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Abstract: We consider the optimal control problem of a class of integral equations with initial and final state constraints, as well as running state coinstraints. We prove Pontryagin's principle, and study the continuity of the optimal control and of the measure associated with first order state constraints. We also establish the Lipschitz continuity of these two functions of time for problems with only first order state constraints.

Key-words: Optimal control, state constraint, Pontryagin's principle, Lipschitz continuity, integral equations, Ekeland's principle.

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Commande optimale d'équations intégrales avec contraintes sur l'état

Résumé : Nous considérons un problème de commande optimale d'une classe d'équations intégrales avec contraintes sur l'état initial et final, ainsi que des contraintes sur l'état à tout instant. Nous prouvons un principe de Pontryaguine, et étudions la continuité de la commande optimale et des mesures associées aux contraintes du premier ordre. Nous établissons également la continuité lipschitz de ces fonctions dans le cas où toutes les contraintes sur l'état sont du premier ordre.

Mots-clés : commande optimale, contraintes sur l'état, principe de Pontryaguine, continuité lipschitz, équations intégrales, principe d'Ekeland.

1 Introduction

Integral equations occur in a natural way in the study of economic problems or population dynamics, see for instance Hritonenko and Yatsenko [22] and Kamien and Schwartz [24]. The optimal control of such systems has been already discussed in a number of papers. Vinokurov [32] provides a maximum principle for a problem with constraints over the sum of integral and final cost functions. His proof has been questioned in Neustadt and Warga [28]. Existence of an optimal control for such problems is studied in Angell [1, 2, 3]. Several variants of the maximum principle for an optimal control problem with integral or final constraints were obtained in Bakke [4], Carlson [12], de la Vega [13], and in the book by Neustadt [29].

The novelty in this paper is that we discuss optimal control problems of integral equations with running state constraints as well as constraints on the initial and final states. We prove a version of Pontryagin's principle, and analyze the Lipschitz continuity (over time) of the control and of some of the multipliers.

Significant advances in the study of optimal control problems with running state constraints have been obtained in recent years. See in particular Bonnans and Hermant [8, 6, 7, 9], Malanowski [25, 26].

There is a specific literature about Lipschitz continuity of the optimal control for state constrained problems: see the pioneering paper Hager [20], Galbraith and Vinter [18, 19], Shvartsman and Vinter [31], Do Rosario de Pinho and Shvartsman [16], Hermant [21] in the case of second order state constraints, and more recently Bonnans [5] in the case of state constraints of arbitrary order. An important tool is the use of alternative optimality systems, motivated by reformulations in which the Hamiltonian function includes some time derivative at appropriate order of the state constraint: see Bryson, Denham and Dreyfus [10], and Jacobson, Lele and Speyer [23]. A clarification of the theory was brought in Maurer [27].

The paper is organized as follows. We set the problem and state Pontryagin's principle in section 2.1. The proof of Pontryagin's principle is provided in section 3. The continuity of the control (and of the multipliers associated to first order state constraints) is analyzed in section 4. The alternative optimality system is introduced in section 5, allowing to obtain (under appropriate hypotheses) the Lipschitz continuity of the control (and of the multipliers associated to first order state constraints). We conclude in section 6 by discussing open problems.

The norm in Euclidean spaces will be denoted by $|\cdot|$. The projection onto a closed convex subset K of an Euclidean space is denoted by $P_K(\cdot)$.

2 Setting and statement of Pontryagin's principle

2.1 Setting

In this paper we consider a state constrained optimal control problem of the following type:

$$(P) \qquad \begin{cases} \mathrm{Min} \displaystyle\int_0^T \ell(u_t, y_t)\mathrm{d}t + \phi(y_0, y_T); \\ \text{(i)} \quad y_t = y_0 + \int_0^t f(t, s, u_s, y_s)\mathrm{d}s; \;\; t \in (0,T); \\ \text{(ii)} \quad g(y_t) \le 0; \;\; t \in [0,T], \\ \text{(iii)} \quad \Phi(y_0, y_T) \in K, \end{cases}$$

with $\ell : I\!R^m \times I\!R^n \to I\!R$, $\phi : I\!R^n \times I\!R^n \to I\!R$, $f : I\!R \times I\!R \times I\!R^m \times I\!R^n \to I\!R^n$, $g : I\!R^n \to I\!R^{n_g}$, $n_g \ge 1$, $\Phi : I\!R^n \times I\!R^n \to I\!R^{n_\Phi}$, and K is a closed and non empty convex subset of $I\!R^{n_\Phi}$. All data f, g, ℓ, ϕ, Φ are assumed to be of class C^∞, and f is supposed to be Lipschitz. Set, for $q \in [1, \infty]$

$$\mathcal{U}_q := L^q(0,T,I\!R^m); \quad \mathcal{Y}_q := W^{1,q}(0,T,I\!R^n). \tag{1}$$

The control and state space are $\mathcal{U} := \mathcal{U}_\infty$, $\mathcal{Y} := \mathcal{Y}_\infty$. We call *trajectory* a pair $(u,y) \in \mathcal{U} \times \mathcal{Y}$, solution of the *state equation* (P)(i). We set τ as the symbol for the first variable of f, so that for instance $D_\tau f$ denotes the partial derivative of f w.r.t. the first variable. Observe that in the case when f does not depend on τ, we recover the classical state constrained optimal control problem.

2.2 Statement of Pontryagin's principle

Denote by $C([0,T])$ the set of continuous functions on $[0,T]$, and by $\widetilde{BV}([0,T])$ the set of functions of bounded variations on $[0,T]$. Elements of $\widetilde{BV}([0,T])$ have left and right limits over $(0,T)$. The jump of a function η with left and right limits at time t is denoted by $[\eta_t] := \eta_{t+} - \eta_{t-}$. When $\eta \in \widetilde{BV}([0,T])$, we define its jumps at time 0 and T as $[\eta_0] = \eta_{0+} - \eta(0)$ and $[\eta_T] = \eta_T - \eta_{T-}$, resp.

If η and λ belong to $\widetilde{BV}([0,T])$, we say that $\eta \mathcal{R} \lambda$ if η and λ have the same value at times 0 and T, and same left and right limits over $(0,T)$. This defines an equivalence relation. We denote by $BV([0,T])$ the quotient space $\widetilde{BV}([0,T])/\mathcal{R}$, and by $BV_T([0,T])$, the set of elements of $BV([0,T])$ for which the elements of the equivalence class have zero value at time $T+$.

We may identify the dual of $C([0,T])$ with $BV_T([0,T])$, the linear form associated with $\eta \in BV_T([0,T])$ being $y \mapsto \int_0^T y_t \mathrm{d}\tilde{\eta}_t$, where $\tilde{\eta}_t$ is an element of the equivalence class of η; in the sequel we will write this integral as $\int_0^T y_t \mathrm{d}\eta_t$.

By $I\!R^{n*}$ we denote the dual of $I\!R^n$, represented as a set of horizontal vectors. More generally all vector-valued dual variables will be seen as horizontal vectors function of time. For instance, the dual of $C([0,T])^n$ will be identified to $BV_T([0,T])^{n*}$, the set of functions of bounded variation over $[0,T]$ with values in $I\!R^{n*}$.

We denote by $F(P)$ the set of $(u,y) \in \mathcal{U} \times \mathcal{Y}$ that satisfy the constraints of problem (P). Set $\mathcal{M} := BV_T([0,T])^{n_g*}$. Let $\bar{\alpha} \in I\!R_+$, $\bar{\Psi} \in I\!R^{n_\Phi}$ and $\bar{\eta} \in \mathcal{M}$.

