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Abstract: We study a weighted improper colouring problem on graph, and in particular of triangular and hexagonal grid graphs. This problem is motivated by a frequency allocation problem. We propose approximation algorithms to compute such colouring.

Key-words: Improper colouring, Weighted colouring, Approximation algorithms

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Allocation de fréquences et coloration impropre des graphes hexagonaux pondérés

Résumé : Motivés par un problème d'allocation de fréquences, nous étudions la coloration impropre des graphes pondérés et plus particulièrement des sous-graphes pondérés de la grille et du réseau triangulaire. Nous donnons des algorithmes d'approximation pour trouver de telles colorations.

Mots-clés : Coloration impropre, Coloration pondérée, Algorithmes d'approximation

1 Introduction

This paper is motivated by a problem posed by Alcatel Space Technologies (see [1]). A satellite sends informations to receivers on earth, each of which is listening several frequencies, one for each signal it needs to receive. Technically it is impossible to focus a signal sent by the satellite exactly on the destination receiver. So part of the signal is spread in an area around it, creating noise for the other receivers displayed in this area and listening the same frequency. Each receiver is able to distinguish the signal directed to it from the extraneous noises it picks up if the sum of the noises does not become too large, i.e. does not exceed a certain threshold T. The problem is to assign frequencies to the signals in such a way that each receiver gets its dedicated signals properly, while minimizing the total number of frequencies used.

Generally the "noise relation" is symmetric, that is if a receiver u is in the noise area of a receiver v then v is in the noise area of u. Hence, interferences may be modelled by a *noise graph* $G = (V(G), E(G))$ whose vertices are the receivers and where two vertices are joined by an edge if and only if they interfer. Moreover, to the graph is attached a *weight function* $p : V(G) \to \mathbb{N}$, where the weight $p(v)$ of the vertex v is equal to the number of signals it has to receive. Hence we have a *weighted graph*, that is a pair (G, p), where G is a graph and p a weight function on the vertex set of G. We can model the frequencies by colours. Therefore, to each vertex, we associate a set $C(v)$ of $p(v)$ distinct colours. If in total l colours are used, the mapping C is called an *l-colouring* of (G, p).

In a simplified version, the intensity I of the noise created by a signal is independent of the frequency and the receiver. Hence to distinguish its signal from noises, a receiver must be in the noise area of at most $k = \lfloor \frac{T}{I} \rfloor$ receivers listening signals on the same frequency. In terms of colouring this property is equivalent to say that for any colour c, the set of vertices having one colour c induces a graph of degree at most k. Such a colouring is called *k-improper*. The *k-improper chromatic number* of (G, p), denoted $\chi_k(G, p)$, is the smallest l such that (G, p) admits a k-improper l-colouring. Note that a 0-improper colouring corresponds to a proper colouring.

In [1] this problem is studied via linear programming. The objective of this paper is to build algorithms giving k-improper colourings of weighted graphs of a certain class with as few colours as possible. An algorithm that gives a k-improper colouring of each weighted graph (G, p) in this class with at most $c_1 \times \chi_k(G, p) + c_2$ colours for some constants c_1 and c_2, is said to be *c_1-approximate* or to have *approximation ratio c_1*.

For any integer q we denote by $\mathbf{q}$ the constant weight function $\mathbf{q}(v) = q$ for all v. For any weight function p, we set $p_{\max} = \max_v(p(v))$.

Proposition 1 *If there exists a k-improper colouring of $(G, \mathbf{q})$ with r colours, then there exists a k-improper colouring of (G, p) with $r \left\lceil \frac{p_{\max}}{q} \right\rceil$ colours.*

In particular, if $\chi_k(G, \mathbf{q}) \le r$, then $\chi_k(G, p) \le r \left\lceil \frac{p_{\max}}{q} \right\rceil$.

Proof. Observe that if we have a k-improper r_1-colouring of (G, p_1) and a k-improper r_2-colouring of (G, p_2), one can easily derive a k-improper $(r_1 + r_2)$-colouring of $(G, p_1 + p_2)$ by using the union of these colourings on two disjoint sets of colours. Doing this repeatedly λ times with $p = \mathbf{q}$, we obtain a λr-colouring of $(G, \lambda\mathbf{q})$. For $\lambda = \left\lceil \frac{p_{\max}}{q} \right\rceil$, we obtain a k-improper colouring of (G, p) since $p(v) \le p_{\max} \le \lambda q$ for every vertex $v \in V(G)$. □

If we have a k-improper r-colouring $(G, \mathbf{q})$ then the above proposition yieldsan immediate (r/q)-approximate algorithm for k-improper (G, p)-colouring because $\chi_k(G, p) \ge p_{\max}$. In this paper, we present improvements of Proposition 1. For any graph G, we denote $\rho_k(H, G, p)$, the maximum of $\chi_k(H', p)$ over the subgraphs H' of G isomorphic to H. For example, $\rho_k(K_1, G, p) = p_{\max}$ and $\rho_0(K_2, G, p) =$

$\max\{p(u) + p(v) \mid uv \in E(G)\}$ so $\rho_0(K_2, G, p) \le 2p_{\max}$. If $k \ge 1$, then $\rho_k(K_2, G, p) = p_{\max}$. By extension, if $\mathcal{H}$ is a family of graphs (finite or not), $\rho_k(\mathcal{H}, G, p)$ is the maximum of $\rho_k(H, G, p)$ over all graphs $H \in \mathcal{H}$. Obviously, for any family $\mathcal{H}$, $\rho_k(\mathcal{H}, G, p) \le \chi_k(G, p)$. The idea to design approximate algorithms for k-improper colouring graphs of some given class consists of finding a finite family of graphs $\mathcal{H}_k$ such that any weighted graph (G, p) in the class, satisfies $\chi_k(G, p) \le c_1.\rho_k(\mathcal{H}_k, G, p) + c_2$ with c_1 a small constant (ideally 1) and c_2 another constant. Hence, by computing $\chi_k(H', p)$ for all the subgraphs H' isomorphic to a graph in $\mathcal{H}_k$, we obtain a c_1-approximate algorithm for $\chi_k(G, p)$. Moreover, we also exhibit algorithms that produce the corresponding c_1-approximate k-improper colouring.

We first show approximate algorithms for general graphs. We then make further improvements for specific graphs, namely the *grid graphs* and the *hexagonal graphs*. The next subsections recall the results known for grid graphs and hexagonal graphs and present ours.

1.1 Grid graphs

The (two-dimensionnal) *grid* is the graph GL defined as follows: the vertices are all integer linear combinations $a\mathbf{f_1} + b\mathbf{f_2}$ of the two vectors $\mathbf{f_1} = (1, 0)$ and $\mathbf{f_2} = (0, 1)$. Thus, we may identify the vertices with the ordered pairs (a, b) of integers. Two vertices are adjacent when the Euclidean distance between them is 1. Hence each vertex $x = (a, b)$ has four neighbours: its *left neighbour* $(a-1, b)$, its *right neighbour* $(a+1, b)$, its *top neighbour* $(a, b+1)$ and its *down neighbour* $(a, b-1)$. A *grid graph* is an induced subgraph of the two-dimensionnal grid.

