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# Clark-Ocone type formula for non-semimartingales with finite quadratic variation

## Formule de Clark-Ocone généralisée pour non-semimartingales à variation quadratique finie

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### Abstract

We provide a suitable framework for the concept of finite quadratic variation for processes with values in a separable Banach space  $B$  using the language of stochastic calculus via regularizations, introduced in the case  $B = \mathbb{R}$  by the second author and P. Vallois. To a real continuous process  $X$  we associate the Banach valued process  $X(\cdot)$ , called *window process*, which describes the evolution of  $X$  taking into account a memory  $\tau > 0$ . The natural state space for  $X(\cdot)$  is the Banach space of continuous functions on  $[-\tau, 0]$ . If  $X$  is a real finite quadratic variation process, an appropriated Itô formula is presented, from which we derive a generalized Clark-Ocone formula for non-semimartingales having the same quadratic variation as Brownian motion. The representation is based on solutions of an infinite dimensional PDE.

### Résumé

Nous présentons un cadre adéquat pour le concept de variation quadratique finie lorsque le processus de référence est à valeurs dans un espace de Banach séparable  $B$ . Le langage utilisé est celui de l'intégrale via régularisations introduit dans le cas réel par le second auteur et P. Vallois. À un processus réel continu  $X$ , nous associons le processus  $X(\cdot)$ , appelé processus *fenêtre*, qui à l'instant  $t$ , garde en mémoire le passé jusqu'à  $t - \tau$ . L'espace naturel d'évolution pour  $X(\cdot)$  est l'espace de Banach  $B$  des fonctions continues définies sur  $[-\tau, 0]$ . Si  $X$  est un processus réel à variation quadratique finie, nous énonçons une formule d'Itô appropriée de laquelle nous déduisons une formule de Clark-Ocone relative à des non-semimartingales réelles ayant la même variation quadratique que le mouvement brownien. La représentation est basée sur des solutions d'une EDP infini-dimensionnelle.

*Keywords:* Calculus via regularization, Infinite dimensional analysis, Clark-Ocone formula, Itô formula, Quadratic variation, Hedging theory without semimartingales.

*2010 MSC:* 60H05, 60H07, 60H30, 91G80.

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### Version française abrégée

Dans cette Note nous développons un calcul stochastique via régularisation de type progressif (*forward*) lorsque le processus intégrateur  $\mathbb{X}$  est à valeurs dans un espace de Banach séparable  $B$ . Ceci est basé sur une notion sophistiquée de *variation quadratique* que nous appellerons  $\chi$ -variation quadratique, où le symbole  $\chi$  correspond à un sous-espace

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$\chi$  du dual du produit tensoriel projectif  $B\hat{\otimes}_\pi B$ . Le calcul via régularisation a été introduit lorsque  $B = \mathbb{R}$  dans [13] et depuis il a été étudié par de nombreux auteurs qui ont fait avancer la théorie et ont produit plusieurs applications. Le lecteur peut consulter [14] pour une revue incluant une liste assez complète de références. Dans ce contexte, les auteurs introduisent une notion de covariation entre deux processus réels  $X$  et  $Y$ , notée  $[X, Y]$  qui généralise le crochet droit usuel lorsque  $X$  et  $Y$  sont des semimartingales. Un vecteur de processus  $\underline{X} = (X^1, \dots, X^n)$  est dit admettre tous ses crochets mutuels si  $[X^i, X^j]$  existe pour tous entiers  $1 \leq i, j \leq n$ .

Lorsque  $B = \mathbb{R}^n$ ,  $\mathbb{X}$  possède une  $\chi$ -variation quadratique avec  $\chi = (B\hat{\otimes}_\pi B)^*$  si et seulement si  $\mathbb{X}$  admet tous ses crochets mutuels. On peut voir qu'un processus à valeurs dans un espace de Banach *localement semi-sommable*  $\mathbb{X}$  au sens de [7], admet une  $\chi$ -variation quadratique avec  $\chi = (B\hat{\otimes}_\pi B)^*$ . Dans ce travail nous traçons une ébauche du calcul stochastique via la formule d'Itô énoncée au Théorème 5.1. Une attention spéciale est consacrée au cas où  $B$  est l'espace  $C([-\tau, 0])$  des fonctions continues définies sur  $[-\tau, 0]$ , pour un certain  $\tau > 0$ , qui est typiquement un espace de Banach non-réflexif, et à une formule de Clark-Ocone généralisée. Soit  $T > 0$ ; tout processus réel continu  $X = (X_t)_{t \in [0, T]}$  est prolongé par continuité pour  $t \notin [0, T]$ .

Soit  $0 < \tau \leq T$  et  $X$  un processus réel continu; nous appelons **fenêtre** le processus à valeurs dans  $C([-\tau, 0])$  défini par

$$X(\cdot) = (X_t(\cdot))_{t \in [0, T]} = \{X_t(u) := X_{t+u}; u \in [-\tau, 0], t \in [0, T]\}.$$

La théorie de l'intégration infini-dimensionnelle par rapport à des martingales (ou des semimartingales, [5, 11, 7]) n'est pas applicable, même lorsque l'intégrateur est la fenêtre  $W(\cdot)$  associée au mouvement brownien standard  $W$ . Au-delà des difficultés qui viennent du fait que  $C([-\tau, 0])$  n'est pas réflexif,  $W(\cdot)$  n'est d'aucune manière une semimartingale à valeurs dans  $C([-\tau, 0])$ .

