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L(2,1)-labelling of graphs*

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Abstract

An $L(2,1)$ -labelling of a graph is a function f from the vertex set to the positive integers such that $|f(x) - f(y)| \geq 2$ if $\text{dist}(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $\text{dist}(x, y) = 2$, where $\text{dist}(u, v)$ is the distance between the two vertices u and v in the graph G . The *span* of an $L(2,1)$ -labelling f is the difference between the largest and the smallest labels used by f plus 1. In 1992, Griggs and Yeh conjectured that every graph with maximum degree $\Delta \geq 2$ has an $L(2,1)$ -labelling with span at most $\Delta^2 + 1$. We settle this conjecture for Δ sufficiently large.

1 Introduction

In the channel assignment problem, transmitters at various nodes within a geographic territory must be assigned channels or frequencies in such a way as to avoid interferences. A model for the channel-assignment problem developed wherein channels or frequencies are represented with nonnegative integers, “close” transmitters must be assigned different integers and “very close” transmitters must be assigned integers that differ by at least 2. This quantification led to the definition of an $L(p, q)$ -labelling of a graph $G = (V, E)$ as a function f from the vertex set to the positive integers such that $|f(x) - f(y)| \geq p$ if $\text{dist}(x, y) = 1$ and $|f(x) - f(y)| \geq q$ if $\text{dist}(x, y) = 2$, where $\text{dist}(u, v)$ is the distance between the two vertices u and v in the graph G . The notion of $L(2,1)$ -labelling first appeared in 1992 [12]. Since then, a large number of articles has been published devoted to the study of $L(p, q)$ -labellings. We refer the interested reader to the surveys of Calamoneri [6] and Yeh [24].

Generalizations of $L(p, q)$ -labellings in which for each $i \geq 1$, a minimum gap of p_i is required for channels assigned to vertices at distance i , have also been studied (see for example the recent survey of Griggs and Král’ [11], and consult also [18, 15, 3, 16]).

In the context of the channel-assignment problem, the main goal is to minimise the number of channels used. Hence, we are interested in the *span* of an $L(p, q)$ -labelling f , which is the difference between the largest and the smallest labels of f plus 1. The $\lambda_{p,q}$ -number of G is $\lambda_{p,q}(G)$, the minimum span over all $L(p, q)$ -labellings of G . In general, determining the $\lambda_{p,q}$ -number of a graph is NP-hard [9]. In their seminal paper, Griggs and Yeh [12] observed that a greedy algorithm yields $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta + 1$, where Δ is the maximum degree of the graph G . Moreover, they conjectured that this upper bound can be decreased to $\Delta^2 + 1$.

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Conjecture 1 ([12]). For every $\Delta \geq 2$ and every graph G of maximum degree Δ , $\lambda_{2,1}(G) \leq \Delta^2 + 1$.

This upper bound would be tight: there are graphs with degree Δ , diameter 2 and $\Delta^2 + 1$ vertices, namely the 5-cycle, the Petersen graph and the Hoffman-Singleton graph. Thus, their square is a clique of order $\Delta^2 + 1$, so the span of every $L(2, 1)$ -labelling is at least $\Delta^2 + 1$.

Jonas [14] improved slightly on Griggs and Yeh's upper bound by showing that every graph of maximum degree Δ admits a $(2, 1)$ -labelling with span at most $\Delta^2 + 2\Delta - 3$. Subsequently, Chang and Kuo [7] provided the upper bound $\Delta^2 + \Delta + 1$ which remained the best general upper bound for about a decade. Král' and Škrekovski [17] brought this upper bound down by 1 as the corollary of a more general result. And, using the algorithm of Chang and Kuo [7], Gonçalves [10] decreased this bound by 1 again, thereby obtaining the upper bound $\Delta^2 + \Delta - 1$. Note that Conjecture 1 is true for planar graphs of maximum degree $\Delta \neq 3$. For $\Delta \geq 7$ it follows from a result of van den Heuvel and McGuinness [23], and Bella et al. [4] proved it for the remaining cases.

In this paper, we show Conjecture 1 for sufficiently large Δ .

Theorem 2. *There is a Δ_0 such that for every graph G of maximum degree $\Delta \geq \Delta_0$,*

$$\lambda_{2,1}(G) \leq \Delta^2 + 1.$$

Using the greedy algorithm, or the previous mentioned bounds, we obtain the following corollary.

Corollary 3. *For every graph G of maximum degree Δ , $\lambda_{2,1}(G) \leq \Delta^2 + \mathcal{O}(1)$.*

Generalize for (p, q) -labelling

Actually, we consider a more general setup. We are given a graph G_1 with vertex-set V , along with a spanning subgraph G_2 . We want to assign integers from 1 to k to the elements of V so that vertices adjacent in G_1 receive different colours and vertices adjacent in G_2 receive colours which differ by at least 2. Typically the maximum degree of G_1 is much larger than the maximum degree of G_2 . In the case of $L(2, 1)$ -labelling, G_1 is the square of G_2 . We impose the condition that for some integer Δ , G_1 has maximum degree at most Δ^2 and G_2 has maximum degree Δ . We show that under these conditions there exists a colouring for $k = \Delta^2 + 1$ provided that Δ is large enough. This is best possible since G_1 may be a clique of size $\Delta^2 + 1$. Formally, we prove the following result.