For $y \in \mathcal{Y}$, denote the *end points Lagrangian* as the function $[\mathbb{R}_+ \times \mathbb{R}^{n_\Phi *}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ whose expression is

$$\Phi[\alpha, \Psi](y_0, y_T) := \alpha\phi(y_0, y_T) + \Psi\Phi(y_0, y_T). \tag{2}$$

We adopt here (as in some of the Russian literature, see e.g. Dmitruk [15]) the convention of denoting the multipliers as parameters of functions having primal and dual variables, such as Hamiltonian of Lagrangian functions. Let $(\bar{u}, \bar{y}) \in F(P)$. The associated costate, whenever it exists, is defined as the solution in $\mathcal{P} := BV([0,T])^{n*}$ of

$$\begin{cases} -\mathrm{d}\bar{p}_t &= \bar{\alpha} D_y \ell(\bar{u}_t, \bar{y}_t)\mathrm{d}t + \bar{p}_t D_y f(t,t,\bar{u}_t,\bar{y}_t)\mathrm{d}t + \displaystyle\sum_{i=1}^{n_g} g_i'(\bar{y}_t)\mathrm{d}\bar{\eta}_{i,t} \\ & \quad + \displaystyle\int_t^T \bar{p}_s D^2_{\tau,y} f(s,t,\bar{u}_t,\bar{y}_t)\mathrm{d}s, \\ (-\bar{p}_{0-}, \bar{p}_{T+}) &= \Phi'[\bar{\alpha}, \bar{\Psi}](\bar{y}_0, \bar{y}_T). \end{cases} \tag{3}$$

By standard contraction arguments, it can be shown that the variant of (3) obtained by removing the initial condition on the costate has a unique solution in $\mathcal{P}$. Next we introduce the *Hamiltonian function*

$$H[\alpha, p](t, u, y) := \alpha\ell(u, y) + p_t f(t, t, u, y) + \int_t^T p_s D_\tau f(s, t, u, y)\mathrm{d}s. \tag{4}$$

The Hamiltonian is a function parameterized by $(\alpha, p) \in \mathbb{R}_+ \times BV([0,T])^{n*}$, from $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ into $\mathbb{R}$. Note that the dynamics can be written as

$$-\mathrm{d}\bar{p}_t = D_y H[\bar{\alpha}, \bar{p}](t, \bar{u}_t, \bar{y}_t)\mathrm{d}t + (\mathrm{d}\bar{\eta}_t)\, g'(\bar{y}_t). \tag{5}$$

Definition 2.1. Let $(\bar{u}, \bar{y}) \in F(P)$. We say that $(\bar{\alpha}, \bar{\eta}, \bar{\Psi}, \bar{p})$ in $\mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}^{n_\Phi *} \times \mathcal{P}$, is a *Pontryagin multiplier* associated with $(\bar{u}, \bar{y}) \in F(P)$ if the costate equation (3) is satisfied, as well as the four following conditions: *non triviality*

$$\bar{\alpha} + \|\bar{\eta}\| + |\bar{\Psi}| > 0, \tag{6}$$

complementarity

$$\bar{\eta} \geq 0; \quad \sum_{i=1}^{n_g} \int_0^T g_i(y_t)\mathrm{d}\eta_{i,t} = 0. \tag{7}$$

transversality condition

$$\bar{\Psi} \in N_K(\Phi(\bar{y}_0, \bar{y}_T)), \tag{8}$$

and *Hamiltonian inequality*

$$H[\bar{\alpha}, \bar{p}](t, \bar{u}_t, \bar{y}_t) \leq H[\bar{\alpha}, \bar{p}](t, u, \bar{y}_t), \quad \text{for all } u \in \mathbb{R}^m, \text{ for a.a. } t \in (0,T). \tag{9}$$

We say that $(\bar{u}, \bar{y}) \in F(P)$ is a *Pontryagin extremal*, or that it satisfies *Pontryagin's principle*, if the set of associated Pontryagin multipliers is not empty.

The set of Pontryagin multipliers, that we denote by $\Lambda(\bar{u}, \bar{y})$, is a convex cone not containing zero. When $(\bar{\alpha}, \bar{\eta}, \bar{\Psi}, \bar{p}) \in \Lambda(\bar{u}, \bar{y})$ is such that $\bar{\alpha} > 0$, we say that the multiplier is *regular* and we may identify the multiplier with the one of the same direction with $\alpha = 1$. In the latter case we say that $(\bar\eta, \bar\Psi, \bar p)$ is a regular multiplier. When $\bar\alpha = 0$ we say that $(\bar\eta, \bar\Psi, \bar p)$ is a singular multiplier.

We say that $(\bar u, \bar y) \in F(P)$, is a local solution of (P) *in the L^1 norm* if the following holds:

$$\left\{ \begin{array}{l} \displaystyle\int_0^T \ell(\bar u_t, \bar y_t)\mathrm{d}t + \phi(\bar y_0, \bar y_T) \leq \int_0^T \ell(u_t, y_t)\mathrm{d}t + \phi(y_0, y_T), \\ \text{for all } (u,y) \in F(P) \quad \text{such that } \|u - \bar u\|_1 + |y_0 - \bar y_0| \text{ is small enough.} \end{array} \right. \tag{10}$$

Our main theorem follows.

Theorem 2.2. *Any local solution of problem (P), in the L^1 norm, is a Pontryagin extremal.*

In the subsequent sections we will prove this theorem and analyse some consequences, as the analysis of continuity and Lipschitz continuity of the control. We first establish the well-posedness of the state equation.

3 Proof of Pontryagin's principle

3.1 Study of the state equation

We recall that the function f is supposed to be Lipschitz. The lemma below is of course well-known.

Lemma 3.1. *If f is Lipschitz, then given $(u, y_0) \in \mathcal{U}_1 \times \mathbb{R}^n$, the state equation $(P)(i)$ has a unique solution in $\mathcal{Y}_1$, denoted $y[u, y_0]$, and for all $(u', y'_0) \in \mathcal{U}_1 \times \mathbb{R}^n$, we have that*

$$\|y[u', y'_0] - y[u, y_0]\|_\infty = O(\|u' - u\|_1 + |y'_0 - y_0|). \tag{11}$$

Proof. a) Existence and uniqueness of the state is obtained using the usual technique of contraction operators for the Cauchy-Lipschitz theorem.
b) Denote $y := y[u, y_0]$ and $y' := y[u', y'_0]$. The estimate (11) is a consequence of Gronwall's lemma, once we observe that the state equation implies

$$\|y' - y\|_\infty \leq |y'_0 - y_0| + L_f \int_0^T (|u'_s - u_s| + |y'_s - y_s|)\, \mathrm{d}s. \tag{12}$$

□

Let $(\bar u, \bar y)$ be a trajectory. The *classical linearized system* is the following equation, where $(v, z) \in \mathcal{U} \times \mathcal{Y}$:

$$z_t = y_0 - \bar y_0 + \int_0^t D_{(u,y)} f(t, s, \bar u_s, \bar y_s)(v_s, z_s)\mathrm{d}s, \quad t \in (0, T). \tag{13}$$

We next introduce a variant that we will call *Pontryagin linearization*, since it is strongly related to the Pontryagin maximum principle, and whose expression is as follows:

$$z_t = y_0 - \bar y_0 + \int_0^t \left[D_y f(t, s, \bar u_s, \bar y_s) z_s + f(t, s, u_s, \bar y_s) - f(t, s, \bar u_s, \bar y_s) \right] \mathrm{d}s, \quad t \in (0, T). \tag{14}$$

Lemma 3.2. *Let (u, y_0) and $(\bar{u}, \bar{y}_0)$ belong to $\mathcal{U} \times I\!R^n$, with associated states denoted by y and $\bar{y}$, resp. Let z be the solution of the Pontryagin linearization (14). If $D_y f$ is Lipschitz, then for some $C_1 > 0$ depending only on the data of (P), we have that*

$$\|\bar{y} + z - y\|_\infty \leq C_1 \left(\|u - \bar{u}\|_1^2 + |y_0 - \bar{y}_0|^2\right). \tag{15}$$

