

# On the Holonomy or Algebraicity of Generating Functions Counting Lattice Walks in the Quarter-Plane

Guy Fayolle, Kilian Raschel

► **To cite this version:**

Guy Fayolle, Kilian Raschel. On the Holonomy or Algebraicity of Generating Functions Counting Lattice Walks in the Quarter-Plane. Markov Processes and Related Fields, Polymath, 2010, 16 (3), pp.485-496. <inria-00469603>

**HAL Id: inria-00469603**

**<https://hal.inria.fr/inria-00469603>**

Submitted on 2 Apr 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

*On the Holonomy or Algebraicity of Generating  
Functions Counting Lattice Walks in the  
Quarter-Plane*

Guy Fayolle — Kilian Raschel

N° 7242

Avril 2010

Thème NUM



*R*apport  
de recherche



# On the Holonomy or Algebraicity of Generating Functions Counting Lattice Walks in the Quarter-Plane

Guy Fayolle\* , Kilian Raschel †

Thème NUM — Systèmes numériques  
Équipe-Projet Imara

Rapport de recherche n° 7242 — Avril 2010 — 12 pages

**Abstract:** In two recent works [2, 1], it has been shown that the counting generating functions (CGF) for the 23 walks with small steps confined in a quarter-plane and associated with a finite group of birational transformations are holonomic, and even algebraic in 4 cases – in particular for the so-called Gessel’s walk. It turns out that the type of functional equations satisfied by these CGF appeared in a probabilistic context almost 40 years ago. Then a method of resolution was proposed in [4], involving at once algebraic tools and a reduction to boundary value problems. Recently this method has been developed in a combinatorics framework in [11], where a thorough study of the explicit expressions for the CGF is proposed. The aim of this paper is to derive the nature of the bivariate CGF by a direct use of some general theorems given in [4].

**Key-words:** Algebraic, holonomic, generating function, piecewise homogeneous lattice walk, quarter-plane, universal covering, Weierstrass elliptic function.

\* INRIA Paris-Rocquencourt, Domaine de Voluceau, BP 105, 78153 Le Chesnay Cedex, France. [Guy.Fayolle@inria.fr](mailto:Guy.Fayolle@inria.fr)

† Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris Cedex 05, France. [kilian.raschel@upmc.fr](mailto:kilian.raschel@upmc.fr)

## Sur l'holonomie ou l'algébricité de fonctions génératrices comptant des marches aléatoires dans le quart de plan

**Résumé :** Dans deux articles récents [2, 1], il a été montré que les fonctions génératrices de comptage (CGF) pour 23 marches avec pas unité, évoluant dans le quart de plan et associées à un groupe fini de transformations bi-rationnelles, sont holonomes et même algébriques dans 4 cas – notamment celui de la marche dite de Gessel. Or il s'avère que le type d'équations fonctionnelles vérifiées par ces CGF est apparu il y a environ 40 ans, dans le contexte probabiliste des marches aléatoires. Une méthode de résolution fut proposée dans [4], basée à la fois sur des techniques algébriques et sur la réduction à des problèmes aux limites. Récemment cette méthode a été développée dans un cadre combinatoire dans [11], où une étude fouillée des formes explicites des CGF a été menée. L'objet du présent article est de retrouver la nature des CGF bivariées par une application directe de théorèmes donnés dans [4].

**Mots-clés :** Algébrique, holonome, fonction génératrice, marche discrète homogène par morceaux, quart de plan, couverture universelle, fonction elliptique de Weierstrass.

## 1 Introduction

The enumeration of planar lattice walks is a classical topic in combinatorics. For a given set  $\mathcal{S}$  of allowed jumps (or steps), it is a matter of counting the number of paths starting from some point and ending at some arbitrary point in a given time, and possibly restricted to some regions of the plane. A first basic and natural question arises: how many such paths exist? A second question concerns the nature of the associated counting generating functions (CGF): are they rational, algebraic, holonomic (or D-finite, i.e. solution of a linear differential equation with polynomial coefficients)?

For instance, if no restriction is made on the paths, it is well-known that the CGF are rational and easy to obtain explicitly. As an other example, if the walks are supposed to remain in a half-plane, then the CGF can also be computed and turn out to be algebraic in this case, see e.g. [3].

Next it is quite natural to consider walks evolving in a domain formed by the intersection of two half-planes, for instance the positive quarter-plane  $\mathbb{Z}_+^2$ . In this case, the problems become more intricate and multifarious results appeared: indeed, some walks admit of an algebraic generating function, see e.g. [7] and [8] for the walk with step set  $\mathcal{S} = \{(-1, 0), (1, 1), (0, -1)\}$  and starting from  $(0, 0)$ , whereas some others admit a CGF which is even not holonomic, see e.g. [3] for the walk with  $\mathcal{S} = \{(-1, 2), (2, -1)\}$  and starting from the point  $(1, 1)$ .

