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# Performance bound for Approximate Optimistic Policy Iteration

Christophe Thiery, Bruno Scherrer

We provide here a proof of the performance bound theorem published in Thiery and Scherrer (2010). This theorem applies to Least-Squares  $\lambda$  Policy Iteration and more generally approximate, optimistic Policy Iteration algorithms.

## Theorem 1 (Performance bound for Approximate Optimistic Policy Iteration)

Let  $(\lambda_n)_{n \geq 1}$  be a sequence of positive weights such that  $\sum_{n \geq 1} \lambda_n = 1$ . Let  $Q_0$  be an arbitrary initialization. We consider an iterative algorithm that generates the sequence  $(\pi_k, Q_k)_{k \geq 1}$  with

$$\begin{aligned}\pi_{k+1} &\leftarrow \text{greedy}(Q_k), \\ Q_{k+1} &\leftarrow \sum_{n \geq 1} \lambda_n (B_{\pi_{k+1}})^n Q_k + \epsilon_{k+1}.\end{aligned}$$

$\epsilon_{k+1}$  is the approximation error made when estimating the next value function. Let  $\epsilon$  be a uniform majoration of that error, i.e. for all  $k$ ,  $\|\epsilon_k\|_\infty \leq \epsilon$ . Then

$$\limsup_{k \rightarrow \infty} \|Q^* - Q^{\pi_k}\|_\infty \leq \frac{2\gamma}{(1-\gamma)^2} \epsilon.$$

## Proof

**Notations and main idea of the proof** We will use the following notations:

- $b_k = Q_k - B_{\pi_{k+1}} Q_k$  is the Bellman error,
- $d_k = Q^* - (Q_k - \epsilon_k)$  is the difference between the optimal value function and the  $Q_k$  iterate (before error),
- $s_k = Q_k - \epsilon_k - Q^{\pi_k}$  is the difference between the  $Q_k$  iterate (before error) and the (true) value of the policy  $\pi_k$ ,
- $\beta = \sum_{n \geq 1} \lambda_n \gamma^n$  (note that  $0 \leq \beta \leq \gamma$ ).

The distance between the value of the optimal policy and the value of the current policy can be formulated as

$$\begin{aligned}
\|Q^* - Q^{\pi_k}\|_\infty &= \max(Q^* - Q^{\pi_k}) \\
&= \max(Q^* - Q_k + \epsilon_k + Q_k - \epsilon_k - Q^{\pi_k}) \\
&= \max(d_k + s_k) \\
&\leq \max d_k + \max s_k
\end{aligned} \tag{1}$$

The idea of the proof is to compute upper bounds on  $d_k$  and  $s_k$ . As we will see, the bounds we will obtain will both depend on an upper bound on the Bellman error  $b_k$ , that we derive first.

**An upper bound on the Bellman error  $b_k$ :** As  $\pi_{k+1}$  is the greedy policy with respect to  $Q_k$ , we have  $B_{\pi_k} Q_k \leq B_{\pi_{k+1}} Q_k$ , which allows us to write

$$\begin{aligned}
b_k &= Q_k - B_{\pi_{k+1}} Q_k \\
&= Q_k - B_{\pi_k} Q_k + B_{\pi_k} Q_k - B_{\pi_{k+1}} Q_k \\
&\leq Q_k - B_{\pi_k} Q_k \\
&= (Q_k - \epsilon_k + \epsilon_k) - B_{\pi_k} (Q_k - \epsilon_k + \epsilon_k) \\
&= (Q_k - \epsilon_k) - B_{\pi_k} (Q_k - \epsilon_k) + \epsilon_k - \gamma P_{\pi_k} \epsilon_k \\
&= \sum_{n \geq 1} \lambda_n [(B_{\pi_k})^n Q_{k-1}] - \sum_{n \geq 1} \lambda_n [(B_{\pi_k})^{n+1} Q_{k-1}] + (I - \gamma P_{\pi_k}) \epsilon_k \\
&= \sum_{n \geq 1} \lambda_n [(B_{\pi_k})^n Q_{k-1}] - (B_{\pi_k})^{n+1} Q_{k-1}] + (I - \gamma P_{\pi_k}) \epsilon_k \\
&= \sum_{n \geq 1} \lambda_n (\gamma P_{\pi_k})^n (Q_{k-1} - B_{\pi_k} Q_{k-1}) + (I - \gamma P_{\pi_k}) \epsilon_k \\
&= \sum_{n \geq 1} \lambda_n (\gamma P_{\pi_k})^n b_{k-1} + (I - \gamma P_{\pi_k}) \epsilon_k.
\end{aligned}$$