Motivés par des applications liées à la couverture d'options dépendant de toute la trajectoire, nous discutons une formule de type Clark-Ocone visant à décomposer une classe significative  $h$  de v.a. dépendant de la trajectoire d'un processus  $X$  dont la variation quadratique vaut  $[X]_t = t$ . Cette formule généralise des résultats inclus dans [15, 1, 3] visant à déterminer des formules de valorisation et de couverture d'options vanille où asiatique dans un modèle de prix d'actif ayant la même variation quadratique que le modèle de Black-Scholes. Si le bruit dans un environnement stochastique est modélisé par la dérivée d'un mouvement brownien  $W$ , le théorème de représentation des martingales et la formule classique de Clark-Ocone sont deux outils fondamentaux de calcul. Le théorème 7.1 et les considérations à la fin de la section 7 montrent que dans une certaine mesure une formule de type Clark-Ocone reste valable lorsque la loi du processus sous-jacent n'est plus la mesure de Wiener mais le processus conserve la même variation quadratique que  $W$ . Il est en fait possible de représenter des variables aléatoires  $h = H(X_T(\cdot))$ , où  $H : C([-T, 0]) \rightarrow \mathbb{R}$ , comme

$$h = H_0 + \int_0^T \xi_t d^- X_t \tag{0.1}$$

sous des conditions suffisantes raisonnables sur la fonctionnelle  $H$ , où  $H_0$  est un nombre réel et  $\xi$  est un processus adapté à la filtration associée à  $X$  qui sont donnés de façon quasi-explicite. Ici  $d^- X_t$  symbolise l'intégration progressive ("forward") via régularisations définie dans [14]. Ces quantités sont exprimées à l'aide d'une fonctionnelle  $u : [0, T] \times C([-T, 0]) \rightarrow \mathbb{R}$  de classe  $C^{1,2}([0, T] \times C([-T, 0]))$  qui est solution d'une équation aux dérivées partielles; la représentation (0.1) de  $h$  a lieu avec  $H_0 = u(0, X_0(\cdot))$  et  $\xi_t = D^{\delta_0} u(t, X_t(\cdot))$ , où  $D^{\delta_0} u(t, \eta) := Du(t, \eta)(\delta_0)$ ,  $Du$  symbolisant la dérivée de Fréchet par rapport à  $\eta \in C([-T, 0])$ ;  $Du(t, \eta)$  est donc une mesure signée finie.

Si  $X$  est un mouvement brownien standard  $W$  et  $h \in \mathbb{D}^{1,2}$ , l'expression (0.1) coïncide avec la formule de Clark-Ocone classique.

## 1. Introduction

In the whole paper  $(\Omega, \mathcal{F}, \mathbb{P})$  is a fixed probability space, equipped with a given filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  fulfilling the usual conditions,  $B$  will be a separable Banach space and  $\mathbb{X}$  a  $B$ -valued process. If  $K$  is a compact set,  $\mathcal{M}(K)$  will

denote the space of Borel (signed) measures on  $K$ .  $C([-\tau, 0])$  will denote the space of continuous functions defined on  $[-\tau, 0]$  whose topological dual space is  $\mathcal{M}([-\tau, 0])$ .  $W$  will always denote an  $(\mathcal{F}_t)$ -real Brownian motion. Let  $T > 0$  be a fixed maturity time. All the processes  $X = (X_t)_{t \in [0, T]}$  are prolonged by continuity for  $t \notin [0, T]$  setting  $X_t = X_0$  for  $t \leq 0$  and  $X_t = X_T$  for  $t \geq T$ .

We first recall the basic concepts of forward integral and covariation and some one-dimensional results concerning calculus via regularization, a fairly complete survey on the subject being [14]. For simplicity, all the considered integrator processes will be continuous.

**Definition 1.1.** Let  $X$  (respectively  $Y$ ) be a continuous (resp. locally integrable) process.

The **forward integral of  $Y$  with respect to  $X$**  (resp. the **covariation of  $X$  and  $Y$** ), whenever it exists, is defined as

$$\int_0^t Y_s d^- X_s := \lim_{\epsilon \rightarrow 0^+} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds \quad \text{in probability for all } t \in [0, T], \quad (1.1)$$

$$\left( \text{resp. } [X, Y]_t = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) ds \quad \text{in the ucp sense with respect to } t \right), \quad (1.2)$$

provided that the limiting process admits a continuous version. If  $\int_0^t Y_s d^- X_s$  exists for any  $0 \leq t < T$ ;  $\int_0^T Y_s d^- X_s$  will symbolize the **improper forward integral** defined by  $\lim_{t \rightarrow T} \int_0^t Y_s d^- X_s$ , whenever it exists in probability.

If  $[X, X]$  exists then  $X$  is said to be a **finite quadratic variation** process.  $[X, X]$  will also be denoted by  $[X]$  and it will be called **quadratic variation of  $X$** . If  $[X] = 0$ , then  $X$  is said to be a **zero quadratic variation process**. If  $\mathbb{X} = (X^1, \dots, X^n)$  is a vector of continuous processes we say that it has all its **mutual covariations** (brackets) if  $[X^i, X^j]$  exists for any  $1 \leq i, j \leq n$ .