In this framework, M. Bousquet-Mélou and M. Mishna have initiated in [2] a systematic study of the nature of the walks confined to  $\mathbb{Z}_+^2$ , starting from the origin and having steps of size 1, which means that  $\mathcal{S}$  is included in the set  $\{(i, j) : |i|, |j| \leq 1\} \setminus \{(0, 0)\}$ . Examples of such walks are shown in figures 1.1 and 1.2 below.

A priori, there are  $2^8$  such models. In fact, after eliminating trivial cases and models equivalent to walks confined to a half-plane, and noting also that some models are obtained from others by symmetry, it is shown in [2] that one is left with 79 inherently different problems to analyze.

A common starting point to study these 79 walks relies on the following analytic approach. Let  $q(i, j, k)$  denote the number of paths in  $\mathbb{Z}_+^2$  starting from  $(0, 0)$  and ending at  $(i, j)$  at time  $k$ . Then the corresponding CGF

$$F(x, y, z) = \sum_{i, j, k \geq 0} f(i, j, k) x^i y^j z^k \tag{1.1}$$

satisfies the functional equation obtained in [2]

$$K(x, y)F(x, y, z) = c(x)F(x, 0, z) + \tilde{c}(y)F(0, y, z) + c_0(x, y), \tag{1.2}$$

where

$$\begin{cases} K(x, y) = xy \left[ \sum_{(i, j) \in \mathcal{S}} x^i y^j - 1/z \right], \\ c(x) = \sum_{(i, -1) \in \mathcal{S}} x^{i+1}, \\ \tilde{c}(y) = \sum_{(-1, j) \in \mathcal{S}} y^{j+1}, \\ c_0(x, y) = -\delta F(0, 0, z) - xy/z, \end{cases}$$

with  $\delta = 1$  if  $(-1, -1) \in \mathcal{S}$  [i.e. a south-west jump exists],  $\delta = 0$  otherwise.

For  $z = 1/|\mathcal{S}|$ , (1.2) plainly belongs to the generic class of functional equations (arising in the probabilistic context of random walks) studied and solved in the book [4], see section A

of our appendix. For general values of  $z$ , the analysis of (1.2) for the 79 above-mentioned walks has been carried out in [11], where the integrand of the integral representations is studied in detail, via a complete characterization of ad hoc conformal gluing functions.

It turns out that one of the basic tools to solve equation (1.2) is to consider the group exhaustively studied in [4], and originally proposed in [10] where it was called the *group of the random walk*. Starting with this approach, the authors of [2] consider the group of birational transformations leaving invariant the generating function  $\sum_{(i,j) \in \mathcal{S}} x^i y^j$ , which is precisely the group  $W = \langle \xi, \eta \rangle$  generated by

$$\xi(x, y) = \left( x, \frac{1}{y} \frac{\sum_{(i,-1) \in \mathcal{S}} x^i}{\sum_{(i,+1) \in \mathcal{S}} x^i} \right), \quad \eta(x, y) = \left( \frac{1}{x} \frac{\sum_{(-1,j) \in \mathcal{S}} y^j}{\sum_{(+1,j) \in \mathcal{S}} y^j}, y \right).$$

Clearly  $\xi \circ \xi = \eta \circ \eta = \text{id}$ , and  $W$  is a dihedral group of even order larger than 4. In [2] this order is calculated for each of the above-mentioned 79 cases: 56 walks admit an infinite group, while the group of the 23 remaining ones is finite.

It is also proved in [2] that among these 23 walks,  $W$  has order 4 for 16 walks (the ones with a step set having a vertical symmetry),  $W$  has order 6 for 5 walks (the 2 at the left in figure 1.1 and the 3 ones at the left in figure 1.2), and  $W$  has order 8 for the 2 walks on the right in figures 1.1 and 1.2. Moreover, for these 23 walks, the answers to both main questions (explicit expression *and* nature of the CGF (1.1)) are known. In particular, the following results exist.

**Theorem 1.1** ([2]). *For the 16 walks with a group of order 4 and for the 3 walks in figure 1.1, the formal trivariate series (1.1) is holonomic non-algebraic. For the 3 walks on the left in figure 1.2, the trivariate series (1.1) is algebraic.*

The proof of this theorem relies on skillful algebraic manipulations together with the calculation of adequate *orbit* and *half-orbit* sums.