Theorem 4. *There is a Δ_0 , such that for every $\Delta \geq \Delta_0$, and $G_2 \subseteq G_1$ with $\Delta(G_1) \leq \Delta^2$ and $\Delta(G_2) \leq \Delta$, there exists a $(\Delta^2 + 1)$ -colouring of $V(G_1)$ such that no edge of G_1 is monochromatic and for every edge $xy \in E(G_2)$, $|c(x) - c(y)| \geq 2$.*

In the next section we give an outline of the proof. In the section following that, we present some probabilistic tools we need. We then turn to the gory details.

In what follows, we use G_1 -neighbour to mean a neighbour in G_1 and G_2 -neighbour to indicate a neighbour in G_2 . For every vertex v and every subgraph H of G_1 , we let $\deg_H^1(v)$ be the number of G_1 -neighbours of v in H . We omit the subscript if $H = G_1$.

Moreover, lots of inequalities are correct only when Δ is large enough. In such inequalities, we will use the symbols \leq^* , \geq^* , $<^*$ and $>^*$ instead of \leq , \geq , $<$ and $>$, respectively. We note that 10^{69} is a suitable value for Δ_0 (we make no attempt to optimise this value).

2 A Sketch of the Proof

We consider a counter-example to Theorem 4 chosen so as to minimize V . Thus, for every proper subset X of the vertices of G_1 , there is a $(\Delta^2 + 1)$ -colouring c of X such that every edge of G_1 within

X is non-monochromatic, and for every edge xy of G_2 contained within X , $|c(x) - c(y)| \geq 2$. Such a colouring of X is a *good colouring*. In particular, as $G_2 \subseteq G_1$, this implies that every vertex v has more than $\Delta^2 - 2\Delta$ G_1 -neighbours as otherwise we could complete a good colouring of $V(G_1) - v$ greedily. Indeed for each vertex, a coloured G_2 -neighbour forbids 3 colours, which is 2 more as being only a G_1 -neighbour.

The next lemma follows by setting $d = 1000\Delta$ and applying to G_1 a decomposition result due to Reed [21, Lemma 15.2].

Lemma 5. *There is a partition of V into disjoint sets D_1, \dots, D_ℓ, S such that*

- (a) *every D_i has between $\Delta^2 - 8000\Delta$ and $\Delta^2 + 4000\Delta$ vertices;*
- (b) *there are at most $8000\Delta^3$ edges of G_1 leaving any D_i ;*
- (c) *a vertex has at least $\frac{3}{4}\Delta^2$ G_1 -neighbours in D_i if and only if it is in D_i ; and*
- (d) *for each vertex v of S , the neighbourhood of v in G_1 contains at most $\binom{\Delta^2}{2} - 1000\Delta^3$ edges.*

We let H_i be the subgraph of G_1 induced by D_i and $\overline{H_i}$ its complementary graph. An *internal neighbour* of a vertex of D_i we mean a neighbour in H_i . An *external neighbour* of a vertex of D_i is a neighbour that is not internal. One can prove the following.

Lemma 6. *For every i , $\overline{H_i}$ has no matching of size at least $10^3\Delta$.*

For each $i \in \{1, 2, \dots, \ell\}$, we let M_i be a maximum matching of $\overline{H_i}$, and K_i be the clique $D_i - V(M_i)$. By Lemmas 5(a) and 6, $|K_i| \geq \Delta^2 - 10^4\Delta$. We let B_i be the set of vertices in K_i that have more than $\Delta^{5/4}$ G_1 -neighbours outside D_i , and we set $A_i := K_i \setminus B_i$. Considering Lemma 5(b) we can make the following observation.

Observation 7. *For every index $i \in \{1, 2, \dots, \ell\}$,*

$$|B_i| \leq 8000\Delta^{7/4} \text{ and so } |A_i| \geq \Delta^2 - 9000\Delta^{7/4}.$$

We are going to colour the vertices in three steps. We first colour $V_1 := V \setminus \cup_{i=1}^{\ell} A_i$ except some vertices of S . Then we colour the vertices of $V_2 := \cup_{i=1}^{\ell} A_i$. We finish by colouring the uncoloured vertices of S greedily.

In order to extend the (partial) colouring of V_1 to V_2 , we need some properties. We will prove the following.