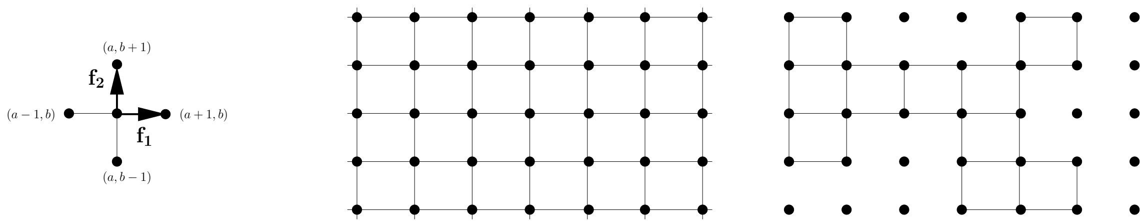

Figure 1: The two-dimensionnal grid and a grid graph.

As a grid graph G has maximum degree 4, k-improper colourings of G are trivial for $k \ge 4$: it suffices to give any set of $p(v)$ colours to every vertex v to produce a k-improper colouring. Hence, for any $k \ge 4$, its k-improper chromatic number equals its maximum weight $\chi_k(G, p) = p_{\max}$.

Regarding proper (0-improper) colourings, by Proposition 1, $\chi_0(G, p) \le 2p_{\max}$ as a grid graph is bipartite. This upper bound is tight when there is an edge uv such that $p(u) = p(v) = p_{\max}$. But such an edge may not exist. However, one can find the weighted chromatic number of a grid graph and more generally of any weighted bipartite graph.

Theorem 2 (McDiarmid and Reed [8]) *Let $G = ((A, B), E)$ be a bipartite graph. Then for any weight p, $\chi_0(G, p) = \rho_0(\{K_1, K_2\}, G, p)$.*

Proof. Let us colour every vertex a of A with $\{1, 2, \ldots, p(a)\}$ and every vertex b of B with $\{l, l-1, \ldots, l - p(b) + 1\}$ where $l = \rho_0(\{K_1, K_2\}, G, p)$. □

In Section 3, for $1 \le k \le 3$, we provide α_k-approximate polynomial algorithms that compute a k-improper colouring of a weighted grid graph with $\alpha_1 = 13/9$, $\alpha_2 = 27/20$ and $\alpha_3 = 19/16$.

It would be nice to solve the following problem:

Problem 3 For any fixed $1 \le k \le 3$, is it $\mathcal{NP}$-complete to find the k-improper chromatic number of a weighted grid graph?

1.2 Hexagonal graphs

The triangular lattice graph TL may be described as follows. The vertices are all integer linear combinations $a\mathbf{e_1} + b\mathbf{e_2}$ of the two vectors $\mathbf{e_1} = (1,0)$ and $\mathbf{e_2} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus we may identify the vertices with the ordered pairs (a,b) of integers. Two vertices are adjacent when the Euclidean distance between them is 1. Therefore, each vertex $x = (a,b)$ has six neighbours: its *left neighbour* $(a-1,b)$, its *right neighbour* $(a+1,b)$, its *leftup neighbour* $(a-1,b+1)$, its *rightup neighbour* $(a,b+1)$, its *leftdown neighbour* $(a,b-1)$ and its *rightdown neighbour* $(a+1,b-1)$. A *hexagonal graph* is an induced subgraph of the triangular lattice.

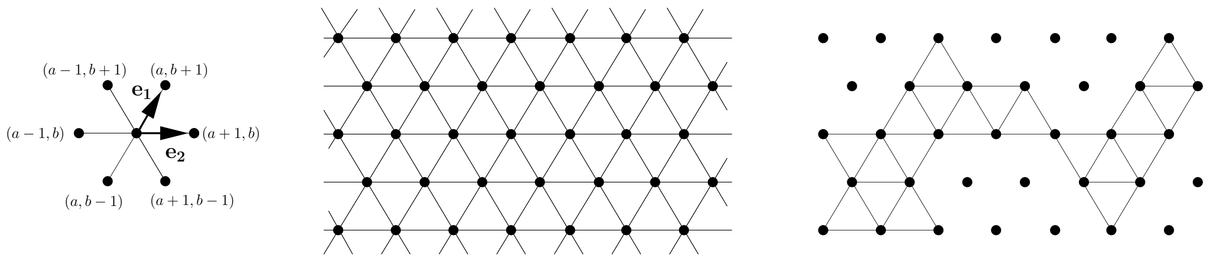

Figure 2: The triangular lattice and a hexagonal graph.

For any $k \geq 6$, the k-improper chromatic number of a hexagonal graph is its maximum weight because it has maximum degree 6.

McDiarmid and Reed [8] showed that it is $\mathcal{NP}$-complete to decide whether the chromatic number of a weighted hexagonal graph is 3 or 4. Hence, there is no polynomial time algorithm for finding the weighted chromatic number of hexagonal graphs (unless $\mathcal{P} = \mathcal{NP}$). Therefore, one has to find approximate algorithms. The better known so far has approximation ratio 4/3 and is based on the following result:

Theorem 4 (McDiarmid and Reed [8]) *For any weigthed hexagonal graph* (G,p)*,*

$$\chi_0(G,p) \leq \frac{4}{3}\rho_0(\{K_1,K_2,K_3\},G,p).$$

A distributed algorithm which guarantees the $\frac{4}{3}\rho_0(\{K_1,K_2,K_3\},G,p)$ bound is reported by Narayanan and Schende [7]. However, one expects to have approximate algorithms with ratios better than 4/3. In particular, Reed and McDiarmid conjecture that, for big weights, the ratio may be decreased to almost 9/8.

Conjecture 5 (McDiarmid and Reed [8]) *There exists a constant c such that for any weigthed hexagonal graph* (G,p)*,*

$$\chi_0(G,p) \leq \frac{9}{8}\rho_0(\{K_1,K_2,K_3\},G,p) + c.$$

Note that the ratio 9/8 in the above conjecture is the best possible. Indeed, consider a 9-cycle C_9 with constant weight q. A colour can be assigned to at most 4 vertices, so $\chi_0(C_9,q) \geq \frac{9q}{4}$. Clearly, $\rho_0(\{K_1,K_2,K_3\},C_9,q) = 2q$. So $\chi_0(C_9,q) \geq \frac{9}{8}\rho_0(\{K_1,K_2,K_3\},C_9,q)$. An evidence for this conjecture has been given by Havet [2], who proved that if a hexagonal graph G is triangle-free (i.e. has no K_3) then $\chi_0(G,p) \leq \frac{7}{6}\rho_0(\{K_1,K_2\},G,p) + 5$. See also [10] for an alternative proof and [5] for a distributed algorithm for colouring triangle-free hexagonal graphs with $\frac{5}{4}\rho_0(\{K_1,K_2\},G,p) + 3$ colours.

Regarding improper colouring, Havet, Kang and Sereni [3, 4, 9] generalized the above mentionned $\mathcal{NP}$-completeness result of McDiarmid and Reed:

Theorem 6 (Havet, Kang and Sereni [3, 4, 9]) *For $0 \leq k \leq 5$, the following problem is $\mathcal{NP}$-complete:*
Instance: *a weighted hexagonal graph (G, p).*
Question: *is (G, p) k-improper 3-colourable?*

Hence one cannot expect to have polynomial algorithm to find the k-improper chromatic number of weighted hexagonal graphs. In Section 4, for $1 \leq k \leq 5$, we provide α_k-approximate polynomial algorithms that compute a k-improper colouring of a weighted hexagonal graph with $\alpha_1 = 20/11$, $\alpha_2 = 12/7$, $\alpha_3 = 18/13$, $\alpha_4 = 80/63$ and $\alpha_5 = 41/36$.