**Theorem 1.2** ([1]). *For the so-called Gessel's walk on the right in figure 1.2, the formal trivariate series (1.1) is algebraic.*

The proof given in [1] involves a powerful computer algebra system (Magma), allowing to carry out dense calculations.

The main goal of our paper will be to present another proof of theorems 1.1 and 1.2 for the bivariate generating function  $(x, y) \mapsto F(x, y, z)$ , by means of the general and powerful approach proposed in the book [4]. In particular, we are going to show how chapter 4 of this book (which deals with walks having a finite group) yields rather directly the nature of the bivariate generating functions (1.1) for the 23 walks associated with a finite group. The rest of the paper has a simple organization. Indeed, section 2 recalls the results of [4], used in Section 3, where we find the nature of the counting generating functions coming in (1.2).

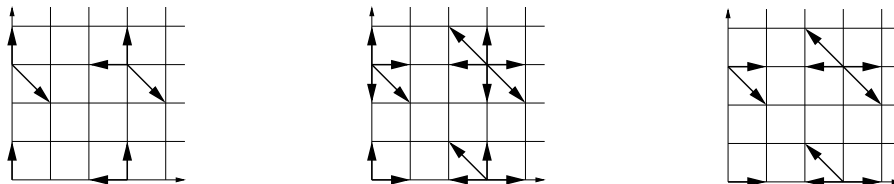


Fig. 1.1: On the left, 2 walks with a group of order 6. On the right, 1 walk with a group of order 8

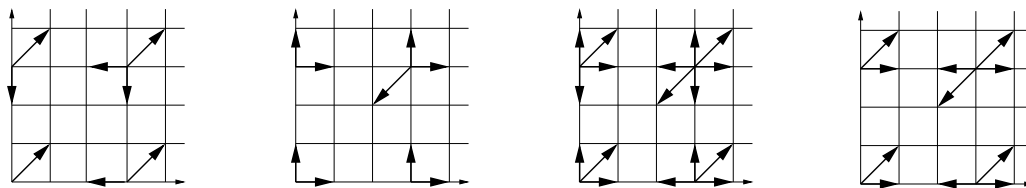


Fig. 1.2: On the left, 3 walks with a group of order 6. On the right, 1 walk with a group of order 8

## 2 Some results of [4] with miscellaneous extensions

In this section we use the notation of appendix A, where is presented the generic functional equation (A.1) studied in [4]. It clearly contains (1.1) as a particular case, since the coefficients  $q$ ,  $\tilde{q}$  and  $q_0$  occurring in (A.1) may depend on the *two* variables  $x, y$ . Moreover, although this be not crucial, the unknown function  $\pi(x)$  [resp.  $\tilde{\pi}(y)$ ] is not necessarily equal to  $\pi(x, 0)$  [resp.  $\tilde{\pi}(0, y)$ ].

Chapter 5 of [4] explains how to solve (A.1) by reduction to boundary value problems of Riemann-Hilbert-Carleman type, and explicit expressions for the unknown functions are obtained.

Independently, *when the group of the walk is finite*, chapter 4 of [4] provides another approach allowing to characterize completely the solutions of (A.1), and also to give *necessary and sufficient conditions* for these solutions *to be rational or algebraic*. We shall proceed along these lines in the sequel.

### 2.1 The group and the genus

Let  $\mathbb{C}(x, y)$  be the field of rational functions in  $(x, y)$  over  $\mathbb{C}$ . Since  $Q$  is assumed to be irreducible in the general case, the quotient field  $\mathbb{C}(x, y)$  denoted by  $\mathbb{C}_Q(x, y)$  is also a field.

**Definition 2.1.** *The group of the random walk is the Galois group  $\mathcal{H} = \langle \xi, \eta \rangle$  of automorphisms of  $\mathbb{C}_Q(x, y)$  generated by  $\xi$  and  $\eta$  given by*

$$\xi(x, y) = \left( x, \frac{1}{y} \frac{\sum_i p_{i,-1} x^i}{\sum_i p_{i,+1} x^i} \right), \quad \eta(x, y) = \left( \frac{1}{x} \frac{\sum_j p_{-1,j} y^j}{\sum_j p_{+1,j} y^j}, y \right).$$

Let

$$\delta = \eta\xi.$$

Then  $\mathcal{H}$  has a normal cyclic subgroup  $\mathcal{H}_0 = \{\delta^i, i \in \mathbb{Z}\}$ , which is finite or infinite, and  $\mathcal{H}/\mathcal{H}_0$  is a group of order 2. When the group  $\mathcal{H}$  is finite of order  $2n$ , we have  $\delta^n = \text{id}$ . We shall write  $f_\alpha = \alpha(f)$  for any automorphism  $\alpha \in \mathcal{H}$  and any function  $f \in \mathbb{C}_Q(x, y)$ .

