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Abstract: We consider the problem of numerically approximating the solution of an elliptic partial differential equation with random coefficients and homogeneous Dirichlet boundary conditions. We focus on the case of a lognormal coefficient, we have then to deal with the lack of uniform coercivity and uniform boundedness with respect to the randomness. This model is frequently used in hydrogeology. We approximate this coefficient by a finite dimensional noise using a truncated Karhunen-Loève expansion. We give then estimates of the corresponding error on the solution, both a strong error estimate and a weak error estimate, that is to say an estimate of the error commited on the law of the solution. We obtain a weak rate of convergence wich is twice the strong one. Besides this, we give a complete error estimate for the stochastic collocation method in this case, where neither coercivity nor boundedness are stochastically uniform. To conclude, we apply these results of s trong and weak convergence to two classical cases of covariance kernel choices: the case of an exponential covariance kernel on a box and the case of an analytic covariance kernel, yielding explicit weak and strong convergence rates.

Key-words: uncertainty quantification, elliptic PDE with random coefficients, Karhunen-Loève expansion, strong error estimate, weak error estimate, lognormal distribution

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Estimations d'erreurs forte et faible pour des équations aux dérivées partielles elliptiques à coefficients aléatoires

Résumé : On s'intéresse à l'approximation numérique de la solution d'une équation aux dérivées partielles elliptique à coefficients aléatoires, avec des conditions de Dirichlet homogènes au bord. On se concentre sur le cas d'un coefficient lognormal, on est ainsi confronté au fait que ce coefficient n'est ni uniformément borné, ni uniformément coercif par rapport à l'aléatoire. Ce modèle est fréquemment utilisé en hydrogéologie. On approche ce coefficient dans un espace aléatoire de dimension finie, en utilisant un développement de Karhunen-Loève. On donne alors des estimations pour l'erreur qui en découle sur la solution, une estimation d'erreur forte mais également une estimation d'erreur faible, c'est à dire une estimation de l'erreur commise sur la loi de la solution. On obtient alors un taux de convergence faible double du taux de convergence forte. De plus, on donne une estimation d'erreur complète pour la méthode de collocation appliquée dans ce cas où le coefficient n'est ni uniformément borné, ni uniformément coercif par rapport à l'aléatoire. Pour conclure, on applique ces résultats à deux choix particuliers de noyaux de covariance: le cas d'une covariance exponentielle sur un pavé et le cas d'une covariance analytique, donnant des taux de convergence forte et faible explicites dans chaque cas.

Mots-clés : quantification des incertitudes, EDP elliptique à coefficients aléatoires, développement de Karhunen-Loève, estimation d'erreur forte, estimation d'erreur faible, distribution lognormale

1 Introduction

Many engineering applications involve uncertainty on the input data, such as material properties. This uncertainty results from heterogeneity of the medium and incomplete knowledge on the medium properties and can be modelled by partial differential equations with random coefficients. This work addresses elliptic partial differential equations with random coefficients and focuses on application to hydrogeology, namely the prediction of flow in porous media, but there are many other applications, e.g., random vibrations, composite materials, seismic activity and deformations of inhomogeneous materials such as wood or biomaterials. The aim is to compute the law of the solution, but in practice we are usually interested only in some moments.
Several methods have been developed: Monte-Carlo and Monte-Carlo based methods, moment equations, perturbation methods which are adapted to the case of small uncertainty, homogenization, multiscale analysis and stochastic spectral methods [1, 2, 3, 4, 7, 9, 10, 11, 12, 15, 16, 19, 20, 21, 22, 25, 26, 27, 28], which regroup stochastic galerkin methods and stochastic collocation methods.
As in many previous works, we consider the model equation

$$-\text{div}(a(\omega,x)\nabla_x u(\omega,x)) = f(\omega,x).$$

Both stochastic Galerkin methods and stochastic collocation methods are based on the approximation of a in a finite dimensional probability space, i.e. using a finite number of random variables. These methods are therefore adapted to the case where the probability space has a low dimensionality, i.e. in the case where we have a good approximation a_N of a such that a_N is a function of N random variables with N small. Such approximations a_N of a can be obtained by using either a Karhunen-Loève or a polynomial chaos expansion. We compute then the solution u_N of the approximated equation resulting from replacing a by a_N. In this paper we focus on the convergence of u_N to u.
More precisely, we work here with homogeneous Dirichlet boundary conditions and a homogeneous lognormal random field a. This is a frequently used model for flow equation in porous media. The truncated Karhunen-Loève expansion of $\log(a)$ at order N provides then an approximation a_N of a. It is important to notice that in such a case, unlike what is frequently assumed, neither the random filed a nor its approximations a_N are uniformly coercive with respect to ω. However such non-uniformly bounded and coercive field have been considered in [8, 13] and in the recent work [6].
Up to our knowledge, the convergence of u_N to u has never been studied under realistic assumptions on a, in particular for a lognormal field. This is the main goal of this article. We first give a strong convergence result of u_N to u, i.e. a bound for the error in $L^p(H_0^1)$-norm. Then a weak convergence result is obtained, i.e. we bound the error commited on the law of u. We find a bound for the weak error whose order is twice the strong order, which presents a significant interest since the number N of random variables has to be small in order to be able to compute u_N and since the law of u is what we are interested in. For simplicity we assume that f is deterministic, but all the results can be easily extended to the case when f is random, under adequate assumptions.
To begin with, we prove the existence and uniqueness of the solution u, remarking once again that in the considered case, the random field a is not uniformly coercive with respect to ω. Then we make assumptions on the eigenvalues and on the eigenfunctions of the Karhunen-Loève expansion, which enable us to prove two preliminary results: the strong convergence of a_N to a and the existence of a uniform bound in L^p-norm for a_N^{max} and $\frac{1}{a_N^{min}}$.
We can then give a strong convergence result of u_N to u, with almost sure convergence and

L^p convergence. The strong error is basically bounded by the squared root of the remainder of the series of the eigenvalues. We prove next a weak convergence result, showing that the error commited on the law of u is bounded by the remainder of the series of the eigenvalues. Besides this work gives a complete convergence analysis of the collocation method, adapting the results of I. Babuska, F. Nobile and R. Tempone in [1], in which uniform coercivity of a with respect to ω is assumed. Finally we give examples of covariance kernel for which these results apply. In particular the exponential case on a box and the gaussian case are studied, these covariance kernels being frequently used to model the hydraulic conductivity for the flow equation in porous media.

2 Equation, existence and uniqueness of the solution

In this section, we first define the homogeneous lognormal random field a and make some regularity assumptions on its covariance kernel, then we define a linear elliptic partial differential equation with random coefficients, namely a, and finally show the existence and uniqueness of the solution of this equation. Let D be an open bounded domain in $\mathbb{R}^d$ with $\mathcal{C}^2$ boundary and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a function $k \in \mathcal{C}^{0,1}(\mathbb{R}^+, \mathbb{R})$, and $g : \Omega \times \bar{D} \to \mathbb{R}$ a mean-free gaussian field with covariance kernel $cov[g](x, y) = k(\|x - y\|)$.

Proposition 2.1. *Under these assumptions, g admits a version whose trajectories belong to $\mathcal{C}^{0,\alpha}(\bar{D})$ a.s. for $\alpha < 1/2$.*

Proof. Let us denote by L the Lipschitz constant of k.

$$\begin{aligned} \mathbb{E}[|g(x) - g(y)|^2] &= \mathbb{E}[g(x)^2] - 2\mathbb{E}[g(x)g(y)] + \mathbb{E}[g(y)^2] \\ &= 2(k(0) - k(\|x - y\|)) \\ &\leq 2L\|x - y\|. \end{aligned}$$

We recall the existence, for any positive integer p, of a constant $c_p = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{2p} e^{-\frac{x^2}{2}} dx$ such that for all mean-free gaussian random variable X,

$$E[|X|^{2p}] \leq c_p E[|X|^2]^p.$$

Therefore, since $g(x) - g(y)$ is a mean-free gaussian random variable, we have, for any positive integer p,

$$E[|g(x) - g(y)|^{2p}] \leq c_p (2L)^p \|x - y\|^p.$$

According to the Kolmogorov continuity theorem [5], there exists a version of g which is a.s. Hölder-continuous with any exponent $\beta < \frac{p-d}{2p}$. Since this hold for any positive integer p, letting $p \to +\infty$, it follows that there exists a version of g which is a.s. Hölder continuous with any exponent $\beta < 1/2$. □

We define the lognormal homogeneous random field $a : \Omega \times \bar{D} \to \mathbb{R}$ as $a(\omega, x) = e^{g(\omega,x)}$. By Proposition 2.1, the trajectories of a are a.s. continuous on the compact set $\bar{D}$, we can then define a.s, $a^{min}(\omega) = \min_{x \in \bar{D}} a(\omega, x)$ and $a^{max}(\omega) = \max_{x \in \bar{D}} a(\omega, x)$. These random variables have then the following integrability properties:

Proposition 2.2. $\frac{1}{a^{min}(\omega)} \in L^p(\Omega)$ *and* $a^{max}(\omega) \in L^p(\Omega)$ *for any* $p > 0$.

Proof. We have $a^{min}(\omega) \geq e^{-||g(\omega)||_{\mathcal{C}^0(\bar{D})}}$ and $a^{max}(\omega) \leq e^{||g(\omega)||_{\mathcal{C}^0(\bar{D})}}$. By Fernique's theorem [5], since g defines a mean-free gaussian measure on the Banach space $\mathcal{C}^0(\bar{D})$, there exists $\lambda > 0$ such that $\mathbb{E}[e^{\lambda||g(\omega)||^2_{\mathcal{C}^0(\bar{D})}}] < +\infty$. Thus, for any $p > 0$,

$$\mathbb{E}[e^{p||g(\omega)||_{\mathcal{C}^0(\bar{D})}}] \leq \mathbb{E}[e^{\lambda||g(\omega)||^2_{\mathcal{C}^0(\bar{D})}+\frac{p^2}{4\lambda}}] < +\infty.$$

For more details, see the proof of Proposition 3.4. Therefore $e^{||g(\omega)||_{\mathcal{C}^0(\bar{D})}} \in L^p(\Omega)$, and finally $\frac{1}{a^{min}(\omega)} \in L^p(\Omega)$ and $a^{max}(\omega) \in L^p(\Omega)$, for any $p > 0$. □

Proposition 2.3. *Let f in $L^2(D)$, then the equation:*

$$\begin{aligned} -\operatorname{div}(a(\omega,x)\nabla u(\omega,x)) &= f(x) \text{ on } D, \\ u &= 0 \text{ on } \partial D, \end{aligned} \tag{1}$$

admits a unique solution u, which belongs to $L^p(\Omega, H^1_0(D))$, for any $p > 0$.

Remark: All the following results hold in the case where the forcing term f is stochastic, under adequate assumptions.

Proof. For almost all $\omega \in \Omega$, the equation admits a unique solution $u(\omega) \in H^1_0(D)$, the mapping $\omega \mapsto u(\omega)$ is measurable and we have, a.s. :

$$\|u(\omega,x)\|_{H^1_0(D)} \leq C_D \frac{\|f\|_{L^2(D)}}{a^{min}(\omega)} \tag{2}$$

where C_D is the constant given by Poincaré inequality. For every $p > 0$, by Proposition 2.2, $\frac{1}{a^{min}} \in L^p(\Omega)$, so for any $p > 0$,

$$\mathbb{E}\left[\|u(\omega,x)\|^p_{H^1_0(D)}\right] \leq C^p_D \mathbb{E}\left[\left(\frac{1}{a^{min}(\omega)}\right)^p\right] \|f\|^p_{L^2(D)} < +\infty.$$

□

3 Strong convergence of a_N to a

In this section, we define the approximated random field a_N and study the strong convergence of a_N to a, i.e. the $L^p(\Omega, \mathcal{C}^0(\bar{D}))$-convergence, and the almost-sure convergence. Let $\{(\lambda_n, b_n)\}$ denote the sequence of eigenpairs associated with the compact self-adjoint operator that maps

$$f \in L^2(D) \mapsto \int_D \operatorname{cov}[g](x,.)f(x)dx \ \in L^2(D),$$

where $\lambda_1 \geq \lambda_2 \geq ... \geq 0$, and the eigenfunctions are orthonormal. We recall that $\mathbb{E}[g(\omega, x)] = 0$ and that $\operatorname{cov}[g](x,y) = k(\|x-y\|)$. Therefore $\sum_{n\geq 1} \lambda_n = |D|k(0)$. Then the truncated Karhunen-Loève expansion [17, 18] g_N of the stochastic gaussian process g and its exponential a_N are defined by

$$g_N(\omega,x) = \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega), \quad a_N(\omega,x) = e^{\sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)}.$$

where the real random variables $(Y_n)_{n\geq 1}$ are uniquely determined by

$$Y_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_D g(\omega, x) b_n(x) dx.$$

They are independent gaussian random variables with mean zero and unit variance. Mercer theorem [23] gives the following convergence result:

$$\sup_{x\in D} \mathbb{E}[(g - g_N)^2](x) \to 0 \text{ as } N \to +\infty.$$

This convergence result does not actually enable us to conclude on the convergence of the solution.
From now on, we make the following assumptions:

- The eigenfunctions b_n are continuously differentiable.
- There exists constants C and $a > 0$ such that, for any $n \in \mathbb{N}$,

$$\|b_n\|_\infty \leq C \text{ and } \|\nabla b_n\|_\infty \leq C n^a.$$

- There exists $b > 0$ such that $\sum_{n\geq 1} \lambda_n n^b$ is convergent.

