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The Strong Perfect Graph Conjecture: 40 years of Attempts, and its Resolution

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Abstract

The Strong Perfect Graph Conjecture (SPGC) was certainly one of the most challenging conjectures in graph theory. During more than four decades, numerous attempts were made to solve it, by combinatorial methods, by linear algebraic methods, or by polyhedral methods. The first of these three approaches yielded the first (and to date only) proof of the SPGC; the other two remain promising to consider in attempting an alternative proof.

This paper is an unbalanced survey of the attempts to solve the SPGC; unbalanced, because (1) we devote a significant part of it to the ‘primitive graphs and structural faults’ paradigm which led to the Strong Perfect Graph Theorem (SPGT); (2) we briefly present the other “direct” attempts, that is, the ones for which results exist showing one (possible) way to the proof; (3) we ignore entirely the “indirect” approaches whose aim was to get more information about the properties and structure of perfect graphs, without a direct impact on the SPGC.

Our aim in this paper is to trace the path that led to the proof of the SPGT as completely as possible. Of course, this implies large overlaps with the recent book on perfect graphs [81], but it also implies a deeper analysis (with additional results) and another viewpoint on the topic.

1 Introduction

The theory of perfect graphs started in the early sixties, when Claude Berge [2], motivated by Shannon’s work on communication theory (and more particularly by the notion of zero-error capacity of a noisy channel), started to investigate certain invariants of graphs. These, in turn, led him to the study of various classes of graphs. Berge’s repeated questions, and his student Ghouila-Houri’s (partial) answers, concerning the relations between these graph classes led Berge to propose two fundamental conjectures, the so-called Weak Perfect Graph Conjecture and Strong Perfect Graph Conjecture (WPGC and SPGC), which prompted an impressive amount of research. Lovász gave in 1971 a proof of the WPGC [65], but the SPGC remained open until 2002, when Chudnovsky, Robertson, Seymour and Thomas [13] proved it using combinatorial methods.

To formulate these two conjectures, we first need several definitions.

Definition 1.1 A *hole* is a chordless cycle with at least four vertices, while an *anti-hole* is the complement of a hole. An *odd hole* (respectively *odd anti-hole*) is a hole (respectively anti-hole) with an odd number of vertices.

Definition 1.2 Let G be a graph.

- the *clique number* $\omega(G)$ is the size of its largest pairwise adjacent set of vertices (or *clique*);
- the *stability number* (or *independence number*) $\alpha(G)$ is the size of its largest pairwise non-adjacent set of vertices (or *stable set*);
- the *chromatic number* $\chi(G)$ is the minimum number of stable sets (also called *colours* in this context) needed to partition the vertices of G .
- the *clique covering number* $\theta(G)$ is the minimum number of cliques needed to partition the vertices of G .

Let \overline{G} denote the complement of a graph G . Observe that $\omega(G) = \alpha(\overline{G})$ and $\chi(G) = \theta(\overline{G})$.

Definition 1.3 A graph is called a *Berge graph* if it contains neither an odd hole nor an odd anti-hole as an induced subgraph.

Definition 1.4 A graph G is called *perfect* if, for each of its induced subgraphs G' , $\chi(G') = \omega(G')$. It is called *minimal imperfect* if it is not perfect but all its proper induced subgraphs are perfect.

It is easy to see that a perfect graph is necessarily a Berge graph. The converse is the well known Strong Perfect Graph Conjecture:

Conjecture 1.5 (SPGC, Berge [2]) *A graph is perfect if (and only if) it is a Berge graph.*

As noted by Berge, the SPGC implies the weaker conjecture:

Conjecture 1.6 (WPGC, Berge [2]) *A graph is perfect if and only if its complement is perfect.*

Further questions may be asked of perfect graphs: can one test in polynomial time whether a graph is perfect or not? Do perfect graphs have an easily identified structure? Can one find a largest clique of a perfect graph in polynomial time? Can one colour a perfect graph in polynomial time? and many others [18, 15]. To answer these questions, several approaches have been used from the very beginning: combinatorial, polyhedral or algebraic methods. Colouring a perfect graph or finding its largest clique can be done in polynomial time using the ellipsoid method [50]; the proof of the WPGC can be obtained using any one of these approaches [65, 45, 47]; the proof of the SPGC [13] and the polynomial algorithms to recognize perfect graphs [12] use combinatorial methods exclusively. Many important features of perfect or minimal imperfect graphs have been found using alternative approaches (described in the subsequent sections). The theory of perfect graphs is therefore not *one* approach, but a collection of approaches developed and applied in order to reach *one* goal: to understand what makes a graph perfect from the point of view of its structure and its properties.

As noted above, the efficiency of different approaches to solve the perfect graph conjectures became obvious when Lovász [65] and Fulkerson [45] obtained simultaneously (almost) the same result. While Lovász used exclusively combinatorial methods to prove the WPGC, Fulkerson used the theory of anti-blocking polyhedra to prove a theorem close to the WPGC (the Pluperfect Theorem) and formulated the missing result (see Section 2) which would have allowed him to deduce the WPGC. Like the WPGC, the SPGC stimulated and encouraged the development of more than one approach before a proof was eventually found. For many years, completely different viewpoints simultaneously co-existed, going from the ‘just do it’ version (where it was assumed that the existing tools were sufficient to prove the SPGC, and it remained to glue the “bricks” together in a clever order) to the ‘nothing is done yet’ version (where much more powerful methods were considered necessary for a successful approach of the SPGC).

The first (and to date only) proof [13] showed that the first viewpoint (using combinatorial methods) could be successful, and even that one could do without many of the existing tools ... assuming much skill in ‘gluing the bricks together’. In the spirit of Kronecker’s Decomposition Theorem for finite Abelian groups [90] and of Seymour’s Decomposition Theorem for regular matroids [96], an idea was suggested in [18]: that every Berge graph could be constructible from ‘primitive’ Berge graphs, easily seen to be perfect, by perfection-preserving operations. The proof in [13] does not go so far: ‘primitive’ Berge graphs are effectively used to build larger graphs, but not all the operations are perfection-preserving.

This paper is organized as follows. We present in Section 2 the polyhedral approach and in Section 3 the study of partitionable graphs. Even if these approaches have not (yet) led to a proof of the SPGC, they deserve to be highlighted for the alternative insights they offer. Many other approaches (including forbidden subgraphs, even pairs, edge orientations, colouring

algorithms, P_4 -structure) have produced many deep results about perfect graphs. We choose not to present them here, since they can be found in a recent volume of articles on perfect graphs [81]. Section 4 is devoted to the approach which led to the proof of the SPGT, namely the ‘primitive graphs and structural faults’ paradigm. We present not only its early applications to perfect graphs and the main theorems based on this paradigm, but also the diverse conjectures (and counterexamples) to which it has given rise. In a word, we retrace the progression from idea to fulfillment. In Section 5 we list statements equivalent to the SPGT. Finally, in Section 6 we mention further lines of research on graphs without holes and on the relationship between the four parameters $\alpha, \chi, \omega, \theta$ in these graphs.

In order to point out the similarities and differences between our paper and the volume [81], every citation of a result which is also mentioned in [81] will be written in italics (*[65]* instead of [65]).

Throughout the paper, we will use the notation $G = (V, E)$ to denote a graph with vertex set V (of cardinality n) and edge set E (of cardinality m). When no confusion is possible, $\omega(G)$ will be simply denoted ω (and similarly for other parameters). Moreover, a clique (stable set, respectively) of size k will be called a k -*clique* (*stable k -set*, respectively).

2 The polyhedral approach

The use of polyhedral methods to study combinatorial problems, and more specifically properties of graphs, was developed during the forties and the fifties by Dantzig, Ford, Fulkerson, Hoffman, Johnson, Kruskal, and Kuhn on problems such as assignment problems, flows, and the traveling salesman problem. The field was extended and popularized by the works of Edmonds in the sixties and seventies, particularly by his characterization of the matching polytope [38, 39]. He also emphasized the links among polyhedra, min-max relations, good characterizations, and polynomial time solvability. For more information the reader can refer to the survey of Schrijver [91]. The efficiency of the polyhedral methods is strongly illustrated by the algorithmic results obtained using the ellipsoid method. Since our goal here is to discuss the proof of the SPGT as a characterization of perfect graphs, we do not focus our attention on the algorithmic aspects (we only mention them in the appropriate context).

2.1 Matrices, graphs and polyhedra

In this section, we give the main definitions that allow us to transform matrices, graphs and polyhedra into each other, all this in relation to linear programming (LP) and integer programming (IP) problems. We will assume that the reader already has a basic knowledge about these topics. More details and further properties concerning polyhedra can be found in [53].

We limit our presentation to $\{0,1\}$ -matrices and to non-negative vectors, even if most of the definitions hold in greater generality. By convention, the vectors are column vectors, except if explicitly stated otherwise.

Let \mathbf{A} be an $m \times n$ $\{0,1\}$ -matrix (that is with m rows and n columns), let $\mathbf{1}_m$ be the m -vector having all components equal to 1 and let \mathbf{w} and \mathbf{b} be non-negative vectors of sizes n and m , respectively. Consider the following IP-problem (\mathbf{w}^t is the transpose of \mathbf{w}):

$$\text{Find Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{A} \mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbf{Z}^n, \mathbf{x} \geq \mathbf{0}\},$$

consider its LP-relaxation, the dual of its LP-relaxation, and the IP-problem associated with this dual. Provided that the optima involved exist, we have the following inequalities:

$$\begin{aligned} \text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{A} \mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbf{Z}^n, \mathbf{x} \geq \mathbf{0}\} &\leq \text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{A} \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \\ &= \text{Min } \{\mathbf{y}^t \mathbf{b} : \mathbf{A}^t \mathbf{y} \geq \mathbf{w}, \mathbf{y} \geq \mathbf{0}\} \leq \text{Min } \{\mathbf{y}^t \mathbf{b} : \mathbf{A}^t \mathbf{y} \geq \mathbf{w}, \mathbf{y} \in \mathbf{Z}^m, \mathbf{y} \geq \mathbf{0}\} \end{aligned} \quad (1)$$

Definition 2.1 The linear system $\{\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is *totally dual integral* (TDI) if, for every integral vector \mathbf{w} for which the linear program $\text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ has a finite optimum, the dual $\text{Min } \{\mathbf{y}^t \mathbf{b} : \mathbf{A}^t \mathbf{y} \geq \mathbf{w}, \mathbf{y} \geq \mathbf{0}\}$ has an integral optimal solution \mathbf{y} (that is, the second inequality in (1) becomes an equality).

Definition 2.2 A *polyhedron* P is the intersection of finitely many affine half-spaces. A *supporting hyperplane* of P is a hyperplane containing at least one point of P , and such that all points in P lie on one of the closed half-spaces defined by the hyperplane. A *face* of a polyhedron is either the polyhedron itself, or the intersection of P with a supporting hyperplane of P . A *facet* of P is a maximal (relative to inclusion) face distinct from P .

Edmonds and Giles [40] showed that if a linear system $\{\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is TDI and b is integral, then the polyhedron $P(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is an integral polyhedron (that is, each face of $P(\mathbf{A})$ contains an integer point), but the converse is not true in general. It is co-NP-complete to check in general whether a polyhedron $\{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{Ax} \leq \mathbf{b}\}$ is integral [76], even if \mathbf{A} is a $\{0, 1\}$ -matrix.

Consider now the case where $\mathbf{b} = \mathbf{1}_m$. We are going to show (Theorem 2.6) that the four optima in (1) are strongly related.

To do this, we first define (following Fulkerson [45]):

Definition 2.3 The *anti-blocking polyhedron* of the matrix \mathbf{A} , written $P_{\leq}(\mathbf{A})$, is defined by

$$P_{\leq}(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{Ax} \leq \mathbf{1}_m, \mathbf{x} \geq \mathbf{0}\}.$$

We will assume throughout that $P_{\leq}(\mathbf{A})$ is bounded, that is, \mathbf{A} contains no zero column. As a result $P_{\leq}(\mathbf{A})$ can be written as the convex hull of its extreme points $\mathbf{x}^1, \dots, \mathbf{x}^r$.