In [2], the group  $W$  is introduced only in terms of two birational transformations. The difference between the two approaches is not only of a formal character. In fact  $W$  is defined in all of  $\mathbb{C}^2$ , whereas  $\mathcal{H}$  in [4] would act only on an algebraic curve of the type

$$\left\{ (x, y) \in \mathbb{C}^2 : R(x, y) = xy \left[ \sum_{i,j} r_{i,j} x^i y^j - 1/z \right] = 0 \right\}.$$

An immediate question arises: do the notions of order and finiteness coincide in the respective approaches? In general, the answer is clearly no. Indeed (adapting [4], lemma



4.1.1) for the group to be of order 4 it is necessary and sufficient to have

$$\begin{vmatrix} r_{1,1} & r_{1,0} & r_{1,-1} \\ r_{0,1} & r_{0,0} - 1/z & r_{0,-1} \\ r_{-1,1} & r_{-1,0} & r_{-1,-1} \end{vmatrix} = 0.$$

and this condition depends on  $z$ , while  $W$  is independent of  $z$ . On the other hand, if  $W$  is finite so is  $\mathcal{H}$ , and conversely if  $\mathcal{H}$  is infinite so is  $W$ . In addition, with an obvious notation,

$$\text{Order}(\mathcal{H}) \leq \text{Order}(W). \quad (2.1)$$

In the sequel, we shall encounter groups of order not larger than 8.

Besides, in [4], the algebraic curve is associated with a Riemann surface which is of *genus*  $g = 0$  (the sphere) or  $g = 1$  (the torus). Accordingly, the *universal covering* of this surface is

- the Riemann sphere  $\mathbb{P}^1$  if  $g = 0$ ;
- the finite complex plane  $\mathbb{C}_\omega$  if  $g = 1$ .

All 23 random walks we consider in this paper correspond indeed to  $g = 1$ , and an efficient uniformization is briefly described in appendix B. Moreover, they have a finite group, say of order  $2n$  (this will be made precise in section 3).

## 2.2 Algebraicity and holonomy

For any  $h \in \mathbb{C}_Q(x, y)$ , let the *norm*  $N(h)$  be defined as

$$N(h) \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} h_{\delta^i}. \quad (2.2)$$

Written on  $Q(x, y) = 0$ , equation (A.1) yields the system

$$\begin{cases} \pi = \pi_\xi, \\ \pi_\delta - f\pi = \psi, \end{cases} \quad (2.3)$$

where

$$f = \frac{q\tilde{q}_\eta}{\tilde{q}q_\eta}, \quad \psi = \frac{q_0\tilde{q}_\eta}{q_\eta\tilde{q}} - \frac{(q_0)_\eta}{q_\eta}.$$

It turns out that the analysis of (A.1) given in [4] deeply differs, according  $N(f) = 1$  or  $N(f) \neq 1$ . In section 3, as for the above-mentioned walks, it will be shown that we are in the case  $N(f) = 1$  and we need the following important theorem stated in a terse form.

**Theorem 2.2.** *If  $N(f) = 1$ , then the general solution of the fundamental system (2.3) has the form*

$$\pi = w_1 + w_2 + \frac{w}{r},$$

where

- the function  $w_1$  and  $r$  are rational;
- the function  $w_2$  is given by

$$w_2 = \frac{\tilde{\Phi}}{n} \sum_{k=0}^{n-1} \left( \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} \right); \quad (2.4)$$

– on the universal covering  $\mathbb{C}_\omega$ , we have

$$\tilde{\Phi}\left(\omega + \frac{\omega_2}{2}\right) \stackrel{\text{def}}{=} \frac{\omega_1}{2i\pi} \zeta(\omega; \omega_1, \omega_3) - \frac{\omega}{i\pi} \zeta\left(\frac{\omega_1}{2}; \omega_1, \omega_3\right), \quad (2.5)$$

where  $\zeta(\omega; \omega_1, \omega_3) \stackrel{\text{def}}{=} \zeta_{1,3}(\omega)$  stands for the classical Weierstrass  $\zeta$ -function (see [9]) with quasi periods  $\omega_1, \omega_3$ ;

– the function  $w$  is algebraic and satisfies the automorphy conditions  $w = w_\xi = w_\delta$ . ■

The quantities  $\omega_1, \omega_2, \omega_3$  have explicit forms (see [4], lemmas 3.3.2 and 3.3.3) and the automorphism  $\delta$  writes simply as the translation

$$\delta(\omega) = \omega + \omega_3, \quad \forall \omega \in \mathbb{C}_\omega.$$

In addition, when the group is finite of order  $2n$ , there exists an integer  $k$  relatively prime with  $n$  satisfying the relation

$$n\omega_3 = k\omega_2,$$

which in turn allows to prove (see [4], lemma 4.3.5) that  $\tilde{\Phi}(\omega)$  defined in (2.5) is *not* algebraic in  $x(\omega)$ . We are now in a position to derive the following corollary.

