Graphs

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**Abstract**—Given an undirected graph  $G = (V, E)$  and a weight function  $w : V \rightarrow \mathbb{R}^+$ , a vertex coloring of  $G$  is a partition of  $V$  into independent sets, or color classes. The *weight* of a vertex coloring of  $G$  is defined as the sum of the weights of its color classes, where the weight of a color class is the weight of a heaviest vertex belonging to it. In the WEIGHTED COLORING problem, we want to determine the minimum weight among all vertex colorings of  $G$  [1]. This problem is NP-hard on general graphs, as it reduces to determining the chromatic number when all the weights are equal. In this article we study the WEIGHTED COLORING problem on  $P_4$ -sparse graphs, which are defined as graphs in which every subset of five vertices induces at most one path on four vertices [2]. This class of graphs has been extensively studied in the literature during the last decade, and many hard optimization problems are known to be in P when restricted to this class. Note that cographs (that is,  $P_4$ -free graphs) are  $P_4$ -sparse, and that  $P_4$ -sparse graphs are  $P_5$ -free. The WEIGHTED COLORING problem is in P on cographs [3] and NP-hard on  $P_5$ -free graphs [4]. We show that WEIGHTED COLORING can be solved in polynomial time on a subclass of  $P_4$ -sparse graphs that strictly contains cographs, and we present a 2-approximation algorithm on general  $P_4$ -sparse graphs. The complexity of WEIGHTED COLORING on  $P_4$ -sparse graphs remains open.

## I. I

In this paper, we adopt the graph terminology defined in [5]. Additionally, the definition and classical results about the modular decomposition of graphs can be found in [6].

The classical  $V$   $C$  problem is one of the most studied problems in graph theory, due to its many applications in both theoretical and practical domains. Given a graph  $G = (V, E)$ , a (vertex)  $k$ -coloring of  $G$  is a function  $c : V \rightarrow \{1, \dots, k\}$  that associates to each vertex  $v \in V$  a color  $c(v)$  such that if  $(u, v) \in E$ , then  $c(u) \neq c(v)$ . The minimum integer  $k$  such that a graph  $G$  admits a  $k$ -coloring is the *chromatic number* of  $G$ , denoted by  $\chi(G)$ . A  $k$ -coloring can also be seen as a partition  $S = (S_1, \dots, S_k)$  of the vertex set into color classes.

Given a vertex weighted graph  $G = (V, E, w)$ , the weight of a vertex coloring of  $G$  is given by the sum of the weights of its color classes, where the weight of a color class is the weight of a heaviest vertex belonging to it. In the  $W$   $C$  problem, we want to determine the minimum weight among all the colorings of  $G$ . The *weighted chromatic number*, denoted by  $\chi_w(G)$ , of a graph  $G$  is the value of the minimum weight of a coloring of  $G$ . The definition of this problem

was motivated by the Distributed Dual Bus Network Media Access Control Protocol, which is a standard IEEE802.6 for metropolitan networks [1].

The  $W$   $C$  problem is a generalization of the classical  $V$   $C$  problem and hence it is also NP-hard. In fact, it is hard even for classes where  $V$   $C$  can be easily solved [3], [4], [7].

In this paper, we consider vertex-weighted  $P_4$ -sparse graphs. A graph  $G$  is called  $P_4$ -sparse if every 5 vertices of  $V(G)$  induce at most one  $P_4$  [2]. This class of graphs was first studied by Jamison and Olariu [8]–[11].

The class of  $P_4$ -sparse graphs is strictly contained in the class of  $P_5$ -free graphs (for which  $W$   $C$  is NP-hard [3]) and strictly contains  $P_4$ -free graphs, or cographs (for which  $W$   $C$  is polynomial [4]).

There are a number of optimization problems that can be solved in polynomial time on  $P_4$ -sparse graphs [10], [12]. The algorithms that solve these problems usually calculate the desired parameter in a simple post-order traversal in the modular decomposition tree of the graph, which can be found in linear time [13] for any graph. We use the same approach to determine the weighted chromatic number for a subclass of  $P_4$ -sparse graphs. Recall that in a modular decomposition tree of any graph, each node either is *series*, which means that there is a *complete join* between the modules defined by its children, or is *parallel*, which means that there is a *disjoint union* between the modules defined by its children, or is *neighborhood*, which means that the quotient graph of the modules defined by its children and its complement are connected.

$P_4$ -sparse graphs can be characterized by their modular decomposition. In order to present this result, we need to define a *spider* graph.