Definition 2.4 Let $\mathbf{x}^1, \dots, \mathbf{x}^r$ be the extreme points of $P_{\leq}(\mathbf{A})$, and let \mathbf{X} be the matrix whose rows are $(\mathbf{x}^1)^t, \dots, (\mathbf{x}^r)^t$. The *anti-blocker* of $P_{\leq}(\mathbf{A})$ is the polyhedron $\overline{P_{\leq}(\mathbf{A})}$ defined by

$$\overline{P_{\leq}(\mathbf{A})} = \{\mathbf{a} \in \mathbf{R}^n : \mathbf{Xa} \leq \mathbf{1}_r, \mathbf{a} \geq \mathbf{0}\}.$$

Notice that \mathbf{X} has no zero column and that the anti-blocker of $\overline{P_{\leq}(\mathbf{A})}$ is $P_{\leq}(\mathbf{A})$ [45]. The theory of anti-blocking polyhedra, developed by Fulkerson [45], can be used (but this is only one of its numerous applications) to attack the SPGC, as will be seen in subsections 2.2, 2.3.

We are mainly interested in matrices whose associated anti-blocking polyhedron is integral:

Definition 2.5 An $m \times n$ $\{0, 1\}$ -matrix \mathbf{A} with no zero column is *perfect* if the anti-blocking polyhedron $P_{\leq}(\mathbf{A})$ is integral, that is, all vertices of $P_{\leq}(\mathbf{A})$ are $\{0, 1\}$ -vectors.

Thus, if \mathbf{A} is perfect, then the linear program $\text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{Ax} \leq \mathbf{1}_m, \mathbf{x} \geq \mathbf{0}\}$ has an integral optimal solution for all (non-negative) vectors $\mathbf{w} \in \mathbf{R}^m$.

The following result, due to Lovász, gives the relation we were looking for concerning the optima:

Theorem 2.6 (Lovász [65])

For a $m \times n$ $\{0, 1\}$ -matrix \mathbf{A} with no zero column, the following statements are equivalent:

- i) the linear system $\{\mathbf{Ax} \leq \mathbf{1}_m, \mathbf{x} \geq \mathbf{0}\}$ is TDI,*
- ii) the matrix \mathbf{A} is perfect,*
- iii) $\text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{x} \geq \mathbf{0}, \mathbf{Ax} \leq \mathbf{1}_m\}$ has an integral optimal solution for all $\mathbf{w} \in \{0, 1\}^m$.*

Every $m \times n$ $\{0,1\}$ -matrix is associated with a pair of polyhedra. This theorem states an important result involving the matrix itself, (one of the) associated polyhedra, and the four problems in (1), when $\mathbf{b} = \mathbf{1}_m$. In order to apply such tools to graphs, and particularly to perfect graphs, matrices are associated with graphs. To do this, let G be a graph on n vertices and associate with its vertex set V the vector space \mathbf{R}^n , where the indices $1, 2, \dots, n$ of a vector correspond to the vertices of G . The *incidence vector* of some $U \subseteq V$ is defined as the n -vector \mathbf{x}^U having value 1 for each index corresponding to a vertex in U , and value 0 for all the other indices. The row vector $(\mathbf{x}^U)^t$ is called the *row incidence vector* of U .

Definition 2.7 For a graph G , the *clique matrix* associated with G is the $\{0,1\}$ -matrix \mathbf{C}_G whose rows are the row incidence vectors of the cliques in G . Similarly, the *stable set matrix* associated with G is the $\{0,1\}$ -matrix \mathbf{S}_G whose rows are the row incidence vectors of the stable sets in G .

When no confusion is possible, we will simply write \mathbf{C} instead of \mathbf{C}_G and \mathbf{S} instead of \mathbf{S}_G . Now we are ready to investigate perfect graphs using the theory of polyhedra.

2.2 How to deal with perfect graphs?

One of Fulkerson's main goals in developing the theory of anti-blocking polyhedra was the proof of the WPGC. To attack it, he expressed perfection in terms of integer programming.

Let G be a graph on n vertices, and let \mathbf{C} and \mathbf{S} be its associated clique and stable set matrices. We assume \mathbf{C} has size $m \times n$ and \mathbf{S} has size $r \times n$. Consider an arbitrary $\{0,1\}$ -vector \mathbf{w} of size n , which defines the subgraph G' of G for which the equality $\chi(G') = \omega(G')$ is requested: the vertices in G' correspond to the 1 entries in \mathbf{w} . The following statement is then equivalent to the perfection of G :

$$\text{for all } \mathbf{w} \in \{0,1\}^n, \\ \text{Min } \{\mathbf{y}^t \mathbf{1}_r : \mathbf{S}^t \mathbf{y} \geq \mathbf{w}, \mathbf{y} \in \mathbf{Z}^r, \mathbf{y} \geq \mathbf{0}\} = \text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{S} \mathbf{x} \leq \mathbf{1}_r, \mathbf{x} \in \mathbf{Z}^n, \mathbf{x} \geq \mathbf{0}\}. \quad (2)$$

It is easy to see that, in either optimization problem, an optimal solution using nonnegative-integer values is equivalent to an optimal solution using values in $\{0,1\}$. The number of 1s in \mathbf{y} expresses the number of stable sets which are chosen to cover G' , while the number of 1s in \mathbf{x} corresponds to the size of the clique represented by \mathbf{x} .

The perfection of \overline{G} can therefore be expressed similarly to (2), by simply replacing the stable sets of G with the cliques of G (and thus \mathbf{S} by \mathbf{C}):

$$\text{for all } \mathbf{w} \in \{0,1\}^n, \\ \text{Min } \{\mathbf{y}^t \mathbf{1}_m : \mathbf{C}^t \mathbf{y} \geq \mathbf{w}, \mathbf{y} \in \mathbf{Z}^m, \mathbf{y} \geq \mathbf{0}\} = \text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{C} \mathbf{x} \leq \mathbf{1}_m, \mathbf{x} \in \mathbf{Z}^n, \mathbf{x} \geq \mathbf{0}\}. \quad (3)$$

Now, consider this version of perfection proposed by Fulkerson [45]:

Definition 2.8 A graph G is *pluperfect* if the inequality (2) holds for every non-negative integer vector \mathbf{w} .

Fulkerson proved that this property is preserved under complementation.

Theorem 2.9 (Pluperfect Graph Theorem, Fulkerson [45]) *A graph G is pluperfect if and only if \overline{G} is pluperfect.*

This is the main result on (plu)perfection that Fulkerson obtained, and he also noticed that to deduce the WPGC from the Pluperfect Graph Theorem it would suffice to show that perfection implies pluperfection (the converse is obviously true). To do this, it is sufficient to prove that when one replaces a vertex by a 2-clique in a perfect graph G , the new graph is also

perfect. This is a particular case of the Substitution Lemma, used by Lovász [65] to prove the WPGC (which becomes therefore the Perfect Graph Theorem, or PGT). In order to present it, let H and G be two vertex disjoint graphs. Let v be a vertex of G , and let $N(v)$ be the set of vertices adjacent to v in G . The *substitution of H for v in G* creates a new graph by removing v and its incident edges from G , and adding an edge between each vertex of H and each vertex in $N(v)$.

Lemma 2.10 (Substitution Lemma, Lovász [65]). *The graph obtained from a perfect graph G by substitution of a perfect graph H for a vertex of G is a perfect graph.*

An alternative proof of the PGT, also given by Lovász, is a result of the following characterization of perfect graphs:

Theorem 2.11 (Lovász [64])

A graph G is perfect if and only if for every induced subgraph G' of G the following holds:

$$\omega(G')\alpha(G') \geq |V(G')|.$$

As a corollary, we get the next theorem, which introduces the first two of many fascinating properties of maximum cliques and maximum stable sets in minimal imperfect graphs. See Theorem 2.19 for some other such properties.

Theorem 2.12 (Lovász [65]) *Every minimal imperfect graph G satisfies:*

- i) it has exactly $\alpha(G)\omega(G) + 1$ vertices;*
- ii) for every vertex v of G , the graph $G \setminus \{v\}$ can be partitioned into $\alpha(G)$ $\omega(G)$ -cliques, and into $\omega(G)$ stable α -sets.*

Many years later, Gasparyan [47] obtained a very short proof of the PGT exclusively based on elementary linear algebra. A feature of the proof is that it does not use the Substitution Lemma, but only simple deductions concerning structural properties of the maximum cliques and maximum stable sets in minimal imperfect graphs.

Let us turn now towards the SPGT. A way to attack it using anti-blocking polyhedra consists in characterizing perfect graphs with the help of properties of their associated polyhedra.

Definition 2.13 The *stable set polytope* $STAB(G)$ associated with a graph G with n vertices is the convex hull of the incidence vectors of all stable sets in G , that is

$$STAB(G) = \text{ConvHull} \{ \mathbf{x}^S \in \mathbf{R}^n : S \text{ is a stable set of } G \}.$$

Definition 2.14 The *fractional stable set polytope* $QSTAB(G)$ associated with G is the anti-blocking polyhedron of the clique matrix \mathbf{C} , that is

$$QSTAB(G) = P_{\leq}(\mathbf{C}) = \{ \mathbf{x} \in \mathbf{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{1}_m, \mathbf{x} \geq \mathbf{0} \}.$$

Note that $QSTAB(G)$ contains $STAB(G)$ and that

$$\text{ConvHull} (QSTAB(G) \cap \{0, 1\}^n) = STAB(G). \tag{4}$$

When one seeks a defining linear system for $STAB(G)$, there are two types of inequalities that are essential (they always define facets of $STAB(G)$). The first type is given by the nonnegativity of the variables (equivalently, $\mathbf{x} \geq \mathbf{0}$), while the other type is given by the *clique constraints*:

$$\{\mathbf{q}^t \mathbf{x} \leq 1 : \mathbf{q} \text{ is the incidence vector of some clique of } G\}$$

which enforces that a clique of G and a stable set of G can meet in at most one vertex. We can write all the clique constraints concisely as $\mathbf{C}\mathbf{x} \leq \mathbf{1}_m$ (recall that m is the number of rows in \mathbf{C} , and the number of cliques in G).

For a general graph G , these constraints do not specify $STAB(G)$ (that is, $STAB(G)$ is included in, but not necessarily equal to, $QSTAB(G)$). For a perfect graph, the inequalities of (1) in the notation of (3) state for every $\mathbf{w} \in \{0, 1\}^n$ that:

$$\begin{aligned} \text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{C}\mathbf{x} \leq \mathbf{1}_m, \mathbf{x} \in \mathbf{Z}^n, \mathbf{x} \geq \mathbf{0}\} &\leq \text{Max } \{\mathbf{w}^t \mathbf{x} : \mathbf{C}\mathbf{x} \leq \mathbf{1}_m, \mathbf{x} \geq \mathbf{0}\} \\ &= \text{Min } \{\mathbf{y}^t \mathbf{1}_m : \mathbf{C}^t \mathbf{y} \geq \mathbf{w}, \mathbf{y} \geq \mathbf{0}\} \leq \text{Min } \{\mathbf{y}^t \mathbf{1}_m : \mathbf{C}^t \mathbf{y} \geq \mathbf{w}, \mathbf{y} \in \mathbf{Z}^m, \mathbf{y} \geq \mathbf{0}\} \end{aligned} \quad (5)$$

The first and fourth optima in (5) are exactly the two optima in (3), so they are equal. We deduce that all the inequalities become equalities, and thus iii) in Theorem 2.6 holds. Consequently, the linear system $\{\mathbf{C}\mathbf{x} \leq \mathbf{1}_m, \mathbf{x} \geq \mathbf{0}\}$ is TDI (see Definition 2.1), and \mathbf{C} is a perfect matrix. This latter conclusion means that $QSTAB(G)$ is an integer polyhedron. Therefore, using (4) and recalling that a polyhedron is integral if and only if it is the convex hull of the integral vectors it contains, we obtain equality between $STAB(G)$ and $QSTAB(G)$. The converse also holds [14], so (see also [81]):

Theorem 2.15 (Lovász [65], Chvátal [14]) *A graph is perfect if and only if $STAB(G) = QSTAB(G)$.*

We deduce that (1) the nonnegativity constraints and clique constraints specify $STAB(G)$ if and only if G is perfect, and (2) G is perfect if and only if C is a perfect matrix.

Another way to study the properties of perfect graphs, equally involving the polytopes $STAB(G)$ and $QSTAB(G)$, was initiated by Lovász [66]. He introduced a new geometric representation of graphs linking perfectness to semidefinite programming.