Proof. We have that $\zeta := \bar{y} + z - y$ is solution of

$$\zeta_t = \int_0^t \left(D_y f(t, s, \bar{u}_s, \bar{y}_s)\zeta_s + \Delta(t, s)\right) \mathrm{d}s, \quad t \in (0, T), \tag{16}$$

where

$$\Delta(t, s) = f(t, s, u_s, \bar{y}_s) - f(t, s, u_s, y_s) + D_y f(t, s, \bar{u}_s, \bar{y}_s)(y_s - \bar{y}_s), \tag{17}$$

so that, setting $y_s^\sigma := \bar{y}_s + \sigma(y_s - \bar{y}_s)$:

$$\Delta(t, s) = \int_0^1 \left(D_y f(t, s, u_s, y_s^\sigma) - D_y f(t, s, \bar{u}_s, \bar{y}_s)\right) (y_s - \bar{y}_s)\mathrm{d}\sigma. \tag{18}$$

It follows that, denoting by $L_{D_y f}$ the Lipschitz constant of $D_y f$:

$$|\Delta(t, s)| \leq L_{D_y f}(|u_s - \bar{u}_s| + \|y - \bar{y}\|_\infty)\|y - \bar{y}\|_\infty \tag{19}$$

so that $\|\Delta(t, \cdot)\|_1 = O\left(\|u - \bar{u}\|_1^2 + |y_0 - \bar{y}_0|^2\right)$. We conclude with Gronwall's lemma. ☐

3.2 The penalized problem

In this section we provide a proof for Pontryagin's principle (theorem 2.2). The first step consiste in proving a variant of this result in the case when the control is subject to the constraint of belonging to a certain compact set. So, given a compact set $U \subset I\!R^m$, consider the problem obtained by adding to the formulation of (P) the control constraint that the control a.a. belongs to U:

$$(P_U) \qquad \begin{cases} \text{Min} \displaystyle\int_0^T \ell(u_t, y_t)\mathrm{d}t + \phi(y_0, y_T); \\ \text{(i)} \quad y_t = y_0 + \int_0^t f(t, s, u_s, y_s)\mathrm{d}s; \ \ t \in (0, T); \\ \text{(ii)} \quad g(y_t) \leq 0; \ \ t \in [0, T], \\ \text{(iii)} \quad \Phi(y_0, y_T) \in K, \\ \text{(iv)} \quad u_t \in U, \quad \text{for a.a. } t \in (0, T). \end{cases}$$

The set $\Lambda_U(\bar{u}, \bar{y})$ of Pontryagin multipliers is defined as in definition (2.1), replacing the Hamiltonian inequality (9) by

$$H[\bar{\alpha}, \bar{p}](t, \bar{u}_t, \bar{y}_t) \leq H[\bar{\alpha}, \bar{p}](t, u, \bar{y}_t), \quad \text{for all } u \in U, \text{ for a.a. } t \in (0, T). \tag{20}$$

Theorem 3.3. *Any local solution of problem (P_U), in the L^1 norm, is a Pontryagin extremal.*

Proof. Let $(\bar{u}, \bar{y})$ be a local solution of problem (P_U). We denote $\mathcal{U}_U := L^\infty(0, T, U)$.

The Banach space $C[0,T]^{n_g}$ being separable, there exists an equivalent norm denoted $\|\cdot\|_e$ such that the dual unit ball is stricly convex, see e.g. [14]. Since $C := C[0,T]_-^{n_g}$ is convex, the associated distance function denoted by $d_C(.)$, which is non expansive, has out of C unit norm subgradients, and is therefore differentiable out of C. We note

$$J(u, y_0) := \int_0^T \ell(u(t), y[u, y_0](t))\mathrm{d}t + \phi(y_0, y_T[u, y_0]). \tag{21}$$

Consider the cost function

$$\begin{aligned} J_\varepsilon(u, y_0) := \quad & \Big((J(u, y_0) - J(\bar{u}, \bar{y}_0) + \varepsilon^2)_+^2 + (d_C(g(y[u, y_0])))^2 \\ & \qquad + (d_K(\Phi(y_0, y_T[u, y_0]))^2 \Big)^{\frac{1}{2}}, \end{aligned} \tag{22}$$

and the problem

$$\operatorname*{Min}_{(u,y_0)} \quad J_\varepsilon(u, y_0); \quad (u, y_0) \in \mathcal{U}_U \times I\!R^n. \tag{P_ε}$$

Since J_ε is a nonnegative function and $J_\varepsilon(\bar{u}, \bar{y}_0) = \varepsilon^2$, we have that $(\bar{u}, \bar{y}_0)$ is an ε^2 solution of P_ε. Since U is bounded, we have that the function $(u, y_0) \to J_\varepsilon(u, y_0)$ is continuous for the *augmented Ekeland metric*

$$\rho_A((u, y_0), (u', y_0')) := |y_0 - y_0'| + \rho_E(u, u'), \tag{23}$$

where ρ is the Ekeland metric given by

$$\rho_E(u, u') := \text{meas}(\{t \in (0, T) : u_t \neq u_t'\}). \tag{24}$$

Hence, by Ekeland's principle [17], there exists $(u^\varepsilon, y_0^\varepsilon) \in \mathcal{U}_U \times I\!R^n$ such that

$$|y_0^\varepsilon - \bar{y}_0| + \rho_E(u^\varepsilon, \bar{u}) \leq \varepsilon, \tag{25}$$

and

$$J_\varepsilon(u^\varepsilon, y_0^\varepsilon) \leq J_\varepsilon(u, y_0) + \varepsilon\left(|y_0 - y_0^\varepsilon| + \rho_E(u, u^\varepsilon)\right), \quad \text{for all} \quad (u, y_0) \in \mathcal{U} \times I\!R^n. \tag{26}$$

Let $y^\varepsilon = y[u^\varepsilon, y_0^\varepsilon]$ denote the state associated with control u^ε and initial condition y_0^ε. We have that $J_\varepsilon(u^\varepsilon, y_0^\varepsilon) > 0$ (otherwise we would have $(u^\varepsilon, y^\varepsilon) \in F(P)$ and $J(u^\varepsilon, y_0^\varepsilon) < J(\bar{u}, \bar{y}_0)$, which would contradict for ε small enough the local optimality of $(\bar{u}, \bar{y})$). Set

$$\alpha_\varepsilon = \frac{\left(J(u^\varepsilon, y_0^\varepsilon) - J(\bar{u}, \bar{y}_0) + \varepsilon^2\right)_+}{J_\varepsilon(u^\varepsilon, y_0^\varepsilon)} \quad ; \quad \Psi^\varepsilon = \frac{P_K(\Phi(y_0^\varepsilon, y_T^\varepsilon)) - \Phi(y_0^\varepsilon, y_T^\varepsilon)}{J_\varepsilon(u^\varepsilon, y_0^\varepsilon)}, \tag{27}$$

and

$$\psi^\varepsilon = \begin{cases} \dfrac{d_C(g(y^\varepsilon)) D d_C(g(y^\varepsilon))}{J_\varepsilon(u^\varepsilon, y_0^\varepsilon)} & \text{if } g(y^\varepsilon) \notin C, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

We have that $|\Psi^\varepsilon| = J_\varepsilon(u^\varepsilon, y_0^\varepsilon)^{-1} d_K(\Phi(y_0^\varepsilon, y_T^\varepsilon))$ and since $||D d_C(g(y^\varepsilon))||_e = 1$, we deduce that $||\psi^\varepsilon||_e = J_\varepsilon(u^\varepsilon, y_0^\varepsilon)^{-1} d_C(g(y^\varepsilon))$. Therefore

$$\alpha_\varepsilon^2 + ||\psi^\varepsilon||_e^2 + |\Psi^\varepsilon|^2 = 1. \tag{29}$$

In addition, since d_C is a convex function, we have

$$\langle \psi^\varepsilon, z - g(y^\varepsilon)\rangle \le 0, \quad \text{for all} \;\; z \in C, \tag{30}$$

and from the definition of projection

$$\Psi^\varepsilon(w - P_K(\Phi(y_0^\varepsilon, y_T^\varepsilon))) \le 0, \quad \text{for all} \;\; w \in K. \tag{31}$$