*Definition 1.1:* A *spider* is a graph whose vertex set can be partitioned into disjoint sets  $S$ ,  $K$ , and  $R$  such that:

- 1)  $|S| = |K| \geq 2$ ,  $S$  is a stable set,  $K$  is a clique;
- 2) Every vertex in  $R$  is adjacent to all the vertices in  $K$  and to no vertex in  $S$ ;
- 3) There exists a bijection  $f : S \rightarrow K$  such that either the spider is of Type 1 (called *thick spider*), i.e.:

$$N_G(s) \cap K = K - \{f(s)\}, \text{ for all vertices } s \in S;$$

or it is of Type 2 (called *thin spider*), i.e.:

$$N_G(s) \cap K = \{f(s)\}, \text{ for all vertices } s \in S.$$

Observe that the unique non-trivial maximal strong submodule of a spider is exactly the set  $R$ .

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*Theorem 1.1 ([14]):*  $G$  is a  $P_4$ -sparse graph if, and only if, the quotient graph of each neighborhood node of its modular decomposition tree  $T(G)$  is isomorphic to a spider  $H = (S \cup K \cup R, E)$ .

In Section II, we present the main results of this paper. In Section III, we show that there exists a 2-approximation algorithm for  $\chi_w$  on  $P_4$ -sparse graphs. Finally, we propose a conjecture in Section IV.

## II. APPROXIMATION

It is not difficult to see that:

*Remark 2.1:* Given the weighted chromatic numbers of two graphs  $G_1$  and  $G_2$ , the weighted chromatic number of the graph  $G$  obtained by the complete join of  $G_1$  and  $G_2$  is equal to  $\chi_w(G) = \chi_w(G_1) + \chi_w(G_2)$ .

Our algorithm will traverse the modular decomposition tree of the graph in a post-order way, in order to calculate its weighted chromatic number. Remark 2.1 implies that it is easy to deal with the *series* nodes. The rest of this section is dedicated to the *neighborhood* nodes (spiders) and *parallel* nodes (disjoint union) of  $P_4$ -sparse graphs.

### A. Spiders

From now on we suppose, unless said otherwise, that  $G = (V = S \cup K \cup R, E, w)$  is a spider. We prove that an optimal weighted coloring of  $G$  can be obtained in polynomial time, provided that we have an optimal weighted coloring of  $R$ . We start by making some remarks.

*Remark 2.2:* We can assume that  $w(v) > 0$ , for all  $v \in V(G)$ , since given any coloring  $c$  of  $G$ , we can put each vertex  $v$  with weight zero in a color class consisting only of  $v$ , without increasing the weight of  $c$ .

*Remark 2.3:* Without loss of generality, we can suppose that all the non-neighbors of each vertex  $s_i$  of  $S$ , for all  $i = 1, \dots, |S|$ , have weight strictly smaller than  $w(s_i)$ . Otherwise, given a coloring  $c$  of  $G$  such that there exists a vertex  $s_i \in S$  that does not belong to a color class of one of its heavier non-neighbors, then we can find a coloring  $c'$  of  $G$  such that  $w(c') \leq w(c)$  by recoloring  $s_i$  with a color from one of its heavier non-neighbors.

By the definition of a spider, all the edges between the vertices of  $K$  and  $R$  exist. By consequence, for any  $l$ -coloring  $\mathcal{S} = \{S_1, \dots, S_l\}$  of  $G$ , there is no class  $S_i$  containing vertices from both  $K$  and  $R$ . We can then define  $CK$  (resp.  $CR$ ), the set of *colors of  $K$*  (resp. *colors of  $R$* ), as the set whose elements are the color classes that contain at least one vertex of  $K$  (resp. one vertex of  $R$ ). Observe that the sets  $CK$  and  $CR$  are disjoint.

*Lemma 2.1:* Given an optimal weighted coloring  $\mathcal{S} = \{S_1, \dots, S_l\}$  of a spider  $G$ , the following holds: if  $R = \emptyset$ , then there exists at most one color class  $S_i$  of  $\mathcal{S}$ , such that  $S_i \notin CK$ . Otherwise, there is no color class  $S_i$  in the set  $S \setminus (CK \cup CR)$ .