Let G be a graph with n vertices, and let $\mathbf{u}^0, \mathbf{u}^1, \dots, \mathbf{u}^n$ be a set of vectors in \mathbf{R}^{n+1} such that \mathbf{u}^0 is special and \mathbf{u}^i corresponds to the vertex v_i of G , for $1 \leq i \leq n$. We define the following set of constraints, dependent on G :

- $(\mathbf{u}^0)^t \cdot \mathbf{u}^0 = 1$;
- $(\mathbf{u}^0)^t \cdot \mathbf{u}^i = (\mathbf{u}^i)^t \cdot \mathbf{u}^i$, for each $i \in \{1, 2, \dots, n\}$;
- $(\mathbf{u}^i)^t \cdot \mathbf{u}^j = 0$, for each pair of indices $i, j \in \{1, 2, \dots, n\}$ such that $v_i v_j$ is an edge of G .

Definition 2.16 The *theta-body* $THETA(G)$ of a graph G is the set

$$\{\mathbf{z} \in \mathbf{R}^n : \text{there exist vectors } \mathbf{u}^0, \mathbf{u}^1, \dots, \mathbf{u}^n \text{ satisfying the constraints above and such that } (\mathbf{u}^i)^t \cdot \mathbf{u}^i = z_i \text{ for each } i \in \{1, 2, \dots, n\}\}$$

It can be shown [53] that $STAB(G) \subseteq THETA(G) \subseteq QSTAB(G)$. In general $THETA(G)$ is not a polytope, but Lovász proved

Theorem 2.17 (Grötschel, Lovász, Schrijver [53]) *Let G be a graph. The following statements are equivalent:*

- i) G is perfect;
- ii) $THETA(G)$ is a polytope;
- iii) $THETA(G) = STAB(G)$;
- iv) $THETA(G) = QSTAB(G)$.

These characterizations of perfect graphs did not yet furnish a proof of the SPGT, but they provided important information about the properties of perfect graphs with impact on another very difficult problem: deciding whether the parameters $\omega(G)$, $\chi(G)$, $\theta(G)$ can be computed

in polynomial time for perfect graphs. This problem has been solved using polytopes, in an even more general form. Consider a graph G of order n whose vertices have prescribed weights w_i , for $1 \leq i \leq n$. Let $\mathbf{w} = (w_1, \dots, w_n)^t$. Computing the *weighted stability number* $\alpha(G, \mathbf{w})$ (that is the maximum, over all stable sets S in G , of the sum of the vertex weights in S) means to maximize the linear function $\mathbf{w}^t \mathbf{x}$ over the stable set polytope $STAB(G)$ (the restriction to $\mathbf{w} \in \{0, 1\}^n$ corresponds to computing $\alpha(G)$). Grötschel, Lovász and Schrijver [50] showed that this problem (as well as computing the parameters $\omega(G, \mathbf{w})$, $\chi(G, \mathbf{w})$, $\theta(G, \mathbf{w})$ defined similarly to $\omega(G)$, $\chi(G)$, $\theta(G)$ on a weighted graph G) can be solved in polynomial time for perfect graphs using the ellipsoid method and the polytopes $STAB(G)$, $QSTAB(G)$, $THETA(G)$.

2.3 How to deal with minimal imperfect and partitionable graphs?

We now investigate minimal imperfect graphs and a larger class, defined in this section, the partitionable graphs.

Recall that, by Theorem 2.15, the non-negativity constraints $\mathbf{x} \geq \mathbf{0}$ and the clique constraints $\{\mathbf{q}^t \mathbf{x} \leq 1 : \mathbf{q} \text{ is the incidence vector of a clique of } G\}$ are sufficient to describe $STAB(G)$ if and only if G is perfect.

When G is not perfect, these two types of constraints remain necessary, but they are no longer sufficient to describe $STAB(G)$. The problem of finding a complete list of constraints defining $STAB(G)$ becomes much more difficult. Two other types of constraints seem promising.

- the *odd hole constraints*: if $C \subseteq V$ induces a chordless odd cycle in G with incidence vector \mathbf{c} , then $\mathbf{c}^t \mathbf{x} \leq \frac{1}{2}(|\mathbf{c}| - 1)$ is satisfied for every incidence vector \mathbf{x} of a stable set. Thus, the odd hole constraints are valid for $STAB(G)$.

- the *odd anti-hole constraints*: if $D \subseteq V$ induces a chordless odd cycle in the complement \bar{G} of G , and \mathbf{d} is the incidence vector of the anti-hole D in G , then $\mathbf{d}^t \mathbf{x} \leq 2$ is satisfied for every incidence vector \mathbf{x} of a stable set. Thus, the odd anti-hole constraints are valid for $STAB(G)$.

The graphs for which the non-negativity constraints, the clique constraints, and the odd hole constraints are sufficient to specify $STAB(G)$ are called *h-perfect*. Chvátal [14] was the first author to be interested in *h-perfect* graphs, and it was shown by Grötschel, Lovász and Schrijver [52] that the weighted stable set problem can be solved in polynomial time for these graphs too.

The clique constraints, the odd hole constraints, and the odd anti-hole constraints are special cases of the *rank constraints* defined as follows: if $U \subseteq V$ and \mathbf{u} denotes the incidence vector of U then

$$\mathbf{u}^t \mathbf{x} \leq \alpha(G[U])$$

(where $G[U]$ denotes the subgraph of G induced on the vertices of U).

A general rank constraint may be useless since in general we do not know how to calculate $\alpha(G[U])$ efficiently. At the same time, many rank constraints follow from others as in the case of perfect graphs, where all the rank constraints follow from the clique constraints. Some other results are known; for instance Chvátal [14] showed that if G is a connected α -critical graph (that is, $\alpha(G - e) = \alpha(G) + 1$ for each edge e of G), then the *full rank constraint* $(\mathbf{1}_n)^t \mathbf{x} \leq \alpha(G)$ defines a facet of $STAB(G)$. However, no complete description of the rank constraints that define facets of $STAB(G)$ is known. In fact, the set of all rank constraints (over all $U \subseteq V$) is *not* sufficient in combination with the nonnegativity constraints to describe $STAB(G)$ for general graphs (an example may be found in [68]).

Padberg [74, 75] obtained many important results on minimal imperfect graphs and in particular on $STAB(G)$ and $QSTAB(G)$ when G is a minimal imperfect graph G . Recall that (see Definition 2.14 and (4)):

$$QSTAB(G) = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{1}_m, \mathbf{x} \geq \mathbf{0}\}, \text{ and}$$

$$STAB(G) = \text{ConvHull}(QSTAB(G) \cap \{0, 1\}^n).$$

Moreover, we can consider two similar polyhedra using the matrix \mathbf{S} , which coincides with the clique matrix for \overline{G} .

$$QSTAB(\overline{G}) = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{S}\mathbf{x} \leq \mathbf{1}_r, \mathbf{x} \geq \mathbf{0}\}, \text{ and}$$

$$STAB(\overline{G}) = \text{ConvHull}(QSTAB(\overline{G}) \cap \{0, 1\}^n)$$

As noticed by Padberg, there is an (incomplete) ‘duality’ relation between the vertices of $STAB(G)$ and the facets of $QSTAB(\overline{G})$ (and hence between the vertices of $STAB(\overline{G})$ and the facets of $QSTAB(G)$). This remark is confirmed by the following relation between these polytopes:

Theorem 2.18 (Padberg [74]) *If G is minimal imperfect, then $QSTAB(G)$ is the anti-blocker of $STAB(\overline{G})$ and $QSTAB(\overline{G})$ is the anti-blocker of $STAB(G)$.*

The study of the vertices and facets of these polyhedra allowed Padberg to obtain the following structural features of minimal imperfect graphs:

Theorem 2.19 (Padberg [74]) *Every minimal imperfect graph G on n vertices satisfies:*

- (P1) *it has exactly n stable $\alpha(G)$ -sets and n $\omega(G)$ -cliques;*
- (P2) *the incidence vectors of the n stable $\alpha(G)$ -sets are linearly independent, as are those of the n $\omega(G)$ -cliques ;*
- (P3) *each vertex of G lies in exactly $\alpha(G)$ stable $\alpha(G)$ -sets and in exactly $\omega(G)$ $\omega(G)$ -cliques;*
- (P4) *for every $\omega(G)$ -clique Q there exists a unique stable $\alpha(G)$ -set S such that $Q \cap S = \emptyset$, and conversely ;*
- (P5) *for every vertex v , the set $V \setminus \{v\}$ has a unique partition into $\alpha(G)$ $\omega(G)$ -cliques, and a unique partition into $\omega(G)$ stable $\alpha(G)$ -sets.*

Unfortunately, minimal imperfect graphs are not characterized by the nice properties in Theorem 2.12 and Theorem 2.19. A larger class, defined by Bland, Huang and Trotter [5], has all these remarkable properties:

Definition 2.20 A graph G on n vertices is called (r, s) -partitionable if there exist integers $r, s > 1$ such that:

- $n = rs + 1$, and
- for every vertex v of G , $V \setminus \{v\}$ can be partitioned into r s -cliques and into s stable r -sets.

By Lovász’s Theorem 2.12, every minimal imperfect graph G is an $(\alpha(G), \omega(G))$ -partitionable graph. For this reason, partitionable graphs were intensively studied, both from a polyhedral point of view and from a combinatorial or linear algebraic point of view (see Section 3).

>From a polyhedral point of view, since for an imperfect graph G the stable set polytope $STAB(G)$ is strictly contained in its fractional stable set polytope $QSTAB(G)$, the difference between these two polytopes can be used as a tool to decide whether (intuitively) a graph is “close to” or “far from” a perfect graph. In this direction, it would hopefully be possible to distinguish a minimal imperfect graph from a partitionable graph which is not minimal imperfect by studying their associated polytopes.

To this end, Padberg [75] introduced the notion of almost integral polytope, defined with respect to a $\{0, 1\}$ -matrix \mathbf{A} with no zero column. To motivate this definition, first consider the following polytopes:

$$P_{\leq}(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{1}_m, \mathbf{x} \geq \mathbf{0}\}$$

$$P_I(\mathbf{A}) = \text{ConvHull}(P_{\leq}(\mathbf{A}) \cap \{0, 1\}^n)$$

The first polytope is obviously the anti-blocking polyhedron associated to \mathbf{A} , while $P_I(\mathbf{A})$ is the polytope defined as the convex hull of the integral vectors in $P_{\leq}(\mathbf{A})$. When A is the clique matrix \mathbf{C} of a graph G , we have $QSTAB(G) = P_{\leq}(\mathbf{C})$ and $STAB(G) = P_I(\mathbf{C})$.

Furthermore, for $1 \leq j \leq n$, define the polytopes obtained from the preceding ones by fixing the j -th coordinate to 0:

$$P_{\leq}^j(\mathbf{A}) = P_{\leq}(\mathbf{A}) \cap \{\mathbf{x} \in \mathbf{R}^n : x_j = 0\}$$

$$P_I^j(\mathbf{A}) = P_I(\mathbf{A}) \cap \{\mathbf{x} \in \mathbf{R}^n : x_j = 0\}.$$

Now, following Padberg [75], define

Definition 2.21 An $m \times n$ $\{0, 1\}$ -matrix \mathbf{A} with no zero column is *almost perfect* if

- $P_I(\mathbf{A}) \neq P_{\leq}(\mathbf{A})$, and
- $P_I^j(\mathbf{A}) = P_{\leq}^j(\mathbf{A})$, for every integer $j \in \{1, 2, \dots, n\}$.

Definition 2.22 A polyhedron $P \subset \mathbf{R}^n$ is an *almost integral polytope* if there exists an almost perfect matrix \mathbf{A} such that $P = P_{\leq}(\mathbf{A})$.

The difference between almost integral polytopes and integral polytopes concerns exactly one vertex:

Theorem 2.23 (Padberg [75]) *If $P_{\leq}(\mathbf{A})$ is almost integral then*

- i) *Every fractional vertex has exactly n adjacent integral vertices;*
- ii) *$P_{\leq}(\mathbf{A})$ has exactly one fractional vertex.*

Moreover, Padberg proved that A is almost perfect if and only if it is the clique matrix of a minimal imperfect graph G and deduced this remarkable result:

Theorem 2.24 (Padberg [75]) *A graph G on n vertices is minimal imperfect if and only if the fractional stable set polytope $QSTAB(G)$ has a unique vertex \mathbf{u} with non-integral coefficients. This n -vector \mathbf{u} is equal to $\frac{1}{\omega(G)}\mathbf{1}_n$, is contained in exactly n facets, and is adjacent to exactly n other vertices, namely the incidence vectors of the n maximum stable sets of G . Also, $STAB(G) = QSTAB(G) \cap \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \geq \mathbf{0}, (\mathbf{1}_n)^t \mathbf{x} \leq \alpha(G)\}$.*

Remark 2.25 Theorem 2.24 states that G is minimal imperfect if and only if $QSTAB(G)$ has precisely one fractional vertex, which can be cut off exactly without introducing new fractional extreme points by the cutting plane $(\mathbf{1}_n)^t \mathbf{x} = \alpha(G)$ (this plane corresponds to V. Chvátal's full rank constraint). This gives one of the very rare properties satisfied by each minimal imperfect graph and by no other partitionable graph.