Let $\eta^\varepsilon \in \mathcal{M}$ be such that $\mathrm{d}\eta^\varepsilon = \psi^\varepsilon$. The Pontryagin linearization (14) at the trajectory $(u^\varepsilon, y^\varepsilon)$ is

$$z_t^\varepsilon = y_0 - y_0^\varepsilon + \int_0^t (D_y f(t,s,u_s^\varepsilon, y_s^\varepsilon) z_s^\varepsilon + f(t,s,u_s,y_s^\varepsilon) - f(t,s,u_s^\varepsilon, y_s^\varepsilon))\mathrm{d}s, \quad t \in (0,T). \tag{32}$$

Let us compute the directional derivative of the perturbed cost w.r.t. y_0 at $(u^\varepsilon, y_0^\varepsilon)$ in an arbitrary direction $w_0 \in I\!R^n$. Let us denote by w the directional derivative of the state w.r.t. the initial condition, at the point $(u^\varepsilon, y_0^\varepsilon)$, in the direction w_0. We have that

$$w_t = w_0 + \int_0^t D_y f(t,s,u_s^\varepsilon, y_s^\varepsilon) w_s \mathrm{d}s. \tag{33}$$

We obtain

$$\left\{ \begin{array}{rcl} D_{y_0} J_\varepsilon(u^\varepsilon, y_0^\varepsilon) w_0 & = & \alpha_\varepsilon \displaystyle\int_0^T D_y \ell(u_t^\varepsilon, y_t^\varepsilon) w_t \mathrm{d}t + \sum_{i=1}^{n_g} \int_0^T g_i'(y_t^\varepsilon) w_t \mathrm{d}\eta_{i,t}^\varepsilon \\ & & + \Phi'[\alpha_\varepsilon, \Psi^\varepsilon](y_0^\varepsilon, y_T^\varepsilon)(w_0, w_T). \end{array} \right. \tag{34}$$

Let $p_t^\varepsilon \in \mathcal{P}$ be the unique solution of the costate equation

$$\begin{array}{rcl} -\mathrm{d}p_t^\varepsilon & = & \alpha^\varepsilon D_y \ell(u_t^\varepsilon, y_t^\varepsilon)\mathrm{d}t + p_t^\varepsilon D_y f(t,t,u_t^\varepsilon, y_t^\varepsilon)\mathrm{d}t + \displaystyle\sum_{i=1}^{n_g} g_i'(y_t^\varepsilon)\mathrm{d}\eta_{i,t}^\varepsilon \\ & & + \displaystyle\int_t^T p_s^\varepsilon D_{\tau,y}^2 f(s,t,u_t^\varepsilon, y_t^\varepsilon)\mathrm{d}s, \quad s \in [0,T]; \\ p_{T+}^\varepsilon & = & D_{y_T} \Phi[\alpha^\varepsilon, \Psi^\varepsilon](y_0^\varepsilon, y_T^\varepsilon). \end{array} \tag{35}$$

After an integration by parts, we see that (34) reduces to

$$D_{y_0} J_\varepsilon(u^\varepsilon, y_0^\varepsilon) w_0 = \left(D_{y_0}\Phi[\alpha_\varepsilon, \Psi^\varepsilon](y_0^\varepsilon, y_T^\varepsilon) + p_{0-}^\varepsilon \right) w_0. \tag{36}$$

Since (26) implies $|D_{y_0} J_\varepsilon(u^\varepsilon, y_0^\varepsilon)| \le \varepsilon$, we deduce that

$$|p_{0-}^\varepsilon + D_{y_0}\Phi[\alpha^\varepsilon, \Psi^\varepsilon](y_0^\varepsilon, y_T^\varepsilon)| \le \varepsilon. \tag{37}$$

We next claim that, for any trajectory $(u,y) \in \mathcal{U} \times \mathcal{Y}$ we have that

$$\begin{array}{rcl} J_\varepsilon(u, y_0) - J_\varepsilon(u^\varepsilon, y_0^\varepsilon) = & \int_0^T \left(H[\alpha^\varepsilon, p^\varepsilon](t, u_t, y_t^\varepsilon) - H[\alpha^\varepsilon, p^\varepsilon](t, u_t^\varepsilon, y_t^\varepsilon) \right) \mathrm{d}t \\ & + O(\|u - u^\varepsilon\|_1^2 + |y_0 - y_0^\varepsilon|^2). \end{array} \tag{38}$$

Indeed, set $R_\varepsilon := \|u - u^\varepsilon\|_1^2 + |y_0 - y_0^\varepsilon|^2$. By lemma (3.2), denoting by z^ε the Pontryagin linearization defined in (32), we have that

$$g(y) - g(y^\varepsilon) = g'(y^\varepsilon) z^\varepsilon + O(R_\varepsilon), \tag{39}$$

$$\Phi(y_0, y_T) - \Phi(y_0^\varepsilon, y_T^\varepsilon) = \Phi'(y_0^\varepsilon, y_T^\varepsilon)(z_0^\varepsilon, z_T^\varepsilon) + O(R_\varepsilon) \tag{40}$$

$$\left\{ \begin{array}{rl} J(u,y_0) - J(u^\varepsilon, y_0^\varepsilon) = & \displaystyle\int_0^T (\ell(u_t, y_t^\varepsilon) - \ell(u_t^\varepsilon, y_t^\varepsilon) + D_y\ell(u_t^\varepsilon, y_t^\varepsilon) z_t^\varepsilon)\,\mathrm{d}t \\ & \qquad + \phi'(y_0^\varepsilon, y_T^\varepsilon)(z_0^\varepsilon, z_T^\varepsilon) + O(R_\varepsilon). \end{array} \right. \tag{41}$$

By the chain rule we deduce that

$$\left\{ \begin{array}{rl} J_\varepsilon(u,y_0) - J_\varepsilon(u^\varepsilon, y_0^\varepsilon) = & \alpha_\varepsilon \left(J(u,y) - J(u^\varepsilon, y^\varepsilon)\right) + \int_0^T g'(y_t^\varepsilon) z_t^\varepsilon \mathrm{d}\eta_t^\varepsilon \\ & + (\Psi^\varepsilon)^t \left(\Phi(y_0, y_T) - \Phi(y_0^\varepsilon, y_T^\varepsilon)\right) + O(R_\varepsilon). \end{array} \right. \tag{42}$$

Applying (35)-(41) to (42) and integrating by parts, our claim follows.

Given $\varepsilon > 0$, let $(u^\varepsilon, y_0^\varepsilon)$ be the control and initial state obtained by Ekeland's Principle, satisfying (25)-(26). Let y^ε be the associated state, and denote by p^ε the associated costate, solution of (35). Since $||u - u^\varepsilon||_1 = O(\rho_E(u, u^\varepsilon))$, (26) and (38) imply

$$\left\{ \begin{array}{l} \displaystyle\frac{1}{\rho_A((u,y_0),(u^\varepsilon, y_0^\varepsilon))} \int_0^T (H[\alpha^\varepsilon, p^\varepsilon](t, u_t^\varepsilon, y_t^\varepsilon) - H[\alpha^\varepsilon, p^\varepsilon](t, u_t, y_t^\varepsilon))\,\mathrm{d}t \\ \qquad\qquad\qquad \le \varepsilon + O(||u - u^\varepsilon||_1 + |y_0 - y_0^\varepsilon|). \end{array} \right. \tag{43}$$

By classical arguments for unconstrained problems we obtain that

$$H[\alpha^\varepsilon, p^\varepsilon](t, u_t^\varepsilon, y_t^\varepsilon) \le H[\alpha^\varepsilon, p^\varepsilon](t, u, y_t^\varepsilon) + \varepsilon, \quad \text{for all } u \in U \text{ a.a. } t \in (0,T). \tag{44}$$

Since U is bounded, $\bar{u}_t$ and u_t^ε belong to U for a.a. t, and $\rho_A((\bar{u}, \bar{y}_0), (u^\varepsilon, y_0^\varepsilon)) < \varepsilon$, we have that, when $\varepsilon \downarrow 0$, $\|u^\varepsilon - \bar{u}\|_1 + |y_0^\varepsilon - \bar{y}_0| \to 0$ and therefore, by lemma 3.1, $y^\varepsilon \to \bar{y}$ uniformly. In view of (29)-(31), we can extract a subsequence ε_k such that, denoting by "$\stackrel{*}{\to}$" the weak $*$ convergence,

$$\left\{ \begin{array}{l} \alpha_{\varepsilon_k} \to \bar{\alpha} \in [0,1], \\ \Psi^{\varepsilon_k} \to \bar{\Psi}; \quad \bar{\Psi}(w - \Phi(\bar{y}_0, \bar{y}_T)) \le 0, \ \text{ for all } w \in K, \\ \psi^{\varepsilon_k} \stackrel{*}{\to} \bar{\psi}; \quad \left\langle \bar{\psi}, z - g(\bar{y}) \right\rangle \le 0, \ \text{for all } z \in C, \\ p^{\varepsilon_k} \stackrel{*}{\to} \bar{p}. \end{array} \right. \tag{45}$$

The complementarity and transversality conditions (7)-(8) conditions follow, as well as the costate equation (3) (passing to the limit in (37) for the initial condition), and the Hamiltonian inequality (9) follows by passing to the limit in (44).