*Proof:* If  $R = \emptyset$ , for otherwise, one could obtain a coloring  $\mathcal{S}'$ , with weight strictly smaller than  $\mathcal{S}$ , by merging the color classes that have only vertices from  $S$ . In the case  $R \neq \emptyset$ , again by contradiction, one could merge  $S_i$ , for some color

$S_i \in S \setminus (CK \cup CR)$ , with some color of  $CR$  to obtain a coloring with weight strictly smaller than  $\mathcal{S}$ . ■

We will denote by the *color of  $S$* , or simply  $cS$ , the unique possible color class which does not belong to  $CK \cup CR$ .

*Lemma 2.2:* There exists at most one color class  $S_j$  from every optimal weighted coloring  $\mathcal{S} = \{S_1, \dots, S_l\}$  of  $G$  such that  $S_j$  intersects both  $S$  and  $R$ .

*Proof:* Suppose, by contradiction, that there are two colors from  $CR$ ,  $S_j$  and  $S_{j'}$ ,  $j \neq j'$ , such that  $S_j$  and  $S_{j'}$  contain vertices of  $S$ . Moreover, without loss of generality, suppose that  $w(S_j) \geq w(S_{j'})$ . Again, By Remark 2.3, the vertices with the greatest weight in each color class  $S_j$  and  $S_{j'}$  belong to  $S$ . Thus, the coloring  $\mathcal{S}'$  obtained from  $\mathcal{S}$  by moving the vertices of  $S \cap S_j$  to  $S_{j'}$  would have weight strictly smaller than  $w(\mathcal{S})$ , a contradiction. ■

Now we prove the following lemma to be used in the sequel:

*Lemma 2.3:* If  $R \neq \emptyset$ , then given an optimal weighted coloring  $\mathcal{S}_R$  of the subgraph of  $G$  induced by  $R$ , there exists an optimal weighted coloring  $\mathcal{S}$  of  $G$  that is an extension of  $\mathcal{S}_R$ .

*Proof:* Let  $\mathcal{S}' = \{S'_1, \dots, S'_k\}$  be an optimal weighted coloring of  $G$  and let  $S'_i$ , by Lemma 2.2, be the unique possible color of  $\mathcal{S}'$  that contains vertices from both  $S$  and  $R$ , for some  $i \in \{1, \dots, k\}$ .

Observe that  $S'_i$  contains a vertex  $r^*$  with the maximum weight of a vertex in  $R$  (for otherwise, by recoloring all the vertices of  $S \cap S'_i$  with the color of a vertex with maximum weight in  $R$ , we would obtain a coloring with weight strictly smaller than  $w(\mathcal{S}')$ ).

Let  $\mathcal{S}$  be a coloring of  $G$  such that the partition of the vertices of  $R$  agrees with the partition given by  $\mathcal{S}_R$ , the vertices of  $S \cap S'_i$  are assigned to the same element of the partition of  $r^*$ , and the vertices of  $\{S \cup K\} - \{S \cap S'_i\}$  maintain the partition given by  $\mathcal{S}'$ .

Since  $\mathcal{S}_R$  is an optimal weighted coloring to  $G[R]$ , observe that  $\mathcal{S}$  is an optimal weighted coloring to  $G$ , because in both colorings  $\mathcal{S}'$  and  $\mathcal{S}$ , the color classes of  $CK$  and  $cS$ , if the latter one exists, are the same. ■

Suppose now that the vertices of  $S$  are labeled  $S = \{s_1, \dots, s_m\}$  satisfying  $w(s_1) \leq \dots \leq w(s_m)$ . We are ready to prove that:

*Lemma 2.4:* There exists an optimal weighted coloring  $c'$  of  $G$  such that exactly one of the following statements holds:

- 1) There exists an integer  $j$ , such that the vertices  $s_1, \dots, s_{j-1}$  are either assigned to color  $cS$  or to the color of a heaviest vertex of  $R$ , while the vertices  $s_j, \dots, s_m$  are assigned, each one individually, to colors of their non-neighbors in  $K$ ;
- 2) The vertices of  $S$  are all assigned to  $cS$ , or to the color of a heaviest vertex of  $R$ , or, each one individually, to a color of one of its non-neighbors in  $K$ .

*Proof:* Observe that, for any vertex  $s_i \in S$ , either it belongs to  $cS$  or it has a color of one of its non-neighbors in  $K$  or  $R$ .