2 General algorithms

If S is a set of vertices of G, we denote by $G\langle S \rangle$ the subgraph of G induced by the vertices of S.

We extend the weight function to sets of vertices and to subgraphs in the following intuitive way: the *weight of a set of vertices* $X \subset V(G)$ is $p(X) = \sum_{x \in X} p(x)$ while the *weight of a subgraph* $H \subset G$ is $p(H) = p(V(H))$.

In this section, we improve Proposition 1. To do so, instead of considering only $p_{\max}$, we consider the number of colours that a vertex and its neighbours may require. As shown by the following proposition, this number may be larger than $p_{\max}$. The graph $K_{1,k+1}$ is the graph with $k+2$ vertices and $k+1$ edges linking one vertex, called the *centre* to the $k+1$ others, called *spikes*.

Proposition 7 *For every weight function p, $\chi_k(K_{1,k+1}, p) \geq \dfrac{1}{k+1} \displaystyle\sum_{v \in V(K_{1,k+1})} p(v) = \dfrac{p(K_{1,k+1})}{k+1}$.*

Proof. Let u be the centre of $K_{1,k+1}$ and $v_1, \dots, v_{k+1}$ its spikes. Consider a k-improper colouring C of $K_{1,k+1}$. For $1 \leq i \leq k+1$, set $q(v_i) = |C(v_i) \setminus C(u)|$. The colouring C uses at least $M = \max_{1 \leq i \leq k+1}\{q(v_i) + p(u)\} \geq p(u) + \frac{1}{k+1} \sum_{i=1}^{k+1} q(v_i)$ colours. But a colour in $C(u)$ is assigned to at most k of the spikes because the colouring is k-improper. Thus $\sum_{i=1}^{k+1} q(v_i) \geq \sum_{i=1}^{k+1} p(v_i) - kp(u)$. It follows $M \geq \frac{1}{k+1}\left(p(u) + \sum_{i=1}^{k+1} p(v_i)\right)$. □

We call $(k+1)$-*star*, or simply *star*, a subgraph of G isomorphic to $K_{1,k+1}$.

We set $\theta_k(G,p) = \max\{p(H)/(k+1) \mid H \ (k+1)\text{-star of } G\}$ and $\omega_k(G,p) = \max\{p_{\max}, \theta_k(G,p)\}$. According to Proposition 7, $\omega_k(G,p) \leq \rho_k(\{K_1, K_{1,k+1}\}, G, p) \leq \chi_k(G,p)$.

The idea of our general algorithm is to, step by step, decrease $\omega_k(G,p)$ by priorizing the colouring of the vertices with *high* weights, which directly affect this parameter. Once fixed a k-improper colouring C of $(G, \mathbf{q})$ with r colours, at each round of the algorithm, C is repeated several times, spending so, at each round, a multiple of r new colours, in order to assign packets of q colours to these heavy vertices. Also, the colouring of the vertices with *small* weights is postponed, in such way that, at the last step, the number of colours required to finish the k-improper colouring of G is under control. The next theorem gives this general algorithm.

Theorem 8 *Let G be a graph with bounded degree Δ fixed. Let C be a k-improper colouring C of $(G, \mathbf{q})$ with r colours, for some integer q. Let $a_k(r,q) = (k+1)r - q$, $\alpha_k(r,q) = \frac{a_k(r,q).r + rq}{a_k(r,q).q + rq} = \frac{(k+1)r^2}{(k+2)rq - q^2}$, $\gamma_k(r,q) = \max\{(k+2)(rq-1), (k+1)rq + kq^2, a_k(r,q)q + rq\}$ and $\beta_k(r,q) = r\left\lceil \frac{\gamma_k(r,q)}{q} \right\rceil$.*

There is a polynomial algorithm which returns a k-improper colouring of (G,p) with at most $\alpha_k(r,q) \times \omega_k(G,p) + \beta_k(r,q)$ colours.

In particular, if $\chi_k(G, \mathbf{q}) \leq r$, then $\chi_k(G,p) \leq \alpha_k(r,q) \times \omega_k(G,p) + \beta_k(r,q)$.

Proof. To make the proof more readable, we omit the parameters k, q and r which are fixed. So $a = a_k(r,q)$, $\alpha = \alpha_k(r,q)$, $\gamma = \gamma_k(r,q)$, $\beta = \beta_k(r,q)$ and $\omega(G,p) = \omega_k(G,p)$. Consider the following algorithm :

Algorithm 1 0. Initialisation: $(G^0,p^0) := (G,p)$, $S := \emptyset$ and $i = 0$.

1. Add the vertices of low weight to S that will be treated at the end (Step 3):
   $S^i := \{v \in V(G) \mid p^i(v) \leq \gamma\}$, $S := S \cup S^i$ and for all $v \in S^i$, $s(v) := p^i(v)$. $G^{i+1} := G^i - S^i$.
2. If G^{i+1} is not empty:
   - 2.1. Give to each vertex v of G^{i+1} a certain number $n^i(v)$ of colours among a set of $ar+rq$ colours, in such a way that $\omega(G^{i+1}, p^i - n^i) \leq \omega(G^i, p^i) - (aq + rq)$.
   - 2.2. Set $p^{i+1} := p^i - n^i$ and $i := i + 1$ and go to Step 1.
3. Colour $(G\langle S\rangle, s)$ with β colours. This is possible by Proposition 1 using $\left\lceil \frac{\gamma}{q} \right\rceil$ times the colouring C, since $s_{\max} \leq \gamma$.

At each Step 2, $ar + rq$ colours are used and $\omega(G^i,p^i)$ decreases by at least $aq + rq$. So after $m = \lfloor \frac{\omega(G,p)}{aq+rq} \rfloor$ steps, the remaining $\omega(G^m,p^m)$ is at most $aq + rq \leq \gamma$ (by the choice of γ). Therefore, Algorithm 1 yields a k-improper colouring of (G,p) with at most $\alpha\omega(G,p) + \beta$ colours and the theorem will be proved if we can perform step 2.1 as indicated.

Let us now describe precisely Step 2.1. Set $\omega^i = \omega_k(G^i,p^i)$. We distinguish several kinds of vertices depending on their own weight and the one of their neighbours. A *big vertex* is a vertex such that $p^i(v) > \omega^i - rq$. A *small vertex* is a non-big vertex. Moreover, a small vertex is *goofy* if it is adjacent to a big vertex, and *regular*, otherwise.

At each step 2.1, we first use a times the colouring C : with ar colours, each vertex receives aq of them. Then we use rq additional colours in the following way: we assign all of them to the big vertices, and we use q times colouring C on the graph induced by the regular vertices. As a result, each big vertex receives rq additional colours and each regular vertex receives q^2 ones. Hence $n^i(v) = aq + rq$ if v is big, $n^i(v) = aq + q^2$ if v is regular, and $n^i(v) = aq$, if v is goofy.

Let us first check that this colouring is k-improper.