Lemma 8. *There is a good colouring c of a subset Y of V_1 such that*

- (i) *every uncoloured vertex of V_1 is in S ;*
- (ii) *for each edge xy of every M_i , $c(x) = c(y)$;*
- (iii) *for every uncoloured vertex v of V_1 there are at least 2Δ colours that appear on two G_1 -neighbours of v ; and*
- (iv) *for every colour j and clique A_i there are at most $\frac{4}{5}\Delta^2$ vertices of A_i that have either a G_1 -neighbour outside D_i coloured j or a G_2 -neighbour coloured using $j - 1, j$ or $j + 1$.*

We then establish that a colouring that verifies the conditions of Lemma 8 can be extended to $Y \cup V_2$.

Lemma 9. *Every good colouring of a subset Y of V_1 satisfying conditions (i)–(iv) of Lemma 8 can be completed to a good colouring of $Y \cup V_2$.*

By Lemma 8(iii), we can then complete the colouring by colouring the vertices of $V_1 - Y$ greedily.

Thus to prove our theorem, we need only prove Lemmas 8 and 9. We use several probabilistic tools, namely the Lovász Local Lemma, the Chernoff Bound, the Simple Concentration Bound, Talagrand’s Inequality and McDiarmid’s Inequality. Each of these tools is presented in the book of Molloy and Reed [21], and most are presented in many other places. Forthwith the details.

3 The Proof of Lemma 8

In this section, we want to find a good colouring for an appropriate subset Y of $G[V_1]$, which satisfies conditions (i)–(iv) of Lemma 8. We actually construct new graphs G_1^* and G_2^* and consider good colourings of these graphs. This will help us to ensure that the conditions of Lemma 8 hold.

3.1 Forming G_1^* and G_2^*

For $j \in \{1, 2\}$, we obtain G'_j from G_j by contracting each edge of each M_i into a vertex (that is, we consider these vertex pairs one by one, replacing the pair xy with a vertex adjacent to all of the neighbours of both x and y in the graph). We let C_i be the set of vertices obtained by contracting the pairs in M_i . We set $V^* := V_1 - \cup_{i=1}^{\ell} V(M_i) + \cup_{i=1}^{\ell} C_i$. For each $i \in \{1, 2, \dots, \ell\}$, let Big_i be the set of vertices of V^* not in $B_i \cup C_i$ that have more than $\Delta^{9/5}$ neighbours in A_i . We construct G_1^* from G'_1 by removing the vertices of $\cup_{i=1}^{\ell} A_i$ and adding for each i an edge between every pair of vertices in Big_i . And G_2^* is obtained from G'_2 by removing the vertices of $\cup_{i=1}^{\ell} A_i$.

Note that $G_2^* \subseteq G_1^*$. Our aim is to colour the vertices of V^* except some of S such that vertices adjacent in G_1^* are assigned different colours, and vertices adjacent in G_2^* are assigned colours at distance at least 2. Such a colouring is said to be *nice*. To every partial nice colouring of V^* is associated the good colouring of V_1 obtained as follows: each coloured vertex of $V \cap V^*$ keeps its colour, and for each index i , every pair of matched vertices of M_i is assigned the colour of the corresponding vertex of C_i . So this partial good colouring satisfies condition (ii) of Lemma 8.

Definition 10. For every vertex u and every subset F of V^* ,

- the number of G_1^* -neighbours of u in F is $\delta_F^1(u)$;
- the number of G_2^* -neighbours of u in F is $\delta_F^2(u)$; and
- we set $\delta_F^*(u) := \delta_F^1(u) + 2\delta_F^2(u)$.

For all these notations, we omit the subscript if $F = V^*$.

The next lemma bounds these parameters, and we omit its (short) proof here.

Lemma 11. *Let v be a vertex of V^* . The following hold.*

- (i) $\delta^2(v) \leq 2\Delta$, and if $v \notin \cup_{i=1}^{\ell} C_i$ then $\delta^2(v) \leq \Delta$;
- (ii) if $v \in S \cap \text{Big}_i$ for some i , then $\delta^1(v) \leq \Delta^2 - 6\Delta$;
- (iii) $\delta^1(v) \leq \Delta^2$, and if $v \notin S$ then $\delta^1(v) \leq \frac{3}{4}\Delta^2$.

Our construction of G'_1 and G'_2 is designed to deal with condition (ii) of Lemma 8. The edges we add between vertices of Big_i are designed to help with condition (iv). The bound of $\frac{3}{4}\Delta^2$ on the degree of the vertices of $V^* \setminus S$ in the last lemma, helps us to ensure that condition (i) holds.

To ensure that condition (iii) holds, we would like to use condition (i) and the fact that sparse vertices have many non-adjacent pairs of G_1 -neighbours. However, in constructing G_1^* , we contracted some pairs of non-adjacent vertices and added edges between some other pairs of non-adjacent vertices. As a result, possibly some vertices in S are no longer sparse. We have to treat such vertices carefully.

We define \hat{S} to be those vertices in S that have at least 90Δ neighbours outside S . Then \hat{S} contains all the vertices which may no longer be sufficiently sparse, as we note next.