Let us consider the following third polytope associated to G :

$$FSTAB(G) = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \geq \mathbf{0}, \mathbf{C}\mathbf{x} \leq \mathbf{1}_m, (\mathbf{1}_n)^t \mathbf{x} \leq \alpha(G)\}.$$

Clearly $STAB(G) \subseteq FSTAB(G) \subseteq QSTAB(G)$. Shepherd [98] defined:

Definition 2.26 A graph G is *near-perfect* if $STAB(G) = FSTAB(G)$.

Minimal imperfect graphs and perfect graphs are near-perfect. For a near-perfect graph G distinct from a perfect graph, only the full rank constraint is used to cut off all fractional vertices of $QSTAB(G)$ (note that the full rank constraint is not a facet of $STAB(G)$ for a perfect graph G distinct from a complete graph). Shepherd [98] showed that the subclass of near-perfect graphs closed under complementation consists exactly of all perfect graphs and all minimal imperfect graphs. We know from Bland, Huang, and Trotter [5] that for every partitionable graph G distinct from an odd hole and from an odd anti-hole, at most one of G and \overline{G} is near-perfect. More precisely, Wagler proved:

Theorem 2.27 (Wagler [106]) *A partitionable graph G is minimal imperfect if and only if G is near-perfect.*

Remark 2.28 If G is partitionable but not minimal imperfect, then by Theorem 2.24 we know that $QSTAB(G)$ has at least two fractional vertices. Wagler noticed that, by [5], every partitionable graph G produces the full rank facet, but by Theorem 2.27 the full rank facet does not suffice to cut off all fractional vertices of $QSTAB(G)$, so at least two cutting planes are required to obtain $STAB(G)$.

Wagler [105, 106] extended her analysis of polytopes associated with graphs by considering two different relaxations of $STAB(G)$ (which are polytopes defined by the non-negativity constraints and (weak) rank constraints) and by studying the corresponding versions of perfection obtained by requiring the equality between one of these polytopes and $STAB(G)$. However, for an alternative approach to the SPGT, the class of near-perfect graphs remains the most interesting class.

The main and very important contribution of the polyhedral approach to the theory of perfect graphs consists of an alternative proof of the PGT (using Fulkerson's Pluperfect Graph Theorem and Lovász's Substitution Lemma) and of two algorithmic results. The first one, already mentioned, was obtained by Grötschel, Lovász, Schrijver [50] in 1981 and concerns the existence of polynomial algorithms to solve the four (weighted) optimization problems on perfect graphs (maximum weighted stable set etc.). The other one, more recent, is due to Shepherd [99] (2001): there exists a polynomial algorithm recognizing partitionable graphs.

Concerning the recognition of perfect graphs, it was known since the beginning of the eighties that deciding whether a graph is *not* perfect is in NP ([9], [51], [67]). Until 2002, it was not known whether testing the perfectness of a graph is a polynomial problem, and more generally even if this problem is in NP. Unfortunately the polyhedral approach brought no answer to the recognition problem. After the SPGT was proved by Chudnovsky, Robertson, Seymour, and Thomas [13], the polynomiality of recognizing perfect graphs was proved by Chudnovsky, Cornuejols, Liu, Seymour, and Vušković [12]. This method is combinatorial, but is not an immediate application of the graph structural decomposition theorem for Berge graphs, 'which was a big surprise' as explained by Seymour in [97].

Deducing a proof of the SPGT from the study of the polytopes $STAB(G)$ and $QSTAB(G)$ for a Berge graph G seems to be quite difficult. Another possible way is to study $STAB(G)$ and $FSTAB(G)$ for any partitionable graph, following the approach suggested by Shepherd and Wagler. It is not clear that this way is easier than the previous one.

3 Partitionable graphs

Lovász's Theorem 2.12 can be seen as a first quantitative and qualitative result on minimal imperfect graphs, which Padberg (Theorem 2.19) completed with a long list of remarkable properties. The structure of minimal imperfect graphs seemed, at that moment, to be very clear; the maximum cliques and stable sets in these graphs had very strong properties; yet, a

proof of the SPGT could not be found in this way. The reason is that almost all of the structural results on minimal imperfect graphs are also valid for the larger class of partitionable graphs.

As already noticed, by Theorem 2.12 a minimal imperfect graph G is an $(\alpha(G), \omega(G))$ -partitionable graph. Moreover, Bland, Huang and Trotter proved that:

Theorem 3.1 (Bland, Huang, Trotter [5]) *Let G be an (r, s) -partitionable graph. Then*

- i) \overline{G} is (s, r) -partitionable;*
- ii) $r = \alpha(G), s = \omega(G)$;*
- iii) G is not perfect;*
- iv) the properties of minimal imperfect graphs (P1), (P2), (P3), (P4), (P5) in Theorem 2.19 also hold for G .*

Due to (ii), from now on we will use the term (α, ω) -partitionable graph G instead of (r, s) -partitionable graph.

Partitionable graphs form a very complex graph class, and the next subsections attempt to justify this assertion by presenting the following facts: for each $\alpha, \omega > 1$, there exist (α, ω) -partitionable graphs, and the number of partitionable graphs on n vertices grows exponentially with n [6]. Various types of partitionable graphs have been identified, but there is no exhaustive description or classification of them. The results obtained up to about partitionable graphs cast only a little light on a subject where much more remains to be learned.

3.1 Equivalent statements

Different relaxations of the definition of partitionability were proposed. The first, proposed by Shepherd, is used in the proof of the existence of a polynomial algorithm to recognize (α, ω) -partitionable graphs (see also [99]):

Theorem 3.2 (Shepherd [98]) *A graph G with n vertices is an (α, ω) -partitionable graph if and only if*

- i) $n = \alpha\omega + 1$, and*
- ii) G has a family of n stable α -sets such that:*
 - 1. each vertex is contained in exactly α of these sets;*
 - 2. for every set S in the family, there exists at least one ω -clique Q such that $Q \cap S = \emptyset$.*

This theorem shows that we can replace a part of the definition of an (α, ω) -partitionable graph (that is the condition that $V - \{v\}$ has both a partition into α ω -cliques and a partition into ω stable α -sets, for every vertex v) with that part of the properties (P1), (P3), (P4) that concern the stable α -sets. The symmetry between stable sets and cliques, which was required in the original definition, seems therefore not to be necessary.

Indeed, this idea is supported by a result due to Boros, Gurvich, and Hougardy [6], which shows that in the definition of an (α, ω) -partitionable graph it is sufficient to demand partitionability of $V - \{v\}$ using only one of the families of maximum cliques and maximum stable sets.

Definition 3.3 Let V be a finite set of n elements, and \mathcal{C} be a family of its subsets. The family \mathcal{C} is *partitionable* if $|\mathcal{C}| \leq |V|$ and, for every $v \in V$, the set $V - \{v\}$ is the union of some family P_v of pairwise disjoint sets from \mathcal{C} .

In this more general context, we have:

Theorem 3.4 (Boros, Gurvich, Hougardy [6]) *If \mathcal{C} is a partitionable family of subsets of a finite set V of size n , such that $2 \leq |C| \leq n - 2$ for each $C \in \mathcal{C}$, then there exist unique integers α and ω , both at least 1, such that:*

- i) $n = \alpha\omega + 1$;
- ii) $|\mathcal{C}| = n$ and $|C| = \omega$, for all $C \in \mathcal{C}$;
- iii) $|P_v| = \alpha$, for all $v \in V$;
- iv) there exists an (α, ω) -partitionable graph whose family of ω -cliques is \mathcal{C} and whose family of stable α -sets is $\{P_v : v \in V\}$.

In other words, the set of maximum stable sets of a partitionable graph can be uniquely determined if one knows its family of maximum clique sets, and if this family is partitionable. The analogous statement holds when cliques and stable sets are interchanged. However, once the two families are known, the partitionable graph is not uniquely determined. It has this property only if we ask for a *normalized* (α, ω) -partitionable graph, that is if we ask that every edge belongs to some ω -clique. Every other (α, ω) -partitionable graph contains the normalized (α, ω) -partitionable graph and can be obtained from this one by adding or removing edges between *indifferent pairs*, that is, pairs of vertices that do not belong to the same ω -clique or to the same stable α -set.

Another property of minimal imperfect graphs, initially noticed by Padberg [74], turned out to be true for partitionable graphs, and even to give a characterization of them. For a graph G with clique number ω and stability number α , let \mathbf{S}_α and \mathbf{C}_ω respectively denote the incidence matrices of the maximum stable sets and the maximum cliques in G (each column is the incidence vector of such a set). Let \mathbf{J} be the $n \times n$ matrix with all entries 1. Combining results from [5, 19, 47, 23] yields the following theorem:

Theorem 3.5 *For a graph G with n vertices, and integers $\alpha, \omega > 1$, the following statements are equivalent:*

- i) G is an (α, ω) -partitionable graph;
- ii) $\alpha = \alpha(G)$ and, for each vertex v in G and stable set $S \subseteq V$, $\omega = \omega(G \setminus S) = \chi(G \setminus \{v\})$.
- iii) $\mathbf{J} - (\mathbf{S}_\alpha)^t \mathbf{C}_\omega$ is a permutation $n \times n$ matrix;
- iv) $\mathbf{J} - (\mathbf{S}_\alpha)^t \mathbf{C}_\omega$ has a submatrix that is a permutation $n \times n$ matrix.

All these results show that many of the properties initially discovered for minimal imperfect graphs extend to partitionable graphs. In the following section, we discuss the advantages and drawbacks of this situation.

3.2 Minimal imperfect graphs versus partitionable graphs

As long as the SPGC was open, every new property of minimal imperfect graphs was possibly a way to prove the SPGT. Two remarkable results in this direction are that minimal imperfect graphs have no star-cutset [16] (see Section 4.5) and no even pair [43, 73] (see Section 4.3); in fact, minimal imperfect graphs share these two properties with the larger class of partitionable graphs [16, 3]. Examples of properties possessed by all minimal imperfect graphs but not by all partitionable graphs include the absence of a homogeneous pair [20] (see Section 4.5) and the absence of a small transversal [18] (see Section 3.2).

Now that the SPGT is proved, several other properties have become especially significant, because of their capacity to express the SPGT in different, but equivalent, ways.

For $n, k \geq 2$ let C_n^k denote the graph with vertices v_1, v_2, \dots, v_n whose edges $v_i v_j$ correspond to the pairs i, j such that $|i - j| \leq k \pmod{n}$. A *critical clique* in a (α, ω) -partitionable graph G is an ω -clique that intersects only $2\omega - 2$ other ω -cliques of G .

Three immediate corollaries of the SPGT are:

- (A) Every partitionable graph contains an odd hole or an odd antihole.
- (B) Every minimal imperfect graph G contains a spanning subgraph isomorphic to $C_{\alpha(G)\omega(G)+1}^{\omega(G)-1}$.
- (C) Every minimal imperfect graph contains a critical clique.

Conversely, the SPGT is an immediate corollary of (A); in [17] and in [80], it was deduced from (B) and (C) respectively.

Statements (A) and (B) are related to several attempts to build specific partitionable graphs with the aim of finding among them an imperfect graph containing no odd hole and no odd anti-hole. We discuss them next, and we briefly discuss (C) in the context of uniquely colourable graphs (next subsection).