Suppose by way of contradiction that it is not k-improper. The only possibility is that some additional colour has been assigned to a vertex v and $k+1$ of its neighbours since otherwise we were using C which is k-improper. Hence v is the centre of a star H whose vertices are all big. Then $p^i(H) \geq (k+2)(\omega^i - rq+1) = (k+1)\omega^i + \omega^i - (k+2)(rq-1)$. But, since G^{i+1} is not empty, $\omega^i \geq p^i_{\max} > \gamma \geq (k+2)(rq-1)$. So $p^i(H) > (k+1)\omega^i$ which contradicts the definition of ω^i.

Let us now check that $\omega^{i+1} \leq \omega^i - (aq + rq)$.

For a vertex v, $p^{i+1}(v) \leq p^i(v) - n^i(v)$. If v is big $n^i(v) = aq + rq$ and, if v is small, $n^i(v) \geq aq$ and $p^i(v) \leq \omega^i - rq$. In both cases, $p^{i+1}(v) \leq \omega^i - (aq + rq)$.

So $p^{i+1}_{\max} \leq \omega^i - (aq + rq)$.

Consider now a star H of G^{i+1}. It is *big* if $p^i(H) \geq (k+1)\omega^i - q^2$.

Claim 1 *A big star H has a vertex which is not goofy.*

Proof. By the contrapositive. Let H be a star whose vertices are all goofy. Let u be its centre and $v_1, \dots, v_{k+1}$ its spikes with $p^i(v_1) \geq \dots \geq p^i(v_{k+1})$.

There exists a big vertex v_0 adjacent to u. The vertex v_0 is not one of the v_i, because all of them are small. Let H' be the star with centre u and spikes $v_0, \dots, v_k$. Set $S = \sum_{j=1}^{k} p^i(v_j)$. We have $p^i(H') = p^i(u) + S + p^i(v_0) \leq (k+1)\omega^i$ and $p^i(v_0) > \omega^i - rq$. Hence $S < k\omega^i + rq - p^i(u)$. But $p^i(v_{k+1}) \leq \frac{S}{k}$, so $p^i(v_{k+1}) < \omega^i + \frac{rq}{k} - \frac{p^i(u)}{k}$.

Observe that $p^i(H) = p^i(u) + S + p^i(v_{k+1})$ and so, by consequence of the above inequalities, $p^i(H) < (k+1)\omega^i + rq(\frac{k+1}{k}) - \frac{p^i(u)}{k}$. As $u \in G^{i+1}$, we have $p^i(u) > \gamma \geq (k+1)rq + kq^2$, thus $p^i(H) < (k+1)\omega^i - q^2$. Hence H is not big. □

Each vertex of H receives at least aq colours. Hence if H is not big, $p^{i+1}(H) \leq p^i(H) - (k+2)aq \leq (k+1)\omega^i - q^2 - (k+2)aq$. If H is big, at least one vertex of H is not goofy and so receives at least $aq + q^2$ colours. Thus in both cases

$$p^{i+1}(H) \leq (k+1)\omega^i - (k+2)aq - q^2 \leq (k+1)(\omega^i - (aq + rq)),$$

because $a = (k+1)r - q$.

Thus $\theta_k(G^{i+1}, p^{i+1}) \leq \omega^i - (aq + rq)$. Hence, since $p^{i+1}_{\max} \leq \omega^i - (aq+rq)$, we get $\omega^{i+1} \leq \omega^i - (aq+rq)$.

Algorithm 1 requires to compute the value of $\omega_k(G, p)$. Since there are at most $n\binom{\Delta}{k+1}$ $k+1$-stars in a graph on n vertices with maximum degree Δ, it can be done in $O\left(n\binom{\Delta}{k+1}\right)$ operations. □

For 1-improper colouring, changing slightly the parameters of Algorithm 1 yields a better approximation ratio. Replacing $\beta_1(r,q)$ by a slightly higher value $\beta'_1(r,q)$ allows us to replace $a_1(r,q)$ by a smaller value $a'_1(r,q)$ which yields a better approximation ratio $\alpha'_1(r,q) = \frac{a'_1(r,q).r+rq}{a'_1(r,q).q+rq}$.

Theorem 9 *Let r and q be two integers. Set $a'_1(r,q) = 2r - 2q$ if $r \geq 2q$, $a'_1(r,q) = r$ if $r \leq 2q$, $\alpha'_1(r,q) = \frac{a'_1(r,q).r+rq}{a'_1(r,q).q+rq}$, $\gamma'_1(r,q) = 3rq$ and $\beta'_1(r,q) = 3r^2$. There is a polynomial algorithm that, given a weighted graph G and a 1-improper colouring C of $(G, \mathbf{q})$ with r colours, produces a 1-improper colouring of (G,p) with at most $\alpha'_1(r,q) \times \omega_1(G,p) + \beta'_1(r,q)$ colours.*

In particular, if $\chi_1(G, \mathbf{q}) \leq r$, then $\chi_1(G,p) \leq \alpha'_1(r,q) \times \omega_1(G,p) + \beta'_1(r,q)$.

Proof. We use Algorithm 1 with $\beta = \beta'_1(r,q)$, $a = a'_1(r,q)$, $\alpha = \alpha'_1(r,q)$ and $\gamma = \gamma'_1(r,q)$.

Note that $\gamma \geq 3rq$ implies $\gamma \geq aq + rq$ as $aq + rq = 3rq - 2q^2$ if $r \geq 2q$ and $aq + rq = 2rq$ if $r \leq 2q$.

The beginning of the proof is the same as that of Theorem 8. Note that the condition needed for the k-improper property is $\gamma \geq 3(rq-1)$ which is satisfied. Like in Theorem 8, we have $p^{i+1}_{\max} \leq \omega^i - (aq+rq)$. (recall that $\omega^i = \omega_1(G^i, p^i)$). So in order to show that $\omega^{i+1} \leq \omega^i - (aq + rq)$, it remains to show that $\theta_k(G^{i+1}, p^{i+1}) \leq \omega^i - (aq + rq)$, that is for any 2-star H of G^{i+1}, $\rho^{i+1}(H) \leq 2(\omega^i - (aq + rq))$.

For that let us redefine a *big star* in G^i as a 2-star with $p^i(H) > 2\omega^i - 2rq + aq$.

Claim 2 *A big star contains either a big vertex or two regular vertices.*

Proof. Suppose H is a 2-star with centre u and spikes v_1 and v_2. Assume, moreover, that two of these vertices are goofy and none is big. W.l.o.g. there is a big vertex b_1 adjacent to v_1 and a big vertex b_2 adjacent to u or v_2. As b_1, v_1 and u form a 2-star, by the definition of ω^i, $p^i(b_1) + p^i(v_1) + p^i(u) \leq 2\omega^i$. Similarly, $p^i(b_2)+p^i(u)+p^i(v_2) \leq 2\omega^i$. Summing these two inequalities, we obtain $p^i(v_1)+p^i(u)+p^i(v_2) \leq 4\omega^i - p^i(b_1) - p^i(b_2) - p^i(u) \leq 2\omega^i - p^i(u) + 2rq - 2$. Since u is in G^{i+1}, $p^i(u) \geq \gamma \geq 4rq - aq - 1$ because $4rq - aq = 2rq + 2q^2 \leq 3rq$ if $r \geq 2q$, and $4rq - aq = 3rq$ if $r \leq 2q$. So $p^i(H) \leq 2\omega^i - 2rq + aq$. □

Each vertex receives at least aq colours. Hence, if H is not big, then $p^{i+1}(H) \leq p^i(H) - 3aq \leq 2\omega^i - 2rq - 2aq$

Suppose now that H is big. Then, by Claim 2, H contains a big vertex or two regular ones. Hence $p^{i+1}(H) \leq p^i(H) - 3aq - \min\{rq, 2q^2\}$. By our choice of a, we get $p^{i+1}(H) \leq 2(\omega^i - (aq + rq))$. □

When $\Delta(G) = k+1$, one can also get a better approximation ratio. To do so, we need to change the parameters $a_k(r,q)$, $\alpha_k(r,q)$ and $\beta_k(r,q)$ but also to modify slightly Step 2.1 of Algorithm 1. We take advantage of the condition $\Delta(G) = k+1$ to more precisely allow the rq additional colours.