Consider now an optimal weighted coloring  $c$  of  $G$ . Let  $j \in \{1, \dots, m, m+1\}$  be the highest index of a heaviest vertex

of  $S$  that is colored either with a color of  $R$  or with the color of  $S$  (consider that if  $j-1=0$  then there is no vertex with these colors, and if  $j-1=m$  then all the vertices of  $S$  are colored by colors of  $R$  or  $S$ ). Observe that we can obtain a coloring  $c'$  such that  $w(c') \leq w(c)$  by assigning to all  $s_1, \dots, s_{j-1}$  the color of  $s_j$ . Moreover, if  $s_j \in CR$ , using similar arguments to those used in Lemma 2.3, we may recolor  $s_1, \dots, s_{j-1}$  with the color of a heaviest vertex of  $R$ . ■

Now, denote by  $k^*$  a heaviest vertex of  $K$ , by  $k^{**}$  a second heaviest vertex of  $K$  and by  $s^*$  the neighbor of  $k^*$ , if the spider  $G$  is of type 2.

*Lemma 2.5:* Let  $G = (S \cup K \cup R, E, w)$  be a spider of type 2 and let  $c'$  be an optimal weighted coloring of  $G$  as described in Lemma 2.4. We can construct a coloring  $c''$  satisfying  $w(c'') \leq w(c')$  and such that either:

- the color of the vertices  $s_j, \dots, s_m$  is equal to the color of a heaviest vertex  $k^*$  of  $K$ , except possibly  $s^*$  that would have a color of a second heaviest vertex  $k^{**}$ ; or
- the color of the vertices  $s_j, \dots, s_m$  is equal to the color of  $k_i \neq k^*$ , for some vertex  $k_i \in K$ , except possibly the vertex  $s_i$ , the only neighbor of  $k_i$  in  $S$ , that would have the color of  $k^*$ .

*Proof:* We need to show that we can obtain from an optimal weighted coloring  $c'$ , a coloring  $c''$  such that  $w(c'') \leq w(c')$  and  $c''$  satisfies the lemma conditions. If in the coloring  $c'$  no vertex of  $S$  has colors of  $K$ , then the lemma is trivially true. Otherwise, let  $j-1$  be the highest index of a heaviest vertex of  $S$  that is colored either with a color of  $R$  or with the color of  $S$ . To prove the lemma we distinguish the following cases:

1)  $c'(s_m) = c'(k^*)$

a)  $s^* \notin \{s_j, \dots, s_m\}$

Observe that in this case all the vertices with colors of  $K$  in  $S$  are not adjacent to  $k^*$  and, consequently, they could all receive the color of  $k^*$ . Let  $c''$  be the coloring obtained from  $c'$  by assigning to all the vertices in the set  $\{s_j, \dots, s_m\}$  the color  $c'(k^*)$  of  $k^*$ . At first, observe that  $w(c'(k^*)) = w(c''(k^*))$ , because  $s_m$  is a heaviest vertex of  $S$  and by hypothesis  $c'(s_m) = c'(k^*)$ . Moreover, all the other color classes have not increased their weight, because they have just lost some vertices. Then,  $w(c'') \leq w(c')$  and  $c''$  satisfies the conditions of the lemma.

b)  $s^* \in \{s_j, \dots, s_m\}$

Observe that  $c'(s_m) = c'(k^*)$ , so  $s^* \neq s_m$ . Let us construct the coloring  $c''$  in two steps. At first, observe that if we put all the vertices of  $\{s_j, \dots, s_m\} \setminus \{s^*\}$  in the color class  $c'(k^*)$  we will not increase the weight of the coloring, because by hypothesis  $s_m$  already belongs to  $c'(k^*)$ .

If, after this first change, the color of  $s^*$  is equal to the color of  $k^{**}$ , we have already obtained a coloring satisfying the conditions of the lemma, otherwise assume  $c'(s^*) = c'(k_i)$ , for some vertex  $k_i \neq k^*$ ,  $k_i \in K$ .

In this case, we claim that if we recolor  $s^*$  with the color  $c'(k^{**})$  we will create a coloring  $c''$  such that  $w(c'') \leq w(c')$ . To show this fact, observe that the color classes that may change their weight by recoloring  $s^*$  with  $c'(k^{**})$  are  $c'(k_i)$  and  $c'(k^{**})$ . However, by Remark 2.3,  $w(c'(k_i)) = w(c''(k^{**}))$ , and observe that  $w(c'(k^{**})) = w(k^{**}) \geq w(k_i) = w(c''(k_i))$ . Finally,  $c''$  satisfies the conditions of the lemma.