The first method for constructing partitionable graphs was proposed by Chvátal [17]: consider a graph with vertices v_1, v_2, \dots, v_n and whose edges $v_i v_j$ correspond to the pairs i, j such that $|i - j| \leq k \pmod{n}$. We obtain C_n^k , the graphs defined above. Then, for arbitrary $\alpha \geq 2$ and $\omega \geq 2$ the graph $C_{\alpha\omega+1}^{\omega-1}$ is an (α, ω) -partitionable graph. Clearly, for $\alpha \geq 2$, $C_{2\alpha+1}^1$ is an odd hole, and $C_{2\alpha+1}^{\alpha-1}$ is an odd anti-hole. Chvátal showed that a minimal imperfect graph G with $\alpha(G) > 2$ and $\omega(G) > 2$ cannot contain a spanning subgraph $C_{\alpha(G)\omega(G)+1}^{\omega(G)-1}$ thus deducing that the SPGT is equivalent to (B).

Bland, Huang, and Trotter [5] were the first to construct a partitionable graph without a spanning subgraph isomorphic to $C_{\alpha\omega+1}^{\omega-1}$.

The two methods proposed in [19] by Chvátal, Graham, Perol, and Whitesides give infinite classes of normalized (α, ω) -partitionable graphs that we will call (using the first letter of each author's name) $CGPW_1$ and $CGPW_2$. The class $CGPW_1$ is obtained by recursive local replacements of a piece of an (α, ω) -partitionable graph in order to obtain an $(\alpha + 1, \omega)$ -partitionable graph (and similarly in the complement of the graph, thus getting an $(\alpha, \omega + 1)$ -graph). The condition of the following theorem is satisfied by the graphs in $CGPW_1$ (a *small transversal* of a graph G with stability number α and clique number ω is a set of $\alpha + \omega - 1$ vertices which intersects every stable α -set and every ω -clique of G).

Theorem 3.6 (Sebö [92]) *Let G be an (α, ω) -partitionable graph, and assume there exist vertices $v_{-(\omega-1)}, \dots, v_{-1}, v_0, v_1, \dots, v_\omega$ such that, for all $i \in \{-\omega, \dots, 0\}$, the vertices $v_{i+1}, \dots, v_{i+\omega}$ form an ω -clique. Then G is an odd hole, is an odd anti-hole, or contains a small transversal.*

This theorem confirms what the SPGT ensured many years later, namely that there is no counterexample to the SPGC in the class $CGPW_1$.

The class $CGPW_2$ contains graphs with circular symmetries obtained from prescribed factorizations of ω and α . Grinstead [49] proved that every graph in the class $CGPW_2$ contains either an odd hole or an odd anti-hole (thus deducing that this class could not contain counterexamples to the SPGC). Other results on partitionable graphs with circular symmetries can be found in [1] and [79].

The construction $CGPW_1$ was generalized by Boros, Gurvich, and Hougardy [6]. Their method was suggested by a result of Sebő [92] and builds (α, ω) -partitionable graphs that all contain a critical clique. The computations show that there exists a unique partitionable graph with at most 25 vertices that is not an odd hole, is not an odd anti-hole, and contains no small transversal. As it can be shown that this unique candidate is *not* a counterexample to the SPGC, the construction in [6] ensured that such a counterexample must have at least 26 vertices, thus slightly improving the previous known lower bound of 25 obtained by Gurvich and Udalov [54].

3.3 Uniquely colourable graphs

The approach by uniquely colourable graphs combines two types of methods: linear algebraic methods and combinatorial methods.

By Padberg's result (Theorem 2.19), in a minimal imperfect graph G , there are exactly n ω -cliques, their incidence vectors are linearly independent, and for every $v \in V$, the set $V - \{v\}$ is uniquely partitioned into α maximum cliques and into ω maximum stable sets. The same holds for every partitionable graph, by Theorem 3.1.

Definition 3.7 Every graph that has exactly one partition into the fewest stable sets is said to be *uniquely colourable*.

We deduce that in every partitionable graph G , for every $v \in V$, the graph $G - \{v\}$ is uniquely colourable.

Tucker [102] firstly proposed to attack the proof of the SPGT by studying uniquely colourable graphs. Based on the idea that two vertices adjacent to the same $(\omega - 1)$ -clique must have the same colour (the colour of each vertex is therefore “forced” by the colour of the other vertex), Tucker proposed a two-step approach:

Step 1. Describe a combinatorial “forcing” procedure to colour uniquely colourable perfect graphs (and more particularly graphs of the form $G \setminus \{v\}$, where G is a minimal imperfect graph).

Step 2. Show that this forcing procedure can be applied to every minimal imperfect graph (because of its uniquely colourable subgraphs), and that in this case the minimal imperfect graph is necessarily an odd hole or an odd anti-hole.

Fonlupt and Sebö [42] investigated the structure of the graphs to which such a forcing procedure could be applied. Let G be an arbitrary graph. For every vertex x of G , the *neighbourhood* of x , denoted $N(x)$, is the set of all vertices adjacent to x .

Definition 3.8 Two non-adjacent vertices x and y of G form a *co-critical nonedge* if the common neighbourhood of x and y in G contains an $(\omega - 1)$ -clique. A co-critical nonedge (x, y) of the complementary graph \bar{G} is called a *critical edge* of G .

It is easy to see that two vertices that form a co-critical nonedge must have the same colour in every ω -colouring of G (if such a colouring exists). Transitivity is introduced by replacing an edge with a path:

Definition 3.9 A path of G whose edges are all critical is a *critical path* of G . A critical path of \bar{G} is a *co-critical path* of G . A maximal subgraph of G , the vertices of which are connected by critical paths, is a *critical component* of G . A critical component of \bar{G} is a *co-critical component* of G .

Definition 3.10 Two vertices x and y are said to be *forced* in G if there exists a co-critical path joining x and y in G .

It is easy to see that two forced vertices also must have the same colour in every ω -colouring of G (if such a colouring exists).

All these definitions treat the necessity of two vertices to have the same colour. In particular, the notion of co-critical nonedge allows us to easily imagine a forcing colouring procedure (given below), which will work only if the graph H resulting from G is ‘simple’, that is, it belongs to a graph class for which a colouring algorithm is available:

Forcing colouring procedure

```

 $H := G$ ;
while (a co-critical nonedge  $(x, y)$  exists in the graph  $H$ ) do
  let  $H_{xy}$  be the graph obtained from  $H$  by contracting  $x, y$  into a single vertex with
  neighbourhood  $N(x) \cup N(y)$ 
   $H := H_{xy}$ 
endwhile;
if ( $H$  belongs to a class of graphs for which a colouring algorithm  $\mathcal{A}$  is known) then
  colour  $H$  using algorithm  $\mathcal{A}$ ;

```

```

for all  $v \in V$  do
    give to  $v$  the colour of the vertex  $w \in H$  which “contains”  $v$ , by contraction
endfor
endif

```

For an arbitrary graph G , the resulting graph H could still be difficult to colour, as shown for example by the case where $H = G$ since G contains no co-critical nonedge. But for uniquely colourable perfect graphs, it is conjectured that the graph H is very easy to colour:

Conjecture 3.11 (Fonlupt, Sebő [42]) *If G is a perfect graph, then G is uniquely colourable if and only if the graph H obtained by repeatedly contracting all co-critical nonedges is a clique.*

This conjecture is equivalent to a conjecture earlier formulated by Tucker [102]. If it were true, the preceding forcing procedure would be a good (or at least a natural) candidate for the forcing procedure in Step 1 above.

Turning our attention to the SPGT, consider the two following conjectures:

Conjecture 3.12 (Fonlupt, Sebő [42]) *Every uniquely colourable perfect graph contains a co-critical nonedge.*

Conjecture 3.13 (Sebő [93]) *If G is a partitionable graph that has both a co-critical nonedge and a critical edge, then it is an odd hole or an odd anti-hole or has a small transversal.*

The SPGT can be deduced from Conjectures 3.12 and 3.13 as follows. Let G be a minimal imperfect graph. Apply Conjecture 3.12 to $G \setminus \{v\}$ and to $\overline{G} \setminus \{v\}$, where $v \in V$, to obtain that $G \setminus \{v\}$ contains a co-critical nonedge and a critical edge. Apply Chvátal’s lemma [17] stating that minimal imperfect graphs do not contain small transversals to deduce, using Conjecture 3.13, that G must be an odd hole or an odd anti-hole. A related conjecture, implying the SPGT, was proposed by G. Bacsó and was invalidated in [88].

Several efforts [42, 93, 92] concentrated on Conjecture 3.13 and several partial results were found:

Theorem 3.14 (Sebő [93]) *If G is a partitionable graph such that (v_1, v_2) is a co-critical nonedge with respect to the $(\omega - 1)$ -clique K , and there exist $u_1, u_2 \in K$ (not necessarily distinct) such that (u_1, v_1) and (u_2, v_2) are critical edges, then G is an odd hole or an odd anti-hole or has a small transversal.*

Theorem 3.15 (Sebő [92]) *If G is a partitionable graph having a vertex v such that v is incident to two critical edges and one co-critical nonedge of G , then G is an odd hole or an odd anti-hole or has a small transversal.*

Furthermore, it can be proved [80] that any critical component that is not a complete graph contains a *critical clique*, that is an ω -clique which meets exactly $2(\omega - 1)$ other ω -cliques, so that (C) (see previous subsection) is equivalent to the SPGT.

Critical cliques are related to critical edges by the following result:

Theorem 3.16 (Sebő [92]) *If G is an (α, ω) -partitionable graph, and Q is an ω -clique of G , then the following statements are equivalent:*

- i) Q is a critical clique;*
- ii) the critical edges in Q form a spanning tree of Q ;*
- iii) the subgraph $G - Q$ is uniquely colourable.*

The equivalence between *i)* and *ii)* in Theorem 3.16 is the result used in the construction of Boros, Gurvich, and Hougardy [6], as noted in the preceding subsection.

More particular applications, related to proving the SPGT for particular classes of graphs, can be found in [69] (split-neighbourhood graphs), [70] ((P_5, K_5) -free graphs) [89] (chair-free graphs), [100] (3-chromatic graphs), [101] ($K_{1,3}$ -free graphs). For additional information on partitionable graphs, see [80].

The results and conjectures presented in this section, devoted to partitionable graphs, amply explain the interest of this approach to the SPGT. More statements showing possibilities to prove the SPGT using partitionable graphs and small transversals can be found in Section 5.

4 Primitive graphs and structural faults

4.1 Preliminary remarks

Say that an operation ϕ defined on p graphs G_1, \dots, G_p and yielding a graph $\phi(G_1, \dots, G_p)$ (denoted G) is *perfection preserving* if the perfection of G_1, \dots, G_p implies the perfection of G . Say that a partition (defined by some specific properties) of a graph G is *friendly* if no minimal imperfect graph admits such a partition.

Here are two main ways to prove that a graph class \mathcal{C} only contains perfect graphs.

Find a decomposition theorem: Let G be a graph in \mathcal{C} . Show that either G belongs to a class of already known perfect graphs, or show that G can be built from smaller perfect graphs in \mathcal{C} using some perfection preserving operation.

Find a friendly partition: Let G be a graph in \mathcal{C} . Show that either G belongs to a class of already known perfect graphs, or G admits a friendly partition.

The first of these two methods is particularly interesting since it permits successively decomposing the graph G until all the pieces are already known perfect graphs in \mathcal{C} . Meanwhile, as long as we only require the *existence* of smaller perfect graphs in \mathcal{C} allowing us to build G , we need to prove *two* different statements: that G can be obtained using the perfection-preserving operation, and that this can be done using perfect pieces. Therefore, the particular case of a perfection-preserving operation which ensures that G is perfect *if and only if* the pieces G_1, G_2, \dots, G_p are perfect is even more interesting (but also more difficult to obtain).

The second method permits showing that \mathcal{C} is a class of perfect graphs in the following way. If the contrary holds, then \mathcal{C} contains a minimal imperfect graph G . Since G must then admit a friendly partition, this is impossible.

In both cases, the already known perfect graphs usually form a small, well known class of perfect graphs. We will call them *primitive* graphs.

Remark 4.1 Sometimes, it is possible to deduce from a friendly partition a perfection preserving operation yielding a decomposition theorem for the class \mathcal{C} . However, this is not always the case. Many friendly partitions yield operations that allow perfect graphs to be glued together to produce a graph that is not a Berge graph and hence is not perfect.

The SPGT was proved using friendly partitions, and no decomposition theorem is available yet. In this section we present the evolution of the concepts related to primitive graphs from the very first results until the SPGT was proved.

4.2 The first decomposition theorems

Historically, the first results concerning the decomposition of each graph in a prescribed class into primitive graphs are due to Dirac [37] (1961) and Gallai [46] (1962). They concern trian-

gulated and i -triangulated graphs respectively (a graph is *triangulated* or *chordal* if every cycle of length larger than 3 has a chord). Since triangulated graphs are a subclass of i -triangulated graphs, we give here the most general result.