When  $X$  is a (continuous) semimartingale (resp. Brownian motion) and  $Y$  is an adapted cadlag process (resp. such that  $\int_0^T Y_s^2 ds < \infty$  a.s.), the integral  $\int_0^t Y_s d^- X_s$  exists and coincides with classical Itô's integral  $\int_0^t Y_s dX_s$ , see Proposition 6 in [14]. Stochastic calculus via regularization is a theory which allows, in many specific cases to manipulate those integrals when  $Y$  is anticipating or  $X$  is not a semimartingale. If  $X, Y$  are  $(\mathcal{F}_t)$ -semimartingales then  $[X, Y]$  coincides with the classical bracket  $\langle X, Y \rangle$ , see Corollary 2 in [14]. Finite quadratic variation processes will play a central role in this note: this class includes of course all  $(\mathcal{F}_t)$ -semimartingales. However that class is much richer. Typical examples of finite quadratic variation processes are  $(\mathcal{F}_t)$ -Dirichlet processes.  $D$  is called  $(\mathcal{F}_t)$ -Dirichlet process if it admits a decomposition  $D = M + A$  where  $M$  is an  $(\mathcal{F}_t)$ -local martingale and  $A$  is a zero quadratic variation process. It holds in that case  $[D] = [M]$ . This class of processes generalizes the semimartingales since a locally bounded variation process has zero quadratic variation. A slight generalization of that notion is the notion of weak Dirichlet, which was introduced in [8].  $X$  is called  $(\mathcal{F}_t)$ -weak Dirichlet if it admits a decomposition  $X = M + A$  where  $M$  is an  $(\mathcal{F}_t)$  local martingale and  $A$  is a process such that  $[A, N] = 0$  for any continuous  $(\mathcal{F}_t)$  local martingale  $N$ . An  $(\mathcal{F}_t)$ -weak Dirichlet process is not necessarily a finite quadratic variation process. On the other hand if  $A$  has finite quadratic variation then it holds  $[X] = [M] + [A]$ . Another interesting example is the bifractional Brownian motion  $B^{H,K}$  with parameters  $H \in (0, 1)$  and  $K \in (0, 1]$  which has finite quadratic variation if and only if  $HK \geq 1/2$ , see [12]. Notice that if  $K = 1$ , then  $B^{H,1}$  coincides with a fractional Brownian motion with Hurst parameter  $H \in (0, 1)$ . If  $HK = 1/2$  it holds  $[B^{H,K}] = 2^{1-K}t$ ; if  $K \neq 1$  this process is not even Dirichlet with respect to its own filtration. Other significant examples are the so-called weak  $k$ -order Brownian motions, for fixed  $k \geq 1$ , constructed by [9], which are in general not Gaussian.  $X$  is a weak  $k$ -order Brownian motion if for every  $0 \leq t_1 \leq \dots \leq t_k < +\infty$ ,  $(X_{t_1}, \dots, X_{t_k})$  is distributed as  $(W_{t_1}, \dots, W_{t_k})$ . If  $k \geq 4$  then  $[X]_t = t$ .

One central object of this work will be the generalization to infinite dimensional valued processes of the stochastic integral via regularization, see Definition 3.1. A stochastic calculus for Banach valued martingales was considered by [2, 16] and references therein, generalizing the classical stochastic calculus of [5, 11, 7].

We introduce now a particular Banach valued process. Given  $0 < \tau \leq T$  and a real continuous process  $X$ , we will call **window process** associated with  $X$ , the  $C([- \tau, 0])$ -valued process denoted by  $X(\cdot)$  defined as

$$X(\cdot) = (X_t(\cdot))_{t \in [0, T]} = \{X_t(u) := X_{t+u}; u \in [-\tau, 0], t \in [0, T]\}.$$

The window process  $W(\cdot)$  associated with the classical Brownian motion  $W$  will be called **window Brownian motion**. We observe that  $W(\cdot)$  is not a  $B = C([- \tau, 0])$ -valued semimartingale even in the (weak) sense that  ${}_{B^*} \langle \mu, W_t(\cdot) \rangle_B$  is a real semimartingale for any  $\mu \in B^*$ . In fact setting  $\mu = \delta_0 + \delta_{-\tau/2}$ , we get

$$Y_t := {}_{\mathcal{M}([- \tau, 0])} \langle \mu, W_t(\cdot) \rangle_{C([- \tau, 0])} = \int_{-\tau}^0 W_t(u) d\mu(u) = W_t + W_{t-\frac{\tau}{2}}$$

which is not a semimartingale. In fact its canonical filtration is the filtration  $(\mathcal{F}_t)$  associated with  $W$ . Taking into account Corollary 3.14 of [4]  $Y$  is an  $(\mathcal{F}_t)$ -weak Dirichlet process with martingale part  $W$ . By uniqueness of the decomposition of a weak Dirichlet process (see Proposition 16 of [14])  $Y$  cannot be an  $(\mathcal{F}_t)$ -semimartingale.