2)  $c'(s_m) \neq c'(k^*)$

a)  $s^* \notin \{s_j, \dots, s_m\}$

Suppose  $c'(s_m) = c'(k_i)$ . We claim that if we put all the vertices of  $S$  with color  $c'(k_i)$  in the color class  $c'(k^*)$  we will create a coloring  $c$  such that  $w(c) \leq w(c')$ . Again this verification is simple because only the color classes  $c'(k_i)$  and  $c'(k^*)$  may have their weights modified. Observe that  $w(c'(k_i)) = w(s_m) = w(c(k^*))$  and  $w(c'(k^*)) \geq w(k^*) \geq w(k_i) = w(c(k_i))$ .

At last, observe that in the coloring  $c$  we have  $c(s_m) = c(k^*)$  and we are again in the case 1a.

b)  $s^* \in \{s_j, \dots, s_m\}$

i)  $c'(s^*) \neq c'(s_m)$

We can repeat the steps of case 2a to find a coloring  $c$  from  $c'$  such that  $w(c) \leq w(c')$  by recoloring all the vertices in  $S$  colored  $c'(s_m)$  with the color  $c'(k^*)$ . Then, we obtain a coloring as in the case 1b.

ii)  $c'(s^*) = c'(s_m)$

Suppose that  $c'(s^*) = c'(s_m) = c'(k_i)$ , for some  $k_i \neq k^*$ .

In this case, observe that we cannot modify the color of  $s_m$  to the color  $c'(k^*)$ , because  $s^*$  and  $s_m$  have the same color and  $s^*$  and  $k^*$  are neighbors. We cannot use Remark 2.3 to compare the weights of these vertices and, consequently, to be sure that the weight of the coloring will not increase after moving the vertices in  $S \setminus \{s^*\}$  with color  $c'(k_i)$  to the color  $c'(k^*)$ .

However, as in the case 1, if the only neighbor of  $k_i$  in  $S$ , say  $s_i$ , does not belong to the set  $\{s_j, \dots, s_m\}$ , then we can put all the vertices from  $s_j$  to  $s_m$  in the color  $c'(k_i)$  obtaining a coloring  $c''$  satisfying the condition of the lemma.

If  $s_i \in \{s_j, \dots, s_m\}$ , we can use the arguments of the case 1b to conclude that we can assign to all the vertices  $\{s_j, \dots, s_m\} \setminus \{s_i\}$  the color  $c'(k_i) = c'(s_m)$  without increasing the weight of the coloring. Moreover, observe that we can assign to  $s_i$  the color  $c'(k^*)$  generating a coloring  $c''$  satisfying the condition of the lemma, because  $w(c'(k_i)) = w(s_i) = w(c''(k^*))$  and  $w(c'(k^*)) = w(k^*) \geq w(k_i) = w(c''(k_i))$ .

**Input:** Spider  $G = (S \cup K \cup R, E)$  and an optimal weighted coloring  $c_R$  of  $G[R]$   
**Output:** Optimal weighted coloring of  $G$

```

1  $m \leftarrow |S|$ ;
2 Create artificial vertices  $s_0$  and  $s_{m+1}$  in  $S$  and order them such that
 $w(s_0) \leq \dots \leq w(s_{m+1})$ ;
3 Choose  $k^*$ ,  $k^{**}$  and  $r^*$  and define  $c, c' \leftarrow \emptyset$ ;
4 foreach  $r \in R$  do
5    $c'(r) := c_R(r)$ ;
6 foreach  $k \in K$  do
7    $c'(k) :=$  a color among the  $|K|$  colors of  $K$ ;
8 for  $j = 1, \dots, m+1$  do
9   for  $i = 0, \dots, j-1$  do
10    if  $R \neq \emptyset$  then
11       $c'(s_i) \leftarrow c'(r^*)$ ;
12    else
13       $c'(s_i) \leftarrow cS$ ;
14  if Spider  $G$  is of type 1 then
15    for  $i = j, \dots, m$  do
16       $c'(s_i) \leftarrow$  the color of its non-neighbor in  $K$  ( $c'(f(s_i))$ );
17    if  $w(c') < w(c)$  then
18       $c \leftarrow c'$ ;
19  else
20    for  $i = j, \dots, m$  do
21      if  $(s_i, k^*) \notin E(G)$  then
22         $c'(s_i) \leftarrow c'(k^*)$ ;
23      else
24         $c'(s_i) \leftarrow c'(k^{**})$ ;
25    if  $w(c') < w(c)$  then
26       $c \leftarrow c'$ ;
27    foreach  $k_i \in K \setminus \{k^*\}$  do
28      for  $i = j, \dots, m$  do
29        if  $(s_i, k_i) \notin E(G)$  then
30           $c'(s_i) \leftarrow c'(k_i)$ ;
31        else
32           $c'(s_i) \leftarrow c'(k^*)$ ;
33      if  $w(c') < w(c)$  then
34         $c \leftarrow c'$ ;