In remains to check the nontriviality condition (6). If $\bar{\alpha} > 0$ or $\bar{\Psi} \neq 0$, it holds. Otherwise, by (29), $\|\psi^{\varepsilon_k}\|_e \to 1$. By (30), ψ^ε is nonnegative, which means that η^ε is non decreasing, so that its classical dual norm of η^ε, equal to its total variation, is $|\eta_0^\varepsilon|$ (in view of the zero terminal condition). Since $\|\psi^{\varepsilon_k}\|_e \to 1$, we have that $\liminf_k |\eta_0^\varepsilon| > 0$. Passing to the limit we obtain that the element $\bar{\eta}$ of $BV_T([0,T])$ associated to ψ is also nondecreasing and that $\bar{\eta}_0 \neq 0$ (since by the definition of weak$*$ convergence, $\eta_0^\varepsilon = \psi^{\varepsilon_k}(\mathbf{1}) \to \bar{\psi}(\mathbf{1}) = \bar{\eta}_0$). The conclusion follows. □

Proof of theorem 2.2. Let $(\bar{u}, \bar{y})$ be a local solution in L^1 of problem (P). Let $R > ||\bar{u}||_{L^\infty(0,T,U)}$. Denote $U_R = \bar{B}_R$, and $\mathcal{U}_R = L^\infty(0,T,U_R)$. Obviously $(\bar{u}, \bar{y})$ is a local solution in L^1 of the problem (P_R) obtained by adding to (P) the constraint $u_t \in U_R$ a.a. By theorem 3.3, there exists some $(\alpha_R, \eta^R, \Psi^R, p^R) \in$

$\Lambda_R(\bar{u},\bar{y})$, where by $\Lambda_R(\bar{u},\bar{y})$ we denote the set of Pontryagin multipliers associated with $(\bar{u},\bar{y})$ for problem (P_R), and we may assume that

$$\alpha_R^2 + ||\psi^R||_e^2 + |\Psi^R|^2 = 1. \tag{46}$$

The Hamiltonian inequality for problem (P_R) writes

$$H[\alpha_R, p^R](t,\bar{u}_t,\bar{y}_t) \leq H[\alpha_R, p^R](t,u,\bar{y}_t), \quad \text{for all } u \in \bar{B}(0,R), \text{ for a.a. } t \in (0,T). \tag{47}$$

We next pass to the limit when $R \uparrow +\infty$, quite in the same way than passing to the limit when $\varepsilon \downarrow 0$ in the proof of theorem 3.3, so there is no need to repeat the arguments. The conclusion follows. □

4 Continuity of the control and multipliers

In this section we will establish some results of continuity and Lipschitz continuity for the control and the multipliers associated with state constraints of first order (having in mind that those associated with state constraints of higher order typically have jumps). A delicate question is to understand how should be defined the order of a state constraint in our setting.

4.1 Order of the state constraint

Let (u,y) be a trajectory. Then the time derivative of the state is

$$\dot{y}_t = f(t,t,u_t,y_t) + \int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s. \tag{48}$$

This leads to the definition of the total derivative of a function $t \mapsto G(t,y_t)$, along the trajectory (y,u), as $G^{(1)}(t,u_t,y_t,u,y)$, where $G^{(1)} : I\!R \times I\!R^m \times I\!R^n \times \mathcal{U} \times \mathcal{Y} \to I\!R$ is defined by

$$\begin{aligned} G^{(1)}(t,\tilde{u},\tilde{y},u,y) := \quad & D_t G(t,\tilde{y}) + D_{\tilde{y}} G(t,\tilde{y}) f(t,t,\tilde{u},\tilde{y}) \\ & + D_{\tilde{y}} G(t,\tilde{y}) \int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s. \end{aligned}$$

In other words, the total derivative of $G(t,y_t)$ is

$$\begin{aligned} G^{(1)}(t,u_t,y_t,u,y) := \quad & D_t G(t,y_t) + D_{\tilde{y}} G(t,y_t) f(t,t,u_t,y_t) \\ & + D_{\tilde{y}} G(t,y_t) \int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s. \end{aligned}$$

In particular, the total derivative of the ith state constraint is $g_i^{(1)}(t,u_t,y_t,u,y)$, where

$$g_i^{(1)}(t,\tilde{u},\tilde{y},u,y) = g_i'(\tilde{y}) f(t,t,\tilde{u},\tilde{y}) + g_i'(\tilde{y}) \int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s. \tag{49}$$

We say that the ith state constraint is of *first order* if the dependence w.r.t. $\tilde{u}$ of the above expression is non trivial, i.e., if

$$g_i'(\tilde{y}) D_u f(t,t,\tilde{u},\tilde{y}) \neq 0, \quad \text{for some } (t,\tilde{u},\tilde{y}) \in I\!R \times I\!R^m \times I\!R^n. \tag{50}$$

Otherwise we say that the ith state constraint is of higher order. In that case, we have

$$g_i'(\tilde{y})D_u f(t,t,\tilde{u},\tilde{y}) = 0, \quad \text{for all } (t,\tilde{u},\tilde{y}) \in I\!R \times I\!R^m \times I\!R^n, \tag{51}$$

We can then write $g_i^{(1)}(t,u_t,y_t,u,y)$ under the form $g_i^{(1)}(t,y_t,u,y)$, and define $g^{(2)}$, the total derivative of $g^{(1)}$, as

$$g_i^{(2)}(t,u_t,y_t,u,y) = D_t g_i^{(1)}(t,y_t,u,y) + D_{\tilde{y}} g_i^{(1)}(t,y_t,u,y)\dot{y}_t. \tag{52}$$

Note that

$$\begin{aligned} D_t g_i^{(1)}(t,y_t,u,y) = \quad & g_i'(y_t)\left(2D_\tau f(t,t,u_t,y_t) + D_s f(t,t,u_t,y_t)\right) \\ & + g_i'(y_t)\int_0^t D^2_{\tau\tau} f(t,s,u_s,y_s)\mathrm{d}s. \end{aligned} \tag{53}$$

From (48) and (52) we get

$$\begin{aligned} g_i^{(2)}(t,u_t,y_t,u,y) \quad &= \quad D_t g_i^{(1)}(t,y_t,u,y) \\ &+ \quad D_{\tilde{y}} g_i^{(1)}(t,y_t,u,y)\left(f(t,t,u_t,y_t) + \int_0^t D_\tau f(t,s,u_s,y_s)ds\right). \end{aligned}$$

Using

$$D_{\tilde{u}}\left(D_t g_i^{(1)}(t,y_t,u,y)\right) = D_t\left(D_{\tilde{u}} g_i^{(1)}(t,y_t,u,y)\right) = 0 \tag{54}$$

$$D_{\tilde{u}}\left(D_{\tilde{y}} g_i^{(1)}(t,y_t,u,y)\right) = D_{\tilde{y}}\left(D_{\tilde{u}} g_i^{(1)}(t,y_t,u,y)\right) = 0 \tag{55}$$

we obtain

$$D_{\tilde{u}} g_i^{(2)}(t,u_t,y_t,u,y) = D_{\tilde{y}} g_i^{(1)}(t,y_t,u,y) D_u f(t,t,u_t,y_t). \tag{56}$$