Theorem 10 *Let r and q be two integers. Set $a''_k(r,q) = (k+1)r - 2q$ if $r \geq 2q$, and $a''_k(r,q) = kr$ if $r \leq 2q$, and $\alpha''_k(r,q) = \frac{a''_k(r,q).r+rq}{a''_k(r,q).q+rq}$, $\gamma''_k(r,q) = \max\{(k+2)(rq-1), a''_k(r,q)q + rq\}$ and $\beta''_k(r,q) = r\left\lceil \frac{\gamma''(r,q)}{q} \right\rceil$. There exists a polynomial algorithm that given a graph G with maximum degree $k+1$ and a k-improper colouring C of $(G, \mathbf{q})$ with r colours, produces a k-improper colouring of (G,p) with at most $\alpha''_k(r,q) \times \omega_k(G,p) + \beta''_k(r,q)$ colours.*

In particular, if $\chi_k(G, \mathbf{q}) \leq r$, then $\chi_k(G,p) \leq \alpha''_k(r,q) \times \omega_k(G,p) + \beta''_k(r,q)$.

Proof. We use an algorithm globally identical to Algorithm 1 with $\beta = \beta''_k(r,q)$, $a = a''_1(r,q)$, $\gamma = \gamma''_k(r,q)$ and $\alpha = \alpha''_1(r,q)$ in place of $\beta_k(r,q)$, $a_k(r,q)$ and $\alpha_k(r,q)$ respectively.

Only Step 2.1 is slightly modified in the following way.

As before $\omega^i = \omega_k(G^i, p^i)$. *Big*, *small*, *goofy* and *regular* vertices are defined as in the proof of Theorem 8. A regular vertex is *isolated* if it is adjacent to no regular vertex.

As in the proof of Theorem 8, at each Step 2.1, we first use a times the colouring C : with ar colours, each vertex receives aq of them. But the rq additional colours are assigned in a slightly different way: we give all of them to the big and isolated regular vertices, and we use q times colouring C on the non-isolated regular vertices, so that each them get q^2 additional colours. Hence $n^i(v) = aq + rq$ if v is big or isolated regular, $n^i(v) = aq + q^2$ if v is non-isolated regular, and $n^i(v) = aq$ if v is goofy.

One shows that the obtained colouring is k-improper in the same way as in the proof of Theorem 8. We need to check that for each i, $\omega^{i+1} \leq \omega^i - (aq + rq)$.

Identically to the proof of Theorem 8, as $\gamma \geq (k+2)(rq-1)$, one shows that $p^{i+1}_{\max} \leq \omega^i - (aq + rq)$.

Claim 3 *A star H contains a big vertex or an isolated regular vertex x or two non-isolated regular vertices.*

Proof. Let u be the centre of H and $v_1, \ldots, v_{k+1}$ its spikes. Since G has maximum degree $k+1$ the only neighbours of u are the v_i's. Hence, if no vertex of H is big, then u is regular. Moreover, if u is not isolated, one of the v_i's is also regular. □

Consider a star H of G^{i+1}. Observe that each vertex of H receives at least aq colours. Moreover, by Claim 3, H contains a vertex that receives at least $aq + rq$ colours or two vertices receiving at least $aq + q^2$ colours each. Consequently, $p^{i+1}(H) \leq p^i(H) - (k+2)qa - \min\{rq, 2q^2\}$. By our choice of a, we get $p^{i+1}(H) \leq (k+1)(\omega^i - (aq + rq))$.

Thus $\theta_k(G^{i+1}, p^{i+1}) \leq \omega^i - aq - rq$, and so $\omega^{i+1} \leq \omega^i - aq - rq$. □

3 Grid graphs

3.1 General algorithms applied to grid graphs

In order to apply Theorems 8 and 10 to grid graphs, we determine $\chi_k(GL, \mathbf{q})$ for every positive integer q and $1 \leq k \leq 3$.

We first prove a preliminary lemma. Let C be a colouring of a weighted graph (G, p). We denote by $c_{u,v}$ the number of colours assigned to both u and v, that is $c_{u,v} = |C(u) \cap C(v)|$. We use the standard notation $N(u)$ for the *neighborhood* of u.

Lemma 11 *Let C be a k-improper colouring of a weighted graph (G, p).*

(i) $\sum_{v \in N(u)} c_{u,v} \leq kp(u)$*;*

(ii) $\forall u, v$*, C uses at least* $p(u) + p(v) - c_{u,v}$ *colours;*

(iii) $\forall u, v, w$*, C uses at least* $p(u) + p(v) + p(w) - c_{u,v} - c_{u,w} - c_{v,w}$*.*

Proof. (i) A colour assigned to u is assigned to at most k neighbours of u because C is k-improper.

(ii) and (iii) follow from the Inclusion-Exclusion Formula:

$$|C(u) \cup C(v)| = |C(u)| + |C(v)| - |C(u) \cap C(v)| = p(u) + p(v) - c_{u,v};$$

$$\begin{aligned} |C(u) \cup C(v) \cup C(w)| &= |C(u)| + |C(v)| + |C(w)| - |C(u) \cap C(v)| - |C(u) \cap C(w)| - |C(v) \cap C(w)| \\ &\quad + |C(u) \cap C(v) \cap C(w)| \\ &\geq p(u) + p(v) + p(w) - c_{u,v} - c_{u,w} - c_{v,w}. \end{aligned}$$

□

Theorem 12 *For the grid GL, we have:*

(i) $\chi_1(GL, \mathbf{q}) = 2q$*,*

(ii) $\chi_2(GL, \mathbf{q}) = \left\lceil \frac{3q}{2} \right\rceil$*, and*

(iii) $\chi_3(GL, \mathbf{q}) = \left\lceil \frac{5q}{4} \right\rceil$*.*

Proof. Let us first show the k-improper colourings of $(GL, \mathbf{q})$ with the required number of colours.

(i) The grid is bipartite so $(GL, \mathbf{q})$ has a 0-improper (and thus also 1-improper) colouring with $2q$ colours.

(ii) For $1 \leq j \leq 3$, let $U_j = \{(a, b) \mid a + b = j \mod 3\}$. Assign the colours $\left\lceil \frac{(j-1)q}{2} \right\rceil + 1, \ldots, \left\lceil \frac{jq}{2} \right\rceil$ to vertices which are not in U_j. See Figure 3 for $q = 2$.