Remark: Such assumptions are fulfilled in the case of an exponential covariance kernel on a rectangular domain. Although these assumptions are not fulfilled in the case of a gaussian covariance kernel, similar results as the following ones hold in this case, and more generally in the case of an analytic covariance kernel, for more details see section 6.

Definition 1. *For $\alpha \leq b$, we define*

$$R_N^\alpha = \sum_{n>N} \lambda_n n^\alpha.$$

We notice that $R_N^\alpha \leq N^{\alpha-b}(\sum_{n>N} \lambda_n n^b)$, and therefore, $R_N^\alpha = o(N^{\alpha-b})$ when $N \to +\infty$.

Proposition 3.1. *For any p,α, β such that $0 < \alpha \leq \min\{b, 2a\}$ and $\beta < \frac{\alpha}{2a}$, there exists a constant $A_{\alpha,\beta,p}$ such that for all N in $\mathbb{N}$:*

$$\|g_N - g\|_{L^p(\Omega, \mathcal{C}^{0,\beta}(\bar{D}))} \leq A_{\alpha,\beta,p} (R_N^\alpha)^{\frac{1}{2}}.$$

In particular, for all p, for any $0 < \alpha \leq \min\{b, 2a\}$, there exists a constant $A_{\alpha,p}$ such that for all N in $\mathbb{N}$,

$$\|g_N - g\|_{L^p(\Omega, \mathcal{C}^{0}(\bar{D}))} \leq A_{\alpha,p} (R_N^\alpha)^{\frac{1}{2}}.$$

Proof. For all $x, y \in D$, and $\theta \in]0,1[$ such that $\theta \leq \frac{b}{2a}$,

$$\begin{aligned} (b_n(x) - b_n(y))^2 &\leq (2\|b_n\|_\infty)^{2(1-\theta)} (\|b_n'\|_\infty \|x-y\|)^{2\theta} \\ &\leq C_\theta n^{2a\theta} \|x-y\|^{2\theta}, \end{aligned}$$

where $C_\theta = 2^{2(1-\theta)}C^2$.

$$\begin{aligned}\mathbb{E}\left[((g_N-g)(x)-(g_N-g)(y))^2\right] &= \sum_{n>N}\lambda_n(b_n(x)-b_n(y))^2\\ &\le C_\theta\left(\sum_{n>N}\lambda_n n^{2a\theta}\right)\|x-y\|^{2\theta}\\ &\le C_\theta R_N^{2a\theta}\|x-y\|^{2\theta}.\end{aligned}$$

For all $x, y \in D$ and N in $\mathbb{N}$, $(g_N-g)(x)-(g_N-g)(y)$ is a mean-free gaussian random variable, as a limit in L^2 of a linear combination of independent gaussian random variables, $(g_N-g)(x)-(g_N-g)(y) = \lim_{p\to+\infty}\sum_{n=N+1}^{p}\sqrt{\lambda_n}Y_n(\omega)(b_n(x)-b_n(y))$.
Arguing as in the proof of Proposition 2.1 yields

$$\begin{aligned}\mathbb{E}\left[((g_N-g)(x)-(g_N-g)(y))^{2p}\right] &\le c_p\mathbb{E}\left[((g_N-g)(x)-(g_N-g)(y))^2\right]^p\\ &\le C_\theta^p c_p(R_N^{2a\theta})^p\|x-y\|^{2\theta p}.\end{aligned}$$

Thus, for any $0<\alpha\le\min\{b,2a\}$, for any positive integer p, there exists a constant $C_{\alpha,p}$ such that:

$$\mathbb{E}\left[((g_N-g)(x)-(g_N-g)(y))^{2p}\right] \le C_{\alpha,p}(R_N^\alpha)^p\|x-y\|^{\frac{\alpha p}{a}}.$$

We now are going to use the Kolmogorov continuity theorem [5]. For any $0<\alpha\le\min\{b,2a\}$, let ν be such that $\alpha\frac{p}{a}-d-2p\nu>-d$ i.e. $\nu<\frac{\alpha}{2a}$, then

$$\begin{aligned}\mathbb{E}[\|g_N-g\|_{W^{\nu,2p}}^{2p}] &= \int_\Omega\int_D\int_D\frac{|(g_N-g)(x)-(g_N-g)(y)|^{2p}}{\|x-y\|^{d+2p\nu}}\\ &\le C_{\alpha,p}(R_N^\alpha)^p\int_D\int_D\|x-y\|^{\alpha\frac{p}{a}-d-2p\nu}\\ &\le C_{\alpha,p,\nu}(R_N^\alpha)^p.\end{aligned}$$

with $C_{\alpha,p,\nu} = C_{\alpha,p}\int_D\int_D|x-y|^{\alpha\frac{p}{a}-d-2p\nu}<+\infty$.
Next we use that $W^{\nu,2p}$ is continuously embedded in $\mathcal{C}^{0,\beta}(\bar{D})$ for $\beta<\min\{1,\nu-\frac{d}{2p}\}$, let $k_{\beta,\nu}$ denote the norm of this continuous embedding. For any α such that $0<\alpha\le\min\{b,2a\}$, for any β such that $\beta<\frac{\alpha}{2a}$, there exists $p_0\ge1$ such that for all $p\ge p_0$, $\beta+\frac{d}{2p}<\frac{\alpha}{2a}$, then we choose ν such that $\beta+\frac{d}{2p}<\nu<\frac{\alpha}{2a}$, finally $g_N-g\in\mathcal{C}^{0,\beta}(\bar{D})$ for almost all ω, with:

$$\mathbb{E}[\|g_N-g\|_{\mathcal{C}^{0,\beta}(\bar{D})}^{2p}] \le k_{\beta,\nu}^{2p}C_{\alpha,p,\nu}(R_N^\alpha)^p.$$

Therefore, for any $0<\alpha\le\min\{b,2a\}$, for any β such that $\beta<\frac{\alpha}{2a}$, there exists $p_0>0$ such that for any $p\ge p_0$, there exists a constant $A_{\alpha,\beta,p}$ such that for any N in $\mathbb{N}$:

$$\|g_N-g\|_{L^{2p}(\Omega,\mathcal{C}^{0,\beta}(\bar{D}))} \le A_{\alpha,\beta,p}(R_N^\alpha)^{\frac{1}{2}}.$$

Since in a probability space, if $p\le q$, $f\in L^q$ implies $f\in L^p$ with $\|f\|_{L^p}\le\|f\|_{L^q}$, we can conclude that for any $p>0$ and $0<\alpha<b$, for any β such that $\beta<\frac{\alpha}{2a}$, there exists a constant $A_{\alpha,\beta,p}$ such that for all N in $\mathbb{N}$:

$$\|g_N-g\|_{L^p(\Omega,\mathcal{C}^{0,\beta}(\bar{D}))} \le A_{\alpha,\beta,p}(R_N^\alpha)^{\frac{1}{2}}.$$

□

Proposition 3.2. *For any $\beta < \min\{1, \frac{b}{2a}\}$, for almost all ω, $g_N \to g$ in $\mathcal{C}^{0,\beta}(\bar{D})$ as $N \to +\infty$ and so $a_N \to a$ in $\mathcal{C}^0(\bar{D})$ as $N \to +\infty$.*
It follows that a_N^{max} converges a.s. to a^{max} and a_N^{min} converges a.s. to a^{min} as $N \to +\infty$.

Proof. We use the Borel-Cantelli lemma: let $0 < \alpha \leq \min\{b, 2a\}$ such that $\beta < \frac{\alpha}{2a}$ and p such that $p > \frac{2}{b-\alpha}$, by the previous proposition, there exists a constant $A_{\alpha,\beta,p}$ such that for all N in $\mathbb{N}$:

$$\|g_N - g\|_{L^p(\Omega, \mathcal{C}^{0,\beta}(\bar{D}))} \leq A_{\alpha,\beta,p}(R_N^\alpha)^{\frac{1}{2}},$$

which implies that for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(\|g_N - g\|_{C^{0,\beta}} \geq \varepsilon) &\leq \frac{\|g_N - g\|^p_{L^p(\Omega, \mathcal{C}^{0,\beta}(\bar{D}))}}{\varepsilon^p} \\ &\leq \frac{A^p_{\alpha,\beta,p}(R_N^\alpha)^{\frac{p}{2}}}{\varepsilon^p}. \end{aligned}$$

Let $\gamma > 0$ such that $\frac{2}{p} < \gamma < b - \alpha$, then $R_N^\alpha \leq R_N^{\alpha+\gamma} N^{-\gamma} = o(N^{-\gamma})$, when $N \to +\infty$ and since $\sum_{N\geq 1} N^{-\frac{\gamma p}{2}} < +\infty$, we have $\sum_{N\geq 1} \mathbb{P}(\|g_N - g\|_{C^{0,\beta}} \geq \varepsilon) < +\infty$. The Borel-Cantelli lemma yields $\mathbb{P}(\limsup(\|g_N - g\|_{C^{0,\beta}} \geq \varepsilon)) = 0$ for all $\varepsilon > 0$, and so $g_N \to g$ a.s. in $\mathcal{C}^{0,\beta}(\bar{D})$ as $N \to +\infty$. Finally, thanks to the continuity of the exponential function, we conclude that $a_N = e^{g_N} \to a = e^g$ a.s. in $\mathcal{C}^0(\bar{D})$ as $N \to +\infty$. □

The following results, which give uniform bounds for the random variables a_N^{max} and $\frac{1}{a_N^{min}}$ in the L^p-norm will be used to conclude this section and in the next two sections.