Given a trajectory $(u,y) \in \mathcal{U} \times \mathcal{Y}$, let us define $g^{(k+1)}$ as the total derivative of $g^{(k)}$, and the *order of a state constraint* g_i as the smallest positive integer q_i such that (note that for higher orders the partial derivative below depends in general on (u,y) also and not only $(t,\tilde{u},\tilde{y})$)

$$\begin{cases} D_{\tilde{u}} g_i^{(k)}(t,\tilde{u},\tilde{y},u,y) = 0, \text{ for all } (t,\tilde{u},\tilde{y},u,y) \in I\!R \times I\!R^m \times I\!R^n \times \mathcal{U} \times \mathcal{Y}, \\ \text{for all } 0 \le k < q_i. \end{cases} \tag{57}$$

For a state constraint g_i of order q with $k < q$, we can then write $g_i^{(k)}(t,u_t,y_t,u,y)$ under the form $g_i^{(k)}(t,y_t,u,y)$ and we have:

$$g_i^{(k+1)}(t,u_t,y_t,u,y) = D_t g_i^{(k)}(t,y_t,u,y) + D_{\tilde{y}} g_i^{(k)}(t,y_t,u,y)\dot{y}_t \tag{58}$$

Using analogous equations (54)-(55) for $g^{(k)}$ instead of $g^{(1)}$ we obtain

$$D_{\tilde{u}} g_i^{(k+1)}(t,u_t,y_t,u,y) = D_{\tilde{y}} g_i^{(k)}(t,y_t,u,y) D_u f(t,t,u_t,y_t). \tag{59}$$

So we see that, although the expression of high order derivatives of state constraints is rather involved, the partial derivative w.r.t. u_t may be written in a way very similar to the one for ordinary differential equations.

4.2 Continuity of the control

Let $(\bar{u},\bar{y})$ be a Pontryagin extremal. We say that $\bar{u}$ has *side limits* on $[0,T]$ if it has left limits on $(0,T]$ and right limits on $[0,T)$. When $t\in(0,T)$ is such that $\bar{u}_t$ has left and right limits at time t, denoted by $\bar{u}_{t\pm}$, with jump $[\bar{u}_t]:=\bar{u}_{t+}-\bar{u}_{t-}$, we define

$$\bar{u}_t^\sigma := \bar{u}_{t-}+\sigma[\bar{u}_t], \quad \sigma\in[0,1], \tag{60}$$

so that $\bar{u}_t^0=\bar{u}_{t-}$ and $\bar{u}_t^1=\bar{u}_{t+}$. We need to set, for $\sigma\in[0,1]$:

$$H^\sigma[\bar{\alpha},\bar{p}](t,u,y) := \bar{\alpha}\ell(u,y)+\bar{p}_t^\sigma f(t,t,u,y)+\int_t^T \bar{p}_s D_\tau f(s,t,u,y)\mathrm{d}s. \tag{61}$$

The basic hypothesis is

$$\left\{\begin{array}{r} \text{For some } \alpha_H>0,\ \alpha_H|[\bar{u}_t]|^2 \le D^2_{uu}H^\sigma[\bar{\alpha},\bar{p}](t,\bar{u}_t^\sigma,\bar{y}_t)([\bar{u}_t],[\bar{u}_t]), \\ \text{for all } \sigma\in[0,1],\ t\in[0,T]. \end{array}\right. \tag{62}$$

We denote by I_1 (resp. $I_1(t)$) the set of (resp. of active at time t) first order state constraints, and use the hypothesis of *positive linear independence* w.r.t. the control of *first-order* active state constraints along the trajectory $(\bar{u},\bar{y})$:

$$\sum_{i\in I_1(t)} \beta_i D_{\tilde{u}} g_i^{(1)}(t,\bar{u}_t,\bar{y}_t,\bar{u},\bar{y})=0 \ \text{ and } \beta\ge 0 \text{ implies } \beta=0, \text{ for all } t\in[0,T]. \tag{63}$$

Define

$$H[\bar{\alpha},p](t_\pm,u,y) := \alpha\ell(u,y)+p_{t\pm}f(t,t,u,y)+\int_t^T p_s D_\tau f(s,t,u,y)\mathrm{d}s. \tag{64}$$

Theorem 4.1 (Continuity of the control)**.** *Let $(\bar{u},\bar{y})$ be a Pontryagin extremal for (P) with associated Pontryagin multiplier $(\bar{\alpha},\bar{\eta},\bar{\Psi},\bar{p})$.*
(i) *Assume that, for some $R>\|\bar{u}\|_\infty$, $H[\bar{\alpha},\bar{p}](t_\pm,\cdot,\bar{y}_t)$ has, for all $t\in(0,T)$, a unique minimum w.r.t. the control over $B(0,R)$, denoted $\hat{u}_{t\pm}$. Then (a representative of) $\bar{u}$ has side limits on $[0,T]$, equal to $\hat{u}_{t\pm}$.*
(ii) *Assume that $\bar{u}$ has side limits on $[0,T]$ and that (62) holds. Then $\bar{u}$ is continuous.*
(ii) *Assume that the control is continuous and that (63) holds. Then the multipliers η_i associated with components g_i of the state constraint of* first order *$(q_i=1)$ are continuous on $[0,T]$.*

Proof. (i) It suffices to derive the desired property for left limits. So take $\tau\in(0,T)$ and let $t_k\uparrow\tau$ be such that $\bar{u}_{t_k}=\hat{u}_{t_k\pm}$. We can actually take subsequences for which the $\pm$ has constant sign, so for instance assume that $\bar{u}_{t_k}=\hat{u}_{t_k-}$. Let $\tilde{u}$ be a limit point of $\bar{u}_{t_k}$. Then

$$\begin{array}{rl} H[\bar{\alpha},\bar{p}_\tau](\tau_-,\tilde{u},\bar{y}_\tau) = & \lim_k H[\bar{\alpha},\bar{p}_{t_k}](t_{k-},\bar{u}_{t_{k-}},\bar{y}_{t_k}) \\ \le & \lim_k H[\bar{\alpha},\bar{p}_{t_k}](t_{k-},\hat{u}_{\tau-},\bar{y}_{t_k}) = H[\bar{\alpha},\bar{p}_\tau](\tau,\hat{u}_{\tau-},\bar{y}_\tau). \end{array}$$

In view of the hypothesis, this implies $\tilde{u}=\hat{u}_{\tau-}$, as was to be proved.
(ii) Given $t\in[0,T]$ and $\sigma\in[0,1]$, we apply to $F(\sigma):=D_uH^\sigma[\bar{\alpha},\bar{p}](t,\bar{u}_t^\sigma,\bar{y}_t)$ the identity $F(1)-F(0)=\int_0^1 F'(\sigma)\mathrm{d}\sigma$, valid since F is of class C^1. Since

$$F'(\sigma)=D^2_{uu}H^\sigma[\bar{\alpha},\bar{p}](t,\bar{u}_t^\sigma,\bar{y}_t)[\bar{u}_t]-[\eta_t]g'(\bar{y}_t)D_uf(t,t,\bar{u}_t^\sigma,\bar{y}_t), \tag{65}$$

we have

$$\begin{aligned} 0 &= D_u H^1[\bar\alpha,\bar p](t,\bar u_{t+},\bar y_t) - D_u H^0[\bar\alpha,\bar p](t,\bar u_{t-},\bar y_t) = F(1)-F(0) \\ &= \int_0^1 \big(D^2_{uu} H^\sigma[\bar\alpha,\bar p](t,\bar u_t^\sigma,\bar y_t)[\bar u_t] - [\bar\eta_t] g'(\bar y_t) D_u f(t,t,\bar u_t^\sigma,\bar y_t)\big)\mathrm{d}\sigma. \end{aligned} \tag{66}$$