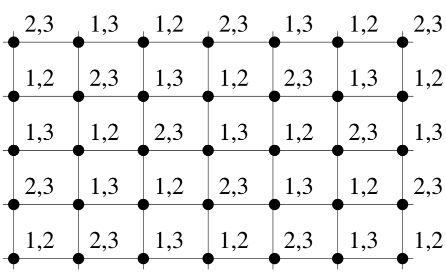

Figure 3: A 2-improper colouring of the square grid.

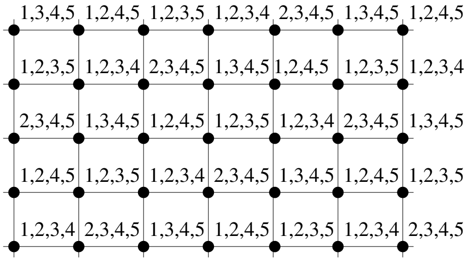

Figure 4: A 3-improper colouring of the square grid.

(iii) For $1 \leq j \leq 5$, let T_j be the set of vertices obtained from the vertex $(j, 0)$ by adding the linear combinations of the vectors $2\mathbf{f_1} + \mathbf{f_2}$ and $5\mathbf{f_1}$. For $1 \leq j \leq 5$, assign the colours $\left\lceil \frac{(j-1)q}{4} \right\rceil + 1, \ldots, \left\lceil \frac{jq}{4} \right\rceil$ to vertices not in T_j. See Figure 4 for $q = 4$.

Let us now show that these colourings are optimal. To do this, let C be a k-improper colouring of $(GL, \mathbf{q})$ with $\chi_k(GL, \mathbf{q})$ colours. If $k = 1$, consider a 4-cycle in GL. A colour may be used on at most two of these vertices. Hence $2q$ colours are needed. Now, suppose that $2 \leq k \leq 3$ and let u be a vertex of GL. Applying Lemma 11 (ii) to the four neighbours of u, we obtain: $4\chi_k(GL, \mathbf{q}) \geq 8q - \sum_{v \in N(u)} c_{u,v}$. Now by Lemma 11 (i), $4\chi_k(GL, \mathbf{q}) \geq (8 - k)q$. Therefore, we have respectively $\chi_2(GL, \mathbf{q}) \geq \frac{3q}{2}$ and $\chi_3(GL, \mathbf{q}) \geq \frac{5q}{4}$. □

Corollary 13 *For $1 \leq k \leq 3$, there are α_k-approximate algorithms for finding a k-improper colouring of a weighted grid graph, where $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{27}{20}$, and $\alpha_3 = \frac{19}{16}$.*

Proof. Theorems 9 and 12 give the result for $k = 1$. Theorems 8 and 12 give the result for $k = 2$. Theorems 10 and 12 give the result for $k = 3$. □

The following theorem improves the last result for 1-improper colouring of weighted grid graphs:

Theorem 14 *There is a $\frac{13}{9}$-approximate algorithm for finding a 1-improper colouring of weighted grid graph.*

Proof. The general idea of the algorithm is still similar, but this time we take advantage of the position of big vertices and quasi-big vertices (a new type of vertex to be described) to 1-improper colour the grid graph (cf Claim 4).

Algorithm 2 0. Initialisation: $(G^0, p^0) := (G, p)$, $S := \emptyset$ and $i = 0$.

1. Add the vertices of low weight to S that will be treated at the end (Step 3): $S^i := \{v \in V(G) \mid p^i(v) < p_{\inf}\}$ with $p_{\inf} = 133$, $S := S \cup S^i$ and for all $v \in S^i$, $s(v) := p^i(v)$. $G^{i+1} := G^i - S^i$.

2. If G^{i+1} is not empty:

   2.1. Give to each vertex v of G^{i+1} a certain number $n^i(v)$ of colours among a set of 156 colours, in such a way that $\omega_k(G^{i+1}, p^i - n^i) \leq \omega_k(G^i, p^i) - 108$.

   2.2. Set $p^{i+1} := p^i - n^i$ and $i := i + 1$, and go to Step 1.

3. Colour $(G\langle S\rangle, s)$ with 264 colours. It is possible as $s_{\max} \le p_{\inf} - 1 = 132$ and $\chi_1(GL, q) = 2q$.

Algorithm 2 yields a 1-improper colouring of (G,p) with at most $\frac{156}{108}\omega_k(G,p) + 264$ colours.

Here again we set $\omega^i = \omega_1(G^i, p^i)$. Let us now describe precisely how to perform Step 2.1. A *big vertex* is a vertex such that $p^i(v) > \omega^i - d_1$ with $d_1 = 24$. A *small vertex* v is a non-big vertex. It is *goofy* if it is adjacent to a big vertex (we note $Goof$ the set of goofy vertices), it is quasi-big if $p^i(v) > \omega^i - d_2$ with $d_2 = 67$, and it is *regular*, otherwise. A 2-star is *big* if its weight is bigger than $2\omega^i - d_3$ with $d_3 = 45$.

Claim 4 *Let u and v be two vertices which are big or quasi-big in G^i. In G^{i+1}, they are at distance at least 3 or form a connected component.*

Proof. Suppose that u and v are at distance at most 2. If they do not form a connected component in G^{i+1}, there is a vertex $w \in V(G^{i+1})$ such that the subgraph induced by u, v and w is a 2-star. Hence $p^i(u)+p^i(v)+p^i(w) \le 2\omega^i$. But $p^i(u)+p^i(v)+p^i(w) > 2\omega^i - 2d_2 + 1 + p_{\inf} = 2\omega^i$, a contradiction. Thus u and v form a connected component. $\square$

Set $a = 48$, $b = 6$ and $c = 3$. At each step 2.1, we give all the $2a + 8b + 4c$ colours to the vertices in the connected components of order 2.

For the vertices not in such small components, we use three different colourings. The two first are based on the following eight sets of vertices:

- $U_1 = \{(0,0) + 2i\mathbf{f_1} + i'(\mathbf{f_1} - 2\mathbf{f_2}), (0,1) + 2i\mathbf{f_1} + i'(\mathbf{f_1} - 2\mathbf{f_2})\}$,
- $U_2 = \{(1,0) + 2i\mathbf{f_1} + i'(\mathbf{f_1} - 2\mathbf{f_2}), (1,1) + 2i\mathbf{f_1} + i'(\mathbf{f_1} - 2\mathbf{f_2})\}$,
- $U_3 = \{(0,1) + 2i\mathbf{f_1} + i'(\mathbf{f_1} - 2\mathbf{f_2}), (0,2) + 2i\mathbf{f_1} + i'(\mathbf{f_1} - 2\mathbf{f_2})\}$,
- $U_4 = \{(1,1) + 2i\mathbf{f_1} + i'(\mathbf{f_1} - 2\mathbf{f_2}), (1,2) + 2i\mathbf{f_1} + i'(\mathbf{f_1} - 2\mathbf{f_2})\}$,
- $U_5 = \{(0,0) + 2i\mathbf{f_2} + i'(2\mathbf{f_1} - \mathbf{f_2}), (1,0) + 2i\mathbf{f_2} + i'(2\mathbf{f_1} - \mathbf{f_2})\}$,
- $U_6 = \{(0,1) + 2i\mathbf{f_2} + i'(2\mathbf{f_1} - \mathbf{f_2}), (1,1) + 2i\mathbf{f_2} + i'(2\mathbf{f_1} - \mathbf{f_2})\}$,
- $U_7 = \{(1,0) + 2i\mathbf{f_2} + i'(2\mathbf{f_1} - \mathbf{f_2}), (2,0) + 2i\mathbf{f_2} + i'(2\mathbf{f_1} - \mathbf{f_2})\}$,
- $U_8 = \{(1,1) + 2i\mathbf{f_2} + i'(2\mathbf{f_1} - \mathbf{f_2}), (2,1) + 2i\mathbf{f_2} + i'(2\mathbf{f_1} - \mathbf{f_2})\}$.