Definition 2. *For $N \in \mathbb{N}$, $\nu \in [0,1]^N$, we define $g_{\nu,N} : \Omega \times D \to \mathbb{R}$ and $a_{\nu,N} : \Omega \times D \to \mathbb{R}$ by: $g_{\nu,N}(\omega, x) = \sum_{n=1}^N \sqrt{\lambda_n} Y_n(\omega) \nu_n b_n(x)$, and $a_{\nu,N}(\omega, x) = e^{g_{\nu,N}(\omega,x)}$.*

Proposition 3.3. *For any $\beta < \min\{\frac{b}{2a}, 1\}$, $p > 0$, there exists a constant $B_{\beta,p}$ such that for any N in $\mathbb{N}$, and ν in $[0,1]^N$.*

$$||g_{\nu,N}||_{L^p(\Omega, \mathcal{C}^{0,\beta}(\bar{D}))} \leq B_{\beta,p}$$

Proof. For any θ such that $0 < \theta < \min\{1, \frac{b}{2a}\}$, for any $N \in \mathbb{N}$, $\nu \in [0,1]^N$

$$\begin{aligned} \mathbb{E}[(g_{\nu,N}(\omega, x) - g_{\nu,N}(\omega, y))^2] &= \sum_{n=1}^N \lambda_n \nu_n^2 (b_n(x) - b_n(y))^2 \\ &\leq C_\theta \left(\sum_{n=1}^N \lambda_n n^{2a\theta} \right) \|x - y\|^{2\theta} \\ &\leq C_\theta R_0^{2a\theta} \|x - y\|^{2\theta}. \end{aligned}$$

Since, for any x, y in D, N in $\mathbb{N}$, $(g^{\nu,N}(x) - g^{\nu,N}(y))$ is a mean-free gaussian random variable, we have:

$$\mathbb{E}[(g_{\nu,N}(\omega, x) - g_{\nu,N}(\omega, y))^{2p}] \leq c_p (R_0^{2a\theta})^p \|x - y\|^{2\theta p}.$$

Therefore, for any $0 < \alpha \leq \min\{b, 2a\}$, for any $p \geq 1$, there exists a constant $M_{\alpha,p}$ such that, for any $N\mathbb{N}$ for any $\nu \in [0,1]^N$, we have:

$$\mathbb{E}[(g_{\nu,N}(\omega,x) - g_{\nu,N}(\omega,y))^{2p}] \quad \leq \quad M_{\alpha,p}\|x-y\|^{\alpha\frac{p}{a}},$$

where $M_{\alpha,p} = c_p(R_0^{2a\theta})^p$, then we use Kolmogorov continuity theorem [5] and conclude as in the proof of Proposition 3.1. $\square$

Definition 3. *By the previous proposition, for any $N \in \mathbb{N}$ and $\nu \in [0,1]^N$, the trajectories of $a_{\nu,N}$ are continuous on the compact set $\bar{D}$ a.s, so we can define, for almost all $\omega \in \Omega$, $a_{\nu,N}^{max}(\omega) = \max\limits_{x\in\bar{D}} a_{\nu,N}(\omega,x)$ and $a_{\nu,N}^{min}(\omega) = \min\limits_{x\in\bar{D}} a_{\nu,N}(\omega,x)$.*

We can finally bound $a_{\nu,N}^{max}$ and $\frac{1}{a_{\nu,N}^{min}}$ in L^p-norm, independently from N and ν.

Proposition 3.4. *For any $p > 0$, $a_{\nu,N}^{max}$ and $\frac{1}{a_{\nu,N}^{min}} \in L^p(\Omega)$, and there exists a constant D_p such that for any $N \in \mathbb{N}$, and $\nu \in [0,1]^N$*

$$\left\|\frac{1}{a_{\nu,N}^{min}}\right\|_{L^p(\Omega)} \leq D_p, \ \textit{and} \quad \|a_{\nu,N}^{max}\|_{L^p(\Omega)} \leq D_p.$$

In particular,

$$\|a_N\|_{L^p(\Omega,\mathcal{C}^0(\bar{D}))} \quad \leq \quad D_p.$$

Proof. We apply Fernique's theorem [5], uniformly with respect to N and ν. There exists $x_0 \in]0,1[$ such that for all $x \in [x_0, 1[$, $\ln(\frac{1-x}{x}) \leq -2$. The previous proposition yields the existence of a constant B_2 such that for any $N \in \mathbb{N}$, and $\nu \in [0,1]^N$:

$$||g_{\nu,N}||_{L^2(\Omega,\mathcal{C}^0(\bar{D}))} \quad \leq \quad B_2.$$

Thus, setting $r_0 = \frac{B_2}{\sqrt{1-x_0}}$, we have, for every $r \geq r_0$,

$$\begin{aligned}\mathbb{P}(\|g_{\nu,N}\|_{\mathcal{C}^0(\bar{D})} \geq r) \quad &\leq \quad \frac{\|g_{\nu,N}\|^2_{L^2(\Omega,\mathcal{C}^0(\bar{D}))}}{r^2} \\ &\leq \quad \frac{B_2^2}{r^2} \leq 1 - x_0.\end{aligned}$$

We now choose λ such that $32\lambda r_0^2 \leq 1$, and we denote by $\mu_{\nu,N}$ the law of $g_{\nu,N} : \Omega \to \mathcal{C}^0(\bar{D})$. Since the $\mu_{\nu,N}$ are centred gaussian measures on the Banach space $\mathcal{C}^0(\bar{D})$, we have then, for any $N \in \mathbb{N}$, and $\nu \in [0,1]^N$,

$$\ln\left(\frac{1-\mu_{\nu,N}(\bar{B}(0,r_0))}{\mu_{\nu,N}(\bar{B}(0,r_0))}\right) + 32\lambda r_0^2 \leq -1.$$

Use Fernique theorem [5], set $k = e^{16\lambda r_0^2} + \frac{e^2}{e^2-1}$, to obtain that for all $N \in \mathbb{N}$, and $\nu \in [0,1]^N$,

$$\mathbb{E}[e^{\lambda\|g_{\nu,N}\|^2_{\mathcal{C}^0(\bar{D})}}] \leq k.$$

Hence, for any $p > 0$, $N \in \mathbb{N}$, and $\nu \in [0,1]^N$,

$$\begin{aligned}\mathbb{E}[e^{p\|g_{\nu,N}\|_{\mathcal{C}^0(\bar{D})}}] \quad &\leq \quad e^{\frac{p^2}{4\lambda}}\mathbb{E}[e^{\lambda\|g_{\nu,N}\|^2_{\mathcal{C}^0(\bar{D})}}] \\ &\leq \quad ke^{\frac{p^2}{4\lambda}}.\end{aligned}$$

Denoting $D_p = (ke^{\frac{p^2}{4\lambda}})^{\frac{1}{p}}$, we conclude that:

$$\left\| \frac{1}{a_{\nu,N}^{min}} \right\|_{L^p(\Omega)} \leq \|e^{\|g_{\nu,N}\|_{\mathcal{C}^0(\bar{D})}}\|_{L^p(\Omega)} \leq D_p,$$

and

$$\|a_{\nu,N}^{max}\|_{L^p(\Omega)} \leq \|e^{\|g_{\nu,N}\|_{\mathcal{C}^0(\bar{D})}}\|_{L^p(\Omega)} \leq D_p.$$

□

Proposition 3.5. *For any $p > 0$, and $0 < \alpha \leq \min\{b, 2a\}$, there exists a constant $E_{\alpha,p}$ such that for any $N \in \mathbb{N}$,*

$$\|a_N - a\|_{L^p(\Omega,\mathcal{C}^0(\bar{D}))} \leq E_{\alpha,p}(R_N^\alpha)^{\frac{1}{2}}$$

Proof. Take $p > 0$, choose $q, r > 0$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then the following inequality

$$\forall x, y \in \mathbb{R} \quad |e^x - e^y| \leq |x - y|(e^x + e^y),$$

together with Hölder's inequality leads to :

$$\|e^{g_N} - e^g\|_{L^r(\Omega,\mathcal{C}^0(\bar{D}))} \leq \|g_N - g\|_{L^p(\Omega,\mathcal{C}^0(\bar{D}))}\|e^{g_N} + e^g\|_{L^q(\Omega,\mathcal{C}^0(\bar{D}))},$$

which we rewrite as

$$\begin{aligned} \|a_N - a\|_{L^r(\Omega,\mathcal{C}^0(\bar{D}))} &\leq \|g_N - g\|_{L^p(\Omega,\mathcal{C}^0(\bar{D}))}\|a_N + a\|_{L^q(\Omega,\mathcal{C}^0(\bar{D}))} \\ &\leq A_{\alpha,p}(R_N^\alpha)^{\frac{1}{2}}(D_q + \|a\|_{L^q(\Omega,\mathcal{C}^0(\bar{D}))}). \end{aligned}$$

We conclude by setting $E_{\alpha,r} = A_{\alpha,p}(D_q + \|a\|_{L^q(\Omega,\mathcal{C}^0(\bar{D}))})$, with $p = q = 2r$ for instance. □

4 Strong convergence of u_N to u

Thanks to the results of the previous section, we can now estimate the strong error commited on the solution u, resulting from the approximation of a by a_N.

Since for all $N \in \mathbb{N}$ and $\nu \in [0,1]^N$, the random variables $a_{\nu,N}^{max}$ and $\frac{1}{a_{\nu,N}^{min}}$ belong to $L^p(\Omega)$ for all $p > 0$, the equation

$$\begin{aligned} -\text{div}(a_{\nu,N}(\omega,x)\nabla u_N^\nu(\omega,x)) &= f(x) \ \text{ on } D, \\ u_N^\nu(\omega,x) &= 0 \ \text{ on } \partial D, \end{aligned}$$

admits a unique solution $u_N^\nu \in L^p(\Omega, H_0^1(D))$ for all $p > 0$. In particular, for $\nu = 1^N$, $a_{\nu,N} = a_N$ and we denote by u_N the solution. Let us set for $(y_1, ..., y_N) \in \mathbb{R}^N$ and $x \in D$, $\tilde{a}_N(y_1, ..., y_N, x) = e^{\sum_{n=1}^N \sqrt{\lambda_i} b_i(x) y_i}$ and $\tilde{u}_N(y_1, ..., y_N, .)$ be the solution of

$$\begin{aligned} -\text{div}_x(\tilde{a}_N(y,x)\nabla_x \tilde{u}_N(y,x)) &= f(x) \ \text{ on } D, \\ \tilde{u}_N(y,x) &= 0 \ \text{ on } \partial D. \end{aligned}$$

It is classical that $\tilde{u}_N$ is a $\mathcal{C}^\infty$ function of $y_1, ..., y_N$. When we need to emphasize the dependance of $\tilde{u}_N$ on $y_1, ..., y_N$, we write $\tilde{u}_N(y_1, ..., y_N)$. We have then $\tilde{u}_N \in \mathcal{C}^\infty(\mathbb{R}^N, H_0^1(D))$. We notice that a.s. $a_N(\omega,x) = \tilde{a}_N(Y_1(\omega), ..., Y_N(\omega), x)$, and $u_N(\omega,x) = \tilde{u}_N(Y_1(\omega), ..., Y_N(\omega), x)$. For convenience, $\tilde{a}_N$ will still be denoted by a_N and $\tilde{u}_N$ by u_N .

We first show the almost sure convergence of u_N to u.

Proposition 4.1. *$u_N(\omega, x)$ converges to $u(\omega, x)$ in $H_0^1(D)$, for almost all ω.*

Proof. By Proposition (3.2), for almost all ω, a_N converges to a in $\mathcal{C}^0(\bar{D})$ i.e. uniformly. Then we use the continuity of the solution u with respect to the coefficient a of the equation, indeed we have a.s.:

$$\begin{aligned}
a_N^{min}\|u-u_N\|^2_{H_0^1(D)} &\le \int_D a_N|\nabla(u-u_N)|^2 \\
&= \int_D (a_N-a)\nabla u\nabla(u-u_N) \\
&\le \|a-a_N\|_{\mathcal{C}^0(\bar{D})}\|u-u_N\|_{H_0^1(D)}\|u\|_{H_0^1(D)}.
\end{aligned}$$

Therefore, thanks to (2), we have, for almost all ω:

$$\|u-u_N\|_{H_0^1(D)} \le \frac{1}{a_N^{min}}\|a-a_N\|_{\mathcal{C}^0(\bar{D})}\|f\|_{L^2(D)}\frac{C_D}{a^{min}}.$$

The right-hand side of this inequality converges a.s. to 0 as $N \to +\infty$ by Proposition 3.2. □

Next we give a convergence result and an estimate error in L^p-norm.

Theorem 4.2. *For all $p > 0$, u_N converges to u in $L^p(\Omega, H_0^1(D))$, and for any $0 < \alpha \le \min\{b, 2a\}$, there exists a constant $F_{\alpha,p}$ such that*

$$\|u-u_N\|_{L^p(\Omega,H_0^1(D))} \le F_{\alpha,p}(R_N^\alpha)^{\frac{1}{2}}.$$

Proof. For all $p > 0$, for almost all ω,

$$\|u-u_N\|_{H_0^1(D)} \le \frac{1}{a_N^{min}}\|a-a_N\|_{\mathcal{C}^0(\bar{D})}\|f\|_{L^2}\frac{C_D}{a^{min}}.$$

Hence, by choosing $q, r, s > 0$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$ it follows from Hölder inequality and Proposition (3.5) that for any $0 < \alpha \le b$

$$\begin{aligned}
\|u-u_N\|_{L^p(\Omega,H_0^1(D))} &\le \left\|\frac{1}{a_N^{min}}\right\|_{L^q(\Omega)}\|a-a_N\|_{L^r(\Omega,\mathcal{C}^0)}\|f\|_{L^2}C_D\left\|\frac{1}{a^{min}}\right\|_{L^s(\Omega)} \\
&\le D_q\|f\|_{L^2}D_sE_{\alpha,r}(R_N^\alpha)^{\frac{1}{2}}.
\end{aligned}$$

□

The following results gives a bound for u_N^ν in $L^p(H_0^1)$-norm independent of N and ν, which will be useful in the next section.