Lemma 12. *Each vertex of $S \setminus \hat{S}$ has at least $450\Delta^3$ pairs of G_1 -neighbours in S that are not adjacent in G_1^* .*

It turns out that we will colour all of \hat{S} , which makes it easier to ensure that condition (iii) holds.

3.2 High Level Overview

Our first step is to colour some of S , including all of \hat{S} . We do this in two phases. In the first one, we consider assigning each vertex of S a colour at random. We show by analyzing this random procedure that there is a partial nice colouring of S such that every vertex of $S - \hat{S}$ satisfies condition (iii) of Lemma 8. In the second phase, we finish colouring the vertices of \hat{S} . We use an iterative quasi-random procedure. In each iteration but the last, each vertex chooses a colour, from those which do not yield a conflict with any already coloured neighbour, uniformly at random. The last iteration has a similar flavour.

We then turn to colouring the vertices in the sets B_i and C_i . Our degree bounds imply that we could do this greedily. However, we will mimic the iterative approach just discussed. We use this complicated colouring process because it allows us to ensure that condition (iv) of Lemma 8 holds for the colouring we obtain. At any point during the colouring process, $\text{Notbig}_{i,j}$ is the set of vertices $v \in A_i$ such that v has either a G'_1 -neighbour $u \notin \text{Big}_i \cup D_i$ that has colour j or a G'_2 -neighbour $u \notin \text{Big}_i$ that has colour $j - 1, j$ or $j + 1$. The challenge is to construct a colouring such that $\text{Notbig}_{i,j}$ remains small for every index i and every colour j .

3.3 Colouring Sparse Vertices

As mentioned earlier, we colour sparse vertices in two phases. The first one provides a partial nice colouring of S satisfying condition (iii) of Lemma 8. The second one extends this nice colouring to all the vertices of \hat{S} , using an iterative quasi-random procedure.

We will need a lemma to bound the size of $\text{Notbig}_{i,j}$. We consider the following setting. We consider a collection of at most Δ^2 subsets of vertices. Each set contains at most Q vertices, and no vertex lies in more than $\Delta^{9/5}$ sets. A random experiment is conducted, where each vertex is marked with probability at most $\frac{1}{Q \cdot \Delta^{2/5}}$. We moreover assume that, for any set of $s \geq 1$ vertices, the probability that all are marked is at most $\left(\frac{1}{Q \cdot \Delta^{2/5}}\right)^s$. Note that this is in particular the case if the vertices are marked independently.

Lemma 13. *Under the preceding hypothesis, the probability that at least $\Delta^{37/20}$ sets contains a marked vertex is at most $\exp(-\Delta^{1/20})$.*

Proof. For every $i \in \{1, 2, \dots, 9\}$, let E_i be the event that at least $\frac{1}{9}\Delta^{37/20}$ sets contain a marked member of T_i , where T_i is the set of vertices lying in between $\Delta^{(i-1)/5}$ and $\Delta^{i/5}$ sets. Note that if at least $\Delta^{37/20}$ sets contain at least one marked vertex, then at least one the events E_i must hold.

The total number of vertices in the sets being at most $\Delta^2 Q$, we deduce that $|T_i| \leq \frac{\Delta^2 Q}{\Delta^{(i-1)/5}}$. Furthermore, if E_i holds then at least $\frac{1}{9}\Delta^{37/20-i/5}$ vertices of T_i must be marked. Therefore,

$$\begin{aligned} \Pr(E_i) &\leq \binom{\Delta^2 Q / \Delta^{(i-1)/5}}{\frac{1}{9}\Delta^{37/20-i/5}} \cdot \left(\frac{1}{Q\Delta^{2/5}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}} \\ &\leq \left(\frac{e\Delta^2 Q / \Delta^{(i-1)/5}}{\frac{1}{9}\Delta^{37/20-i/5} \times Q\Delta^{2/5}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}} \quad (\text{by Stirling formula}) \\ &\leq \left(\frac{9e}{\Delta^{1/20}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}}. \end{aligned}$$

Since $\frac{1}{9}\Delta^{37/20-i/5} \geq \frac{1}{9}\Delta^{1/20}$, the probability that E_i holds is at most $\frac{1}{9} \exp(-\Delta^{1/20})$, and therefore the sought result follows. \square

3.3.1 First Step

Lemma 14. *There exists a nice colouring of a subset H of S with at most $\Delta^2 + 1$ colours such that*

- (i) *every uncoloured vertex v of $S \setminus \hat{S}$ has at least 2Δ colours appearing at least twice in $N_S(v) := N_{G_1}(v) \cap S$;*
- (ii) *every vertex of S has at most $\frac{19}{20}\Delta^2$ coloured neighbours;*
- (iii) *for every index i and every colour j , the size of $\text{Notbig}_{i,j}$ is at most $\Delta^{19/10}$.*

Proof. For convenience, let us set $c := \Delta^2 + 1$. We use the following colouring procedure.