Motivated by the necessity of an Itô formula available also for  $B = C([- \tau, 0])$ -valued processes, we introduce a quadratic variation concept which depends on a subspace  $\chi$  of the dual of the tensor square of  $B$ , equipped with the projective topology, denoted by  $(B \hat{\otimes}_\pi B)^*$ , see Definition 4.3. We recall the fundamental identification  $(B \hat{\otimes}_\pi B)^* \cong \mathcal{B}(B \times B)$ , which denotes the space of  $\mathbb{R}$ -valued bounded bilinear forms on  $B \times B$ . An Itô formula for processes admitting a  $\chi$ -quadratic variation is given in Theorem 5.1. After formulating a theory for  $B$ -valued processes with general  $B$ , in Sections 6 and 7 we fix the attention on window processes setting  $B = C([- \tau, 0])$ . Section 6, in particular Proposition 6.4, is devoted to the evaluation of  $\chi$ -quadratic variation for windows associated with real finite quadratic variation processes. Suppose that  $X$  is a real process such that  $[X]_t = t$ . In Section 7 we give a representation result for a random variable  $h := H(X_T(\cdot))$  where  $H : C([-T, 0]) \rightarrow \mathbb{R}$  is continuous. That is of the type  $h = H_0 + \int_0^T \xi_s d^- X_s$ ,  $H_0 \in \mathbb{R}$  and  $\xi$  adapted process where the integral is considered as the forward integral defined in (1.1). More precisely  $h$  will appear as  $u(T, X_T(\cdot))$  where  $u \in C^{1,2}([0, T] \times C([-T, 0]); \mathbb{R}) \cap C^0([0, T] \times C([-T, 0]); \mathbb{R})$  solves an infinite dimensional partial differential equation of type (7.6). Moreover we will get  $H_0 = u(0, X_0(\cdot))$  and  $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$  where  $D^{\delta_0} u(t, \eta) := Du(t, \eta)(\delta_0)$ ;  $Du$  denotes the Fréchet derivative with respect to  $\eta \in C([-T, 0])$  so  $Du(t, \eta)$  is a signed measure.

## 2. Notations

Symbol  $\mathcal{C}([0, T])$  denotes the linear space of continuous real processes equipped with the ucp (uniformly convergence in probability) topology,  $B^*$  will be the topological dual of the Banach space  $B$ . We introduce now some subspaces of measures that we will frequently use. Symbol  $\mathcal{D}_0([- \tau, 0])$  ( resp.  $\mathcal{D}_{0,0}([- \tau, 0]^2)$ ), shortly  $\mathcal{D}_{0,0}$  ( resp.  $\mathcal{D}_{0,0}$ ), will denote the one dimensional Hilbert space of the multiples of Dirac measure concentrated at 0 ( resp. at  $(0, 0)$ ), i.e.

$$\mathcal{D}_0([- \tau, 0]) := \{\mu \in \mathcal{M}([- \tau, 0]); s.t. \mu(dx) = \lambda \delta_0(dx) \text{ with } \lambda \in \mathbb{R}\} \quad (2.1)$$

( resp.

$$\mathcal{D}_{0,0}([- \tau, 0]^2) := \{\mu \in \mathcal{M}([- \tau, 0]^2); s.t. \mu(dx, dy) = \lambda \delta_0(dx) \delta_0(dy) \text{ with } \lambda \in \mathbb{R}\} ). \quad (2.2)$$

Symbol  $Diag([- \tau, 0]^2)$ , shortly  $Diag$ , will denote the subset of  $\mathcal{M}([- \tau, 0]^2)$  defined as follows:

$$Diag([- \tau, 0]^2) := \left\{ \mu \in \mathcal{M}([- \tau, 0]^2) s.t. \mu(dx, dy) = g(x) \delta_y(dx) dy; g \in L^\infty([- \tau, 0]) \right\}. \quad (2.3)$$

$Diag([- \tau, 0]^2)$ , equipped with the norm  $\|\mu\|_{Diag([- \tau, 0]^2)} = \|g\|_\infty$ , is a Banach space.

### 3. Forward integrals in Banach spaces

In this section we introduce an infinite dimensional stochastic integral via regularization. In this construction there are two main difficulties. The integrator is generally not a semimartingale or the integrand may be anticipative;  $B$  is a general separable, not necessarily reflexive, Banach space.

**Definition 3.1.** Let  $(\mathbb{X}_t)_{t \in [0, T]}$  (respectively  $(\mathbb{Y}_t)_{t \in [0, T]}$ ) be a  $B$ -valued (respectively a  $B^*$ -valued) stochastic process. We suppose  $\mathbb{X}$  to be continuous and  $\mathbb{Y}$  to be strongly measurable (in the Bochner sense) such that  $\int_0^T \|\mathbb{Y}_s\|_{B^*} ds < +\infty$  a.s. For every fixed  $t \in [0, T]$  we define the **definite forward integral of  $\mathbb{Y}$  with respect to  $\mathbb{X}$**  denoted by  $\int_0^t \langle \mathbb{Y}_s, d^- \mathbb{X}_s \rangle_B$  as the following limit in probability:

$$\int_0^t \langle \mathbb{Y}_s, d^- \mathbb{X}_s \rangle_B := \lim_{\epsilon \rightarrow 0} \int_0^t \langle \mathbb{Y}_s, \frac{\mathbb{X}_{s+\epsilon} - \mathbb{X}_s}{\epsilon} \rangle_B ds. \quad (3.1)$$

We say that the **forward stochastic integral of  $\mathbb{Y}$  with respect to  $\mathbb{X}$**  exists if the process

$$\left( \int_0^t \langle \mathbb{Y}_s, d^- \mathbb{X}_s \rangle_B \right)_{t \in [0, T]}$$

admits a continuous version. In the sequel indices  $B$  and  $B^*$  will often be omitted.

### 4. Chi-quadratic variation

**Definition 4.1.** A closed linear subspace  $\chi$  of  $(B \hat{\otimes}_\pi B)^*$ , endowed with its own norm, such that

$$\|\cdot\|_{(B \hat{\otimes}_\pi B)^*} \leq \text{const} \cdot \|\cdot\|_\chi \quad (4.1)$$

will be called a **Chi-subspace (of  $(B \hat{\otimes}_\pi B)^*$ )**.