Lemma 4.3. *For all $p > 0$, there exists a constant G_p such that for all $N \in \mathbb{N}$, and $\nu \in [0,1]^N$*

$$\|u_N^\nu\|_{L^p(\Omega,H_0^1(D))} \le G_p.$$

Proof. For almost all ω, for any $p > 0$, $N \in \mathbb{N}$, and $\nu \in [0,1]^N$, we have by Proposition 3.4:

$$\begin{aligned}
\|u_N^\nu\|_{H_0^1(D)} &\le \frac{C_D}{a_{min}^{\nu,N}}\|f\|_{L^2(D)}, \\
\|u_N^\nu\|_{L^p(\Omega,H_0^1(D))} &\le C_D\left\|\frac{1}{a_{min}^{\nu,N}}\right\|_{L^p}\|f\|_{L^2(D)} \\
&\le C_DD_p\|f\|_{L^2(D)}.
\end{aligned}$$

□

5 Weak convergence of u_N to u

In this section we are interested in the error committed on the law of u, more precisely we show that the order of the bound for the weak convergence of u_N to u is twice the order of the bound for the strong convergence. In order to estimate the weak error, that is to say the expected value of $\varphi(u_N) - \varphi(u)$, for some regular function φ, we need estimates on the growth of the derivatives of $\varphi(u_N)$ with respect to the y_i, which follow from the following estimates on the derivatives of u_N with respect to the y_i.

Proposition 5.1. *For any multi-index $\alpha \in \mathbb{N}^{\mathbb{N}}$ with finite support, we have the following estimate on the growth of the derivatives of u_N with respect to y :*

$$\left\| \frac{\partial^\alpha u_N}{\partial y^\alpha} \right\|_{H_0^1(D)} \leq C_{|\alpha|} \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} ||u_N||_{H_0^1} C^{|\alpha|} \prod_{i \in \mathbb{N}} \sqrt{\lambda_i^{\alpha_i}}.$$

where $C_{|\alpha|}$ only depends of α through his length.

Proof. We recall that for all $y \in \mathbb{R}^N$, $u(y,.)$ solves the following equation

$$\int_D a_N(y,x) \nabla_x u_N(y,x) \nabla_x v(x) = \int_D f(x) v(x), \quad \forall v \in H_0^1(D).$$

We compute the derivatives of a_N with respect to the y_i. For all $1 \leq i,j \leq N$:

$$\frac{\partial a_N}{\partial y_i}(y,x) = \sqrt{\lambda_i} b_i(x) a_N(y,x), \quad \frac{\partial a_N}{\partial y_i \partial y_j}(y,x) = \sqrt{\lambda_i} b_i(x) \sqrt{\lambda_j} b_j(x) a_N(y,x).$$

In every point $y \in \mathbb{R}^N$, the derivatives of u with respect to y_i and with respect to y_i and y_j, for $1 \leq i,j \leq N$ satisfy: $\forall v \in H_0^1(D)$

$$\int_D \frac{\partial a_N}{\partial y_i}(y,x) \nabla u_N(y,x) \nabla v(x) + \int_D a_N(y,x) \nabla \frac{\partial u_N}{\partial y_i}(y,x) \nabla v(x) = 0,$$

$$\int_D \left(\frac{\partial^2 a_N}{\partial y_i \partial y_j} \nabla u_N + \frac{\partial a_N}{\partial y_i} \nabla \frac{\partial u_N}{\partial y_j} + \frac{\partial a_N}{\partial y_j} \nabla \frac{\partial u_N}{\partial y_i} + a_N \nabla \frac{\partial^2 u_N}{\partial y_i \partial y_j} \right)(y,x) \nabla v(x) = 0.$$

Recall that C is a constant such that for every $i \in \mathbb{N}$, $\|b_i\|_\infty \leq C$.
Choosing $v(x) = \frac{\partial u_N}{\partial y_i}(y,x)$ in the first variational formulation, we have:

$$\begin{aligned}
\left\| \sqrt{a_N}(y,x) \nabla \frac{\partial u_N}{\partial y_i} \right\|_{L^2} &\leq \sqrt{\lambda_i} \|b_i\|_\infty ||\sqrt{a_N}(y,x) \nabla u_N||_{L^2} \\
&\leq \sqrt{\lambda_i} C \sqrt{a_{max}^N(y)} ||u_N||_{H_0^1}, \\
\left\| \frac{\partial u_N}{\partial y_i} \right\|_{H_0^1} &\leq \sqrt{\lambda_i} C \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} ||u_N||_{H_0^1}.
\end{aligned}$$

Choosing $v(x) = \frac{\partial^2 u_N}{\partial y_i \partial y_j}(y,x)$ in the second variational formulation, we obtain:

$$\begin{aligned}
\left\| \sqrt{a_N}(y,x) \nabla \frac{\partial^2 u_N}{\partial y_i \partial y_j} \right\|_{L^2} &\leq \sqrt{\lambda_i} \sqrt{\lambda_j} C^2 ||\sqrt{a_N} \nabla u_N||_{L^2} \\
&+ \sqrt{\lambda_i} C ||\sqrt{a_N} \nabla \frac{\partial u_N}{\partial y_j}||_{L^2} + \sqrt{\lambda_j} C ||\sqrt{a_N} \nabla \frac{\partial u_N}{\partial y_i}||_{L^2} \\
&\leq 3 \sqrt{a_{max}^N(y)} \sqrt{\lambda_i} \sqrt{\lambda_j} C^2 ||u_N||_{H_0^1}.
\end{aligned}$$

Therefore:

$$\left\| \frac{\partial^2 u_N}{\partial y_i \partial y_j}(y) \right\|_{H_0^1} \leq 3\sqrt{\frac{a^N_{max}(y)}{a^N_{min}(y)}} \sqrt{\lambda_i}\sqrt{\lambda_j} C^2 ||u_N||_{H_0^1}.$$

The result follows by induction. □

Proposition 5.2. *Let $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$, whose derivatives are bounded by a constant C_φ, then for any multi-index $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $|\alpha| \leq 4$, we have the following estimate on the growth of the derivatives of $\varphi \circ u_N$ with respect to y:*

$$\left\| \frac{\partial^\alpha \varphi \circ u_N}{\partial y^\alpha} \right\|_{H_0^1(D)} \leq C'_{|\alpha|} C_\varphi C^{|\alpha|} \left(1 + \sqrt{\frac{a^N_{max}(y)}{a^N_{min}(y)}} ||u_N||_{H_0^1} \right)^{|\alpha|} \prod_{i \in \mathbb{N}} \sqrt{\lambda_i^{\alpha_i}},$$

where $C'_{|\alpha|}$ only depends of α through his length.

Proof. For $|\alpha| = 1$, we have:

$$\begin{aligned} \frac{\partial \varphi \circ u_N}{\partial y_i}(y) &= \varphi' \circ u_N(y) \frac{\partial u_N}{\partial y_i}(y) \\ \left\| \frac{\partial \varphi \circ u_N}{\partial y_i}(y) \right\|_{H_0^1} &\leq C_\varphi \left\| \frac{\partial u_N}{\partial y_i}(y) \right\|_{H_0^1} \\ &\leq C_\varphi \sqrt{\lambda_i} C \sqrt{\frac{a^N_{max}(y)}{a^N_{min}(y)}} ||u_N||_{H_0^1}. \end{aligned}$$

Then, for $|\alpha| = 2$:

$$\begin{aligned} \frac{\partial^2 \varphi \circ u_N}{\partial y_i \partial y_j}(y) &= \varphi' \circ u_N(y) \frac{\partial^2 u_N}{\partial y_i \partial y_j}(y) + \varphi''(u_N)(y) \frac{\partial u_N}{\partial y_i}(y) \frac{\partial u_N}{\partial y_j}(y) \\ \left\| \frac{\partial^2 \varphi \circ u_N}{\partial y_i \partial y_j}(y) \right\|_{H_0^1} &\leq C_\varphi \left\| \frac{\partial^2 u_N}{\partial y_i \partial y_j}(y) \right\|_{H_0^1} + C_\varphi \left\| \frac{\partial u_N}{\partial y_i}(y) \right\|_{H_0^1} \left\| \frac{\partial u_N}{\partial y_j}(y) \right\|_{H_0^1} \\ &\leq 2C_\varphi \sqrt{\lambda_i}\sqrt{\lambda_j} C^2 \left(1 + \sqrt{\frac{a^N_{max}(y)}{a^N_{min}(y)}} ||u_N(y)||_{H_0^1} \right)^2 \end{aligned}$$

The result follows by induction and Faà di Bruno's formula. □

We are now ready to estimate the weak error, i.e. the quantity $\mathbb{E}[\varphi(u_N) - \varphi(u)]$ in H_0^1-norm. Before stating and proving the estimate on the weak error, we give the basic idea of the proof. To estimate the weak error, we consider the Taylor expansion at order 2 of $\varphi(u_N) - \varphi(u)$ and remark that first order terms and second order terms such that $i \neq j$ are mean-free. In the case where φ is the identity, formally the second order development is:

$$\begin{aligned} u(\omega, x) - u_N(\omega, x) &= u(Y_1(\omega), ..., Y_N(\omega), Y_{N+1}(\omega), ..., x) - u(Y_1(\omega), ..., Y_N(\omega), 0, ..., x) \\ &= \sum_{i > N} \frac{\partial u}{\partial y_i}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x) Y_i(\omega) \\ &+ \frac{1}{2} \sum_{i,j > N} \frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x) Y_i(\omega) Y_j(\omega) + ... \end{aligned}$$

Combining the independence of the Y_i with the fact that the Y_i are mean-free yields that the following terms are mean-free:

$$\begin{aligned}\mathbb{E}\left[\frac{\partial u}{\partial y_i}(Y_1(\omega),...,Y_N(\omega),0,...,x)Y_i(\omega)\right] &= \mathbb{E}\left[\frac{\partial u}{\partial y_i}(Y_1(\omega),...,Y_N(\omega),0,...,x)\right]\mathbb{E}[Y_i(\omega)] \\ &= 0.\end{aligned}$$

Analogously, for $i \neq j$,

$$\begin{aligned}& \mathbb{E}\left[\frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega),...,Y_N(\omega),0,...,x)Y_i(\omega)Y_j(\omega)\right] \\ = \quad & \mathbb{E}\left[\frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega),...,Y_N(\omega),0,...,x)\right]\mathbb{E}[Y_i(\omega)]\mathbb{E}[Y_j(\omega)] \\ = \quad & 0.\end{aligned}$$

The proof below shows that indeed the dominant in the error on the expected value is

$$\sum_{i>N}\mathbb{E}\left[\frac{\partial^2 u}{\partial y_i^2}(Y_1(\omega),...,Y_N(\omega),0,...,x)\right]$$

We now give the general and precise result and its proof.

Theorem 5.3. *There exists a constant c such that for all $N \in \mathbb{N}$, for all $\varphi \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ whose derivatives (excluding φ itself) are bounded by a constant C_φ, we have:*

$$\|\mathbb{E}_\omega[\varphi(u_N)-\varphi(u)]\|_{H_0^1(D)} \leq cC_\varphi \sum_{i>N}\lambda_i$$

Remark: More generally, the result can be extended to the case where the derivatives of φ are bounded by a polynomial, under extra regularity assumptions on f. This is important since it enables to treat the case of the moments of a. However, this generalization requires additionnal technical difficulties.
Remark: The weak error at order N is bounded by R_N^0, whereas the strong error at order N is bounded by $\sqrt{R_N^\alpha}$ for any $0 < \alpha \leq \min\{2a, b\}$. Therefore the weak order is indeed twice the strong order.

