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# Linear time recognition of $P_4$ -indifference graphs

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A graph is a  $P_4$ -indifference graph if it admits an ordering  $<$  on its vertices such that every chordless path with vertices  $a, b, c, d$  and edges  $ab, bc, cd$  has  $a < b < c < d$  or  $d < c < b < a$ . We present a linear time recognition for these graphs.

**Keywords:**  $P_4$ -indifference, algorithm, recognition.

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## 1 Introduction

A  $P_4$  is a chordless path of four vertices. A graph is  $P_4$ -indifference if it admits an ordering  $<$  on its vertex set such that every  $P_4$   $abcd$  has  $a < b < c < d$  or  $d < c < b < a$ . Such an ordering is called a  $P_4$ -indifference ordering. The  $P_4$ -indifference graphs were introduced in [Chv84] as a particular class of *perfectly orderable graphs*. A graph is perfectly orderable if there exists an ordering on its vertex set for which the greedy colouring algorithm produces an optimal colouring.

The first recognition algorithm for  $P_4$ -indifference graphs is due to Hoàng and Reed and has the complexity of  $O(n^6)$  [HR89]. They compute the equivalence classes of some relation on the  $P_4$ 's of the graph. They then check that these classes do not contain a certain subgraph with 6 vertices. Later, Raschle and Simon, studying more carefully the  $P_4$ 's relations, proposed an  $O(n^2m)$  recognition algorithm [RS97].

Recently, Hoàng, Maffray and Noy gave a characterization by forbidden induced subgraphs [HMN99] and raised the question of the existence of a linear time recognition algorithm. We answer their question in the affirmative way using some of their theorems. Moreover our algorithm computes an adequate ordering of the vertices when it concludes that the input graph is  $P_4$ -indifference.

## 2 Theoretical basis

We use the following theorems from [HMN99]:

**Theorem 1** [HMN99] *Any  $P_4$ -indifference graph fulfills the following properties:*

1. *If it contains a  $C_4$  (a chordless cycle of length 4) then it contains an homogeneous set.*
2. *If it contains no  $C_4$  then it is an interval graph.*

These two properties inspire the following recognition algorithm: Compute the modular decomposition tree of the input. For each quotient graph of any node of the tree verify that it is an interval graph. Compute an interval representation of it and use it to test whether it is a  $P_4$ -indifference graph and to compute a good ordering of the vertices if there exists one. The existence of linear time algorithms for modular decomposition and interval graph recognition [MS94, MS99, HM91] make this scheme possible for a linear time recognition algorithm.

To justify such an algorithm, we first need some additional theoretical results linking  $P_4$ -indifference graphs and modular decomposition.

**Theorem 2** [HMN99] *The composition of two graphs is  $P_4$ -indifference iff they are both  $P_4$ -indifference graphs.*

To make the paper self-contained and because the algorithm is strongly based on theorem 2, we present its proof. This result first appeared in [HMN99].

**Proof:** Let  $G$  be the composition of two graphs  $G_1$  and  $G_2$  where  $G$  is obtained from  $G_1$  by replacing a vertex  $u_2$  by  $G_2$  by linking all the vertices of  $G_2$  to all the neighbors of  $u_2$ .

First suppose  $G$  is  $P_4$ -indifference. We prove that  $G_1$  and  $G_2$  are also  $P_4$ -indifference. Let  $z_1 < \dots < z_n$  be a  $P_4$ -indifference ordering of the vertices of  $G$ . The induced order of the vertices of  $G_2$  will obviously fulfill the same condition.  $G_2$  is thus clearly a  $P_4$ -indifference graph. Consider the ordering of the vertices of  $G_1$  obtained from  $z_1 < \dots < z_n$  by erasing all the vertices of  $G_2$  but one that is replaced by  $u_2$ . It is clearly a  $P_4$ -indifference ordering.