**Corollary 2.3.** *Assume the group is finite of order  $2n$  and  $N(f) = 1$ . Then the solution  $\pi(x)$  of (A.1) is holonomic. Moreover it is algebraic if and only if on  $Q(x, y) = 0$ ,*

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = 0. \quad (2.6)$$

In fact, the proof is an immediate consequence of the next lemma.

**Lemma 2.4.** *In the uniformization given in appendix B, let us write*

$$\wp(\omega) \stackrel{\text{def}}{=} g(x(\omega)),$$

where  $g$  is a fractional linear transform. Then the function  $w_2(\wp^{-1}(g(x)))$  defined by (2.4) is holonomic in  $x$ , where  $\wp^{-1}$  denotes the elliptic integral inverse function of  $\wp$ .

**Proof.** Since  $w_2$  is the product of  $\tilde{\Phi}$  by an algebraic function, it suffices to prove that  $\tilde{\Phi}(\wp^{-1}(g))$  is holonomic. It is known that the class of holonomic functions is closed under indefinite integration (see e.g. [6]), thus it is enough to prove that  $[\tilde{\Phi}(\wp^{-1}(g))]'$  is holonomic. In fact, we are going to show that  $[\tilde{\Phi}(\wp^{-1}(g))]'$  is algebraic – we recall that any algebraic function is holonomic. Indeed, use on the one hand that the derivative of  $-\zeta_{1,3}$  is  $\wp_{1,3}$ , the Weierstrass elliptic function with periods  $\omega_1, \omega_3$ , and on the other hand that

$$[\wp^{-1}]'(u) = 1/[4u^3 - g_2u - g_3]^{1/2},$$

$g_2$  and  $g_3$  being the so-called invariants of  $\wp$  – for these two properties see e.g. [9]. In this way we get

$$-2i\pi[\tilde{\Phi}(\wp^{-1}(g))]' = \frac{g'}{[4g^3 - g_2g - g_3]^{1/2}} [\omega_1\wp_{1,3}(\wp^{-1}(g) - \omega_2/2) + 2\zeta_{1,3}(\omega_1/2)].$$

The function  $g$  being rational,  $g'/[4g^3 - g_2g - g_3]^{1/2}$  is algebraic and in order to conclude it is enough to prove that  $\wp_{1,3}(\wp^{-1}(g) - \omega_2/2)$  is algebraic. Since  $n\omega_3 = k\omega_2$ , it is shown in [4], lemma 4.3.3 that both  $\wp$  and  $\wp_{1,3}$  are rational functions of the Weierstrass elliptic function with periods  $\omega_1, k\omega_2$ . Then it is immediate that  $\wp_{1,3}$  is also algebraic function of  $\wp$ . Moreover, by the well-known addition theorem for the Weierstrass elliptic functions (see e.g. [9]),  $\wp$  is a rational function of  $\wp(\cdot + \omega_2/2)$ , so that  $\wp_{1,3}$  is also an algebraic function of  $\wp(\cdot + \omega_2/2)$ . In particular  $\wp_{1,3}(\wp^{-1}(g) - \omega_2/2)$  is algebraic and lemma 2.4 is proved. ■

Of course similar results can be written for  $\tilde{\pi}$ . In particular, it is easy to check that if  $N(f) = 1$  then also  $\tilde{N}(f) = 1$ , where  $\tilde{N}(f) = 1$  is defined in terms of  $\tilde{\delta} \stackrel{\text{def}}{=} \xi\eta$ . Thus we are in a position to state the following general result.