Proof. Let $M > N$, and $x \in D$, the first order Taylor theorem with integral remainder gives:

$$\begin{aligned}& \mathbb{E}_\omega[(\varphi(u_M)-\varphi(u_N))(\omega,x)] \\ = \quad & \mathbb{E}_\omega[\varphi(u_M)(Y_1(\omega),...Y_M(\omega),x)-\varphi(u_M)(Y_1(\omega),...Y_N(\omega),0,...0,x)] \\ = \quad & \mathbb{E}_\omega[D_y(\varphi\circ u_M)(Y_1(\omega),...Y_N(\omega),0,...0,x).(0,...0,Y_{N+1}(\omega),...,Y_M(\omega))] \\ + \quad & \mathbb{E}_\omega\left[\int_0^1(1-t)D_y^2(\varphi\circ u_M)(Y_1,...Y_N,tY_{N+1},...,tY_M,x).(0,...0,Y_{N+1},...,Y_M)^2dt\right]\end{aligned}$$

Since the random variables Y_i are independent, with mean zero and unit variance, the first order term is mean-free:

$$\begin{aligned}& \mathbb{E}_\omega\left[D_y(\varphi\circ u_M)(Y_1(\omega),...Y_N(\omega),0,...0,x).(0,...0,Y_{N+1}(\omega),...,Y_M(\omega))\right] \\ = \quad & \mathbb{E}_\omega\left[\sum_{i=N+1}^{M}\frac{\partial(\varphi\circ u_M)}{\partial y_i}(Y_1(\omega),...Y_N(\omega),0,...0,x)Y_i(\omega)\right] \\ = \quad & \sum_{i=N+1}^{M}\mathbb{E}_\omega\left[\frac{\partial(\varphi\circ u_M)}{\partial y_i}(Y_1(\omega),...Y_N(\omega),0,...0,x)\right]\mathbb{E}_\omega[Y_i(\omega)] \\ = \quad & 0\end{aligned}$$

We now bound the integral remainder term, to begin with we split it into two terms:

$$\begin{aligned}
& \mathbb{E}_\omega \left[(\varphi(u_M) - \varphi(u_N))(\omega, x) \right] \\
= & \sum_{N+1 \le i,j \le M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 (\varphi \circ u_M)}{\partial y_i \partial y_j} (Y_1, ...Y_N, tY_{N+1}, ..., tY_M, x) Y_i Y_j \right] dt \\
= & \sum_{N+1 \le i \le M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 (\varphi \circ u_M)}{\partial y_i^2} (Y_1, ...Y_N, tY_{N+1}, ..., tY_M, x) Y_i^2 \right] dt \\
+ & \sum_{N+1 \le i \ne j \le M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 (\varphi \circ u_M)}{\partial y_i \partial y_j} (Y_1, ...Y_N, tY_{N+1}, ..., tY_M, x) Y_i Y_j \right] dt
\end{aligned}$$

First we give an estimate for the first error contribution. Using the bound of the derivatives of $\varphi \circ u_M$ given in Proposition 5.2, we get for $N+1 \le i \le M$:

$$\begin{aligned}
& \left\| \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 (\varphi \circ u_M)}{\partial y_i^2} (Y_1, ...Y_N, tY_{N+1}, ...tY_M, x) Y_i^2 \right] dt \right\|_{H_0^1(D)} \\
\le & \int_0^1 (1-t) \mathbb{E}_\omega \left[\left\| \frac{\partial^2 (\varphi \circ u_M)}{\partial y_i^2} (Y_1, ...Y_N, tY_{N+1}, ...tY_M, x) \right\|_{H_0^1(D)} Y_i^2 \right] dt \\
\le & C_2' C_\varphi C^2 \lambda_i \int_0^1 \mathbb{E}_\omega \left[\left(1 + \sqrt{\frac{a_N^{max}}{a_N^{min}}} ||u_M||_{H_0^1} \right)^2 (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M, x) Y_i^2 \right] dt \\
\le & 2 C_2' C_\varphi C^2 \lambda_i \int_0^1 \mathbb{E}_\omega \left[\left(1 + \frac{a_N^{max}}{a_N^{min}} ||u_M||_{H_0^1}^2 \right) (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M) Y_i^4 \right] dt.
\end{aligned}$$

We define, for $t \in [0,1]$, $\nu_t \in [0,1]^M$ by $\nu_t(i) = 0$ for $i \le N$ and $\nu_t(i) = t$ for $i > N$, then by Hölder inequality and Propositions 3.4 and 4.3 we have:

$$\begin{aligned}
& \mathbb{E}_\omega \left[\frac{a_M^{max}}{a_M^{min}} \|u_M\|_{H_0^1}^2 (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M, x) Y_i^2 \right] \\
\le & \|a_{\nu_t, M}^{max}\|_{L^4(\Omega)} \left\| \frac{1}{a_{\nu_t, M}^{min}} \right\|_{L^4(\Omega)} \|u_M^{\nu_t}\|_{L^8(\Omega, H_0^1)}^2 \|Y_i\|_{L^4(\Omega)} \\
\le & D_4^2 G_8^2 m_4.
\end{aligned}$$

Where m_4 is the moment of order 4 of a gaussian with mean zero and unit variance. We obtain finally the following bound for the first term of the error contribution.

$$\begin{aligned}
& \left\| \sum_{N+1 \le i \le M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 u_M}{\partial y_i^2} (Y_1, ..., Y_N, tY_{N+1}, ..., tY_M, x) Y_i^2 \right] dt \right\|_{H_0^1(D)} \\
\le & 2 C_2' C_\varphi C^2 (m_4 + D_4^2 G_8^2 m_4) \sum_{N+1 \le i \le M} \lambda_i \\
\le & C_\varphi k_1 \sum_{N+1 \le i \le M} \lambda_i.
\end{aligned}$$

Where $k_1 = 2C_2' C^2 (m_4 + D_4^2 G_8^2 m_4)$. Next we give an estimate for the second term of the error contribution, by using once again the independence of the random variables Y_i, for

$N+1 \leq i < j \leq M$ we get:

$$\begin{aligned}
& \int_0^1 (1-t)\mathbb{E}\left[\frac{\partial^2(\varphi\circ u_M)}{\partial y_i \partial y_j}(X_{i,j}^{t,1,1},x)Y_iY_j\right]dt \\
= & \int_0^1 (1-t)\mathbb{E}\left[\frac{\partial^2(\varphi\circ u_M)}{\partial y_i \partial y_j}(X_{i,j}^{t,1,1},x)Y_iY_j\right]dt \\
- & \int_0^1 (1-t)\mathbb{E}\left[\frac{\partial^2(\varphi\circ u_M)}{\partial y_i \partial y_j}(X_{i,j}^{t,0,1},x)Y_iY_j\right]dt \\
= & \ \mathbb{E}\left[\iint_{[0,1]^2}(1-t)(1-u)t\frac{\partial^3(\varphi\circ u_M)}{\partial y_i^2 \partial y_j}(X_{i,j}^{t,u,1},x)Y_i^2Y_j dtdu\right] \\
= & \ \mathbb{E}\left[\iint_{[0,1]^2}(1-t)(1-u)t\frac{\partial^3(\varphi\circ u_M)}{\partial y_i^2 \partial y_j}(X_{i,j}^{t,u,1},x)Y_i^2Y_j dtdu\right] \\
- & \ \mathbb{E}\left[\iint_{[0,1]^2}(1-t)(1-u)t\frac{\partial^3(\varphi\circ u_M)}{\partial y_i^2 \partial y_j}(X_{i,j}^{t,u,0},x)Y_i^2Y_j dtdu\right] \\
= & \ \mathbb{E}\left[\iiint_{[0,1]^3}(1-t)(1-u)(1-s)t^2\frac{\partial^4(\varphi\circ u_M)}{\partial y_i^2 \partial y_j^2}(X_{i,j}^{t,u,s},x)Y_i^2Y_j^2 dtduds\right].
\end{aligned}$$

Where the random variables $X_{i,j}^{t,u,s}(\omega)$ are defined by

$$X_{i,j}^{t,u,s}(\omega) = (Y_1,...,Y_N,tY_{N+1},...,tuY_i,...,tsY_j,...,tY_M)(\omega).$$

By Proposition 5.2, we have then:

$$\begin{aligned}
& \left\|\mathbb{E}\left[\iiint_{[0,1]^3}(1-t)(1-u)(1-s)t^2\frac{\partial^4(\varphi\circ u_M)}{\partial y_i^2 \partial y_j^2}(X_{i,j}^{t,s,u},x)Y_i^2Y_j^2 dtduds\right]\right\|_{H_0^1(D)} \\
\leq & \iiint_{[0,1]^3}(1-t)(1-u)(1-s)t^2\mathbb{E}\left[\left\|\frac{\partial^4(\varphi\circ u_M)}{\partial y_i^2 \partial y_j^2}(X_{i,j}^{t,s,u},x)\right\|_{H_0^1(D)} Y_i^2Y_j^2\right]dtduds \\
\leq & \ C_4'C^4C_\varphi\lambda_i\lambda_j\iiint_{[0,1]^3}\mathbb{E}\left[\left(1+\|u_M\|_{H_0^1}\sqrt{\frac{a_M^{max}}{a_M^{min}}}\right)^4(X_{i,j}^{t,s,u},x)Y_i^2Y_j^2\right]dtduds \\
\leq & \ 2^3C_4'C^4C_\varphi\lambda_i\lambda_j\left(m_4+\iiint_{[0,1]^3}\mathbb{E}\left[\left(\|u_M\|_{H_0^1}\sqrt{\frac{a_M^{max}}{a_M^{min}}}\right)^4(X_{i,j}^{t,s,u},x)Y_i^2Y_j^2\right]dtduds\right).
\end{aligned}$$

We define, for $t,s,u \in [0,1]$, $\nu_{t,s,u} \in [0,1]^M$ by $\nu_{t,s,u}(n) = 0$ for $n \leq N$, $\nu_{t,s,u}(n) = t$ for $n > N$ such that $n \neq i, n \neq j$, $\nu_{t,s,u}(i) = tu$, and $\nu_{t,s,u}(j) = ts$.
Then, Hölder inequality combined with Propositions 3.4 and 4.3 yields the following esti-

mate:

$$
\begin{aligned}
& \mathbb{E}_\omega\left[\left(\sqrt{\frac{a_M^{max}}{a_M^{min}}}\|u_M\|_{H_0^1}\right)^4 (X_{i,j}^{t,s,u},x)Y_i^2Y_j^2\right] \\
\leq\ & \|a^{max}_{\nu_{t,s,u},M}\|^2_{L^{10}(\Omega)}\left\|\frac{1}{a^{min}_{\nu_{t,s,u},M}}\right\|^2_{L^{10}(\Omega)}\|u_M^{\nu_{t,s,u}}\|^4_{L^{20}(\Omega,H_0^1(D))}\|Y_i^2\|_{L^5(\Omega)}\|Y_j^2\|_{L^5(\Omega)} \\
\leq\ & D_{10}^4\|u_M^{\nu_{t,s,u}}\|^4_{L^{20}(\Omega,H_0^1(D))}m_{10}^{\frac{2}{5}} \\
\leq\ & D_{10}^4G_{20}^4m_{10}^{\frac{2}{5}}.
\end{aligned}
$$

We have finally the following estimate for the second term of the error contribution:

$$
\begin{aligned}
& \left\|\sum_{N+1\leq i\neq j\leq M}\int_0^1(1-t)\mathbb{E}\left[\frac{\partial^2 u_M}{\partial y_i\partial y_j}(Y_1,...Y_N,tY_{N+1},...,tY_M,x)Y_iY_j\right]dt\right\|_{H_0^1(D)} \\
\leq\ & 2^7C_4'C^4C_\varphi(m_4+D_{10}^4G_{20}^4m_{10}^{\frac{2}{5}})\sum_{N+1\leq i\neq j\leq M}\lambda_i\lambda_j \\
\leq\ & C_\varphi k_2\sum_{N+1\leq i\neq j\leq M}\lambda_i\lambda_j.
\end{aligned}
$$

Where $k_2 = 2^7C_4'C^4(m_4+D_{10}^4G_{20}^4m_{10}^{\frac{2}{5}})$.
We have finally the following estimate for the total error:

$$
\begin{aligned}
& \mathbb{E}_\omega\left[(\varphi(u_M)-\varphi(u_N))(\omega,x)\right] \\
=\ & \sum_{N+1\leq i\leq M}\int_0^1(1-t)\mathbb{E}_\omega\left[\frac{\partial^2 u_M}{\partial y_i^2}(Y_1,...Y_N,tY_{N+1},...,tY_M,x)Y_i^2\right]dt \\
+\ & \sum_{N+1\leq i\neq j\leq M}\int_0^1(1-t)\mathbb{E}_\omega\left[\frac{\partial^2 u_M}{\partial y_i\partial y_j}(Y_1,...Y_N,tY_{N+1},...,tY_M,x)Y_iY_j\right]dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \|\mathbb{E}_\omega[(\varphi(u_M)-\varphi(u_N))(\omega,x)]\|_{H_0^1(D)} \\
\leq\ & C_\varphi k_1\sum_{N+1\leq i\leq M}\lambda_i + C_\varphi k_2\sum_{N+1\leq i\neq j\leq M}\lambda_i\lambda_j \\
\leq\ & C_\varphi k_1\sum_{N+1\leq i\leq M}\lambda_i + k_2C_\varphi\left(\sum_{N+1\leq i\leq M}\lambda_i\right)^2 \\
\leq\ & C_\varphi(k_1+k_2|D|k(0))\sum_{i>N}\lambda_i.
\end{aligned}
$$

Indeed we recall that $\sum_{n\geq 0}\lambda_n = |D|k(0)$. We define then $c = k_1 + k_2k(0)|D|$.
We are now ready to conclude, by giving a bound for $\mathbb{E}_\omega[\varphi(u)-\varphi(u_N)]$ in H_0^1-norm.