Second suppose  $G_1$  and  $G_2$  are  $P_4$ -indifference graphs. We show that  $G$  is also a  $P_4$ -indifference graph. Let  $y_1 < \dots < y_p$  be a  $P_4$ -indifference ordering of the vertices of  $G_2$  and  $x_1 < \dots < x_m$  be a  $P_4$ -indifference ordering of the vertices of  $G_1$  where  $u_2 = x_k$ . Then  $x_1 < \dots < x_{k-1} < y_1 < \dots < y_p < x_{k+1} < \dots < x_m$  is clearly a  $P_4$ -indifference ordering of the vertices of  $G$ .  $\square$

An immediate corollary of Theorem 2 is the following:

**Corollary 1** *A graph is a  $P_4$ -indifference graph iff all the quotient graphs of its modular decomposition tree are  $P_4$ -indifference graphs.*

Notice that the second part of the proof of theorem 2 also gives a simple way to compute a  $P_4$ -indifference ordering of the composition of some  $P_4$ -indifference graphs from their  $P_4$ -indifference orderings.

**Corollary 2** *A  $P_4$ -indifference ordering can be computed in linear time from the  $P_4$ -indifference orderings of the quotient graphs.*

**Proof:** Assume you are given a  $P_4$ -indifference ordering for each quotient graph. Then visit the modular tree decomposition in a post-order fashion, and for each node  $N$  do the following:

- if  $N$  is a leaf, then it corresponds to a single vertex and the ordering is trivial.
- if  $N$  is an internal node, then for each son  $S_i$  of  $N$  you have a  $P_4$ -indifference ordering  $\sigma_i$ . Let  $\sigma_H$  the  $P_4$ -indifference ordering of  $H$ , the quotient graph associated to  $N$ . Each  $S_i$  corresponds to a vertex  $x_i$  of  $H$ . Then  $\sigma$  a  $P_4$ -indifference ordering of the graph whose tree decomposition is rooted at  $N$ , can be obtained by substituting each  $\sigma_i$  to  $x_i$  in  $\sigma_H$ .

The linearity of the above algorithm comes from the linearity of the sum of the sizes of the quotient graphs.  $\square$

So now, to complete our recognition algorithm of  $P_4$ -indifference graphs, we just need to compute  $P_4$ -indifference ordering and to recognize prime  $P_4$ -indifference graphs.

### 3 Recognition of prime interval $P_4$ -indifference graphs

Let  $G$  be a prime interval graph. Let  $I_1, \dots, I_n$  be a minimal interval representation of it where each  $I_k$  is an integer interval of  $[1, N] \cap \mathbb{Z}$  (with  $N$  minimal). If  $u$  is a vertex, we denote by  $I_u$  its associated interval. Recall that by definition, two vertices  $u$  and  $v$  of  $G$  are linked iff  $I_u$  intersects  $I_v$ . We say that two intervals *overlap* when they intersect without one being included in the other. When two intervals do not intersect, we say that the one with greater (resp. smaller) elements is *greater* (resp. *smaller*) than the other.

We are now going to show how a minimal interval representation is close to a  $P_4$ -indifference ordering. The following theorem can easily be deduced from the proofs in [HMN99]. It links minimal interval representations and  $P_4$ -indifference orderings.

**Theorem 3** [HMN99] *Consider a prime interval graph  $G$  and a minimal interval representation of it. Let  $\prec$  be the relation satisfying  $x \prec y$  for any vertices  $x, y$  satisfying one of the three following properties:*

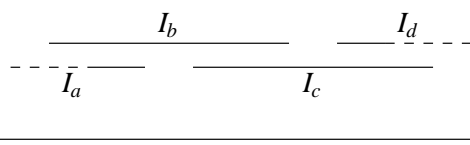
1.  $I_x$  and  $I_y$  overlap and the left bound of  $I_x$  is smaller than the left bound of  $I_y$ .
2.  $I_x$  is included in  $I_y$  and there exists a  $P_4$   $x, y, z, t$  such that  $I_y$  and  $I_z$  overlap and the left bound of  $I_y$  is smaller than the left bound of  $I_z$ .
3.  $I_y$  is included in  $I_x$  and there exists a  $P_4$   $y, x, z, t$  such that  $I_z$  and  $I_x$  overlap and the left bound of  $I_z$  is smaller than the left bound of  $I_x$ .