1. Each vertex of S is activated with probability $\frac{9}{10}$.
2. Each activated vertex is assigned a colour of $\{1, 2, \dots, c\}$, independently and uniformly at random.
3. A vertex which gets a colour creating a conflict—i.e. previously assigned to one of its G_1^* -neighbours, or at distance less than 2 of a colour previously assigned to one of its G_2^* -neighbours—is uncoloured.

We aim at applying the Lovász Local Lemma to prove that, with positive probability, the resulting colouring fulfils the three conditions of the lemma. Let v be a vertex of G . We let $E_1(v)$ be the event that v does not fulfil condition (i), and $E_2(v)$ be the event that v does not fulfil condition (ii). For each i, j , let $E_3(i, j)$ be the event that the size of $\text{Notbig}_{i,j}$ exceeds $\Delta^{19/10}$. It suffices to prove that each of those events occurs with probability less than Δ^{-17} . Indeed, each event is mutually independent of all events involving vertices or dense sets at distance more than 4 in G_1^* or G_1' . Moreover, each vertex of any set A_i has at most $\Delta^{5/4}$ external neighbours in G , and $|A_i| \leq \Delta^2 + 1$. Thus, each event is mutually independent of all but at most Δ^{16} other events. Consequently, the Lovász Local Lemma applies since $\Delta^{-17} \times \Delta^{16} <^* \frac{1}{4}$, and yields the sought result.

Hence, it only remains to prove that the probability of each event is at most Δ^{-17} . Let us start with $E_2(v)$. We define W to be the number of activated neighbours of v . Thus, $\Pr(E_2(v)) \leq$

$\Pr(W > \frac{19}{20}\Delta^2)$. We set $m := |N(v)|$, and we may assume that $m \geq \frac{19}{20}\Delta^2$. The random variable W is just a binomial on m variables with probability $\frac{9}{10}$. In particular, its expected value $\mathbf{E}(W)$ is $\frac{9m}{10}$. Applying Chernoff's bound to W with $t = \frac{m}{20}$, we obtain

$$\begin{aligned} \Pr(W > \frac{19}{20}\Delta^2) &\leq \Pr(|W - \mathbf{E}(W)| > \frac{m}{20}) \\ &\leq 2 \exp\left(-\frac{m^2 \cdot 10}{400 \cdot 27m}\right) \leq^* \Delta^{-17}, \end{aligned}$$

since $\frac{19}{20}\Delta^2 \leq m \leq \Delta^2$.

Let $v \in S \setminus \hat{S}$. We now bound $\Pr(E_1(v))$. We consider the random variable X defined as the number of colours assigned to at least two vertices of $N(v)$, and retained by every vertex of $N(v)$ to which they are assigned. Thus, X is at most the number of colours appearing at least twice in $N(v)$. By Lemma 12, let Ω be a collection of $450\Delta^3$ pairs of G_1 -neighbours of v in S that are not adjacent in G_1^* . The variable X is itself lower bounded by the random variable Y , defined as the number of pairs in Ω that (i) are both assigned the same colour x , (ii) both retain that colour, and (iii) are the only two vertices in $N(v)$ that are assigned x . The probability that some non-adjacent pair of vertices u, w in $N(v)$ satisfies (i) is $\frac{9}{10} \cdot \frac{9}{10} \cdot \frac{1}{c}$. In total, the number of G_1^* -neighbours of v, u, w in H is at most $3\Delta^2$, and the number of G_2^* -neighbours of u and w is at most 4Δ . Therefore, given that they satisfy (i), the vertices u and w also satisfy (ii) and (iii) with probability at least $(1 - \frac{1}{c})^{3\Delta^2} \cdot (1 - \frac{1}{2c})^{4\Delta}$. Consequently,

$$\mathbf{E}(X) \geq \mathbf{E}(Y) \geq 450\Delta^3 \cdot \frac{81}{100c} \cdot \exp\left(-\frac{3\Delta^2}{c}\right) \exp\left(-\frac{2\Delta}{c}\right) > 3\Delta.$$

Hence, if $E_1(v)$ holds then X must differ from its expected value by at least Δ . But we assert that

$$\Pr(|X - \mathbf{E}(X)| > \Delta) \leq \Delta^{-17}, \tag{1}$$

which will yield the desired result.

To establish Equation (1), we apply Talagrand's Inequality, stated in Section 2. We set X_1 to be the number of colours assigned to at least two vertices in $N(v)$, including both members of at least one pair in Ω , and X_2 is the number of colours that (i) are assigned to both members of at least one pair in Ω , and (ii) create a conflict with one of their neighbours, or are also assigned to at least one other vertex in $N(v)$. Note that $X = X_1 - X_2$. Therefore, by what precedes, if $E_1(v)$ holds then either X_1 or X_2 must differ from its expected value by at least $\frac{1}{2}\Delta$. Notice that

$$\mathbf{E}(X_2) \leq \mathbf{E}(X_1) \leq c \cdot 450\Delta^3 \cdot \frac{1}{c^2} \leq 450\Delta.$$