Let  $\chi$  be a Chi-subspace of  $(B \hat{\otimes}_\pi B)^*$ ,  $\mathbb{X}$  be a  $B$ -valued stochastic process and  $\epsilon > 0$ . We denote by  $[\mathbb{X}]^\epsilon$ , the application

$$[\mathbb{X}]^\epsilon : \chi \longrightarrow \mathcal{C}([0, T]) \quad \text{defined by} \quad \phi \mapsto \left( \int_0^t \chi \langle \phi, \frac{J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2)}{\epsilon} \rangle_{\chi^*} ds \right)_{t \in [0, T]} \quad (4.2)$$

where  $J : B \hat{\otimes}_\pi B \longrightarrow (B \hat{\otimes}_\pi B)^{**}$  denotes the canonical injection between a space and its bidual.

#### Remark 4.2.

1. We recall that  $\chi \subset (B \hat{\otimes}_\pi B)^*$  implies  $(B \hat{\otimes}_\pi B)^{**} \subset \chi^*$ .
2. As indicated,  $\langle \cdot, \cdot \rangle_{\chi^*}$  denotes the duality between the space  $\chi$  and its dual  $\chi^*$ ; in fact, by assumption,  $\phi$  is an element of  $\chi$  and element  $J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2)$  naturally belongs to  $(B \hat{\otimes}_\pi B)^{**} \subset \chi^*$ .
3. The real function  $s \rightarrow \langle \phi, J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2) \rangle$  is integrable since  $|\langle \phi, J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2) \rangle| \leq \text{const} \|\phi\|_\chi \|\mathbb{X}_{s+\epsilon} - \mathbb{X}_s\|_B^2$ .
4. With a slight abuse of notation, in the sequel, the application  $J$  will be omitted. The tensor product  $(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2$  has to be considered as the element  $J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2)$  which belongs to  $\chi^*$ .

We give now the definition of the  $\chi$ -quadratic variation of a  $B$ -valued stochastic process  $\mathbb{X}$ .

**Definition 4.3.** Let  $\chi$  be a separable Chi-subspace of  $(B \hat{\otimes}_\pi B)^*$  and  $\mathbb{X}$  a  $B$ -valued stochastic process. We say that  $\mathbb{X}$  **admits a  $\chi$ -quadratic variation** if the following assumptions are fulfilled.

**H1** For every sequence  $(\epsilon_n) \downarrow 0$  there is a subsequence  $(\epsilon_{n_k})$  such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| \chi \left\langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes^2}{\epsilon_{n_k}} \right\rangle_{\chi^*} \right| ds < +\infty, \text{ a.s.}$$

**H2** It exists an application denoted by  $[\mathbb{X}] : \chi \rightarrow \mathcal{C}([0, T])$ , such that

$$[\mathbb{X}]^\epsilon(\phi) \xrightarrow[\epsilon \rightarrow 0_+]{ucp} [\mathbb{X}](\phi) \tag{4.3}$$

for all  $\phi \in \mathcal{S}$ , where  $\mathcal{S} \subset \chi$  such that  $\overline{Span(\mathcal{S})} = \chi$ .

We formulate a technical proposition which is stated in Corollary 4.38 in [6]. Its proof is based on Banach-Steinhaus and separability arguments.

**Proposition 4.4.** Suppose that  $\mathbb{X}$  admits a  $\chi$ -quadratic variation.

1. Relation (4.3) holds for any  $\phi \in \chi$  and  $[\mathbb{X}]$  is a linear continuous application. In particular  $[\mathbb{X}]$  does not depend on  $\mathcal{S}$ .
2. There exists a  $\chi^*$ -valued measurable process  $(\widetilde{[\mathbb{X}]})_{0 \leq t \leq T}$ , cadlag and with bounded variation on  $[0, T]$  such that  $\widetilde{[\mathbb{X}]}_t(\cdot)(\phi) = [\mathbb{X}](\phi)(\cdot, t)$  a.s. for any  $t \in [0, T]$  and  $\phi \in \chi$ .

The existence of  $\widetilde{[\mathbb{X}]}$  guarantees that  $[\mathbb{X}]$  admits a proper version which allows to consider it as a pathwise integral.

**Definition 4.5.** When  $\mathbb{X}$  admits a  $\chi$ -quadratic variation, the  $\chi^*$ -valued measurable process  $(\widetilde{[\mathbb{X}]})_{0 \leq t \leq T}$  appearing in Proposition 4.4, is called  **$\chi$ -quadratic variation** of  $\mathbb{X}$ . Sometimes, with a slight abuse of notation, even  $[\mathbb{X}]$  will be called  $\chi$ -quadratic variation and it will be confused with  $\widetilde{[\mathbb{X}]}$ .

**Definition 4.6.** We say that a continuous  $B$ -valued process  $\mathbb{X}$  admits **global quadratic variation** if it admits a  $\chi$ -quadratic variation with  $\chi = (B \hat{\otimes}_\pi B)^*$ . In particular  $\widetilde{[\mathbb{X}]}$  takes values “a priori” in  $(B \hat{\otimes}_\pi B)^{**}$ .

The natural generalization of quadratic variation for a  $B$ -valued *locally semi summable* process is a  $(B \hat{\otimes}_\pi B)$ -valued process, called the *tensor quadratic variation*, as it was introduced by [7] and [11]. Unfortunately, the tensor quadratic variation does not exist in several contexts. For instance, the window Brownian motion  $W(\cdot)$ , which is our fundamental example, does not admit it, see Remark 6.3. That notion is related to a strong convergence in  $B \hat{\otimes}_\pi B$  while our concept of global quadratic variation is related to a weak star convergence in its bidual. The global quadratic variation generalizes the tensor quadratic variation one: if  $\mathbb{X}$  admits a tensor quadratic variation then it admits a global quadratic variation and those quadratic variations are equal, see Section 6.3 in [6] for details. When  $B$  is the finite dimensional space  $\mathbb{R}^n$ ,  $\mathbb{X}$  admits a tensor quadratic variation and if and only if  $\mathbb{X}$  admits a global quadratic variation. In that case previous properties are also equivalent to the existence of all the mutual brackets in the sense of [14].