```

**Result:**  $c$

---

■ We know, by Lemma 2.3, that there is an optimal weighted coloring of  $G$  that is an extension of an optimal weighted coloring of  $R$ . By Lemmas 2.4 and 2.5 we know that there is an optimal weighted coloring of  $G$  satisfying the conditions of both lemmas. Finally, we proved the following:

*Lemma 2.6:* Let  $G$  be a spider and  $S$  be an optimal weighted coloring of  $G$  satisfying Lemmas 2.4 and 2.5. Then, the coloring  $S$  when restricted to  $R$  is an optimal weighted coloring of  $G[R]$ .

*Proof:* If  $R = \emptyset$ , then the lemma is trivially true. Suppose then, by contradiction that  $S$  does not satisfy the lemma and let  $S_R$  be an optimal weighted coloring to  $R$ .

By Lemma 2.2, there is at most one color  $S_i$  of  $S$  containing vertices from both  $S$  and  $R$ . If there is no such color, then a coloring  $S'$  obtained from  $S$  by recoloring all the vertices of  $R$  like in the coloring  $S_R$  would have weight strictly smaller than the weight of  $S$ . This would be a contradiction to the optimality of  $S$ .

Suppose then that there is a color  $S_i$  containing vertices from  $S$  and  $R$ . Thus, by the same arguments of Lemma 2.3, this color contains a vertex  $r^*$  with the greatest weight of a vertex of  $S$ . Using the same ideas of Lemma 2.3, we may recolor the vertices of  $R$  like in  $S_R$  generating a coloring  $S'$  in such a way that  $w(S') < w(S)$ . It is just necessary to set the color of  $r^*$  to be the same of  $S_i$ . Observe that the colors of  $CK$  and  $cS$  do not change their weights and the sum of the weights of the colors in  $CR$  decreases. This is a contradiction to the optimality of  $S$ . ■

*Proposition 2.1:* Given a spider  $G = (S \cup K \cup R, E, w)$  and an optimal weighted coloring  $c_R$  of  $G[R]$ , then an optimal weighted coloring of  $G$  can be found in  $O(n^3)$  time.

*Proof:* The algorithm that calculates such a coloring is Algorithm 1. Its correctness follows from Lemmas 2.3, 2.4, 2.5 and 2.6. The vertices of  $G$  can be ordered by their weights in  $O(n \log n)$  and the vertices of  $K$  and  $R$  can be colored in linear time, provided we are given an optimal weighted coloring of  $G[R]$ . However, to color the vertices of  $S$ , we have to try all the colorings satisfying Lemmas 2.4 and 2.5 and this can take  $O(n^3)$  in the case we have a spider of type 2. The proposition follows. ■

*Corollary 2.1:* Let  $G$  be a weighted  $P_4$ -sparse graph whose modular decomposition tree  $T(G)$  satisfies the following statement: if  $T(G)$  contains a parallel node  $v$ , then  $v$  represents a module that is a cograph. Then an optimal weighted coloring of  $G$  can be found in  $O(n^3)$  time.

*Proof:* At first, the modular decomposition tree of  $G$ ,  $T(G)$ , can be found in linear time. Then, we do a pre-order traverse in  $T(G)$  by calculating  $\chi_w(G[M])$  at each node parallel node  $m$ , where  $M$  is the module defined by  $m$ . Since  $G[M]$  is a cograph, this can be done by using the already known algorithm for cographs. Finally, we have to visit  $T(G)$  in a post-order way and use Remark 2.1 and Proposition 2.1 to determine  $\chi_w(G[M])$  at each series or neighborhood node  $m$  of  $T(G)$ . ■

Observe that in Corollary 2.1, we present an algorithm to solve the W C problem for a subclass of  $P_4$ -sparse graphs which strictly contains cographs, since its modular decomposition tree may have modules whose the quotient graphs are isomorphic to spiders.