We first assign $2a$ colours such that each vertex of U_1 receive a colours and each vertex of U_2 receives a other colours. Since U_1 and U_2 form a partition of V, each vertex receives a colours, and since $G\langle U_1\rangle$ and $G\langle U_2\rangle$ have maximum degree 1, the colouring we obtain is 1-improper.

Then we use $8b$ colours denoted by $(i,j), 1 \le i \le b, 1 \le j \le 8$. We give all these colours to the big vertices. Then, for $1 \le j \le 8$, we give to each small vertex of $U_j \setminus Goof$ the colours $(i,j), 1 \le i \le b$. A vertex appears in four of the U_j, so each non-goofy vertex receives $4b$ colours. Finally, by Claim 4, a goofy vertex u has a unique big neigbour v. Thus, there is an integer j such that both u and v are in U_j. We assign to u the colours $(i,j), 1 \le i \le b$. Doing so, each goofy vertex receives b colours.

Finally, we use four sets of c colours A_t, A_d, A_l and A_r. Each big or quasi-big vertex receives the colours of all these sets and each top (resp. down, left, right) neighbour receives the colours of A_t (resp. A_d, A_l and A_r). This is 1-improper, by Claim 4.

Let us now check that $\omega^{i+1} \le \omega^i - 108$. Let v be a vertex. If it is big then $n^i(v) = a + 8b + 4c$; if it is quasi-big, then $n^i(v) = a+4b+4c$; if it is goofy, then $n^i(v) = a+b+c$; if it is regular, then $n^i(v) \ge a+4b$, and if, in addition, it is adjacent to a quasi-big vertex, then $n^i(v) \ge a + 4b + c$. Hence, by our choice of a, b, c d_1 and d_2, we have

$$p^{i+1}_{\max} \le \omega^i - \min\{a + 8b + 4c, d_1 + a + 4b + 4c, d_2 + a + b + c, d_2 + a + 4b\} \le p^i_{\max} - 108. \qquad (1)$$

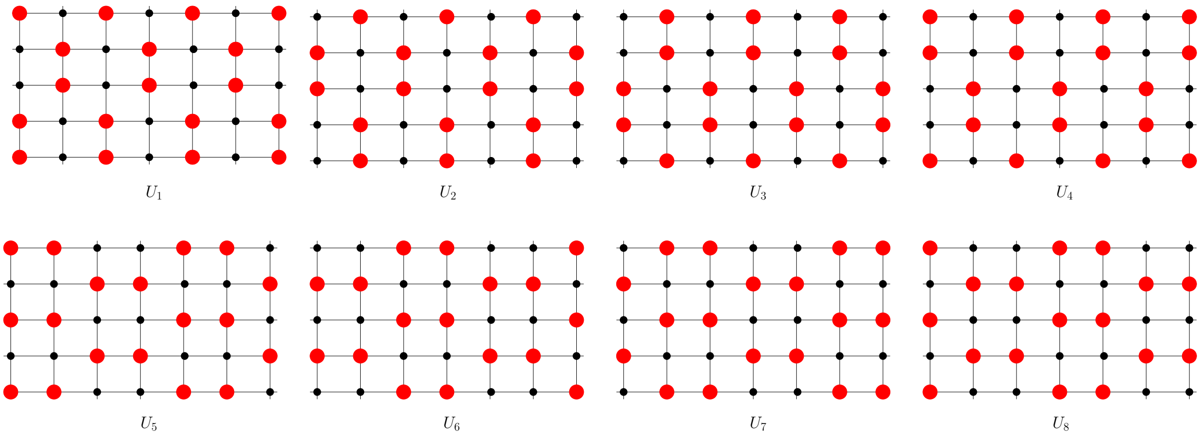

Figure 5: The sets U_i, $1 \leq i \leq 8$. In each figure, the bold vertices are those of U_i and the bottom left vertex is $(0,0)$.

Claim 5 *Let H be a big 2-star in (G^{i+1}, p^i). Then, the following hold:*

(i) if the centre of H is goofy, then H has a big vertex;

(ii) if H has two goofy vertices, then the third one is big;

(iii) if H has a goofy vertex, then it has a big or quasi-big vertex.

Proof. Let u be the centre of H and v_1 and v_2 its spikes.

(i) Suppose that u is goofy. Let w be its big neighbour. Suppose for a contradiction that $w \notin \{v_1, v_2\}$. Considering the two 2-stars with vertex sets $\{w, u, v_1\}$ and $\{w, u, v_2\}$, we obtain that $p^i(v_1) + p^i(u) \leq 2\omega^i - p^i(w) \leq \omega^i + d_1 - 1$ and $p^i(v_2) + p^i(u) \leq 2\omega^i - p^i(w) \leq \omega^i + d_1 - 1$. Hence $2\omega^i + 2d_1 - 2 \geq p^i(v_1) + p^i(v_2) + 2p^i(u) \geq p^i(u) + p^i(H) \geq p^i(u) + 2\omega^i - d_3 + 1$, and so $p^i(u) \leq 2d_1 + d_3 - 3$. But then u is in S^i, which is a contradiction.

(ii) Suppose, by way of contradiction, that H has two goofy vertices and no big vertex. By (i), v_1 and v_2 are the goofy vertices. Let w_1 (resp. w_2) be the big neighbour of v_1 (resp. v_2). (We may have $w_1 = w_2$). As in (i), considering the two 2-stars with vertex sets $\{w_1, v_1, u\}$ and $\{w_2, v_2, u\}$, we obtain $p^i(v_1) + p^i(u) \leq \omega^i + d_1 - 1$ and $p^i(v_2) + p^i(u) \leq \omega^i + d_1 - 1$. These inequalities yield the contradiction as in (i).