If $X_1 \geq t$, then there is a set of at most $4t$ trials whose outcomes certify this, namely the activation and colour assignment for t pairs of variables. Moreover, changing the outcome of any random trial can only affect X_1 by at most 2, since it can only affect whether the old colour and the new colour are counted or not. Thus Talagrand's Inequality applies and we obtain

$$\Pr\left(|X_1 - \mathbf{E}(X_1)| > \frac{1}{2}\Delta\right) \leq 4 \exp\left(-\frac{\Delta^2}{4 \cdot 32 \cdot 4 \cdot 4 \cdot 450\Delta}\right) \leq^* \frac{1}{2}\Delta^{-17}.$$

Analogously, we obtain by Talagrand's Inequality that

$$\Pr\left(|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}\Delta\right) \leq 4 \exp\left(-\frac{\Delta^2}{4 \cdot 32 \cdot 6 \cdot 4 \cdot 450\Delta}\right) \leq^* \frac{1}{2}\Delta^{-17}.$$

Consequently, we infer that $\Pr(|X - \mathbf{E}(X)| > \Delta) \leq^* \Delta^{-17}$, as desired.

It only remains now to deal with $E_3(i, j)$. We use Lemma 13. For each i , every vertex of A_i has at most $\Delta^{5/4}$ external neighbours. Moreover, for each colour j , each such neighbour is activated and assigned a colour in $\{j - 1, j, j + 1\}$ with probability at most $\frac{9}{10} \cdot \frac{3}{c} <^* \frac{1}{\Delta^{5/4} \cdot \Delta^{2/5}}$. As these assignments are made independently, the conditions of Lemma 13 are fulfilled, so we deduce that the probability that $E_3(i, j)$ holds is at most $\exp(-\Delta^{1/20}) \leq^* \Delta^{-17}$. Thus, we obtained the desired upper bound on $\Pr(E_3(i, j))$, which concludes the proof. \square

3.3.2 Second Step

In the second step, we extend the partial colouring of S to all the vertices of \hat{S} , and also to colour the vertices of the sets $B_i \cup C_i$. To do so, we need the following general lemma. Its proof is long and technical, and we omit it here.

Lemma 15. *Let F be a subset of V^* with a partial nice colouring, and H be a set of uncoloured vertices of F . For each vertex u of H , let $L(u)$ be the colours available to colour u , that is that create no conflict with the already coloured vertices of $F \cup H$. We assume that for every vertex u , $|L(u)| \geq 16\Delta^{33/20}$ and $|L(u)| \geq \delta_H^*(u) + X$, where $X \geq 2\Delta$.*

Then, the partial nice colouring of F can be extended to a nice colouring of H such that for every index $i \in \{1, 2, \dots, \ell\}$ and every colour j , the size of $\text{Notbig}_{i,j}$ increases by at most $\Delta^{19/10}$.

Consider a partial nice colouring of S obtained in the first step. In particular, $|\text{Notbig}_{i,j}| \leq \Delta^{19/10}$. We wish to ensure that every vertex of \hat{S} is coloured. This can be done greedily, but to be able to continue the proof we need to have more control on the colouring. We shall apply Lemma 15 to the set H of uncoloured vertices in \hat{S} . For each vertex u , $L(u)$ is initialized as the list of colours that can be assigned to u without creating any conflict. By Lemmas 11 and 14(ii), $|L(u)| \geq \frac{1}{20}\Delta^2 - 4\Delta \geq^* 16\Delta^{33/20}$.

Suppose that u is in no set Big_i . Then $\delta_S^1(u) \leq \deg_S^1(u) \leq \Delta^2 - 90\Delta$, and u has at most Δ G_2^* -neighbours. Hence, we infer that $|L(u)| \geq \deg_H^1(u) + 88\Delta \geq \delta_H^*(u) + 2\Delta$. Assume now that u belongs to some set Big_i . By Lemma 11(ii) and (iii), we have $\delta^1(u) \leq \Delta^2 - 6\Delta$ and $\delta^2(u) \leq \Delta$. So, $|L(u)| \geq \delta_H^1(u) + 6\Delta - 2\Delta \geq \delta_H^*(u) + 2\Delta$.

Therefore, we obtain a nice colouring of $F \cup H$ such that for every index i and every colour j , $|\text{Notbig}_{i,j}| \leq 2\Delta^{19/10}$.

3.4 Colouring the Sets B_i and C_i

Let $H := \bigcup_{i=1}^{\ell} (B_i \cup C_i)$. At this stage, the vertices of H are uncoloured. We first apply Lemma 15 to extend the partial nice colouring of S to the vertices of H in such a way that $\text{Notbig}_{i,j}$ does not grow too much, for every index i and colour j . Next, we will show that the good colouring derived from this nice colouring satisfies the conditions of Lemma 8.

For each vertex u of H , let $L(u)$ be the lists of colours that would not create any conflict with the already coloured vertices. By Lemma 11(i), $\delta^1(u) \leq \frac{3}{4}\Delta^2$. Hence, $|L(u)| \geq \frac{1}{4}\Delta^2 + \delta_H^1(u) - 4\Delta \geq \max(16\Delta^{33/20}, \delta_H^*(u) + 2\Delta)$.