Note that the integral term in the Hamiltonian has no contribution in the above difference. Therefore the remaining analysis is identical to the one of the standard case of the optimal control of an ODE. We give a short proof in order to make the paper self-contained. Observing that $g_i' D_u f = D_{\tilde u} g_i^{(1)} = 0$ if $q_i > 1$, and setting $\nu_i := [\bar\eta_{i,t}]$, we obtain that

$$\int_0^1 D^2_{uu} H^\sigma[\bar\alpha,\bar p](t,\bar u_t^\sigma,\bar y_t)[\bar u_t]\mathrm{d}\sigma \;=\; \sum_{i\in I_1} \nu_i g_i'(\bar y_t) \int_0^1 D_u f(t,t,\bar u_t^\sigma,\bar y_t)\mathrm{d}\sigma. \tag{67}$$

Taking the scalar product of both sides of (67) by $[\bar u_t]$, we get using hypothesis (62) and the relation $\int_0^1 D_u f(t,t,\bar u_t^\sigma,\bar y_t)[\bar u_t]\mathrm{d}\sigma = [f(t,t,\bar u_t^\sigma,\bar y_t)]$ that

$$\alpha_H |[\bar u_t]|^2 \;\le\; \sum_{i\in I_1} \nu_i g_i'(\bar y_t)[f(t,t,\bar u_t^\sigma,\bar y_t)] = \sum_{i\in I_1} \nu_i [g_i^{(1)}(t,\bar u_t,\bar y_t,\bar u,\bar y)]. \tag{68}$$

If $\nu_i > 0$, then $g_i(\bar y_t) = 0$, and since $g_i(\bar y_t)$ attains a local maximum at time t, $[g_i^{(1)}(t,\bar u_t,\bar y_t,\bar u,\bar y)] \le 0$. Therefore, the right-hand side in (68) is a nonpositive. By (62), $[\bar u_t] = 0$. Point (ii) follows.
(iii) Since $[\bar u_t] = 0$, the right-hand side of (67) reduces to

$$\sum_{i\in I_1} \nu_i D_{\tilde u} g_i^{(1)}(t,\bar u_t,\bar y_t,\bar u,\bar y),$$

and is equal to the zero l.h.s. We conclude with (63), using the inequality $\nu \ge 0$. $\square$

5 The alternative optimality system

5.1 First-order alternative system

We next provide an extension of the theory of alternative optimality system to the setting of integral equations. This is a key property for establishing the Lipschitz regularity of the optimal control. Similarly to [20] (see also [27]), we define the *alternative multiplier and costate*, elements of $\mathcal{M}$ and $\mathcal{P}$ resp., as

$$\eta_t^1 := -\bar\eta_t; \quad p_t^1 := \bar p_t - \eta_t^1 g'(\bar y_t), \quad t\in[0,T]. \tag{69}$$

In view of the costate equation (3), we have that

$$\begin{aligned} -\mathrm{d}p_t^1 = &\; -\mathrm{d}\bar p_t + \textstyle\sum_{i=1}^{n_g} g_i'(\bar y_t)\mathrm{d}\eta^1_{i,t} + \eta_t^1 g''(\bar y_t)\dot{\bar y}_t \mathrm{d}t \\ = &\; \big(\bar\alpha D_y \ell(\bar u_t,\bar y_t) + \bar p_t D_y f(t,t,\bar u_t,\bar y_t) + \eta_t^1 g''(\bar y_t) f(t,t,\bar u_t,\bar y_t)\big)\,\mathrm{d}t \\ &+ \Big(\textstyle\int_t^T \bar p_s D^2_{\tau,y} f(s,t,\bar u_t,\bar y_t)\mathrm{d}s + \eta_t^1 g''(\bar y_t)\int_0^t D_\tau f(t,s,\bar u_s,\bar y_s)\mathrm{d}s\Big)\,\mathrm{d}t. \end{aligned} \tag{70}$$

Therefore p^1 is absolutely continuous. Substituting $\bar{p}_t = p^1_t + \eta^1_t g'(\bar{y}_t)$ in the previous r.h.s., using the identity

$$g'(\bar{y}_t) D_y f(t,t,\bar{u}_t,\bar{y}_t) + g''(\bar{y}_t) f(t,t,\bar{u}_t,\bar{y}_t) = \frac{\mathrm{d}}{\mathrm{d}y}\left[g'(\bar{y}_t) f(t,t,\bar{u}_t,\bar{y}_t)\right], \tag{71}$$

and having in mind the expression (49) of $g_i^{(1)}(t,u_t,y_t,u,y)$, we obtain

$$\begin{aligned} -\mathrm{d}p^1_t = \;& \bar{\alpha} D_y \ell(\bar{u}_t,\bar{y}_t) + p^1_t D_y f(t,t,\bar{u}_t,\bar{y}_t) + \int_t^T p^1_s D^2_{\tau,y} f(s,t,\bar{u}_t,\bar{y}_t)\mathrm{d}s \\ & + \eta^1_t D_{\tilde{y}} g^{(1)} + \int_t^T \eta^1_s g'(\bar{y}_s) D^2_{\tau,y} f(s,t,\bar{u}_t,\bar{y}_t)\mathrm{d}s. \end{aligned} \tag{72}$$

This leads to define the *alternative Hamiltonian*, in which $(\tilde{u},\tilde{y}) \in \mathbb{R}^m \times \mathbb{R}^n$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$:

$$\begin{aligned} H^1[\alpha,p^1,\eta^1](t,\tilde{u},\tilde{y},u,y) := \;& H[\alpha,p^1](t,\tilde{u},\tilde{y}) + \eta^1 g^{(1)}(t,\tilde{u},\tilde{y},u,y) + \\ & \int_t^T \eta^1_s g'(\bar{y}_s) D_\tau f(s,t,\tilde{u},\tilde{y})\mathrm{d}s. \end{aligned} \tag{73}$$

Then the dynamics of the alternative costate can be written as

$$-\dot{p}^1_t = D_{\tilde{y}} H^1[\bar{\alpha},p^1,\eta^1](t,\bar{u}_t,\bar{y}_t,\bar{u},\bar{y}), \quad t \in (0,T). \tag{74}$$

The initial-final conditions for the alternative costate are

$$(-p^1_0 - \eta^1_0 g'(\bar{y}_0), p^1_T) = \Phi'[\bar{\alpha},\bar{\Psi}](\bar{y}_0,\bar{y}_T). \tag{75}$$

When analyzing the dependance of the alternative Hamiltonian w.r.t. $\tilde{u}$ we note that

$$H^1[\alpha,p^1,\eta^1](t,\tilde{u},\bar{y}_t,\bar{u},\bar{y}) = H[\alpha,\bar{p}](t,\tilde{u},\bar{y}_t) + \eta^1_t g'(\bar{y}_t)\int_0^t D_\tau f(t,s,\bar{u}_s,\bar{y}_s)\mathrm{d}s. \tag{76}$$

It follows that stationarity or minimality of H w.r.t. u holds iff H^1 has the same property w.r.t. $\tilde{u}$. So the Hamiltonian inequality (9) is equivalent to the corresponding one for the alternative system:

$$\begin{aligned} H^1[\bar{\alpha},p^1,\eta^1](t,\bar{u}_t,\bar{y}_t,\bar{u},\bar{y}) \leq \;& H^1[\bar{\alpha},p^1,\eta^1](t,u,\bar{y}_t,\bar{u},\bar{y}), \\ & \text{for all } u \in \mathbb{R}^m, \text{ for a.a. } t \in (0,T). \end{aligned} \tag{77}$$

5.2 Lipschitz behavior of the control variable

In this section we assume that the control is continuous and that all constraints are of first order, so that we may denote $I(t) = I_1(t)$. Consider the following hypothesis, stronger than (63) (we have removed the hypothesis of nonnegativity of β):

$$\sum_{i \in I(t)} \beta_i D_{\tilde{u}} g_i^{(1)}(t,\bar{u}_t,\bar{y}_t,\bar{u},\bar{y}) = 0 \quad \text{implies } \beta = 0, \quad \text{for all } t \in [0,T]. \tag{78}$$