### B. Disjoint Union

To illustrate the problem tackled in this section, consider the  $P_4$ -sparse graph  $G = A \cup B$  of Fig. 1. An optimal coloring  $c_A$  of  $A$  with weight 5 is given by  $S_1 = \{k_1\}$ ,  $S_2 = \{k_2\}$ ,  $S_3 = \{k_3\}$ , and  $S_4 = \{s_1, s_2, s_3\}$ . An obvious optimal coloring  $c_B$  of  $B$  with weight 6 is given by  $S'_1 = \{u_1\}$ ,  $S'_2 = \{u_2\}$ , and  $S'_3 = \{u_3\}$ . If we combine both colorings by merging the color classes of  $c_A$  and  $c_B$  we obtain a coloring of  $G$  with weight 7. But there exists a better coloring  $c_G$  of  $G$  with weight 6 given

by  $S''_1 = \{s_1, k_1, u_1\}$ ,  $S''_2 = \{s_2, k_2, u_2\}$ , and  $S''_3 = \{s_3, k_3, u_3\}$ . This optimal coloring  $c_G$  restricted to  $A$  has weight 6, which is strictly greater than the weight of  $c_A$ .

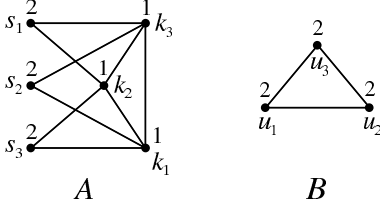


Fig. 1. An optimal weighted coloring of a disjoint union is not given by merging an optimal weighted coloring of each component.

The previous example shows that to compute an optimal weighted coloring of a disjoint union of two graphs, it is not enough to compute an optimal coloring of each component, and then merge the color classes appropriately (as happens for the classical vertex coloring problem). However, we could prove the following:

**Proposition 2.2:** Given a  $k$ -coloring  $S = (S_1, \dots, S_k)$  of a disconnected weighted graph  $G = G_1 \cup G_2 \cup \dots \cup G_m$ , such that each  $G_i$  is a connected component of  $G$ , for all  $i \in \{1, \dots, m\}$ , and  $w(S_1) \geq \dots \geq w(S_k)$ , we can construct a coloring  $S' = (S'_1, \dots, S'_{k'})$  such that  $w(S') \leq w(S)$  and the color class  $S'_i$ , restricted to the component  $G_j$ , is the  $i$ -th heaviest color class of  $G_j$ . Moreover,  $k' \leq k$ .

*Proof:* We need to introduce some extra notation. For  $i = 1, \dots, m$ , let  $S^i_1, \dots, S^i_{k'}$  be the stable sets induced by  $S$  on  $G^i$ , with  $w(S^i_1) \geq \dots \geq w(S^i_{k'})$ . If  $l < k'$ , for  $j = l + 1, \dots, k'$  we also consider, with slight abuse of notation, the empty sets  $S^i_j$  with  $w(S^i_j) = 0$ , for all  $i \in \{1, \dots, m\}$ .

Given this notation, we claim that:

*Claim 1:*

$$w(S_i) \geq \max_{j \in \{1, \dots, m\}} \{w(S^j_i)\}$$

For  $j = 1$  the claim is true, since the weight of  $S_1$  is given by the weight of a heaviest vertex in  $G$ , which equals  $\max\{w(S^1_1), \dots, w(S^m_1)\}$ . Suppose that the claim is not true for some  $j > 2$ , i.e.,  $w(S_j) < \max\{w(S^1_j), \dots, w(S^m_j)\}$ . Suppose, without loss of generality, that  $\max\{w(S^1_j), \dots, w(S^m_j)\} = w(S^1_j)$ . Then, by hypothesis:

$$\begin{aligned} w(S_j) &< \max\{w(S^1_j), \dots, w(S^m_j)\} = \\ w(S^1_j) &\leq w(S^1_{j-1}) \leq \dots \leq w(S^1_1) \leq \dots \leq w(S^1_1). \end{aligned} \quad (1)$$

For  $r = 1, \dots, j$ , let  $S_{q_r} \in \{S_1, \dots, S_k\}$  be the stable set of  $S$  containing  $S^1_r$ . Observe that, by definition, all these sets  $S_{q_r}$  are distinct. Then,

$$w(S^1_r) \leq w(S_{q_r}), \quad r = 1, \dots, j. \quad (2)$$

Combining Equations (1) and (2) we deduce that  $w(S_j) < w(S_{q_r})$ , for each  $r = 1, \dots, j$ . In other words, there exist  $j$  chromatic classes with weight strictly greater than  $w(S_j)$ , a contradiction to the hypothesis that  $w(S_1) \geq \dots \geq w(S_k)$ . Thus, claim follows.