## 5. Itô's formula

The classical Itô formulae for stochastic integrators  $\mathbb{X}$  with values in an infinite dimensional space appear in Section 4.5 of [5] for the Hilbert separable case and in Section 3.7 in [11], see also [7], as far as the Banach case is concerned; they involve processes admitting a tensor quadratic variation. We state now an Itô formula in the general separable Banach space which do not necessarily have a tensor quadratic variation but they have rather a  $\chi$ -quadratic variation, where  $\chi$  is some Chi-subspace where the second order Fréchet derivative lives. This type of formula is well suited for  $C([-\tau, 0])$ -valued integrators as for instance window processes; this will be developed in Sections 6 and 7. In the sequel if  $F : [0, T] \times B \rightarrow \mathbb{R}$  then (if it exists)  $DF$  (resp.  $D^2F$ ) stands for the first (resp. second) order Fréchet derivative with respect to the  $B$  variable.

**Theorem 5.1.** Let  $B$  be a separable Banach space,  $\chi$  be a Chi-subspace of  $(B\hat{\otimes}_\pi B)^*$  and  $\mathbb{X}$  a  $B$ -valued continuous process admitting a  $\chi$ -quadratic variation. Let  $F : [0, T] \times B \rightarrow \mathbb{R}$  of class  $C^{1,2}$  Fréchet. such that

$$D^2F : [0, T] \times B \rightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \text{ is continuous with respect to } \chi . \quad (5.1)$$

Then the forward integral

$$\int_0^t \int_{B^*} \langle DF(s, \mathbb{X}_s), d^- \mathbb{X}_s \rangle_B , \quad t \in [0, T] ,$$

exists and the following formula holds

$$F(t, \mathbb{X}_t) = F(0, \mathbb{X}_0) + \int_0^t \partial_t F(s, \mathbb{X}_s) ds + \int_0^t \int_{B^*} \langle DF(s, \mathbb{X}_s), d^- \mathbb{X}_s \rangle_B + \frac{1}{2} \int_0^t \int_{\chi} \langle D^2F(s, \mathbb{X}_s), d[\widetilde{\mathbb{X}}]_s \rangle_{\chi^*} a.s. \quad (5.2)$$

Its proof is given in Section 8 of [6].

## 6. Evaluation of $\chi$ -quadratic variations for window processes

From this section we fix  $B$  as the Banach space  $C([-\tau, 0])$ . In this section we give some examples of Chi-subspaces and then we give some evaluations of  $\chi$ -quadratic variations for window processes  $\mathbb{X} = X(\cdot)$ . For illustration of possible applications of Itô formula (5.2), consider the following functions. Let  $H : B \rightarrow \mathbb{R}$  and  $\eta \in B$  defined by

$$a) \quad H(\eta) = f(\eta(0)), \quad f \in C^2(\mathbb{R}); \quad b) \quad H(\eta) = \left( \int_{-\tau}^0 \eta(s) ds \right)^2; \quad c) \quad H(\eta) = \int_{-\tau}^0 \eta^2(s) ds. \quad (6.1)$$

Those functions are of class  $C^2(B)$ ; computing the second order Fréchet derivative  $D^2H : B \rightarrow (B\hat{\otimes}_\pi B)^*$  we obtain the following:

$$a) \quad D_{dx dy}^2 H(\eta) = f''(\eta(0)) \delta_0(dx) \delta_0(dy); \quad b) \quad D^2 H(\eta) = 2 \mathbb{1}_{[-\tau, 0]^2}; \quad c) \quad D_{dx dy}^2 H(\eta) = 2 \delta_x(dy) dx. \quad (6.2)$$

In all those examples,  $D^2H(\eta)$  lives in a particular Chi-subspace  $\chi$ . Respectively we have  $D^2H : B \rightarrow \chi$  continuously with

$$a) \quad \chi = \mathcal{D}_{0,0}([-\tau, 0]^2); \quad b) \quad \chi = L^2([-\tau, 0]^2); \quad c) \quad \chi = \text{Diag}([-\tau, 0]^2). \quad (6.3)$$

Other examples of Chi-subspaces are  $\mathcal{M}([-\tau, 0]^2)$  and its subspace  $\chi^0([-\tau, 0]^2)$ , (shortly  $\chi^0$ ), defined by

$$\chi^0([-\tau, 0]^2) := (\mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])) \hat{\otimes}_h^2, \quad (6.4)$$

where  $\hat{\otimes}_h$  stands for the Hilbert tensor product. The latter one will intervene in Theorem 7.1 in relation with the generalized Clark-Ocone formula. We evaluate now some  $\chi$ -quadratic variations of window processes.

**Proposition 6.1.** Let  $X$  be a real valued process with Hölder continuous paths of parameter  $\gamma > 1/2$ . Then  $X(\cdot)$  admits a zero global quadratic variation.

**Example 6.2.** Examples of real processes with Hölder continuous paths of parameter  $\gamma > 1/2$  are fractional Brownian motion  $B^H$  with  $H > 1/2$  or a bifractional Brownian motion  $B^{H,K}$  with  $HK > 1/2$ .