Definition 4.2 Two chords x_1x_2 and y_1y_2 of a cycle are said to *cross* if the vertices x_1, y_1, x_2, y_2 appear in this order along the cycle. A graph G is said to be *i -triangulated* whenever every odd cycle of length at least five has at least two non-crossing chords.

The *join* of disjoint graphs G and H is the graph obtained from the union of G and H by joining each vertex of G to each vertex of H . A graph G is *i -primitive* if it has one of the following properties: (1) either G is the join of a nonempty complete graph and a connected bipartite graph with at least three vertices, or (2) G is a complete k -partite graph (for some $k \geq 1$). Moreover, say that a connected graph G has a *clique cutset* if there exists a clique Q in G such that $G \setminus Q$ is disconnected. It is easy to see how to define a composition using the notion of clique cutset:

Definition 4.3 Let G_1 and G_2 be connected graphs containing cliques C_1 and C_2 of the same size. A graph G obtained from G_1 and G_2 by merging C_1 and C_2 according to a bijection is obtained from them by *clique identification*.

It is easy to show that this operation is perfection-preserving. Gallai proved the following theorem:

Theorem 4.4 (Gallai [46]) *If G is an i -triangulated graph, then either G is i -primitive or G contains a clique cutset.*

Remark 4.5 This theorem is *not* a characterization theorem: every i -triangulated graph can be successively decomposed until only i -primitive graphs are obtained, but the iterative composition of graphs starting with i -primitive graphs does not always give an i -triangulated graph (see [48] for an example). The class of graphs built using the i -primitive graphs and clique identification was studied by Gavril [48] and is called the class of *clique separable graphs*.

Remark 4.6 Dirac's [37] result on *triangulated* graphs states that these graphs are either cliques or have a clique cutset.

Based on these early results, the following question was raised: given a primitive class of perfect graphs and some perfection-preserving operations can we show that every perfect graph decomposes via these operations into primitive graphs? In [18], Chvátal mentions a discussion with Whitesides in the fall of 1977 where she suggested that this might be possible (see also [107]). Several subsequent results supported this belief.

4.3 Extending Gallai's approach: parity and Meyniel graphs

Two decomposition theorems were developed to characterize two related classes of graphs: parity graphs and Meyniel graphs. These two major generalizations of Gallai's approach are due to Burlet and Uhry [8] and to Burlet and Fonlupt [7], respectively.

Definition 4.7 A graph G is a *parity graph* [8] if for any two induced paths joining the same pair of vertices their lengths have the same parity.

Note that G is a parity graph if and only if each of its odd cycles of length at least five has at least two crossing chords (see [72, 87]). These graphs are perfect by a theorem of Olaru [87].

To characterize parity graphs, say that two vertices x and y are *true* (respectively *false*) *twins* if they are adjacent (respectively non-adjacent) and $N(x) = N(y)$. By Lovász's Substitution

Lemma [65], whenever a perfect graph is substituted for a vertex in a perfect graph, the resulting graph is still perfect. In particular adding a true or false twin to a perfect graph also yields a perfect graph. If B is a bipartite graph and $X \cup Y$ is the bipartition of its set of vertices, then the *extension of a graph G by B* is the graph obtained from G and B by (1) considering a subset $\{x_1, \dots, x_p\}$ of X , (2) considering a set $\{t_1, \dots, t_p\}$ of pairwise false twins, and (3) for each i , $1 \leq i \leq p$, contracting x_i and t_i into a unique vertex. This operation creates either a clique cutset in G (in the case where $p = 1$) or an *even pair*, that is a pair of non-adjacent vertices such that every chordless path joining them has an even pair of edges. Since minimal imperfect graphs contain neither clique cutsets, nor even pairs [73], the extension of a graph by a bipartite graph is perfection-preserving.

Theorem 4.8 (Burlet, Uhry [8]) *A graph G is a parity graph if and only if it can be obtained from a single vertex by iteratively applying the following operations: creation of a false twin, creation of a true twin, extension by a bipartite graph.*

As a consequence of the “if and only if” statement and of the algorithmic simplicity (that is, polynomiality) to decide whether a graph is obtained by one of the three indicated compositions, Theorem 4.8 yields a polynomial algorithm to recognize parity graphs. Note that Jansen [63] proved that G is a parity graph if and only if the Cartesian product $G \times K_2$ is a perfect graph.

The class of Meyniel graphs is a larger class that contains the class of parity graphs:

Definition 4.9 A graph G is a *Meyniel graph* if every odd cycle of length at least 5 has at least two chords.

These graphs owe their name to Meyniel [72], a student of Berge who showed in 1976 that the strong perfect graph theorem holds true for Meyniel graphs. They were also studied (and proved to be perfect) independently from Meyniel in [71]. An alternative and very nice proof can be found in [67].

Burlet and Fonlupt in [7] gave a characterization theorem and deduced a polynomial time algorithm to construct Meyniel graphs starting from primitive ones.

Definition 4.10 A graph is *M -primitive* if has disjoint sets S and Q such that S is a stable set, Q is a clique, $G - S - Q$ is a 2-connected bipartite graph B , every vertex of Q is adjacent to every vertex of B , and every vertex of S has at most one neighbour in B .

We next introduce another combining operation.

Definition 4.11 Let G_1 and G_2 be graphs. Suppose that each graph G_i has a vertex v_i whose neighbourhood consists of a clique Q_i and a set R_i such that every vertex of R_i is adjacent to every vertex of Q_i . If also $|Q_1| = |Q_2|$, then the *amalgam G* formed from (G_1, v_1, Q_1) and (G_2, v_2, Q_2) is obtained by deleting v_1 and v_2 , identifying every vertex in Q_1 to a vertex of Q_2 , and making every vertex of R_1 adjacent to every vertex of R_2 .

Burlet and Fonlupt show that the amalgam operation is perfection-preserving, and that:

Theorem 4.12 (Burlet, Fonlupt [7]) *A graph G is a Meyniel graph if and only if either G is M -primitive or G can be obtained from two smaller Meyniel graphs using the amalgam operation.*

As pointed out before, the result of Burlet and Fonlupt goes further, since an $O(n^7)$ algorithm can be derived to recognize a Meyniel graph. This time bound was later improved to $O(m^2)$ by Roussel and Rusu following a different approach [84].

4.4 Alternative attempts to define perfection-preserving operations

The paper [103] by Tucker introduces stable cutsets in a similar way to clique cutsets: we say that a connected graph has a *stable cutset* if it contains a stable set S such that $G \setminus S$ is disconnected. The operation of *stable-set identification* can be defined similar to Definition 4.3, but it is not perfection-preserving:

Theorem 4.13 (Tucker [103]) *No minimal imperfect graph contains a stable cutset, except for the odd holes.*

To obtain a perfection-preserving operation, supplementary conditions related to path parity have to be satisfied by the two initial graphs and their stable sets (see [31]). Unfortunately, these conditions are not natural at all and no further important results support investigating stable cutsets in proving the SPGC.

Cornuéjols and Cunningham in [33] proposed the following operations on graphs. Let G_1 and G_2 be graphs. Define a general operation $\phi_{ik}(G_1, G_2)$ as follows.

Definition 4.14 For $j \in \{1, 2\}$, consider a clique of size $i+k$ in G_j with vertices $\{v_1^j, \dots, v_i^j\} \cup Q_j$, and let U_j be the remaining vertices of G_j . Assume that:

- no vertex of U_j is adjacent to more than one vertex v_h^j ($1 \leq h \leq i$);
- each vertex of U_j that is adjacent to v_h^j for some $h = 1, \dots, i$ is also adjacent to all the vertices in Q_j .

The $\phi_{ik}(G_1, G_2)$ is obtained by:

- one-to-one identifying the vertices in the cliques Q_1 and Q_2 , and
- for each $h \in \{1, \dots, i\}$, deleting v_h^1 and v_h^2 and joining every neighbour of v_h^1 to every neighbour of v_h^2 .

Not all these operations (obtained for different values of i and k) are perfection-preserving. Nevertheless, many of them are already known operations, which are presented now in a common form, showing their common features.

For $i = 0$, we first have (as $\phi_{00}(G_1, G_2)$) the disjoint union of two graphs G_1 and G_2 , which is obviously perfection-preserving. Clique identification is the special case $\phi_{0k}(G_1, G_2)$, where k is the size of the clique.

For $i = 1$, we find (as $\phi_{10}(G_1, G_2)$) the operation introduced by Bixby [4] under the name "*composition*", who also proved that it is a perfection-preserving operation. As pointed out by Chvátal, Bixby obtained this result in 1972 but did not publish it till much later. Cunningham (re-)introduced this operation in [36] (in fact he introduced the so-called *split decomposition* of directed graphs and gave an $O(n^3)$ algorithm to decompose a graph following this split decomposition). Going further, $\phi_{1k}(G_1, G_2)$ is the *amalgam* operation defined by Burllet and Fonlupt [7] in order to characterize Meyniel graphs, and also proved to be perfection-preserving by Chvátal [16]. His short proof is an illustration of the power of the Star-cutset Lemma (Lemma 4.17).

For $i = 2$, the operation $\phi_{2k}(G_1, G_2)$ is a new one that Cornuéjols and Cunningham [33] call *2-amalgam*. Their proof that the 2-amalgam operation is perfection-preserving is independent of the Star Cutset Lemma. Moreover, they give an $O(m^2n^2)$ algorithm to decide whether a graph can be obtained from two other graphs using the 2-amalgam operation.

Definition 4.15 A graph G has a *2-join* if its vertices can be partitioned into sets X_1 and X_2 , each of size at least 3, such that each X_i contains nonempty disjoint subsets A_i and B_i with the properties that all of A_1 is adjacent to all of A_2 , all of B_1 is adjacent to all of B_2 , and these are the only adjacencies involving X_1 and X_2 .

A graph G arising from the operation $\phi_{20}(G_1, G_2)$, which is a particular case of the 2-amalgam, has a 2-join (assuming that the cardinality condition on X_1, X_2 is satisfied). While

the operations $\phi_{20}(G_1, G_2)$ and 2-join are perfection-preserving [33], the authors observe that one cannot claim the stronger statement that the result of these operations is perfect *if and only if* the pieces G_1 and G_2 are perfect. Indeed, two imperfect graphs can yield a perfect graph. Cornuéjols and Cunningham [33, 32] showed that one can define blocks G'_1, G'_2 in a graph G possessing a 2-join such that G is perfect if and only if G'_1, G'_2 are perfect (see [33]).

For $i \geq 3$, if we try to define i -amalgam by $\phi_{ik}(G_1, G_2)$, then we do not obtain a perfection-preserving operation. The reader can easily check that a *7-hole* can be constructed via a 3-amalgam starting from two perfect graphs.

Further perfection-preserving operations have been investigated by Hsu [61], who generalized the amalgam and the 2-amalgam. The generalized amalgam and 2-amalgam allowed Hsu to design a recognition algorithm for planar perfect graphs (see [62]).

Interest in these operations was first to obtain new classes of perfect graphs, and second to seek a structural characterization of perfect graphs. For the following classes, well-specified operations have been found which, when repeatedly applied, build all the graphs of the prescribed class, starting from a restricted list of elementary graphs: P_4 -free graphs [95], triangulated graphs [37], parity graphs [8], Meyniel graphs [7], diamond-free perfect graphs [44], claw-free graphs [21], planar perfect graphs [62], etc. These results are characterizations, and they also provide polynomial recognition algorithms for the graphs in each of these families.

4.5 A major breakthrough: the Star-Cutset Lemma

As indicated in subsection 4.1, the border between friendly partitions and perfection-preserving operations is unclear, since friendly partitions can sometimes, but not always, yield perfection-preserving operations. Chvátal's results [16] on star-cutsets helped both to prove new perfection results and to make this context much more clear.

Definition 4.16 A *star-cutset* of a connected graph G is a nonempty subset C of vertices such that $G - C$ is disconnected and such that some vertex in C is adjacent to all the remaining vertices in C .

Chvátal showed with a short and elegant proof that

Lemma 4.17 (Star-Cutset Lemma, Chvátal [16]) *No minimal imperfect graph has a star-cutset.*

A star-cutset of a graph G thus defines a friendly partition of G . Also, the notion of star-cutset appears in all the various decompositions known at the time, except for the *2-join* operation defined by Cornuéjols and Cunningham [33].