By using the fact that  $P_{\pi_k}$  is a stochastic matrix, we have

$$\max b_k \leq \sum_{n \geq 1} \lambda_n \gamma^n \max b_{k-1} + (1 + \gamma) \epsilon = \beta \max b_{k-1} + (1 + \gamma) \epsilon.$$

We then deduce by induction that

$$\max b_k \leq \sum_{j=0}^{k-1} \beta^j (1 + \gamma) \epsilon + \beta^k \max b_0 = \frac{1 + \gamma}{1 - \beta} \epsilon + O(\gamma^k). \tag{2}$$

**An upper bound on  $d_k$ :** Let us now consider the  $d_k$  term and its evolution.

$$\begin{aligned}
d_{k+1} &= Q^* - (Q_{k+1} - \epsilon_{k+1}) \\
&= Q^* - \sum_{n \geq 1} \lambda_n (B_{\pi_{k+1}})^n Q_k \\
&= \sum_{n \geq 1} \lambda_n [Q^* - (B_{\pi_{k+1}})^n Q_k].
\end{aligned} \tag{3}$$

Since  $\pi_{k+1}$  is the greedy policy with respect to  $Q_k$ , we have  $B_{\pi^*} Q_k \leq B_{\pi_{k+1}} Q_k$ . Therefore

$$\begin{aligned}
& Q^* - (B_{\pi_{k+1}})^n Q_k \\
&= B_{\pi^*} Q^* - B_{\pi^*} Q_k + B_{\pi^*} Q_k - B_{\pi_{k+1}} Q_k + B_{\pi_{k+1}} Q_k - \\
&\quad - (B_{\pi_{k+1}})^2 Q_k + (B_{\pi_{k+1}})^2 Q_k - \dots + (B_{\pi_{k+1}})^{n-1} Q_k - (B_{\pi_{k+1}})^n Q_k \\
&\leq B_{\pi^*} Q^* - B_{\pi^*} Q_k + \gamma P_{\pi_{k+1}} (Q_k - B_{\pi_{k+1}} Q_k) + \\
&\quad + (\gamma P_{\pi_{k+1}})^2 (Q_k - B_{\pi_{k+1}} Q_k) + \dots + (\gamma P_{\pi_{k+1}})^{n-1} (Q_k - B_{\pi_{k+1}} Q_k) \\
&= \gamma P_{\pi^*} (Q^* - Q_k) + \\
&\quad + [\gamma P_{\pi_{k+1}} + (\gamma P_{\pi_{k+1}})^2 + \dots + (\gamma P_{\pi_{k+1}})^{n-1}] (Q_k - B_{\pi_{k+1}} Q_k) \\
&= \gamma P_{\pi^*} (Q^* - (Q_k - \epsilon_k)) - \gamma P_{\pi^*} \epsilon_k + \\
&\quad + [\gamma P_{\pi_{k+1}} + (\gamma P_{\pi_{k+1}})^2 + \dots + (\gamma P_{\pi_{k+1}})^{n-1}] (Q_k - B_{\pi_{k+1}} Q_k) \\
&= \gamma P_{\pi^*} d_k - \gamma P_{\pi^*} \epsilon_k + [\gamma P_{\pi_{k+1}} + (\gamma P_{\pi_{k+1}})^2 + \dots + (\gamma P_{\pi_{k+1}})^{n-1}] b_k.
\end{aligned}$$

As  $P_{\pi^*}$  and  $P_{\pi_{k+1}}$  are stochastic matrices, we deduce

$$\begin{aligned}
\max[Q^* - (B_{\pi_{k+1}})^n Q_k] &\leq \gamma \max d_k + \gamma \epsilon + (\gamma + \gamma^2 + \dots + \gamma^{n-1}) \max b_k \\
&= \gamma \max d_k + \gamma \epsilon + \frac{\gamma - \gamma^n}{1 - \gamma} \max b_k.
\end{aligned}$$

By using Equation 3, we obtain the following induction on  $\max d_k$ :

$$\max d_{k+1} \leq \gamma \max d_k + \gamma \epsilon + \sum_{n \geq 1} \lambda_n \left[ \frac{\gamma - \gamma^n}{1 - \gamma} \max b_k \right].$$

With the help of the Bellman error upper bound obtained earlier (Equation 2) we obtain

$$\begin{aligned}
\max d_{k+1} &\leq \gamma \max d_k + \gamma \epsilon + \sum_{n \geq 1} \lambda_n \left[ \frac{\gamma - \gamma^n}{(1 - \gamma)(1 - \beta)} \right] (1 + \gamma) \epsilon + O(\gamma^k) \\
&= \gamma \max d_k + \gamma \epsilon + \frac{\gamma - \beta}{(1 - \gamma)(1 - \beta)} (1 + \gamma) \epsilon + O(\gamma^k)
\end{aligned}$$

which gives, by taking the limit superior,

$$\limsup_{k \rightarrow \infty} \max d_k \leq \frac{\gamma}{1 - \gamma} \epsilon + \left[ \frac{\gamma - \beta}{(1 - \gamma)^2 (1 - \beta)} \right] (1 + \gamma) \epsilon. \quad (4)$$