**Remark 6.3.** The window Brownian motion  $W(\cdot)$  does not admit a global (and therefore not a tensor) quadratic variation because Condition **H1** is not verified. In fact it is possible to show that

$$\int_0^T \frac{1}{\epsilon} \|W_{u+\epsilon}(\cdot) - W_u(\cdot)\|_B^2 du \geq T A^2(\tilde{\epsilon}) \ln(1/\tilde{\epsilon}) \quad \text{where} \quad \tilde{\epsilon} = \frac{2\epsilon}{T} \quad (6.5)$$

and  $(A(\epsilon))$  is a family of non negative r.v. such that  $\lim_{\epsilon \rightarrow 0} A(\epsilon) = 1$  a.s.

**Proposition 6.4.** Let  $X$  be a real continuous process with finite quadratic variation  $[X]$  and  $0 < \tau \leq T$ . The following properties hold true.

- 1)  $X(\cdot)$  admits zero  $L^2([-\tau, 0]^2)$ -quadratic variation.
- 2)  $X(\cdot)$  admits a  $\mathcal{D}_{0,0}([-\tau, 0]^2)$ -quadratic variation given by

$$[X(\cdot)](\mu) = \mu(\{0, 0\})[X], \quad \forall \mu \in \mathcal{D}_{0,0}([-\tau, 0]^2). \quad (6.6)$$

- 3)  $X(\cdot)$  admits a  $\chi^0([-\tau, 0]^2)$ -quadratic variation which equals

$$[X(\cdot)](\mu) = \mu(\{0, 0\})[X], \quad \forall \mu \in \chi^0([-\tau, 0]^2). \quad (6.7)$$

- 4)  $X(\cdot)$  admits a *Diag*-quadratic variation given by

$$\mu \mapsto [X(\cdot)]_t(\mu) = \int_0^{t \wedge \tau} g(-x)[X]_{t-x} dx \quad t \in [0, T], \quad (6.8)$$

where  $\mu$  is a generic element in  $\text{Diag}([-\tau, 0]^2)$  of type  $\mu(dx, dy) = g(x)\delta_y(dx)dy$ , with associated  $g$  in  $L^\infty([-\tau, 0])$ .

**Remark 6.5.** We remark that in the treated cases, the quadratic variation  $[X]$  of the real finite quadratic variation process  $X$  insures the existence of (and completely determines) the  $\chi$ -quadratic variation. For example if  $X$  is a real finite quadratic variation process such that  $[X]_t = t$ , then  $X(\cdot)$  has the same  $\chi$ -quadratic variation as the window Brownian motion for the  $\chi$  mentioned in the above proposition.

## 7. A generalized Clark-Ocone formula

In this section we will consider  $\tau = T$  and we recall that  $B = C([-T, 0])$ . Let  $X$  be a real stochastic process such that  $X_0 = 0$  and  $[X]_t = t$ . Let  $H : C([-T, 0]) \rightarrow \mathbb{R}$  be a Borel functional; we aim at representing the random variable

$$h = H(X_T(\cdot)). \quad (7.1)$$

The main task will consist in looking for classes of functionals  $H$  for which there is  $H_0 \in \mathbb{R}$  and a predictable process  $\xi$  with respect to the canonical filtration of  $X$  such that  $h$  admits the representation

$$h = H_0 + \int_0^T \xi_s d^- X_s. \quad (7.2)$$

Moreover we look for an explicit expression for  $H_0$  and  $\xi$ . As a consequence of Itô's formula (5.2) for path dependent functionals of the process we will observe that, in those cases, it is possible to find a function  $u$  which solves an infinite dimensional PDE and which gives at the same time the representation result (7.2). One possible representation is the following.

**Theorem 7.1.** Let  $H : C([-T, 0]) \rightarrow \mathbb{R}$  be a Borel functional. Let  $u \in C^{1,2}([0, T] \times C([-T, 0])) \cap C^0([0, T] \times C([-T, 0]))$  such that  $x \mapsto D_x^{ac} u(t, \eta)$  has bounded variation, for any  $t \in [0, T]$ ,  $\eta \in C([-T, 0])$  and  $D^{ac} u(t, \eta)$  is the absolute continuous part of measure  $Du(t, \eta)$ . We suppose moreover that  $(t, \eta) \mapsto D^2 u(t, \eta)$  takes values in  $\chi^0([-T, 0]^2)$  and it is continuous. Suppose that  $u$  is a solution of

$$\begin{cases} \partial_t u(t, \eta) + \int_{]-t, 0]} D^{ac} u(t, \eta) d\eta + \frac{1}{2} D^2 u(t, \eta)(\{0, 0\}) = 0 \\ u(T, \eta) = H(\eta) \end{cases} \quad (7.3)$$

where the integral  $\int_{]-t, 0]} D^{ac} u(t, \eta) d\eta$  has to be understood via an integration by parts as follows:

$$\int_{]-t, 0]} D^{ac} u(t, \eta) d\eta = D^{ac} u(0, \eta)\eta(0) - D^{ac} u(-t, \eta)\eta(-t) - \int_{]-t, 0]} \eta(x) D_{dx}^{ac} u(t, \eta).$$

Then the random variable  $h := H(X_T(\cdot))$  admits the following representation

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(t, X_t(\cdot)) d^- X_t. \quad (7.4)$$

□

Sections 9.8 and 9.9 in [6] provide different reasonable conditions on  $H : C([-T, 0]) \rightarrow \mathbb{R}$  such that there is a function  $u$  solving PDE (7.3) in general situations, i.e. fulfilling the hypotheses of Theorem 7.1. When  $H : C([-T, 0]) \subset L^2([-T, 0]) \rightarrow \mathbb{R}$  is  $C^3(L^2([-T, 0]))$  such that  $D^2 H \in L^2([-T, 0]^2)$  with some other minor technical conditions, Theorem 9.41 in [6] furnishes explicit solutions to (7.3). Another case for which it is possible to do the same is given by Proposition 9.53 in [6], where  $h$  depends (not necessarily smoothly) on a finite number of Wiener integrals of the type  $\int_0^T \varphi(s) d^- X_s$  and  $\varphi \in C^2(\mathbb{R})$ .