$G$  is a  $P_4$ -indifference graph iff  $\prec$  is acyclic. Moreover any extension of  $\prec$  (i.e. for each  $x, y$   $x \prec y$  implies  $x < y$  or for each  $x, y$   $x \prec y$  implies  $x > y$ ) is a  $P_4$ -indifference ordering.

Notice that the previous remarks imply that  $x$  and  $y$  are vertices of some  $P_4$  in each of the three situations of Theorem 3. Moreover any two consecutive vertices of some  $P_4$  are in relation by  $\prec$ . And all above, any  $P_4$   $a, b, c, d$  either verifies  $a \prec b \prec c \prec d$  or  $d \prec c \prec b \prec a$ . See [HMN99] for the details of the proofs.

In order to use theorem 3 to find a  $P_4$ -indifference ordering, in cases 2. and 3., we have to find some  $P_4$  containing  $x$  and  $y$ . Such a work is the bottleneck of complexity issue. The next lemma is the new tool that makes possible the design of a linear time recognition algorithm.

**Lemma 1** *Let  $b$  and  $c$  be two vertices. The corresponding intervals  $I_b$  and  $I_c$  in a minimal interval representation overlap iff  $b$  and  $c$  are the middle vertices of some  $P_4$ .*



**Fig. 1:** Any  $P_4$   $a', b', c', d'$  is embedded in the interval representation as shown. Either  $a' = a, b' = b, c' = c, d' = d$  or  $a' = d, b' = c, c' = b, d' = a$

**Proof:** Consider a  $P_4$   $a, b, c, d$  with edges  $ab, bc, cd$ . We claim that the intervals  $I_b$  and  $I_c$  associated to  $b$  and  $c$  must overlap. They intersect since the two vertices are linked. Since  $I_a$  intersects  $I_b$  but not  $I_c$ ,  $I_b$  cannot be included in  $I_c$ . For a similar reason with  $I_d$ ,  $I_c$  cannot be included in  $I_b$ . Thus any  $P_4$  is of the form illustrated by Figure 3 in the interval representation.

Conversely, when two intervals  $I_b$  and  $I_c$  overlap then  $b$  and  $c$  are the middle vertices of at least one  $P_4$ . This is due to the fact that the interval representation has been chosen minimal. Suppose for example that some elements of  $I_b$  are smaller than those of  $I_c$  (the other case is symmetrical). Let  $i$  be the greatest integer of  $I_b$  that is not in  $I_c$ . There must exist an interval  $I_a$  containing  $i$  without intersecting  $I_c$  otherwise  $i$  could be removed yielding a more compact representation (contradicting the minimality of the present one). The same argument allows to conclude that there must exist some interval  $I_d$  intersecting  $I_c$  but not  $I_b$ .  $a, b, c, d$  is then a  $P_4$ .  $\square$

Therefore theorem 3 can be rewritten as follows (in particular cases 2. and 3.):

**Corollary 3** Consider a prime interval graph  $G$  and a minimal interval representation of it. Let  $\prec$  be the relation satisfying  $x \prec y$  for any vertices  $x, y$  satisfying one of the three following properties (these situations are illustrated by figure 3):