Therefore, by Lemma 15, we complete the partial nice colouring of the vertices of S to the vertices of $\bigcup_{i=1}^{\ell} (B_i \cup C_i)$. Moreover, for each index i and each colour j , the size of each $\text{Notbig}_{i,j}$ is at most $3\Delta^{19/10}$.

Consider now the partial good colouring of V_1 associated to this nice colouring. Let us show that it satisfies the conditions of Lemma 8 (i). By the definition, it satisfies condition (ii) and (i). Condition (iii) follows from Lemma 14. Hence, it only remains to show that condition (iv) holds.

Fix an index i and a colour j . Recall that Big_i is a clique, so there is at most one vertex of Big_i of each colour. Consequently, the number of vertices of A_i with a G'_1 -neighbour in Big_i coloured j is at most $\frac{3}{4}\Delta^2$, by Lemma 5(c). Besides, the number of vertices of A_i with a G'_2 -neighbour in Big_i coloured $j-1$ or $j+1$ is at most 2Δ . Finally, the number of vertices of A_i with either a G'_1 -neighbour not in $\text{Big}_i \cup D_i$ coloured j , or a G'_2 -neighbour not in Big_i coloured $j-1, j$ or $j+1$ is at most $|\text{Notbig}_{i,j}| \leq 3\Delta^{19/10}$. Thus, all together, the number of vertices of A_i with a G_1 -neighbour coloured j , or a G_2 -neighbour coloured $j-1$ or $j+1$ is at most

$$\frac{3}{4}\Delta^2 + 3\Delta^{19/10} + 2\Delta \leq^* \frac{4}{5}\Delta^2,$$

as desired. This concludes the proof of Lemma 8.

4 The Proof of Lemma 9

We want to prove Lemma 9. We consider a good colouring of V satisfying the conditions of Lemma 8. The procedure we apply is comprised of two phases. In the first phase, a random permutation of colours is assigned to the vertices of A_i . In doing so, we might create two kinds of conflicts: a vertex of A_i coloured j might have an external G_1 -neighbour coloured j , or a G_2 -neighbour coloured $j-1$ or $j+1$. We shall deal with these conflicts in a second phase. To be able to do so, we first ensure that the colouring obtained in the first phase fulfils some properties. We first make the following observation.

Proposition 16.

$$|A_i| + |B_i| + \frac{1}{2}|V(M_i)| \leq \Delta^2 + 1$$

Phase 1. For each set A_i , we choose a subset of $a_i := |A_i|$ colours that do not appear on the vertices of $B_i \cup C_i$. Then we assign a random permutation of those colours to the vertices of A_i . This is possible by Proposition 16 and because every edge of M_i is monochromatic by Lemma 8(ii). We let Temp_i be the subset of vertices of A_i with an external G_1 -neighbour of the same colour, or a G_2 -neighbour with a colour at distance less than 2. The following lemma can be proved using the Lovász Local Lemma, McDiarmid's Inequality and Lemma 13.

Lemma 17. *With positive probability, the following hold.*

- (i) For each i , $|\text{Temp}_i| \leq 3\Delta^{5/4}$;
- (ii) for each index i and each colour j , at most $\Delta^{19/10}$ vertices of A_i have a G_1 -neighbour in $\cup_{k \neq i} A_k$ coloured j or a G_2 -neighbour in $\cup_k A_k$ coloured $j-1$ or $j+1$.

Phase 2. We consider a colouring γ satisfying the conditions of Lemma 17. For each set A_i and each vertex $v \in \text{Temp}_i$ we let Swappable_v be the set of vertices u such that

- (a) $u \in A_i \setminus \text{Temp}_i$;
- (b) $\gamma(u)$ does not appear on an external G_1 -neighbour of v ;
- (c) $\gamma(v)$ does not appear on an external G_1 -neighbour of u ;
- (d) $\gamma(u) - 1$ and $\gamma(u) + 1$ do not appear on a G_2 -neighbour of v ;

(e) $\gamma(v) - 1$ and $\gamma(v) + 1$ do not appear on a G_2 -neighbour of u .

Lemma 18. *For every $v \in \text{Temp}_i$, the set Swappable_v contains at least $\frac{1}{10}\Delta^2$ vertices.*

For each index i and each vertex $v \in \text{Temp}_i$, we choose 100 uniformly random members of Swappable_v . These vertices are called *candidates* of v .