**An upper bound on  $s_k$ :** Let us now consider the  $s_k$  term from Equation 1:

$$\begin{aligned}
s_{k+1} &= Q_{k+1} - \epsilon_{k+1} - Q^{\pi_{k+1}} \\
&= \sum_{n \geq 1} \lambda_n [(B_{\pi_{k+1}})^n Q_k] - (B_{\pi_{k+1}})^\infty Q_k \\
&= \sum_{n \geq 1} \lambda_n [(B_{\pi_{k+1}})^n Q_k - (B_{\pi_{k+1}})^\infty Q_k]. \quad (5)
\end{aligned}$$

It can be seen that

$$\begin{aligned}
& (B_{\pi_{k+1}})^n Q_k - (B_{\pi_{k+1}})^\infty Q_k \\
&= (B_{\pi_{k+1}})^n Q_k - (B_{\pi_{k+1}})^{n+1} Q_k + (B_{\pi_{k+1}})^{n+1} Q_k - (B_{\pi_{k+1}})^{n+2} Q_k + \dots \\
&= (\gamma P_{\pi_{k+1}})^n (Q_k - B_{\pi_{k+1}} Q_k) + (\gamma P_{\pi_{k+1}})^{n+1} (Q_k - B_{\pi_{k+1}} Q_k) + \dots \\
&= (\gamma P_{\pi_{k+1}})^n [I + \gamma P_{\pi_{k+1}} + (\gamma P_{\pi_{k+1}})^2 + \dots] b_k.
\end{aligned}$$

As above, by using the stochasticity of  $P_{\pi_{k+1}}$ , we obtain

$$\max[(B_{\pi_{k+1}})^n Q_k - (B_{\pi_{k+1}})^\infty Q_k] \leq \gamma^n (1 + \gamma + \gamma^2 + \dots) \max b_k = \frac{\gamma^n}{1 - \gamma} \max b_k.$$

By using Equation 5, we obtain an upper bound on  $\max s_{k+1}$ :

$$\max s_{k+1} \leq \frac{1}{1 - \gamma} \left[ \sum_{n \geq 1} \lambda_n \gamma^n \max b_k \right].$$

With the help of the Bellman error upper bound (Equation 2) and by taking the limit superior, we have

$$\limsup_{k \rightarrow \infty} \max s_k \leq \frac{1}{1 - \gamma} \left( \sum_{m \geq 1} \lambda_m \gamma^m \frac{1 + \gamma}{1 - \beta} \epsilon \right) = \frac{\beta}{(1 - \gamma)(1 - \beta)} (1 + \gamma) \epsilon. \quad (6)$$

**Conclusion of the proof** Finally, let us get back to Equation 1 and use the upper bounds we just derived for  $d_k$  (Equation 4) and  $s_k$  (Equation 6):

$$\begin{aligned}
\limsup_{k \rightarrow \infty} \|Q^* - Q^{\pi^k}\|_\infty &\leq \limsup_{k \rightarrow \infty} \max d_k + \limsup_{k \rightarrow \infty} \max s_k \\
&= \frac{\gamma}{1 - \gamma} \epsilon + \left[ \frac{\gamma - \beta}{(1 - \gamma)^2 (1 - \beta)} + \frac{\beta}{(1 - \gamma)(1 - \beta)} \right] (1 + \gamma) \epsilon. \\
&= \frac{\gamma}{1 - \gamma} \epsilon + \left[ \frac{\gamma - \beta + (1 - \gamma)\beta}{(1 - \gamma)^2 (1 - \beta)} \right] (1 + \gamma) \epsilon. \\
&= \frac{\gamma}{1 - \gamma} \epsilon + \left[ \frac{\gamma}{(1 - \gamma)^2} \right] (1 + \gamma) \epsilon. \\
&= \frac{\gamma(1 - \gamma) + \gamma(1 + \gamma)}{(1 - \gamma)^2} \epsilon \\
&= \frac{2\gamma}{(1 - \gamma)^2} \epsilon. \quad \blacksquare
\end{aligned}$$

## References

Thiery, C. and B. Scherrer (2010). Least-squares  $\lambda$  policy iteration: Bias-variance trade-off in control problems. In *ICML'10: Proceedings of the 27th Annual International Conference on Machine Learning*.