Let $N \geq 1$, then for all $M > N$, we have:

$$\begin{aligned} ||\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]||_{H_0^1} &\leq ||\mathbb{E}_\omega[(\varphi(u) - \varphi(u_M))]||_{H_0^1} + ||\mathbb{E}_\omega[(\varphi(u_M) - \varphi(u_N)))]||_{H_0^1} \\ &\leq ||\varphi(u) - \varphi(u_M)||_{L^2(\Omega, H_0^1(D))} + cC_\varphi \sum_{i>N} \lambda_i \\ &\leq C_\varphi ||u - u_M||_{L^2(\Omega, H_0^1(D))} + cC_\varphi \sum_{i>N} \lambda_i. \end{aligned}$$

Letting $M \to +\infty$, by Proposition 4.2 we have:

$$||\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]||_{H_0^1} \leq cC_\varphi \sum_{i>N} \lambda_i.$$

$\square$

Remark: Note that the independance of the Y_i is crucial in the proof of 5.3. This is always the case for a lognormal field.

6 An estimate of the total error for the collocation method

In this section, we recall the stochastic collocation method and slightly generalize the proof of the convergence result given in [1] to the case considered here where the assumption of uniform coercivity with respect to ω is not valid. Since we use many preliminary results of [1], we keep their framework and their notations. As we will see later, it is unfortunately difficult to get a relevant estimate of the dependance with respect to N of the constant appearing by bounding the collocation error.
With the same notations and assumptions as above, we have a regularity result for the solution u_N with respect to y. We introduce a weight $\sigma(y) = \prod_{n=1}^N \sigma_n(y_n)$, where $\sigma_n(y_n) = e^{-\alpha_n|y_n|}$, for any $\alpha \in \mathbb{R}^N$ such that $\alpha_n \geq C\sqrt{\lambda_n}$ for all $1 \leq n \leq N$, and the functional space

$$\mathcal{C}^0_\sigma(\mathbb{R}^N, V) = \{v : \mathbb{R}^N \to V,\ v \text{ continuous in } y,\ \sup_{y\in\mathbb{R}^N} \|\sigma(y)v(y)\|_V < +\infty\},$$

for any Banach space V. We denote by ρ the density of $Y = (Y_1, ..., Y_N)$, we have then $\rho(y) = \prod_{n=1}^N \rho_n(y) = \frac{1}{(2\pi)^{N/2}} e^{-\sum_{n=1}^N \frac{y_n^2}{2}}$. The following inclusion holds true:

$$\mathcal{C}^0_\sigma(\mathbb{R}^N, H_0^1(D)) \subset L^2_\rho(\mathbb{R}^N, H_0^1(D)),$$

with continuous embedding. More precisely, for any $v \in \mathcal{C}^0_\sigma(\mathbb{R}^N, H_0^1(D))$, we have

$$\|v\|_{L^2_\rho(\mathbb{R}^N, H_0^1(D))} \leq k_\alpha \|v\|_{\mathcal{C}^0_\sigma(\mathbb{R}^N, H_0^1(D))}$$

where $k_\alpha = \prod_{n=1}^N \frac{1}{\sqrt{2\pi}} \int e^{-\frac{y_n^2}{2} + 2\alpha_n|y_n|} dy_n$.

Proposition 6.1. *The solution u_N of the equation:*

$$\begin{aligned} -\mathrm{div}_x(a_N(y,x)\nabla_x u_N(y,x)) &= f(x) \text{ on } D, \\ u_N(y,x) = 0 \text{ on } \partial D, & \end{aligned}$$

satisfies $u_N \in \mathcal{C}^0_\sigma(\mathbb{R}^N, H_0^1(D))$.

Proof. We recall that $y \mapsto u(y)$ is continuous from $\mathbb{R}^N$ to $H_0^1(D)$. For all $y \in \mathbb{R}^N$, since $\frac{1}{a_N^{min}(y)} \leq e^{C\sum_{n=1}^N \sqrt{\lambda_n} y_n}$, by (2) we have:

$$\begin{aligned}
\sigma(y)\|u_N(y)\|_{H_0^1(D)} &\leq \sigma(y)\frac{C_D}{a_N^{min}(y)}\|f\|_{L^2(D)} \\
&\leq C_D\|f\|_{L^2(D)}e^{\sum_{n=1}^N (C\sqrt{\lambda_n}-\alpha_n)|y_n|} \\
&\leq C_D\|f\|_{L^2(D)}.
\end{aligned}$$

□

We now give an analyticity result of the solution u_N with respect to y, based on one-dimensional arguments in each direction y_n. We introduce the following notation: for any $1 \leq n \leq N$ and $y \in \mathbb{R}^N$, $y = (y_n, y_n^*)$, where $y_n^* \in \mathbb{R}^{N-1}$. We set $\rho_n^*(y) = \prod_{j\neq n}\rho_j(y)$ and $\sigma_n^*(y) = \prod_{j\neq n}\sigma_j(y)$.

Proposition 6.2. *For any $1 \leq n \leq N$, the solution $u_N(y_n, y_n^*, x)$ as a function of y_n, $u : \mathbb{R} \to \mathcal{C}^0_{\sigma_n^*}(\mathbb{R}^{N-1}, H_0^1(D))$ admits an analytic extension $u(z, y_n^*, x)$, $z \in \mathbb{C}$, in the region of the complex plane $\Sigma(\tau) = \{z \in \mathbb{C},\ dist(z, \mathbb{R}) \leq \tau\}$, for $\tau < 1$, moreover, for all $z \in \Sigma(\tau)$,*

$$\|\sigma_n(\text{Re } z)u_N(z)\|_{\mathcal{C}^0_{\sigma_n^*}(\mathbb{R}^{N-1},H_0^1(D))} \leq \frac{C_D e^{e^{C\sqrt{\lambda_n}}}}{1-\tau}\|f\|_{l^2(D)}e^{\tau\alpha_n}. \tag{3}$$

Proof. For any $y \in \mathbb{R}^N$, $u_N(y)$ satisfies the following variational formulation:

$$\int_D a_N(y,x)\nabla u_N(y,x)\nabla v(x)dx = \int_D f(x)v(x)dx \quad \forall v \in H_0^1(D).$$

Therefore, for every $y \in \mathbb{R}^N$, for any $k \geq 1$, the kth derivative of u_N with respect to y_n satisfies the following variational formulation:

$$\begin{aligned}
&\int_D a_N(y,x)\nabla\frac{\partial^k u_N}{\partial y_n^k}(y,x)\nabla v(x)dx \\
= \quad & -\sum_{l=1}^k \binom{k}{l}\int_D \frac{\partial^l a_N}{\partial y_n^l}(y,x)\nabla\frac{\partial^{k-l}u_N}{\partial y_n^{k-l}}(y,x)\nabla v(x)dx \quad \forall v \in H_0^1(D).
\end{aligned}$$

Since $|\frac{\partial^l a_N}{\partial y_n^l}(y,x)| \leq (\sqrt{\lambda_n}C)^l|a_N(y,x)|$, we obtain the recursive inequalities

$$\left\|\sqrt{a_N(y,x)}\nabla\frac{\partial^k u_N}{\partial y_n^k}(y,x)\right\|_{L^2(D)} \leq \sum_{l=1}^k \binom{k}{l}(C\sqrt{\lambda_n})^l\left\|\sqrt{a_N(y,x)}\nabla\frac{\partial^{k-l}u_N}{\partial y_n^k}(y,x)\right\|_{L^2(D)},$$

$$\frac{\left\|\sqrt{a_N(y,x)}\nabla\frac{\partial^k u_N}{\partial y_n^k}(y,x)\right\|_{L^2(D)}}{k!} \leq \sum_{l=1}^k \frac{(C\sqrt{\lambda_n})^l}{l!}\frac{\|\sqrt{a_N(y,x)}\nabla\frac{\partial^{k-l}u_N}{\partial y_n^k}(y,x)\|_{L^2(D)}}{(k-l)!}.$$

A recurrence yields:

$$\begin{aligned}
\frac{\|\sqrt{a_N(y,x)}\nabla\frac{\partial^k u_N}{\partial y_n^k}(y,x)\|_{L^2(D)}}{k!} &\leq \|\sqrt{a_N(y,x)}\nabla u_N(y,x)\|_{L^2(D)}e^{\sum_{l=1}^k \frac{(C\sqrt{\lambda_n})^l}{l!}} \\
&\leq \|\sqrt{a_N(y,x)}\nabla u_N(y,x)\|_{L^2(D)}e^{e^{C\sqrt{\lambda_n}}}.
\end{aligned}$$

And finally:

$$\frac{\|\frac{\partial^k u_N}{\partial y_n^k}(y,.)\|_{H_0^1(D)}}{k!} \leq \frac{C_D}{a_N^{min}(y)}\|f\|_{L^2(D)} e^{e^{C\sqrt{\lambda_n}}}.$$

We now define for every $y_n \in \mathbb{R}$ the power series $u_N \;:\; \mathbb{C} \to \mathcal{C}^0_{\sigma_n^*}(\mathbb{R}^{N-1}, H_0^1(D))$ as

$$u_N(z, y_n^*, x) = \sum_{k=0}^{+\infty} \frac{(z-y_n)^k}{k!}\frac{\partial^k u_N}{\partial y_n^k}(y_n, y_n^*, x).$$

Since

$$\left\|\frac{|z-y_n|^k}{k!}\frac{\partial^k u_N}{\partial y_n^k}(y_n, y_n^*, x)\right\|_{\mathcal{C}^0_{\sigma_n^*}(\mathbb{R}^{N-1}, H_0^1(D))} \leq \frac{C_D}{a_N^{min}(y_n, y_n^*)}\|f\|_{L^2(D)} e^{e^{C\sqrt{\lambda_n}}}|z-y_n|^k$$

the radius of convergence of this series is 1. Moreover, take $\tau \in]0,1[$, since for all $z \in \mathbb{C}$ such that $|z-y_n| \leq \tau$, $\sigma_n(\mathrm{Re}(z)) \leq e^{\alpha_n \tau}\sigma_n(y_n)$ we have the following estimate:

$$\begin{aligned}
\|\sigma_n(\mathrm{Re}(z))u_N(z)\|_{\mathcal{C}^0_{\sigma_n^*}(\mathbb{R}^{N-1}, H_0^1(D))} &\leq \frac{C_D e^{e^{C\sqrt{\lambda_n}}}}{1-\tau}\|f\|_{L^2(D)} e^{\sum_{i=1}^N (C\sqrt{\lambda_i}-\alpha_i)|y_i|} e^{\alpha_n \tau} \\
&\leq \frac{C_D e^{e^{C\sqrt{\lambda_n}}}}{1-\tau}\|f\|_{L^2(D)} e^{\alpha_n \tau}.
\end{aligned}$$

Hence, by a continuation argument, the function $u_N(y)$ can be extended analytically on the whole region $\Sigma(\tau)$ and estimate (3) follows. $\square$

We recall here the stochastic collocation method: we seek a numerical approximation to the exact solution u_N of the equation

$$\begin{aligned}
-\,\mathrm{div}_x(a_N(y,x)\nabla_x u_N(y,x)) &= f(x) \text{ on } D, \\
u_N(y,x) = 0 \text{ on } \partial D,&
\end{aligned} \tag{4}$$

in a finite dimensional subspace $V_{p,h}$ based on a tensor product, $V_{p,h} = R_p(\mathbb{R}^N) \otimes H_h(D)$, where the following hold:

- $H_h(D) \subset H_0^1(D)$ is a standard finite element space, which contains continuous piece-wise polynomials defined on regular triangulations $\mathcal{T}_h$ that have a maximum mesh spacing parameter $h > 0$.
- $R_p(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N)$ is the span of tensor product polynomials with degree at most $p = (p_1, ..., p_N)$ i.e., $R_p(\mathbb{R}^N) = \otimes_{n=1}^N R_{p_n}(\mathbb{R})$, with

$$R_{p_n}(\mathbb{R}) = \mathrm{span}(y_n^m, m = 0, ..., p_n), \quad n = 1, ..., N.$$

We first introduce the semidiscrete approximation $u_N^h : \mathbb{R} \to H_h(D)$, obtained by projecting (4) onto the subspace $H_h(D)$, for each $y \in \mathbb{R}$, i.e.

$$\int_D a_N(y,x)\nabla u_N^h(y,x)\nabla v(x)dx = \int_D f(x)v(x)dx, \quad \forall v \in H_h(D). \tag{5}$$

The next step consists in collocating (5) on the zeros of orthogonal polynomials and building the discrete solution $u_N^{h,p} \in R_p(\mathbb{R}^N) \otimes H_h(D)$ by interpolating in y the collocated solutions.