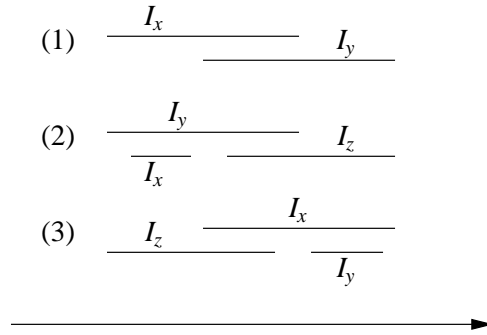
1.  $I_x$  and  $I_y$  overlap and the left bound of  $I_x$  is smaller than the left bound of  $I_y$ .
2.  $I_x$  is included in  $I_y$  and there exists some interval  $I_z$  greater than  $I_x$  overlapping  $I_y$ .
3.  $I_y$  is included in  $I_x$  and there exists some interval  $I_z$  smaller than  $I_y$  overlapping  $I_x$ .

$G$  is a  $P_4$ -indifference graph iff  $\prec$  is acyclic. Moreover any extension of  $\prec$  (i.e. for each  $x, y$   $x \prec y$  implies  $x < y$  or for each  $x, y$   $x \prec y$  implies  $x > y$ ) is a  $P_4$ -indifference ordering.

**Corollary 4** The recognition of prime  $P_4$ -indifference graphs can be done in linear time.

**Proof:** Let us briefly describe the recognition algorithm :

- Test if the input graph is an interval graph and if so compute a minimal interval representation. It can be done in linear time by any linear interval graphs recognition algorithm (see [BL76, KM89, HM91, HMPV97, COS98] for example).
- The  $\prec$  relation can easily be computed in linear time by storing for each interval the two intervals overlapping it which have the rightmost left bound and the leftmost right bound when they exist. During this computation, you can find all the relations corresponding to overlapping intervals (case (1) of figure 3). Then the relations corresponding to case (2) and (3) of figure 3 can be computed.



**Fig. 2:** The three situations of Theorem 3 implying  $x < y$  for a  $P_4$ -indifference ordering.

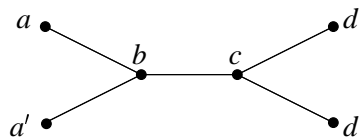
- Using a Depth First Search, the acyclicity of  $\prec$  can be tested and a linear extension of it can be computed (if there exists one). It can be done in linear time.

□

## 4 Conclusions

This paper shows how a linear time algorithm for the  $P_4$ -indifference graphs recognition can be designed. This algorithm strongly relies on modular decomposition as a preprocessing. But linear time modular decomposition algorithms are still complicated to program. So the natural question is: can this preprocessing step be avoided ?

So it has been shown that prime  $P_4$ -indifference graphs are interval graphs. It is well known that Lexicographic Breadth First Search (Lex-BFS) [RTL76] plays an important role on interval graphs [HM91, COS98, HMPV97]. The order Lex-BFS visits the vertices of the input graph can be seen as the output of Lex-BFS: a Lex-BFS ordering. For example in [COS98], 4 sweeps of particular Lex-BFS are used to compute a characteristic ordering of interval graphs (the  $i$ -th sweep starts on the last visited vertex of the previous sweep). One can wonder if Lex-BFS can be used to compute a  $P_4$ -indifference ordering. As illustrated by the graph of figure 4, the answer is no.



**Fig. 3:** A graph such that no Lex-BFS ordering is a  $P_4$ -indifference ordering

On the above graph, no Lex-BFS ordering is a  $P_4$ -indifference ordering. So there is no hope for some special Lex-BFS as those defined in [COS98]. We can remark that this graph contains modules  $\{a, a'\}$  and  $\{d, d'\}$ . It seems that restricted to prime  $P_4$ -indifference graphs, 2 sweeps of Lex-BFS computes a  $P_4$ -indifference ordering (it can be a simplification of the presented algorithm). But up to now, we do not know how to avoid the modular decomposition.

The presented algorithm relies on some properties of prime graphs and also on some  $P_4$  relations. Can these structural results be adapted to other classes of perfectly orderable graphs like for example  $P_4$ -comparability graphs,  $P_4$ -simplicial graphs . . . in order to design efficient recognition algorithms ?

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