Our next hypothesis is of strong Legendre-Clebsch type, reduced to a subspace:

$$\begin{aligned} &\text{For some } \alpha_H > 0: \quad \alpha_H |v|^2 \leq D^2_{uu} H[\bar{\alpha},\bar{p}](t,\bar{u}_t,\bar{y}_t)(v,v), \\ &\text{whenever } D_{\tilde{u}} g_i^{(1)}(t,\bar{u}_t,\bar{y}_t,\bar{u},\bar{y})v = 0, \text{ for all } \; i \in I(t), \; t \in [0,T]. \end{aligned} \tag{79}$$

Theorem 5.1. *Assume that all state constraints are of first order. Let* $(\bar{u}, \bar{y}, \bar{p}, \bar{\eta})$ *be a first-order extremal and associated multipliers, with* $\bar{u}$ *continuous. If hypotheses* (78) *and* (79) *hold, then* $\bar{u}$ *and* $\bar{\eta}$ *are Lipschitz function of time.*

Proof. We adapt the idea of [20]. For $t \in [0, T]$, denote by $\bar{I}(t) := \{1, \ldots, n_g\} \setminus I(t)$ the set of non active first-order constraints. We partition the alternative multiplier at time t into $\eta_t^1 = (\hat{\eta}_t, \tilde{\eta}_t)$, where $\hat{\eta}$ (resp. $\tilde{\eta}$) stands for the components in $I(t)$ (resp. $\bar{I}(t)$). We identify $\tilde{\eta}$ with its extension by zero for the components of η^1 in $I(t)$. Consider the function, where $\tilde{\eta}^1 \in I\!R^{|I(t)|*}$:

$$F[t, \bar{u}, \bar{y}, \alpha, p^1, \eta^1, \tilde{\eta}^1](u) := \quad H[\alpha, p^1](t, u, \bar{y}_t) + \tilde{\eta}^1 g^{(1)}(t, u, \bar{y}_t, \bar{u}, \bar{y}) + \int_t^T \eta_s^1 g'(\bar{y}_s) D_\tau f(s, t, u, \bar{y}_t) \mathrm{d}s, \tag{80}$$

whose expression is close to the one of the alternative Hamiltonian, but with $\tilde{\eta}_t^1$ instead of η_t^1 in the second term of the sum in the r.h.s. Consider the nonlinear programming problem

$$\operatorname*{Min}_{u \in \mathbb{R}^m} F[t, \bar{u}, \bar{y}, \alpha, p^1, \eta^1, \tilde{\eta}^1](u) \text{ subject to } g_i^{(1)}(u, \bar{y}_t, \bar{u}, \bar{y}) = 0, \ i \in I(t). \tag{81}$$

We claim that $\bar{u}_t$ is a local solution of this problem. Indeed, let $g_i(\bar{y}_t)$ be a first-order state constraint. Its total derivative is continuous since $\bar{u}$ is so, and is equal to zero whenever it is active since $g_i(\bar{y}_t)$ reaches a local maximum. It follows that $\bar{u}_t$ is feasible for problem (81).

By the qualification hypothesis (78), there exists a unique Lagrange multiplier. In view of the alternative optimality system, the latter is nothing but $\hat{\eta}_t$. The expression of the first-order optimality conditions (where the variables are $(u, \hat{\eta}_t)$) is

$$D_u H^1[\bar{\alpha}, p^1, \eta_t^1](t, u, \bar{y}_t, \bar{u}, \bar{y}) = 0; \quad g_i^{(1)}(u, \bar{y}_t, \bar{u}, \bar{y}) = 0, \ i \in I(t). \tag{82}$$

Hypothesis (79) is a well-known sufficient condition for local optimality for nonlinear programming problems. It follows that $\bar{u}_t$ is a local solution of (81), as was claimed.

Having (76) in mind, we see that the Jacobian of optimality conditions (82) w.r.t. unknowns $(u, \hat{\eta})$ is

$$\begin{pmatrix} D_{uu}^2 H[\bar{\alpha}, \bar{p}, \eta_t^1](t, \bar{u}_t, \bar{y}_t) & D_{\bar{u}} g_{I(t)}^{(1)}(\bar{u}_t, \bar{y}_t, \bar{u}, \bar{y})^\top \\ D_{\bar{u}} g_{I(t)}^{(1)}(\bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) & 0 \end{pmatrix}. \tag{83}$$

In view of hypotheses (78)-(79), this Jacobian is invertible at $(\bar{u}_t, \hat{\eta}_t)$.

Let (a, b) be a compatible pair, in the sense of section 7.1, for the set $I(t)$. Then $\bar{I}(a) = \bar{I}(b)$. The data of problem (81) satisfy a Lipschitz condition, with a constant not depending on the particular (a, b), since either they are indeed Lipschitz functions of time, or, in the case of $\tilde{\eta}_1$, it has the same value at time a and b. By the implicit function theorem, applied to (82), and standard compactness arguments, there exists $\varepsilon > 0$ and $c > 0$ such that, if $b < a + \varepsilon$, then

$$|\bar{u}_b - \bar{u}_a| + |\eta_b^1 - \eta_a^1| \leq c(b - a), \quad \text{for all compatible pairs } (a, b) \text{ such that } b < a + \varepsilon.$$

By lemma 7.1, $(\bar{u}, \eta^1)$ is Lipschitz over (a, b) whenever $b < a + \varepsilon$. The conclusion follows. $\square$

6 Conclusion

We have performed a partial extension of the theory of optimal control with running and initial-final state constraints problems to the case of integral equations, obtaining a version of Pontryagin's principle as well as continuity properties for the control and the multipliers associated to first order state constraints. We also obtained Lipschitz properties for these variables in the case when all state constraints are of first order.

We leave open the question of second order optimality conditions; see e.g. [8] (without initial-final state constraints) and the analysis of related shooting algorithms in [9]. This involves the analysis of junction points associated to high order state constraints. Of course the shooting algorithm by itself, viewed as the analysis of an autonomous state-costate differential equation, is not valid (think to the case of an unconstrained system). However, the sensitivity analysis for junction points and variations of the state and costate under a perturbation might be extended to the present framework.

Some other types of systems with memory have been considered as in Carlier and Tahraoui [11], Samassi and Tahraoui [30]. It would be of interest to extend the analysis of state constrained problems to these frameworks, as well as for systems with delays.

7 Appendix

7.1 Hager's lemma

We recall Hager's lemma [20]; see [5] for a slightly simplified proof. Let X be a Banach space, and x be a continuous function $[0,1] \to X$. Let $I : [0,1] \to \{1,\dots,n\}$ be upper continuous, i.e.,

$$\text{If } t_n \to t \in [0,1], \text{ and } i \in I(t_n), \text{ then } i \in I(t). \tag{84}$$

We will speak of $I(t)$ as a set of active constraints since this is the case in our application. We say that the pair (a,b) in $[0,1]^2$ is *compatible* if

$$a < b; \quad I(a) = I(b); \quad I(t) \subset I(a), \quad \text{for all } \ t \in (a,b), \tag{85}$$

i.e., the same constraints are active at times a and b, and no other constraint is active for $t \in (a,b)$. We say that $L > 0$ is a Lipschitz constant for x over $E \subset [0,1]^2$ if

$$\|x(a) - x(b)\| \leq L|b-a| \quad \text{whenever } (a,b) \in E. \tag{86}$$

Lemma 7.1. *Assume that $x \in C([0,T],X)$ and that I is upper continuous. Let $L > 0$ be a Lipschitz constant for x over the set of compatible pairs. Then L is a Lipschitz constant for x i.e., we have that*

$$\|x(a) - x(b)\| \leq L|b-a|, \quad \textit{for all } \ (a,b) \in [0,1]^2. \tag{87}$$

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