For each dimension $n = 1, ..., N$, let y_{n,k_n}, $1 \le k_n \le p_n + 1$, be the $p_n + 1$ roots of the Hermite polynomial q_{p_n+1} of degree $p_n + 1$, which then satisfies $\int_{\mathbb{R}} q_{p_n+1}(y)v(y)\rho(y)dy = 0$ for all $v \in R_{p_n}(\mathbb{R})$. To any vector of indexes $[k_1, ..., k_N]$ we associate the global index

$$k = k_1 + p_1(k_2 - 1) + p_1 p_2(k_3 - 1) + ...$$

and we denote by y_k the point $y_k = [y_{1,k_1}, y_{2,k_2}, ..., y_{N,k_N}] \in \mathbb{R}^N$. We also introduce, for each $n = 1, 2, ..., N$ the Lagrange basis $\{l_{n,j}\}_{j=1}^{p_n+1}$ of the space $R_{p_n}(\mathbb{R})$,

$$l_{n,j} \in R_{p_n}(\mathbb{R}), \quad l_{n,j}(y_{n,k}) = \delta_{jk}, \quad j, k = 1, ..., p_n + 1,$$

where δ_{jk} is the Kronecker symbol, and we set $l_k(y) = \prod_{n=1}^{N} l_{n,k_n}(y_n)$. The points y_k are then the nodes of the Gaussian quadrature formula associated to the weight ρ. Hence, the final approximation is given by

$$u_N^{h,p}(y, x) = \sum_{k=1}^{N_p} u_N^h(y_k, x) l_k(y),$$

where $u_n^h(y_k, x)$ is the solution of problem (5) for $y = y_k$.
Equivalently, if we introduce the Lagrange interpolant operator

$$\mathcal{L}_p : \mathcal{C}^0(\mathbb{R}^N, H_0^1(D)) \to R_p(\mathbb{R}^N) \otimes H_0^1(D),$$

such that

$$\mathcal{L}_p v(y) = \sum_{n=1}^{N} v(y_k) l_k(y), \quad \forall v \in \mathcal{C}^0(\mathbb{R}^N, H_0^1(D)),$$

then we have simply $u_N^{h,p} = \mathcal{L}_p u^h$.
Finally, for any continuous function $g : \mathbb{R}^N \to \mathbb{R}$ we introduce the Gauss quadrature formula $E_\rho^p[g]$ approximating the integral $\int_{\mathbb{R}^N} g(y)\rho(y)dy$ as

$$E_\rho^p[g] = \sum_{k=1}^{N_p} \omega_k g(y_k), \quad \omega_k = \prod_{n=1}^{N} \omega_{k_n}, \quad \omega_{k,n} = \int_{\mathbb{R}} l_{k_n}^2(y)\rho_n(y)dy.$$

This can be used to approximate the law of u_N, i.e. we approximate the expected values $\mathbb{E}[\varphi(u_N(Y(\omega), x))]$ as $\mathbb{E}_\rho^p[\varphi(u_N^h)]$, for some function φ.

Our aim is to give an a priori estimate for the total error $\varepsilon = u_N - u_N^{h,p}$ in the natural norm $L_\rho^2(\mathbb{R}^N, H_0^1(D))$. This total error naturally splits into $\varepsilon = (u_N - u_N^h) + (u_N^h - u_N^{h,p})$. The first term is a term of space discretization error and can be estimate easily, indeed, for all $y \in \mathbb{R}^N$, the function $u_N^h(y)$ is the orthogonal projection of $u_N(y)$ onto the subspace $H_h(D)$ with respect to the inner product $(u, v) \to \int_D a_N(y, x)\nabla u(x)\nabla v(x)dx$. Therefore, for all $y \in \mathbb{R}^N$,

$$\begin{aligned}
\|(u_N - u_N^h)(y)\|_{H_0^1(D)} &\le \sqrt{\frac{1}{a_N^{min}(y)}} \inf_{v \in H_h(D)} \left(\int_D a_N(y, x) |\nabla(u_N(y) - v)|^2 dx \right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{a_N^{max}(y)}{a_N^{min}(y)}} \inf_{v \in H_h(D)} \|u_N(y) - v\|_{H_0^1(D)} \\
&\le e^{C \sum_{n=1}^{N} \sqrt{\lambda_n} |y_n|} \inf_{v \in H_h(D)} \|u_N(y) - v\|_{H_0^1(D)}.
\end{aligned}$$

We can finally conclude for the first term, thanks to the standard approximation estimate for the finite element space $H_h(D)$, there exists a constant C_{fe} such that for any $v \in H^2(D)$ and $h > 0$

$$\min_{w \in H_h(D)} \|v - w\|_{H_0^1(D)} \leq C_{fe} h \|v\|_{H^2(D)}.$$

Since $a_N(y)$ is smooth for any $y \in \mathbb{R}^N$, elliptic regularity yields that $u_N(y) \in H^2(D)$ for all $y \in \mathbb{R}^N$, with:

$$\begin{aligned}
\|u_N(y)\|_{H^2(D)} &\leq k \frac{\|f\|_{L^2(D)}}{a_N^{min}(y)} \left(1 + \frac{a_N^{max}(y)}{a_N^{min}(y)}\right) \left(1 + \frac{a_N^{max}(y)}{a_N^{min}(y)} + \frac{\|a_N'(y)\|_{L^\infty(D)}}{a_N^{min}(y)}\right) \\
&\leq k \frac{\|f\|_{L^2(D)}}{a_N^{min}(y)} \left(1 + \frac{a_N^{max}(y)}{a_N^{min}(y)}\right)^2 (1 + \|g_N'(y)\|_{L^\infty(D)})
\end{aligned}$$

where k is a constant independent of f, N and y whose value changes. We can then bound the spacial discretization error, using Hölder inequality and proposition 3.4:

$$\begin{aligned}
&\|(u_N - u_N^h)(y,x)\|_{L^2_\rho(\mathbb{R}^N, H_0^1(D))} \leq \\
&k C_{fe} h \|f\|_{L^2(D)} \left\| \frac{1}{a_N^{min}(y)} \sqrt{\frac{a_N^{max}(y)}{a_N^{min}(y)}} \left(1 + \frac{a_N^{max}(y)}{a_N^{min}(y)}\right)^2 (1 + \|g_N'(y)\|_{L^\infty(D)}) \right\|_{L^2_\rho(\mathbb{R}^N)} \\
&\leq k C_{fe} \|f\|_{L^2(D)} D_8 (1 + D_{16})^5 (1 + \|g_N'(y)\|_{L^{16}(\mathbb{R}^N, \mathcal{C}^0(D))}) h.
\end{aligned}$$

Proposition 6.3. *There exists a constant k independent of N such that for all $h > 0$*

$$\|u_N - u_N^h\|_{L^2_\rho(\mathbb{R}^N, H_0^1(D))} \leq k \|f\|_{L^2(D)} (1 + \|g_N'(y)\|_{L^{16}_\rho(\mathbb{R}^N, \mathcal{C}^0(D))}) h.$$

In particular, if the eigenfunctions b_n have bounded derivatives of order two and if there exists $0 < \theta < 1$ such that the the series

$$\sum_{n \geq 1} \lambda_n \|b_n'\|_\infty^{2(1-\theta)} \|b_n''\|_\infty^{2\theta} < +\infty,$$

then there exists a constant k' independent of N such that for all $h > 0$

$$\|u_N - u_N^h\|_{L^2_\rho(\mathbb{R}^N, H_0^1(D))} \leq k' \|f\|_{L^2(D)} h.$$

Otherwise we have the following rough bound:

$$\|u_N - u_N^h\|_{L^2_\rho(\mathbb{R}^N, H_0^1(D))} \leq k \|f\|_{L^2(D)} h 2^N e^{\sum_{n=1}^N \lambda_n n^{2a}}.$$

Proof. The first inequality follows from what precedes. Under the additional conditions on the eigenpairs (λ_n, b_n) we can prove, similary as in proposition 3.3 that for any $p > 0$ there exists a constant $\tilde{B}_p$ such that for any $N \in \mathbb{N}$, we have

$$\|g_N'(y)\|_{L^p_\rho(\mathbb{R}^N, \mathcal{C}^0(D))} \leq \tilde{B}_p.$$

This bound combined with the previous bound yields a bound for the finite element error independent of N. The last bound, available without additional assumptions follows from the first bound:

$$\begin{aligned}
\|u_N - u_N^h\|_{L^2_\rho(\mathbb{R}^N, H^1_0(D))} &\leq k\|f\|_{L^2(D)}\|1+\sum_{n=1}^N \sqrt{\lambda_n}|y_n|n^a\|_{L^{16}_\rho(\mathbb{R}^N,\mathcal{C}^0(D))}h \\
&\leq k\|f\|_{L^2(D)}h\left(\int_{\mathbb{R}^N}\prod_{n=1}^N (2\pi)^{-1/2}e^{16\sqrt{\lambda_n}|y_n|n^a-\frac{y_n^2}{2}}dy\right)^{1/16} \\
&\leq k\|f\|_{L^2(D)}h2^{\frac{N}{16}}e^{\sum_{n=1}^N \lambda_n n^{2a}}.
\end{aligned}$$

□

Remark: In the general case, the bound explodes as $N \to +\infty$. In particular in the case of an exponential covariance (see further, section 7.1), the bound explodes, which is coherent with the fact that the trajectories of solution u does not belong to H^2 since a does not belong to $\mathcal{C}^1$. However, in the case of an analytic covariance (see further, section 7.2) we can obtain a bound for the finite element error which is independent of N (see section 7.2 combined with the last point of the previous proposition).

The second term $u_N^h - u_N^{h,p}$ is an interpolation error, indeed $u_N^{h,p} = \mathcal{L}_p u_N^h$. First, we recall some known results of approximation theory for functions defined on a one-dimensional domain with values in a Banach space denoted by V. We recall that the mono-dimensional weights ρ_1 and σ_1 are defined on $\mathbb{R}$ by $\rho_1(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$ and $\sigma_1(y) = e^{-\alpha|y|}$ for some $\alpha > 0$, and consider a mono-dimensional weight ν such that there exists a constant C with $\nu(y) \geq Ce^{-\frac{y^2}{4}}$ The proof of the following three propositions can be found in [1] and in the references therein.

Proposition 6.4. *The operator $\mathcal{L}_p : \mathcal{C}^0_\nu(\mathbb{R}, V) \to L^2_{\rho_1}(\mathbb{R}, V)$ is continuous. The norm of this linear continuous operator will be denoted by C_1.*

The following proposition, which is a consequence of a result from Uspensky [24], relates the approximation error $v - \mathcal{L}_p v$ in the L^2_ρ-norm with the best approximation error in the weighted $\mathcal{C}^0_{\sigma_1}$-norm.