Define then a coloring  $S'$  with color classes  $S'_1, \dots, S'_{k'}$  as follows:

$$S'_j := S^1_j \cup \dots \cup S^m_j, \quad j = 1, \dots, k'.$$

By the claim, it is not difficult to conclude that  $S'$  satisfies the Proposition. ■

Consider that  $\omega = \omega(G)$  is the size of a biggest clique of a graph  $G$ . As a consequence of the previous proposition, we can conclude the following corollary:

**Corollary 2.2:** Let  $G = G_1 \cup G_2 \cup \dots \cup G_m$  be a disconnected weighted graph, such that each connected component  $G_i = (S_i, K_i, R_i)$  is a spider with  $R_i = \emptyset$ , for all  $i \in \{1, \dots, m\}$ . Then, there exists an optimal weighted coloring of  $G$  with either  $\omega$  or  $\omega + 1$  colors.

*Proof:* At first, observe that  $\omega(G) = \max_{i \in \{1, \dots, m\}} \{\omega(G_i)\}$ . Suppose that  $S' = (S'_1, \dots, S'_{k'})$  is an optimal weighted coloring of  $G$ . Moreover, observe that each component  $G_i$  has at least  $\omega_i$  colors, for all  $i \in \{1, \dots, m\}$ . Let  $S^*_i$  be the subset of vertices of  $S_i$  colored with colors not belonging to the set of colors used by  $K_i$ . Then we may create a coloring  $S''$  by recoloring all the vertices of  $S^*_i$  with the color of a vertex with greatest weight in  $S^*_i$ , without increasing  $w(S')$ , i.e.,  $w(S'') \leq w(S')$ .

Now, using the previous proposition over the coloring  $S''$ , as we have at most  $\omega(G) + 1$  colors for each component of  $G$ , we may obtain an optimal weighted coloring for  $G$  using at most  $\omega(G) + 1$  colors. ■

### III. A A

To show our approximation algorithm, let us first consider the special partition given by Jamison and Olariu [10], [11]:

**Definition 3.1:** A graph  $G$  has a *special partition* if there exists a family  $\Sigma = \{S_1, \dots, S_q\}$  of disjoint stable sets of  $G$  with  $q \geq 1$  and  $|S_i| \geq 2$ , for all  $i \in \{1, \dots, q\}$ , and there exists an injection  $f : \bigcup_{i=1}^q S_i \rightarrow V - \bigcup_{i=1}^q S_i$  such that the following occurs:

- 1)  $K_i = \{z \mid z = f(s) \text{ for some } s \in S_i\}$  is a clique, for all  $i \in \{1, \dots, q\}$ ;
- 2) A set of vertices  $A$  induces a  $P_4$  in  $G$  if, and only if, there exists a subscript  $i \in \{1, \dots, q\}$  and distinct vertices  $x, y \in S_i$  such that  $A = \{x, y, f(x), f(y)\}$ .

Let us define  $S = \bigcup_{i=1}^q S_i$  and  $K = V - \bigcup_{i=1}^q S_i$ . Observe that the graphs induced by  $S$  and  $K$  are cographs and their weighted chromatic number can be determined in polynomial time [4].

**Theorem 3.1 ([8]):** A graph is a  $P_4$ -sparse graph if, and only if, it is a cograph or it has a special partition.

Then, we can state the following:

**Proposition 3.1:** There exists a linear time approximation algorithm for W C on  $P_4$ -sparse graphs with approximation ratio bounded above by 2.

*Proof:* We claim that if  $G$  and  $H$  are weighted graphs such that  $H \subseteq G$ , then,  $\chi_w(H) \leq \chi_w(G)$ . For otherwise, if  $H \subseteq G$  is a counterexample and  $c$  is an optimal weighted coloring of  $G$ , by restricting  $c$  to the vertices of  $H$ , we would obtain a proper coloring  $c'$  of  $H$  such that  $w(c') < \chi_w(H)$ . As

a consequence of this claim, our approximation algorithm will just color the cographs  $S$  and  $K$  in linear time with disjoint sets of colors. Once  $\chi_w(S) \leq \chi_w(G)$  and  $\chi_w(K) \leq \chi_w(G)$ , the proof is completed. ■

#### IV. F R

We finish the paper with the following conjecture:

*Conjecture 4.1:* There is a polynomial-time algorithm to solve the W C problem on  $P_4$ -sparse graphs.

#### A

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