(iii) If u is goofy, then H has a big vertex by (i). Suppose now that one spike of H, say v_1, is goofy. Let w be the big neighbour of v_1. If $w = u$, we have the result. Assume now that $w \neq u$. Considering the 2-star induced by $\{w, v_1, u\}$, we obtain $p^i(v_1) + p^i(u) \leq \omega^i + d_1 - 1$. Since H is big, $p^i(v_1) + p^i(u) + p^i(v_2) \geq 2\omega^i - d_3 + 1$. So $p^i(v_2) \geq \omega^i - d_1 - d_3 + 2$, that is, v_2 is quasi-big. □

Let H be a 2-star. If it is not big, its weight decreases by at least $3a + 3b + 3c$. Hence

$$p^{i+1}(H) \leq 2\omega^i - 45 - (3a + 3b + 3c) = 2w^i - 216 \tag{2}$$

If H is big, then by Claim 5, it has either three non-goofy vertices, in which case its weight decreases by at least $3a + 12b$, or one big vertex and two goofy vertices, in which case its weight decreases by $3a + 10b + 6c$, or one big vertex, one regular vertex and one goofy vertex, in which case its weight decreases by $3a + 13b + 5c$, or one quasi-big vertex, one regular vertex adjacent to this quasi-big vertex and one goofy vertex, in which case its weight decreases by $3a + 9b + 6c$. Hence, by our choice of a, b, c, for a big star H, we have:

$$p^{i+1}(H) \leq p^i(H) - \min(3a + 12b, 3a + 9b + 6c, 3a + 13b + 5c) \leq p^i(H) - 216. \tag{3}$$

We conclude the proof observing that Inequalities (1), (2) and (3) yield $\omega^{i+1} \leq \omega^i - 108$. □

4 Hexagonal graphs

4.1 General algorithms applied to hexagonal graphs

Theorem 15 *For the triangular lattice TL, we have:*

(i) $\chi_1(TL, \mathbf{q}) = \lceil \frac{5q}{2} \rceil$,

(ii) $\chi_2(TL, \mathbf{q}) = 2q$,

(iii) $\chi_3(TL, \mathbf{q}) = \lceil \frac{3q}{2} \rceil$,

(iv) $\chi_4(TL, \mathbf{q}) = \lceil \frac{4q}{3} \rceil$ *and*

(v) $\chi_5(TL, \mathbf{q}) = \lceil \frac{7q}{6} \rceil$.

Proof. Let us first show the k-improper colourings of $(TL, \mathbf{q})$ with the required number of colours.

(i) For $1 \leq j \leq 5$, let A_j be the set of vertices obtained from the vertex $(j, 0)$ by adding the linear combinations of the vectors $2\mathbf{e_1} + \mathbf{e_2}$ and $5\mathbf{e_1}$. For $1 \leq j \leq 5$, assign the colours $\left\lceil \frac{(j-1)q}{2} \right\rceil + 1, \ldots, \lceil \frac{jq}{2} \rceil$ to vertices of $A_j \cup A_{j+1}$ (with $A_6 = A_1$) (see Figure 4.1 for $q = 2$).

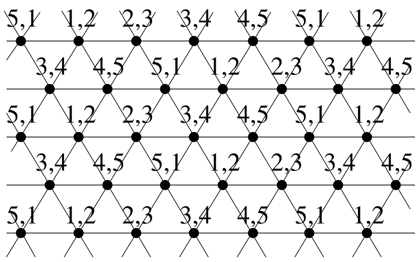

Figure 6: A 1-improper colouring of the triangular lattice.

(ii) Colour a vertex (a, b) with $1, \ldots, q$ if a is even, and with $q + 1, \ldots, 2q$, otherwise.

(iii) For $1 \leq j \leq 3$, let S_j be the set of vertices obtained from the vertex $(j, 0)$, by adding the linear combinations of the vectors $\mathbf{e_1} + \mathbf{e_2}$ and $3\mathbf{e_1}$. Assign the colours $\left\lceil \frac{(j-1)q}{2} \right\rceil + 1, \ldots, \lceil \frac{jq}{2} \rceil$ to the vertices which are not in S_j.

(iv) For $0 \leq j_1 \leq 1$ and $1 \leq j_2 \leq 2$, $T_{2j_1+j_2} = \{(a, b) \mid a \equiv j_1 \mod 2 \text{ and } b \equiv j_2 \mod 2\}$. Assign the colours $\left\lceil \frac{(j-1)q}{3} \right\rceil + 1, \ldots, \lceil \frac{jq}{3} \rceil$ to the vertices which are not in T_j.

(v) For $1 \leq j \leq 7$, let U_j be the set of vertices obtained from the vertex $(j, 0)$ by adding the linear combinations of the vectors $2\mathbf{e_1} + \mathbf{e_2}$ and $7\mathbf{e_1}$. Assign the colours $\left\lceil \frac{(j-1)q}{6} \right\rceil + 1, \ldots, \lceil \frac{jq}{6} \rceil$ to the vertices which are not in U_j.

Let us now show that these colourings are optimal. For this, let C be a k-improper colouring of $(TL, \mathbf{q})$ with $\chi_k(TL, \mathbf{q})$ colours and u be a vertex of TL.

Applying Lemma 11 (i) and (ii) to u and its six neighbours, we obtain that $6\chi_k(TL, \mathbf{q}) \geq (12-k)q$. For $k = 3$, 4 or 5, we have, respectively, $\chi_3(TL, \mathbf{q}) \geq \frac{3q}{2}$, $\chi_4(TL, \mathbf{q}) \geq \frac{4q}{3}$ and $\chi_5(TL, \mathbf{q}) \geq \frac{7q}{6}$.

For $1 \leq k \leq 2$, we need an extra argument. Let u and v be two adjacent vertices and t and w their two common neighbours. Set $f(u, v) = c_{u,t} + c_{u,v} + c_{u,w}$.

Consider now the ordered pair (u, v) which minimizes $f(u, v)$, i.e. $f(u, v) = f_{min}$. Then $f(v, u) = c_{v,t} + c_{u,v} + c_{v,w}$, and therefore, $f(u, v) + f(v, u) = (c_{u,t} + c_{u,v} + c_{v,t}) + (c_{u,w} + c_{u,v} + c_{v,w})$. By Lemma 11 (iii), we have $f(u, v) + f(v, u) \geq 6q - 2\chi_k(TL, \mathbf{q})$.

Let u' be the vertex symmetrical to u compared to v. By Lemma 11 (i), $f(v, u) + f(v, u') \leq kq$, so $f(v, u') \leq kq - f(v, u) \leq kq - 6q + 2\chi_k(TL, \mathbf{q}) + f_{min}$.

For $k = 1$, if $\chi_k(TL, \mathbf{q}) < \frac{5q}{2}$, we would have $f(v, u') < f_{min}$. Similarly, for $k = 2$, if $\chi_k(TL, \mathbf{q}) < 2q$, we would have $f(v, u') < f_{min}$. In both cases, it contradicts the minimality of f in (u, v). □

Corollary 16 *For $1 \leq k \leq 5$, there are α_k-approximate algorithms for finding a k-improper colouring of a weighted hexagonal graph, where $\alpha_1 = \frac{20}{11}$, $\alpha_2 = \frac{12}{7}$, $\alpha_3 = \frac{18}{13}$, $\alpha_4 = \frac{80}{63}$, and $\alpha_5 = \frac{41}{36}$.*

Proof. Theorems 9 and 15 give the result for $k = 1$. Theorems 8 and 15 give the result for $2 \leq k \leq 4$. Finally, Theorems 10 and 15 give the result for $k = 5$. □

5 Conclusion

We have proposed several approximate algorithms whose approximation ratios are summarized in the Table 1.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
Grid	13/9	27/20	19/16	1	1
Hexagonal	20/11	12/7	18/13	80/63	41/36

Table 1: Summary of the approximation ratios.

A natural continuation of this work would be to improve the above ratios. It would be very nice to prove the existence of a Polynomial Time Approximation Scheme or some unapproximability results for these problems. In this direction, the first thing to do regarding grid graphs is to answer Problem 3, i.e. to find whether or not optimally improper colouring a weighted grid graph is NP-complete. We strongly believe that it must be NP-complete.

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