Remark 4.18 Assume we want to make use of the existence of this specific friendly partition in order to define a (possibly) perfection-preserving operation. Consider the natural *star-identification* operation, similar to the one we defined for clique cutsets (Definition 4.3): let G_1 and G_2 be disjoint connected graphs in which C_1 and C_2 are isomorphic stars. Build a new graph G from G_1 and G_2 by merging the corresponding vertices in C_1 and C_2 . As for stable cutsets, the operation defined in this way is *not* a perfection-preserving operation, since an odd hole can be built (as a subgraph of G) by star-identification.

Since the star-identification does not necessarily give a perfect graph, a little more care is necessary to define a perfection-preserving operation involving star-cutsets.

Definition 4.19 Let \mathcal{G} be a class of graphs and P a predicate. The *closure of \mathcal{G} under P* (denoted \mathcal{G}^P) is defined recursively by the rules:

- if $G \in \mathcal{G}$, then $G \in \mathcal{G}^P$;
- if G satisfies P , and $G - v \in \mathcal{G}^P$ for every vertex v , then $G \in \mathcal{G}^P$.

It is easy to see (by induction) that, whenever P is a property that a minimal imperfect graph cannot have, the perfection of every graph in \mathcal{G} implies the perfection of every graph in \mathcal{G}^P . Chvátal considered the particular case of the predicate (denoted $*$) " G or \overline{G} has a star-cutset". With the notation $TRIV$ for the class of graphs with at most two vertices and BIP for the class of all bipartite graphs, he showed that Meyniel graphs are contained in BIP^* while Hayward [56] proved that $TRIV^*$ is the class of *weakly triangulated* graphs (graphs containing no induced cycle of length at least 5 and no complement of such a cycle).

The Star-Cutset Lemma provided a useful way to prove the perfection of some new classes of graphs. See [86] for examples of classes included in \mathcal{G}^* , where \mathcal{G} takes different values and more specifically when \mathcal{G} is the class BIP or the class \overline{BIP} (of graphs whose complement is bipartite). These examples strongly suggest the idea that bipartite graphs are *very* important in the structural analysis of perfect graphs. This idea is confirmed along many results cited in this paper, including the Strong Perfect Graph Theorem.

Returning to star-cutsets, their main value is their combination with other operations to elucidate the structure of perfect graphs.

An early result involving decomposition operations was proposed in [20] by Chvátal and Sbihi, when they proved the SPGT for the class of bull-free graphs (a *bull* is a 5-vertex graph obtained by adding pendant vertices to two distinct vertices of a triangle).

Definition 4.20 A *homogeneous pair* in a graph G is a pair (A_1, A_2) of disjoint vertex subsets such that, with $B = V - A_1 - A_2$, we have:

- $|A_1| + |A_2| \geq 3$ and $|B| \geq 2$;
- if a vertex in B is adjacent to one vertex in A_i then it is adjacent to every vertex in A_i ($i = 1, 2$).

Theorem 4.21 (Chvátal, Sbihi [20]) *No minimal imperfect graph has a homogeneous pair.*

This result leads to the validity of the SPGT for bull-free graphs from the following theorem:

Theorem 4.22 (Chvátal, Sbihi [20]) *Every bull-free Berge graph G satisfies at least one of the following conditions:*

- G or \overline{G} contains a star-cutset;
- G has a homogeneous pair;
- G or \overline{G} is bipartite.

4.6 The first conjectures

In 1986, Reed [82] stated the following conjecture with the goal of decomposing every Berge graph. The *line graph* $L(G)$ of a graph G is the graph whose vertices are the edges of G , such that two vertices in $L(G)$ form an edge if and only if their corresponding edges in G share a vertex.

Conjecture 4.23 (Reed [82]) *Every Berge graph G satisfies at least one of the following conditions:*

- G or \overline{G} contains a star-cutset;
- G or \overline{G} contains an even pair;
- G or \overline{G} is the line graph of a bipartite graph.

Hougardy [59] gave a 20-vertex counterexample to this conjecture, to which he added a 38-vertex counterexample to the same conjecture reduced to C_4 -free graphs (where C_4 is the chordless cycle on four vertices). Several authors then proposed a weaker conjecture (the *diamond* is the graph obtained from K_4 by removing one of its edges):

Conjecture 4.24 [59] *Every Berge graph G satisfies at least one of the following conditions:*

- i) G or \overline{G} contains a star-cutset;*
- ii) G or \overline{G} contains a stable cutset;*
- iii) G or \overline{G} contains an even pair;*
- iv) G or \overline{G} is diamond-free.*

Since the diamond-free Berge graphs are perfect [104], a proof of this conjecture would imply the SPGT. Unfortunately, Conjecture 4.24 also is not true. Rusu [85] disproved it together with various extensions obtained by replacing the diamond with any graph in a prescribed class, and by adding a new condition on odd pairs (an *odd pair* of a graph G is a pair of non-adjacent vertices such that every chordless path joining them has an odd number of edges). The conjecture that no minimal imperfect graph has an odd pair was open until the proof of the SPGT was obtained.

4.7 The skew partition

The large class of counterexamples found in [85] shows that other properties of minimal imperfect graphs are needed before being able to prove the SPGT. The (clique, star) cutsets are certainly of great utility in proving perfection of particular classes of graphs (see [86] for a survey), but they seem to be insufficient to attack the whole class of Berge graphs. Chvátal [16] had this intuition very early (1984) and proposed a more general decomposition; we focus here on this new notion; it is essential for the proof of the SPGT.

Definition 4.25 A *skew partition* of G is a partition (A, B, C, D) of V such that all vertices of A are adjacent to all vertices of B , and all vertices of C are nonadjacent to all vertices of D . Given a skew partition, the set $A \cup B$ is a *skew cutset* of G .

It is easy to see that if (A, B, C, D) is a skew partition of G , then (C, D, A, B) is a skew partition of \overline{G} . Moreover, if $|A|=1$, then $A \cup B$ is a star-cutset. In [41] de Figueiredo, Klein, Kohayakawa, and Reed designed a polynomial-time algorithm to recognize graphs that admit a skew partition and to find a skew partition in such graphs.

An alternative way to define a skew partition of a graph G is to ask for a partition of its vertex set into subsets X and Y such that the subgraph of G induced by X is not connected and the subgraph of \overline{G} induced by Y is not connected. In this view of the skew partition, Y (identical to $A \cup B$ in Definition 4.25) induces a cutset in G while X (identical to $C \cup D$ in Definition 4.25) induces a cutset in \overline{G} . This latter form is used intensively by Chudnovsky, Robertson, Seymour, and Thomas in their proof [13].

Chvátal conjectured:

Conjecture 4.26 (Skew Partition Conjecture, Chvátal [16]) *No minimal imperfect graph admits a skew partition.*

Almost a decade passed between this conjecture and the first results involving skew partitions more general than star-cutsets. One can think that all this time was needed to reach a general consensus that the existing properties were not sufficient to formulate a decomposition statement for Berge graphs. As a result, one began to accept the feeling that a solution to Chvátal's Conjecture 4.26 would be a decisive step toward a solution to Berge's conjecture. This feeling was confirmed later by Chudnovsky, Robertson, Seymour, and Thomas, who solved a particular (but sufficient) case of the Skew Partition Conjecture a few months before they completed their proof of the Strong Perfect Graph Theorem (which in turn implies the skew partition conjecture).

Between 1993 and 2001 various particular cases of the skew partition conjecture were proved (Cornuéjols, Reed [34], Hoàng [57], Roussel and Rubio [83], Conforti, Cornuéjols, Gasparyan, Vušković [23]). We list them here.

Theorem 4.27 (Cornuéjols, Reed [34]) *No minimal imperfect graph contains a skew partition (A, B, C, D) with the property that $A \cup B$ induces a complete multi-partite graph.*

To this first result, Hoàng added a generalization of Chvátal's Star-cutset Lemma:

Theorem 4.28 (Hoàng [57]) *No minimal imperfect graph contains a skew partition (A, B, C, D) with the property*

(H) there exist optimal colourings $\mathcal{C}_1, \mathcal{C}_2$ of $G[A \cup B \cup C], G[A \cup B \cup D]$ respectively, such that $|\mathcal{C}_1(A)| \geq |\mathcal{C}_2(A)|$ and $|\mathcal{C}_1(B)| \geq |\mathcal{C}_2(B)|$.

To see that this is a generalization of the Star-Cutset Lemma, let S be a star-cutset of G with central vertex x , set $A = \{x\}$, and let $B = S \setminus \{x\}$.

Hoàng used Theorem 4.28 to prove two other special cases of the Skew Partition Conjecture. Let G be a graph with a skew partition (A, B, C, D) . The notions of U -cutset and T -cutset defined below follow the terminology in (Hoàng [57]) (in particular, note that U and T are *not* variables). The set $A \cup B$ is a U -cutset if there are distinct vertices $u_1, u_2 \in C$ such that u_1 is adjacent to all the vertices of A and u_2 is adjacent to all the vertices of B . The set $A \cup B$ is a T -cutset if there are vertices $u_1 \in C, u_2 \in D$ such that each of the vertices u_1 and u_2 is adjacent to all the vertices of A . Then we have:

Theorem 4.29 (Hoàng [57]) *No minimal imperfect graph contains a U -cutset (respectively a T -cutset).*

On the road to the Skew Partition Theorem, Roussel and Rubio [83] generalized Theorem 4.27 of Cornuéjols and Reed:

Theorem 4.30 (Roussel, Rubio [83]) *No minimal imperfect graph contains a skew cutset $A \cup B$ such that A induces a stable set.*

The proof of Theorem 4.30 by Roussel and Rubio uses a lemma which, in turn, is crucial in the proof of the SPGT by Chudnovsky, Robertson, Seymour, and Thomas [13].

Lemma 4.31 (Roussel-Rubio Lemma [83]) *Let G be a Berge graph, and let X be a vertex subset such that $\overline{G}[X]$ is connected. Let $P = [v_1, \dots, v_n]$ be a chordless path in $G \setminus X$ with odd length, such that both ends of P are adjacent to all the vertices of X . Then either :*

- i) there exists some vertex $x \neq v_1, v_n$ of P such that x is adjacent to all the vertices of X , or*
- ii) P has length at least 5 and X contains two non adjacent vertices a, b such that $[a, v_2, \dots, v_{n-1}, b]$ is a chordless path of $P \cup \{a, b\}$, or*
- iii) P has length 3 and there is an odd chordless antipath of $X \cup P$ joining the internal vertices of P with interior in X .*

Conforti, Cornuéjols, Gasparyan, and Vušković [13] defined a new partition that contains several special cases of skew partition:

Definition 4.32 A graph G has a *universal 2-amalgam* if its vertex set can be partitioned into nonempty vertex subsets V_A, V_B and (possibly empty) U , such that :

- V_A contains sets A_1 and A_2 such that $A_1 \cup A_2$ is non-empty, V_B contains sets B_1 and B_2 such that $B_1 \cup B_2$ is non-empty, every vertex of A_1 is adjacent to every vertex of B_1 , every vertex of A_2 is adjacent to every vertex of B_2 , and these are the only adjacencies between V_A and V_B ;
- every vertex of U is adjacent to $A_1 \cup A_2 \cup B_1 \cup B_2$ and possibly to other vertices of $V(G)$;
- $|U \cup V_A| \geq 2$, and if $(V_A \cup U) \setminus (A_1 \cup A_2) = \emptyset$ then A_1 or A_2 has size at least 2;
- $|U \cup V_B| \geq 2$, and if $(V_B \cup U) \setminus (B_1 \cup B_2) = \emptyset$ then B_1 or B_2 has size at least 2.

The universal 2-amalgam extends the notions of join, amalgam, 2-amalgam and homogeneous pairs. If U , $V_A \setminus (A_1 \cup A_2)$, and $V_B \setminus (B_1 \cup B_2)$ are not empty, then the universal 2-amalgam is a special case of a skew partition. Before the SPGT was proved, Conforti et al. [23] proved:

Theorem 4.33 (Conforti, Cornuéjols, Gasparyan, Vušković [23]) *If G is a minimal imperfect graph that contains a universal 2-amalgam, then G or \overline{G} is an odd hole.*

This theorem was used by Conforti and Cornuéjols [22] to show the validity of the SPGT for a generalization of Meyniel graphs, and for the line graphs of bipartite graphs. The paper by Conforti and Cornuéjols is important for another reason: it revealed the value of using the decompositions based on 2-joins in the context of perfect graphs (remarkable results involving 2-joins on other classes of graphs were obtained earlier [28], [24], [25]).