Definition 19. A candidate u of v is *unkind* if either

- (a) u is a candidate for some other vertex;
- (b) v has an external neighbour w that has a candidate w' with the same colour as u ;
- (c) v has a G_2 -neighbour w that has a candidate w' coloured $\gamma(u) - 1, \gamma(u)$ or $\gamma(u) + 1$;
- (d) v has an external neighbour w that is a candidate for exactly one vertex w' with the same colour as u ;
- (e) v has a G_2 -neighbour w that is a candidate for exactly one vertex w' coloured $\gamma(u) - 1, \gamma(u)$ or $\gamma(u) + 1$;
- (f) u has an external neighbour w that has a candidate w' with the same colour as v ;
- (g) u has a G_2 -neighbour w that has a candidate w' coloured $\gamma(v) - 1, \gamma(v)$ or $\gamma(v) + 1$;
- (h) u has an external neighbour w that is a candidate for exactly one vertex w' with the same colour as v ; or
- (i) u has a G_2 -neighbour w that is a candidate for exactly one vertex w' coloured $\gamma(v) - 1, \gamma(v)$ or $\gamma(v) + 1$.

A candidate of v is *kind* if it is not unkind.

Lemma 20. *With positive probability, for each index i , every vertex of Temp_i has a kind candidate.*

Proof. For every vertex v in some Temp_i , let $E_1(v)$ be the event that v does not have a kind candidate. Each event is mutually independent of all events involving dense sets at distance greater than 2. So each event is mutually independent of all but at most Δ^9 other events. Hence, we shall prove that the probability of each event is at most Δ^{-10} , and so the conclusion will follow from the Lovász Local Lemma since $\Delta^{-10} \cdot \Delta^9 < \frac{1}{4}$.

Observe that the probability that a particular vertex of Swappable_v is chosen is $100/|\text{Swappable}_v|$, which is at most $1000\Delta^{-2}$.

We wish to upper bound $\Pr(E_1(v))$ for an arbitrary vertex $v \in \text{Temp}_i$, so we can assume that all vertices but v have already chosen candidates. By Lemma 17(i), the number of vertices that satisfy condition (a) of Definition 19 is at most $300\Delta^{5/4}$. Note that the vertex v has at most $\Delta^{5/4}$ external neighbours, each having at most 100 candidates. Since each colour appears on at most one member of Swappable_v , we deduce that the number of vertices satisfying one of the conditions (b) or (d) is at most $101\Delta^{5/4}$. Similarly, the number of vertices satisfying one of the conditions (c) or (e) is at most 303Δ . We define Y to be the number of vertices of Swappable_v that meet conditions (f), (g), (h) or (i) of Definition 19. Each vertex u of Swappable_v has at most $\Delta^{5/4}$ external neighbours, and at most Δ G_2 -neighbours. Each of those neighbours has at most one potential candidate with each colour of $\{\gamma(v) - 1, \gamma(v), \gamma(v) + 1\}$. Also, each neighbour can be chosen as a candidate for at most one vertex coloured j , for each $j \in \{\gamma(u) - 1, \gamma(u), \gamma(u) + 1\}$. So, by Lemma 18,

the probability that u meets one of those conditions is at most $3\Delta^{5/4} \times 1000\Delta^{-2} = 3000\Delta^{-3/4}$. Therefore, $\mathbf{E}(Y) \leq |\text{Swappable}_v| \times 3000\Delta^{-3/4} \leq 3000\Delta^{5/4}$.

We now prove that Y is concentrated thanks to Talagrand's Inequality. If $Y \geq s$ then there is a set of $2s$ candidates which certifies this fact—take one neighbour of each member of Swappable_v counted by Y . Recall that each vertex w of some A_k for $k \neq i$ has at most $\Delta^{5/4}$ neighbours in A_i . Thus changing the choice of candidates for w can affect only which candidates of w meet condition (f) or (g), or which vertices that are neighbours of the old 100 candidates, or the new 100 candidates, meet condition (h) or (i). So it can affect Y by at most $202\Delta^{5/4}$. Hence Talagrand's Inequality implies that

$$\begin{aligned} \Pr\left(Y > 2\Delta^{17/9}\right) &\leq 4 \exp\left(-\frac{\Delta^{34/9}}{64 \times (202\Delta^{5/4})^2 \times 3000\Delta^{5/4}}\right) \\ &\leq^* \frac{1}{2} \Delta^{-10}. \end{aligned}$$

Therefore, with probability at least $1 - \frac{1}{2}\Delta^{-10}$ the number of unkind members of Swappable_v is at most

$$2\Delta^{17/9} + 300\Delta^{5/4} + 101\Delta^{5/4} + 303\Delta <^* 3\Delta^{17/9}.$$

In this case, the probability that no candidate is kind is at most

$$\left(\frac{3\Delta^{17/9}}{\Delta^2/10}\right)^{100} < \frac{1}{2} \Delta^{-10}.$$

Consequently, the probability that $E_1(v)$ holds is at most $\frac{1}{2}\Delta^{-10} + \frac{1}{2}\Delta^{-10} = \Delta^{-10}$, as desired. \square

To conclude, choose candidates satisfying the preceding lemma. For each vertex $v \in \text{Temp}_i$ swap the colour of v and one of its kind candidates. The obtained colouring is the desired one. This concludes the proof of Lemma 9.

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