**Theorem 2.5.** *Assume the group is finite of order  $2n$  and  $N(f) = 1$ . Then the bivariate series  $\pi(x, y)$  solution of (A.1) is holonomic. Furthermore  $\pi(x, y)$  is algebraic if and only if (2.6) and  $(\tilde{2.6})$  hold on  $Q(x, y) = 0$ ,  $(\tilde{2.6})$  denoting the condition for  $\tilde{\pi}(y)$  to be algebraic, obtained by symmetry from (2.6). ■*

### 3 Nature of the counting generating functions

We return now to equation (1.2) and make use of the machinery of section 2 with

$$f = \frac{\tilde{c}\tilde{c}_\eta}{\tilde{c}c_\eta}, \quad \psi = \frac{c_0\tilde{c}_\eta}{c_\eta\tilde{c}} - \frac{(c_0)_\eta}{c_\eta}. \quad (3.1)$$

Beforehand it is important to note that the variable  $z$  plays here the role of a *parameter*. Moreover, the CGF (1.1) is well defined and analytic in a region containing the domain

$$\{(x, y, z) \in \mathbb{C}^3 : |x| \leq 1, |y| \leq 1, |z| < 1/|\mathcal{S}|\}.$$

The analysis in [4] aimed at finding the stationary measure of the random walk, in which case  $z = 1$ . But it is possible to check *mutatis mutandis* that most of the topological results remains valid at least for  $\Re(z) \geq 0$  [see for instance [5] or [11] for time dependent problems].

**Lemma 3.1.** *For any finite  $n$  and  $f$  given by (3.1), we have  $N(f) = 1$ .*

**Proof.** Since  $c$  [resp.  $\tilde{c}$ ] is a function solely of  $x$  [resp.  $y$ ], we have  $c = c_\xi$  and  $\tilde{c} = \tilde{c}_\eta$ . Hence

$$f = \frac{c}{c_\eta} = \frac{c_\delta}{c_{\delta\eta}} = \frac{c_\delta}{c},$$

which yields immediately  $N(f) = 1$  from the definition (2.2). ■

In order to obtain theorems 1.1 and 1.2 for the bivariate function  $(x, y) \mapsto F(x, y, z)$ , it is thus enough, by theorem 2.5 and lemma 3.1, to prove the following final proposition.

**Proposition 3.2.** *For the 4 walks in figure 1.2, (2.6) and  $(\tilde{2.6})$  hold in  $\mathbb{C}^2$ . As for the 19 other walks (3 in figure 1.1 and the 16 ones with a vertically symmetric step set  $\mathcal{S}$ ), (2.6) or  $(\tilde{2.6})$  do not hold on  $\{(x, y) \in \mathbb{C}^2 : K(x, y) = xy[\sum_{(i,j) \in \mathcal{S}} x^i y^j - 1/z] = 0\}$ , for any  $z$ .*

**Proof.** The proof proceeds in three stages.

(i) *The walks in figure 1.2.*

Let us check (2.6) for the 4 walks in figure 1.2. We begin with the popular Gessel's walk, i.e. the rightmost one in figure 1.2. Here  $c = 1$  in (1.2), so that by (3.1)  $f = 1$  and  $\psi = [(xy)_\eta - xy]/z$ . Moreover, the order of the group  $W$  was shown in [2] to be equal to 8. According to (2.1), we know that  $\text{Order}(\mathcal{H}) \leq 8$ . On the other hand, by lemmas 4.1.1 and 4.1.2 of [4], a simple algebra (details are omitted) can show that the order of  $\mathcal{H}$  cannot be equal to 4 neither to 6, which entails  $\text{Order}(\mathcal{H}) = 8$ , hence  $n = 4$ .

Then

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \sum_{k=0}^3 \psi_{\delta^k} = \frac{1}{z} \sum_{k=0}^3 [(xy)_{\eta\delta^k} - (xy)_{\delta^k}] = -\frac{1}{z} \sum_{\rho \in \mathcal{H}} \text{sign}(\rho)(xy)_\rho. \quad (3.2)$$

The last quantity in (3.2) can be understood as  $(-1/z)$  times the orbit sum of  $xy$  through the group  $\mathcal{H}$ , and it was remarked in [2] that this orbit sum equals zero for Gessel's walk, which is equivalent to (2.6) in  $\mathbb{C}^2$ .

For the second walk in figure 1.2,  $Order(W) = Order(\mathcal{H}) = 6$  ( $n = 3$ ) and still  $c = 1$ , so that the above argument applies.

For the walk on the left in figure 1.2, we have

$$\xi = (x, 1/(xy)), \quad \eta = (1/(xy), y),$$

hence  $\xi\eta = (1/(xy), x)$ . Moreover  $c = x$ , thus by using (3.1) we get  $f = x^2y$  and  $\psi = [y - (xy)^2]/z$ . Next one easily computes  $f_\delta = 1/(xy^2)$ ,  $f_{\delta^2} = y/x$ ,  $\psi_\delta = [x - 1/y^2]/z$  and  $\psi_{\delta^2} = [1/(xy) - 1/x^2]/z$ . Finally an immediate calculation shows that (2.6) holds in  $\mathbb{C}^2$ .

As for the last walk in figure 1.2, we could check  $n = 3$  and again (2.6) holds in  $\mathbb{C}^2$ .