4.8 Near the goal

A popular way to attack the SPGT used to be proving it for graphs not containing a prescribed (usually small) graph F as an induced subgraph. When F has four vertices, the validity of the SPGT was proved for P_4 -graphs (also called *cographs*) [95], claw-free graphs [77], diamond-free graphs [104] (see also Parthasarathy and Ravindra [78]), and K_4 -free graphs [100]. The only remaining case was the class of C_4 -free (or *square-free*) graphs.

In 2001, this long standing problem was solved:

Theorem 4.34 (Conforti, Cornuéjols, Vušković [29]) *For every square-free Berge graph G , at least one of the following statements holds:*

- i) G has a star-cutset;*
- ii) G has a 2-join;*
- iii) G is bipartite or is the line graph of a bipartite graph.*

This theorem, which confirmed (once again) the important role played by the bipartite graphs and their line graphs in the structure of perfect graphs, suggested that for many classes of Berge graphs the primitive graphs are the same:

Definition 4.35 A graph is *primitive* if it is

- bipartite, or
- the complement of a bipartite graph, or
- the line graph of a bipartite graph, or
- the complement of a line graph of a bipartite graph.

Conjecture 4.36 (Conforti, Cornuéjols, Vušković [29]) *If G is a Berge graph, then G is primitive, or G or \overline{G} has a 2-join or a skew partition.*

It is known today that this conjecture is true, with a slight modification of the definition of a 2-join. This was pointed out by Zambelli [108] as a consequence of the results obtained by Chudnovsky, Robertson, Seymour, Thomas [13] and Chudnovsky [11].

4.9 The final assault

Conjecture 4.36 could not be proved directly for several reasons: the skew partition conjecture was not yet proved, and the initial definition of a 2-join was not sufficient for the conclusion. To find the complete statement of the Strong Perfect Graph Theorem, three modifications were necessary:

The first one: since the skew partition conjecture was not yet proved, skew partitions in their general form were not so useful. A restricted version was needed.

Definition 4.37 A *balanced skew partition* is a skew partition (A, B, C, D) such that there is no chordless path of odd length joining nonadjacent vertices in $A \cup B$ with the internal vertices in $C \cup D$, and there is no odd chordless path in \overline{G} between nonadjacent vertices in $C \cup D$ with the internal vertices in $A \cup B$.

The result in [13] concerns *smallest imperfect Berge graphs*, that is, Berge graphs that are minimal imperfect and have the fewest vertices (assuming that such graphs exist). It is easy to see that proving such a result for a smallest (instead of minimal) imperfect graph is not a limitation when we are looking for a decomposition theorem with concern to the whole class of Berge graphs.

Theorem 4.38 (Chudnovsky, Robertson, Seymour, Thomas [13]) *If G is a smallest imperfect Berge graph, then G admits no balanced skew partition.*

The second modification: to the class of primitive graphs, a new type of graphs was added:

Definition 4.39 A graph G is a *double split graph* if its vertex set V can be partitioned into four sets $\{a_1, \dots, a_m\}$, $\{b_1, \dots, b_m\}$, $\{c_1, \dots, c_n\}$, $\{d_1, \dots, d_n\}$, for some $m, n \geq 2$, such that:

- a_i is adjacent to b_i for $1 \leq i \leq m$, and c_j is nonadjacent to d_j for $1 \leq j \leq n$;
- there are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i \leq i' \leq m$, and all four edges exist between $\{c_j, d_j\}$ and $\{c_{j'}, d_{j'}\}$ for $1 \leq j \leq j' \leq n$;
- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and these two edges are vertex disjoint.

A graph is *B-primitive* (that is, Berge-primitive), if it is primitive as in Definition 4.35 or is a double split graph. It is not a difficult task to see that all B-primitive graphs are perfect.

The third modification: two new terms are defined, obtained by slightly modifying the definitions of a 2-join and of a homogeneous set. Denote the subgraph of G induced on some set $X \subseteq V$ by $G|X$.

Definition 4.40 A *proper 2-join* in G is a partition (X_1, X_2) of V such that there exist disjoint nonempty sets $A_i, B_i \subseteq X_i$ ($1 \leq i \leq 2$) satisfying:

- every vertex of A_1 is adjacent to every vertex of A_2 , and every vertex of B_1 is adjacent to every vertex of B_2 ;
- there are no other edges between X_1 and X_2 ;
- for $1 \leq i \leq 2$, every component of $G|X_i$ meets both A_i and B_i , and
- for $1 \leq i \leq 2$, if $|A_i| = |B_i| = 1$ and $G|X_i$ is a path joining the members of A_i and B_i , then it has odd length at least equal to 3.

Definition 4.41 A *proper homogeneous pair* in G is a pair (A, B) of disjoint subsets of V , such that $V - (A \cup B)$ has a partition (C, D, E, F) with the following properties:

- every vertex in A has a neighbour in B as well as a nonneighbour, and vice versa;
- the pairs (A, C) , (A, F) , (B, D) and (B, F) are *complete* (every vertex of one set is adjacent to every vertex of the other);
- the pairs (A, D) , (A, E) , (B, C) and (B, E) are *anticomplete* (no edge between the two sets).

A proper 2-join is a special case of a 2-join, and a proper homogeneous pair is a special case of a homogeneous pair. We deduce that no minimal imperfect graph has a proper 2-join or a proper homogeneous pair.

The final (proved) result is now:

Theorem 4.42 (Chudnovsky, Robertson, Seymour, Thomas [13])

Every Berge graph G satisfies at least one of the following conditions:

- i) G or \overline{G} admits a proper 2-join;*
- ii) G admits a proper homogeneous pair;*
- iii) G admits a balanced skew-partition;*
- iv) G is B -primitive.*

Hence, this long standing conjecture of Berge is now a theorem

Theorem 4.43 (SPGT, Chudnovsky, Robertson, Seymour, Thomas [13])

A graph is perfect if (and only if) it is Berge.

A natural question is whether every condition in Theorem 4.42 is really necessary. Since the proper homogeneous sets are needed in only one case of the proof of Theorem 4.42, it would be desirable to eliminate them. Chudnovsky [10, 11] proved that one can eliminate proper homogeneous sets.

Remark 4.44 It should be noticed here that Theorem 4.43 is *not* a decomposition theorem according to the definitions in Subsection 4.1, even if one eliminates proper homogeneous sets. This is due to skew partitions, which permit combining perfect graphs to produce a non-Berge graph.

The algorithmic point of view was the next step toward a better understanding of Berge graphs. Two teams worked separately on a recognition algorithm for Berge graphs: Chudnovsky and Seymour on the one hand, and Cornuéjols, Liu and Vušković on the other hand. Their distinct approaches followed similar ideas: given a graph G , first produce a ‘clean’ graph H or decide that the graph is not a Berge graph; then, test whether H is odd-hole-free or not. The overlap between the results of the two teams was very important, and a joint paper on recognizing perfect graphs was written [12]. The idea of ‘cleaning’ a graph (as well as the precise definition of this notion) was introduced in [30], and it was further developed in [28], [24] and [26].

The problem of recognizing in polynomial time whether a graph contains an odd hole remains open. The particular case of graphs of bounded clique size is solved in [27].

5 Possible new ways to prove the SPGT

The possible new ways to prove the SPGT that we present in this section rely on several results with common features: they are corollaries of the SPGT; they imply (alone or combined with some other results) the SPGT; no direct proof (not using SPGT) is known for them.

The list we give here is not exhaustive; we mainly tried to select results that give different points of view on the SPGT, and therefore we avoided different variants of the same result obtained by simply replacing a condition with an equivalent one. We did not include in this list the different statements implying the SPGT that have been presented in the rest of the paper.

For a graph G with clique number ω and stability number α , the *intersection graph* of G , denoted $I(G)$, is the graph whose vertices are the maximum cliques in G , with two such cliques adjacent if they have a common vertex. Tucker [102] observed that if G is partitionable, then so is $I(G)$ and, moreover, $\omega(I(G)) = \omega$ and $\alpha(I(G)) = \alpha$.

Also recall that an (α, ω) -partitionable graph is normalized if each of its edges belongs to some ω -clique.

A *2-division* of a graph is a partition of its vertex set into two parts neither of which contains an ω -clique. A graph is said to be *2-divisible* if each of its induced subgraphs with at least one edge has a 2-division.

Here are several corollaries of the SPGT:

- (I.) A minimal imperfect graph has cutsets of cardinality $2\omega - 2$, and the neighborhood of every vertex is such a cutset.
- (II.) If G is a minimal imperfect graph, then G has two adjacent forced vertices.
- (III.) If G is a minimal imperfect graph, then G has a co-critical nonedge.
- (IV.) If G is a minimal imperfect graph, then $I(G)$ has a vertex of degree $2\omega - 2$.
- (V.) If G is a normalized minimal imperfect graph, then G is an odd hole or odd anti-hole.
- (VI.) A graph G is perfect if and only if G and \overline{G} are 2-divisible.
- (VII.) No minimal imperfect graph contains an even pair, but every proper induced subgraph of it contains an even pair or is a clique.

It is easy to see that (I.) implies the SPGT, assuming that the following conjecture is true:

Conjecture 5.1 (Sebö [94]) *If G is a partitionable graph such that G (respectively \overline{G}) has a cutset of cardinality $2\omega - 2$ (respectively $2\alpha - 2$), then G is an odd hole, is an odd anti-hole, or contains a small transversal.*

The proof that assertion (II.) implies the SPGT is given in [93]. A similar theorem (with similar behaviour with respect to the SPGT), involving two critical edges and one co-critical nonedge of G , is suggested by Theorem 3.15.

Assertion (III.) implies the SPGT assuming that Conjecture 3.13 is true. Indeed, in this case we can apply (III.) to a minimal imperfect graph G that is not an odd hole or anti-hole (assuming that such a graph exists) to deduce that G contains a co-critical nonedge. Since \overline{G} is minimal imperfect too, we also deduce that \overline{G} contains a co-critical nonedge, so that G contains a critical edge. By Conjecture 3.13, G has a small transversal, and this is impossible.

Assertion (IV.) implies the SPGT as proved in [94].

Concerning assertion (V.), Tucker [102] showed that if G is partitionable, then $I(I(G))$ is the normalized graph of G . So, if G is a minimal imperfect graph, using (V.) we deduce that its normalized graph is an odd hole or an odd anti-hole. Since odd holes and odd anti-holes do not have indifferent pairs, G is identical to its normalized graph, so G is an odd hole or anti-hole. The SPGT then follows from (V.).

The equivalence between each of the assertions (VI.), (VII.) and the SPGT is shown in [58] and [60] respectively.

6 Related problems

We close this paper with several open problems on graphs without odd or even holes.

Hoàng and McDiarmid [58] studied the relationship between the divisibility of graphs and the four parameters $\omega, \alpha, \chi, \theta$. A graph G is k -divisible if, for each induced subgraph H of G with at least one edge, there is a partition of the vertex set of H into sets V_1, V_2, \dots, V_k such that no V_i contains a maximum clique of H . When $k = 2$, this notion reduces to the 2-divisibility defined in the preceding section. It is easy to prove, by induction, that for a k -divisible graph G one has $\chi(G) \leq k^{\omega(G)-1}$. In addition:

Conjecture 6.1 (Hoàng, McDiarmid [58]) *A graph contains no odd holes if and only if it is 2-divisible.*

If this conjecture is true, then $\chi(G) \leq 2^{\omega(G)-1}$ for every odd-hole-free graph. The following three conjectures are due to Gyárfás [55] (see [15]):

Conjecture 6.2 (Gyárfás [55]) *There is a function f such that if G is odd-hole-free, then $\chi(G) \leq f(\omega(G))$.*

Conjecture 6.3 (Gyárfás [55]) *For every positive integers k , there is a function f_k such that, if G has no hole with length exceeding k , then $\chi(G) \leq f_k(\omega(G))$.*

Conjecture 6.4 (Gyárfás [55]) *For every positive integers k , there is a function g_k such that, if G has no hole with length exceeding k , then $\chi(G) \leq g_k(\omega(G))$.*

Further comments on these problems can be found in [15].

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