**Remark 7.2.** In relation to Theorem 7.1 we observe the following.

- Only pathwise considerations intervene and there is no need to suppose that the law of  $X$  is Wiener measure.
- Since  $H(\eta) = u(T, \eta)$ , we observe that  $H$  is automatically continuous by hypothesis  $u \in C^0([0, T] \times C([-T, 0]))$ .
- Let us suppose  $X = W$ .
  1. Making use of probabilistic technology, (7.4) holds in some cases even if  $H$  is not continuous and  $h \notin L^1(\Omega)$ ; we refer to Section 9.6 in [6] for this type of results.
  2. If  $\int_0^T \xi_s^2 ds < +\infty$  a.s., then the forward integral  $\int_0^T \xi_t d^- W_t$  coincides with the Itô integral  $\int_0^T \xi_t dW_t$ .
  3. If the r.v.  $h = H(W_T(\cdot))$  belongs to  $\mathbb{D}^{1,2}$ , by uniqueness of the martingale representation theorem and point 2., we have  $H_0 = \mathbb{E}[h]$  and  $\xi_t = \mathbb{E}[D_t^m h | \mathcal{F}_t]$ , where  $D^m$  is the Malliavin gradient; this agrees with Clark-Ocone formula.
- If  $X$  is not a Brownian motion, in general  $H_0 \neq \mathbb{E}[h]$  since  $\mathbb{E}\left[\int_0^T \xi_t d^- X_t\right]$  does not generally vanish. In fact  $\mathbb{E}[h]$  will specifically depend on the unknown law of  $X$ .

**Remark 7.3.** The assumption  $[X]_t = t$  is not crucial. With some more work it is possible to obtain similar representations even if  $[X]_t = \int_0^t a^2(s, X_s) ds$  for a large class of  $a : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ . As a limiting case we show this possibility when  $[X] = 0$  and  $h = f\left(\int_0^T \varphi^1(s) d^- X_s, \dots, \int_0^T \varphi^n(s) d^- X_s\right)$  with  $\varphi_i \in C^2([0, T])$  and  $f \in C^2(\mathbb{R}^n)$ . We define  $V_t = u(t, X_t(\cdot))$  where  $u : [0, T] \times C([-T, 0]) \rightarrow \mathbb{R}$  defined by

$$u(t, \eta) = f\left(\int_{]-t, 0]} \varphi^1(s+t) d\eta(s), \dots, \int_{]-t, 0]} \varphi^n(s+t) d\eta(s)\right).$$

After an application of the finite dimensional Itô formula for finite quadratic variation processes, see Proposition 2.4 in [10], and some further calculations, we have

$$h = f(0, \dots, 0) + \int_0^t \xi_s d^- X_s \quad (7.5)$$

with  $\xi_t = \sum_{i=1}^n \partial_i f \left( \int_{-t}^0 \varphi^1(s+t) d^- X_s, \dots, \int_{-t}^0 \varphi^n(s+t) d^- X_s \right) \varphi^i(t)$ . On the other hand, we observe that  $u$  solves the PDE  $\partial_t u + \int_{]-t,0]} D^{ac} u(t, \eta) d\eta = 0$ , which is of the same type of (7.3). Representation (7.5) can be also established via Theorem 5.1, taking into account that  $X(\cdot)$  admits zero  $\chi^0$ -quadratic variation.

In chapter 9 in [6] we enlarge the discussion presented in Theorem 7.1. We can give examples where  $u : [0, T] \times C([-T, 0]) \rightarrow \mathbb{R}$  of class  $C^{1,2}([0, T] \times C([-T, 0]); \mathbb{R}) \cap C^0([0, T] \times C([-T, 0]); \mathbb{R})$  with  $D^2 u \in \chi_0$  such that (7.4) holds and  $u$  solves an infinite dimensional PDE of the type

$$\begin{cases} \partial_t u(t, \eta) + \int_{]-t,0]} D^{ac} u(t, \eta) d\eta'' + \frac{1}{2} \langle D^2 u(t, \eta), \mathbb{1}_D \rangle = 0 \\ u(T, \eta) = H(\eta) \end{cases} \quad (7.6)$$

where  $\mathbb{1}_D(x, y) := \begin{cases} 1 & \text{if } x = y, x, y \in [-T, 0] \\ 0 & \text{otherwise} \end{cases}$ . The integral “ $\int_{]-t,0]} D^{ac} u(t, \eta) d\eta$ ” has to be suitably defined and term  $\langle D^2 u(t, \eta), \mathbb{1}_D \rangle$  indicates the evaluation of the second order derivative on the diagonal of the square  $[-T, 0]^2$ . We observe that solution of (7.3) are also solutions of (7.6) since  $\langle D^2 u(t, \eta), \mathbb{1}_D \rangle = D^2 u(t, \eta)(\{0, 0\})$  because  $D^2 u$  takes values in  $\chi^0$ .

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