Concerning equation (2.6), only Gessel's walk needs some special care, since the 3 others give rise to symmetric conditions. In fact, from a direct calculation along the same lines as above, it is not difficult to see that (2.6) holds in  $\mathbb{C}^2$ .

**(ii)** *The 16 walks with a vertical symmetry.*

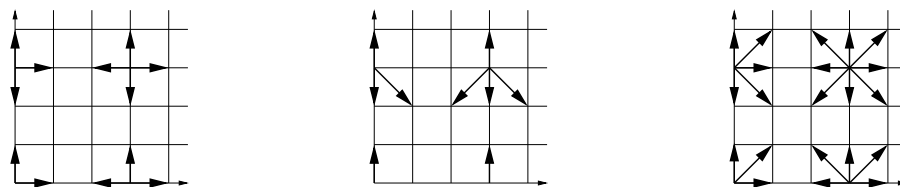


Fig. 3.3: Three examples of walks having a group of order 4, with symmetrical *West* and *East* jumps

We will show the equality

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = -\frac{x^2}{zc}(x - x_\eta)(y - y_\xi), \quad (3.3)$$

which clearly implies that (2.6) will not be satisfied on  $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$ , for any  $z$ .

For these 16 walks we have  $Order(W) = 4 = Order(\mathcal{H})$ , that is  $n = 2$ . Moreover, the vertical symmetry of the jump set  $\mathcal{S}$  has two consequences : first, necessarily  $\eta = (1/x, y)$ ; second, the coefficient  $c$  in (1.2) must be equal to  $x$ ,  $1 + x^2$  or  $1 + x + x^2$ . In addition, for any of these three possible values of  $c$ , we have  $f = c/c_\eta = x^2$ . Hence finally

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \psi + \frac{\psi_\delta}{f_\delta} = \frac{(xy)_\eta - xy}{zc_\eta} + \frac{(xy)_\xi - (xy)_\delta}{zc_\xi f_\delta} = \frac{y(x_\eta - x)}{zc_\eta} + \frac{y_\xi(x - x_\eta)}{zc(x^2)_\delta}.$$

Factorizing by  $x^2/c$  and using again  $c/c_\eta = x^2$ , we obtain (3.3).

**(iii)** *The 3 walks in figure 1.1.*

We are going to show that (2.6) does not hold on  $K(x, y) = 0$ , for any  $z$ .

By similar calculations as in (3.3), we obtain that for the 2 first walks in figure 1.1 (which are such that  $n = 3$ ),

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \frac{(x-y^2)(1-xy)(y-x^2)}{zy^3t},$$

with  $t = y$  for the walk at the left and  $t = x + y$  for the walk in the middle, so that, clearly, (2.6) does not hold on  $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$ , for any  $z$ .

At last, for the walk at the right in figure 1.1, we have  $n = 4$ , and we get

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \frac{(y-1)(x^2-1)(x^2-y)(x^2-y^2)}{xy^4z},$$

which is obviously not identically to zero in the set  $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$ , for any  $z$ . ■

## A The basic functional equations (see [4], chapters 2 and 5)

In a probabilistic framework, one considers a piecewise homogeneous random walk with sample paths in  $\mathbb{Z}_+^2$ . In the interior of  $\mathbb{Z}_+^2$ , the size of the jumps is 1. On the other hand, no assumption is made about the boundedness of the upward jumps on the axes, neither at  $(0, 0)$ . In addition, the downward jumps on the  $x$  [resp.  $y$ ] axis are bounded by  $L$  [resp.  $M$ ], where  $L$  and  $M$  are arbitrary finite integers. Then the invariant measure  $\{\pi_{i,j}, i, j \geq 0\}$  does satisfy the fundamental functional equation

$$Q(x, y)\pi(x, y) = q(x, y)\pi(x) + \tilde{q}(x, y)\tilde{\pi}(y) + q_0(x, y), \quad (\text{A.1})$$

with

$$\left\{ \begin{array}{l} \pi(x, y) = \sum_{i,j \geq 1} \pi_{ij} x^{i-1} y^{j-1}, \\ \pi(x) = \sum_{i \geq 1} \pi_{i0} x^{i-1}, \quad \tilde{\pi}(y) = \sum_{j \geq 1} \pi_{0j} y^{j-1}, \\ Q(x, y) = xy \left[ 1 - \sum_{i,j \in \mathcal{S}} p_{ij} x^i y^j \right], \quad \sum_{i,j \in \mathcal{S}} p_{ij} = 1, \\ q(x, y) = x \left[ \sum_{i \geq -1, j \geq 0} p'_{ij} x^i y^j - 1 \right], \\ \tilde{q}(x, y) = y \left[ \sum_{i \geq 0, j \geq -1} p''_{ij} x^i y^j - 1 \right], \\ q_0(x, y) = \pi_{00} [P_{00}(x, y) - 1]. \end{array} \right.$$

In equation (A.1), the functions  $\pi(x, y), \pi(x), \tilde{\pi}(y)$  are unknown and sought to be analytic in the region  $\{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$  and continuous on their respective boundaries, while the  $q, \tilde{q}, q_0$  (up to a constant) are given probability generating functions supposed to have suitable analytic continuations (as a rule, they are polynomials when the jumps are bounded). The function  $Q(x, y)$  is often referred to as the *kernel* of (A.1).

## B About the uniformization of $Q(x, y) = 0$ (see [4], chapter 3)

When the associated Riemann surface is of genus 1, the algebraic curve  $Q(x, y) = 0$  admits a uniformization given in terms of the Weierstrass  $\wp$  function with periods  $\omega_1, \omega_2$  and its derivatives. Indeed, setting

$$\begin{aligned} Q(x, y) &= a(x)y^2 + b(x)y + c(x), \\ D(x) &= b^2(x) - 4a(x)c(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0, \\ u &= 2a(x)y + b(x), \end{aligned}$$

the following formulae hold (see [4], lemma 3.3.1).

- (i) If  $d_4 \neq 0$  (4 finite branch points  $x_1, \dots, x_4$ ) then  $D'(x_4) > 0$  and

$$\begin{cases} x(\omega) = x_4 + \frac{D'(x_4)}{\wp(\omega) - \frac{1}{6}D''(x_4)}, \\ u(\omega) = \frac{D'(x_4)\wp'(\omega)}{2(\wp(\omega) - \frac{1}{6}D''(x_4))^2}. \end{cases}$$

- (ii) If  $d_4 = 0$  (3 finite branch points  $x_1, x_2, x_3$  and  $x_4 = \infty$ ) then

$$\begin{cases} x(\omega) = \frac{\wp(\omega) - \frac{d_2}{3}}{d_3}, \\ u(\omega) = -\frac{\wp'(\omega)}{2d_3}. \end{cases}$$

## References

- [1] BOSTAN A. AND KAUSERS M. The complete generating function for Gessel walks is algebraic. *Preprint*: <http://arxiv.org/abs/0909.1965> (2009), 1–14.
- [2] BOUSQUET-MÉLOU M. AND MISHNA M. Walks with small steps in the quarter plane. *Preprint*: <http://arxiv.org/abs/0810.4387> (2009), 1–34.
- [3] BOUSQUET-MÉLOU M. AND PETKOVSEK M. Walks confined in a quadrant are not always D-finite. *Theor. Comput. Sci.* 307, 2 (2003), 257–276.
- [4] FAYOLLE G., IASNOGORODSKI R. AND MALYSHEV V. *Random walks in the quarter-plane*, vol. 40 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1999. Algebraic methods, boundary value problems and applications.
- [5] FAYOLLE G., IASNOGORODSKI R. AND MITRANI I. The distribution of sojourn times in a queueing network with overtaking: Reduction to a boundary problem. In *Performance '83: Proceedings of the 9th International Symposium on Computer Performance Modelling, Measurement and Evaluation* (Amsterdam, The Netherlands, The Netherlands, 1983), North-Holland Publishing Co., pp. 477–486.
- [6] FLAJOLET P. AND SEDGEWICK R. *Analytic Combinatorics*. Cambridge Studies in Advances Mathematics. Cambridge University Press, 2008.
- [7] FLATTO L. AND HAHN S. Two parallel queues created by arrivals with two demands. I. *SIAM J. Appl. Math.* 44, 5 (1984), 1041–1053.

- [8] GESSEL I. A probabilistic method for lattice path enumeration. *J. Statist. Plann. Inference* 14, 1 (1986), 49–58.
- [9] HURWITZ A. AND COURANT R. *Funktionen Theorie*, fourth ed. Springer Verlag, 1964.
- [10] MALYSHEV V. Positive random walks and Galois theory. *Uspehi Matem. Nauk* 1 (1971), 227–228.
- [11] RASCHEL K. Counting walks in a quadrant : a unified approach via boundary value problems. *Preprint*: <http://arxiv.org/abs/> (2010), 1–22.



---

Centre de recherche INRIA Paris – Rocquencourt

Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex

Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier

Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq

Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique

615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex

Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex

Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex

Centre de recherche INRIA Sophia Antipolis – Méditerranée : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex

---

Éditeur

INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)

<http://www.inria.fr>

ISSN 0249-6399