Proposition 6.5. *For every function $v \in \mathcal{C}^0_\nu(\mathbb{R}, V)$ the interpolation error satisfies*

$$\|v - \mathcal{L}_p v\|_{L^2_{\rho_1}(\mathbb{R},V)} \leq C_2 \inf_{w \in R_p(\mathbb{R})\otimes V}\|v - w\|_{\mathcal{C}^0_\nu(\mathbb{R},V)}$$

with a constant C_2 independent of p.

We now analyze the best approximation error for a function $v : \mathbb{R} \to V$ which admits an analytic extension in the complex plane, in the region $\Sigma(\tau) = \{z \in \mathbb{C}, \quad \mathrm{dist}(z, \mathbb{R}) \leq \tau\}$ for some $\tau > 0$. We still denote the extension by v; in this case, τ represents the distance between $\mathbb{R}$ and the nearest singularity of $v(z)$ in the complex plane. The following result is a consequence of a result from Hille [14].

Proposition 6.6. *Let v be a function in $\mathcal{C}^0_{\sigma_1}(\mathbb{R}, V)$. We assume that v admits an analytic extension in the strip of the complex plane $\Sigma(\tau)$ for some $\tau > 0$, and that*

$$\forall z = (y + iw) \in \Sigma(\tau), \quad \sigma_1(y)\|v(z)\|_V \leq C_v(\tau).$$

Then, for any $\delta > 0$ there exists a constant C, independent of p, and a function $\Theta(p) = O(\sqrt{p})$ such that

$$\min_{w\in R_p(\mathbb{R})\otimes V}\max_{y\in\mathbb{R}}\left|\|v(y)-w(y)\|_V e^{-\frac{(\delta y)^2}{4}}\right| \leq C\Theta(p)e^{-\tau\delta\sqrt{p}}.$$

We are now ready to prove the following proposition, which gives an estimate of the total interpolation error:

Proposition 6.7. *For any $\tau < 1$, there exists a constant $C_{\tau,N}$, independent of h and p such that*

$$\|u_N^h - u_N^{h,p}\|_{L^2_\rho(\mathbb{R}^N,H^1_0(D))} \leq C_{\tau,N}\sum_{n=1}^{N}\sqrt{p_n}e^{-\frac{\tau}{\sqrt{2}}\sqrt{p_n}}.$$

Remark: We could precise how the constant $C_{\tau,N}$ depends on N, but it would not be really explicit and not necessary relevant.

Proof. From now on, we are back to the the multi-dimensional problem, we are going to split the total interpolation error into N partial errors linked to one-dimensional interpolation errors.
To begin with, repeating the arguments of proposition 6.2, we obtain that u_N^h has the same regularity with respect to y as the exact solution u_N , i.e. more precisely that u_N^h verifies the same properties as proved for u_N in proposition 6.2. We focus on the first direction y_1 and define a one-dimensional interpolation operator
$\mathcal{L}_1 : \mathcal{C}^0_{\sigma_1}(\mathbb{R}, L^2_{\rho_1^*}(\mathbb{R}, H^1_0(D))) \to L^2_{\rho_1}(\mathbb{R}, L^2_{\rho_1^*}(\mathbb{R}, H^1_0(D)))$, by

$$\mathcal{L}_1(y_1, y_1^*, x) = \sum_{k=1}^{p_1+1} v(y_{1,k}, y_1^*, x)l_{1,k}(y_1).$$

Then, the global interpolant $\mathcal{L}_p$ can be written as the composition of two interpolation operators $\mathcal{L}_p = \mathcal{L}_1 \circ \mathcal{L}_p^{(1)}$, where $\mathcal{L}_p^{(1)}$ is the interpolation in all the directions $y_2, y_3, ..., y_N$ except y_1, i.e. $\mathcal{L}_p^{(1)} : \mathcal{C}^0_{\sigma_1^*}(\mathbb{R}^{N-1}, H^1_0(D)) \to L^2_{\rho_1^*}(\mathbb{R}^{N-1}, H^1_0(D))$. We have then

$$\|u_N^h - \mathcal{L}_p u_N^h\|_{L^2_\rho(\mathbb{R}^N,H^1_0)} \leq \|u_N^h - \mathcal{L}_1 u_N^h\|_{L^2_\rho(\mathbb{R}^N,H^1_0)} + \|\mathcal{L}_1(u_N^h - \mathcal{L}_p^{(1)}u_N^h)\|_{L^2_\rho(\mathbb{R}^n,H^1_0)}.$$

Let us bound the first term. We think of u_N^h as a function of y_1 with values in a Banach space V, $u_N^h \in L^2_{\rho_1}(\mathbb{R}, V)$, where $V = L^2_{\rho_1^*}(\mathbb{R}^{N-1}, H^1_0(D))$. The following inclusions hold true:

$$\mathcal{C}^0_{\sigma_1}(\mathbb{R}, V) \subset \mathcal{C}^0_{G_1}(\mathbb{R}, V) \subset L^2_{\rho_1}(\mathbb{R}, V)$$

with $G_1(y_1) = e^{-\frac{y_1^2}{8}}$. Since $G_1 \geq Ce^{-\frac{y^2}{4}}$, Proposition 6.5 yields the following estimate

$$\|u_N^h - \mathcal{L}_1 u_N^h\|_{L^2_\rho(\mathbb{R}^n,H^1_0(D))} \leq C_2 \inf_{w\in R_{p_1}(\mathbb{R})\otimes V}\|u_N^h - w\|_{\mathcal{C}^0_{G_1}(\mathbb{R},V)}.$$

To bound the best approximation error in $\mathcal{C}^0_{G_1}(\mathbb{R}, V)$, we employ the fact that $u_N^h \in \mathcal{C}^0_{\sigma_1}(\mathbb{R}, V)$ and the analyticity result of Proposition 6.2, which yields the analitycity of u_N^h as a function of $\mathbb{R}$ into V on $\Sigma(\tau)$ for every $\tau < 1$ and the following bound, for all $z \in \Sigma(\tau)$:

$$\sigma_1(\mathrm{z})\|u_N^h(z)\|_{\mathcal{C}^0_{\sigma_1^*}(\mathbb{R}^{N-1},H^1_0(D))} \leq \frac{C_D e^{e^{C\sqrt{\lambda_1}}}}{1-\tau}\|f\|_{L^2(D)}e^{\alpha_1\tau},$$

which implies, thanks to the continuous embedding of $\mathcal{C}^0_{\sigma_1^*}(\mathbb{R}^{N-1}, H_0^1(D))$ into $L^2_{\rho_1^*}(\mathbb{R}^{N-1}, H_0^1(D))$, that

$$\sigma_1(\mathrm{z})\|u_N^h(z)\|_{L^2_{\rho_1^*}(\mathbb{R}^{N-1},H_0^1(D))} \leq k_{\alpha_1^*}\frac{C_D e^{e^{C\sqrt{\lambda_1}}}}{1-\tau}\|f\|_{L^2(D)}e^{\alpha_1\tau}.$$

Recalling that we have fixed $V = L^2_{\sigma_1^*}(\mathbb{R}^{N-1}, H_0^1(D))$ in this proof, we are now ready to conclude with the first term by applying proposition 6.6 to $u_N^h \in \mathcal{C}^0_{\sigma_1}(\mathbb{R}, V)$, for $\tau < 1$ and $\delta_1 = \frac{1}{\sqrt{2}}$, which gives the following bound for the best approximation error:

$$\inf_{w\in R_{p_1}(\mathbb{R})\otimes V}\|u_N^h - w\|_{\mathcal{C}^0_{G_1}(\mathbb{R},V)} \leq C\Theta(p_1)e^{-\frac{\tau}{\sqrt{2}}\sqrt{p_1}}.$$

It gives then the following bound for the first term:

$$\|u_N^h - \mathcal{L}_1 u_N^h\|_{L^2_\rho(\mathbb{R}^n,H_0^1(D))} \leq C_2 C\Theta(p_1)e^{-\frac{\tau}{\sqrt{2}}\sqrt{p_1}}.$$

Let us bound the second term. By Proposition 6.4 applied with $\nu = \sigma_1$, we bound the second term:

$$\|\mathcal{L}_1(u_N^h - \mathcal{L}_p^{(1)}u_N^h)\|_{L^2_\rho(\mathbb{R}^n,H_0^1(D))} \leq C_1\|u_N^h - \mathcal{L}_p^{(1)}u_N^h\|_{\mathcal{C}^0_{\sigma_1(\mathbb{R},V)}}.$$

The term on the right-hand size is again an interpolation error. Thus we have to bound the interpolation error in all the other $N-1$ directions, uniformly with respect to y_1 (in the weighted norm $\mathcal{C}^0_{\sigma_1}$). We can proceed iteratively, defining an interpolation L^2, bounding the resulting error in the direction y_2, and so on. □

7 Examples

In this section, we give examples of covariance kernel $cov[g]$ and the strong and weak convergence results corresponding.

7.1 The exponential kernel case on a box

In this subsection, we consider the case where the covariance kernel $cov[g]$ is exponential, i.e. $k(x) = \sigma^2 e^{-\frac{x}{\ell}}$ and D is a box, the length ℓ is called the correlation length and the norm choosen on D is $\|x-y\|_1 = \sum_{i=1}^d |x_i - y_i|$. In this case the eigenvalues and the eigenfunctions can be found analytically, we first show that the assumptions done in section 1 are fulfilled and then give the convergence rate for the strong and weak convergence of u_N to u. We first treat the mono-dimensional case, for convenience we suppose that the domain D is $(0, 1)$. We have then analytic expressions for the eigenvalues and eigenfunctions, the proof can be found in [29]. We consider the characteristic equation

$$(\ell^2 w^2 - 1)\sin(w) = 2\ell w\cos(w)$$

and denote by $(w_n)_{n\geq 1}$ the sequence of its positive roots sorted in an increasing number, then the eigenvalues of the Karhunen-Loève development can be expressed as $\lambda_n = \frac{2\ell\sigma^2}{\ell^2 w_n^2+1}$ and the eigenfunctions as $b_n(x) = \alpha_n(\sin(w_n x) + \ell w_n\cos(w_n x))$ where $\alpha_n = \frac{1}{\sqrt{(\ell^2 w_n^2+1)/2+\ell}}$.

The sequence of the roots (w_n) satisfies $w_n \underset{n\to+\infty}{\sim} n\pi$, which yields the following equivalents for the eigenvalues and eigenfunctions:

$$\lambda_n \underset{n\to+\infty}{\sim} \frac{2\sigma^2}{\ell\pi^2 n^2}$$

$$b_n(x) \underset{n\to+\infty}{\sim} \sqrt{2}\cos(w_n x).$$

We can now show that such a covariance kernel fulfills the assumptions done in section 1: $k \in \mathcal{C}^{0,1}(\mathbb{R})$, the eigenfunctions b_n are continuously differentiable, and we have $\|b_n\|_\infty \leq 2\sqrt{2}$, $\|b'_n\|_\infty \leq 2\sqrt{2}(n+1/2)\pi$ for every $n \geq 0$ and for any $b < 1$, $\sum_{n\geq 1} \lambda_n n^b$ is convergent. We can therefore apply the results of sections 3 and 4. The application of Theorem 4.2 gives the following strong convergence result:

Proposition 7.1. *For every $p > 0$ and $0 < \alpha < 1$, there exists a constant $C_{\alpha,p}$ such that for any $N \in \mathbb{N}$,*

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq C_{\alpha,p} N^{\frac{\alpha-1}{2}}.$$

Proof. Take $0 < \alpha < 1$, we have

$$R_N^\alpha \quad = \quad \sum_{n>N} \lambda_n n^\alpha \leq \frac{2\sigma^2}{\ell\pi^2} \sum_{n>N} n^{\alpha-2} \leq \frac{2\sigma^2}{\ell\pi^2(1-\alpha)} N^{\alpha-1}.$$

□

On the other hand, Theorem 5.3 yields the following weak convergence result:

Proposition 7.2. *There exists a constant C such that for any $N \in \mathbb{N}$ and $\varphi \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ whose derivatives are bounded by a constant C_φ, we have:*

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq C C_\varphi \frac{1}{N}.$$

We now treat the case where the spatial dimension is $d = 2$, the following result can be extended for any dimension d. We choose for the sake of simplicity $D = (0,1)^2$, but it can be immediately generalized to the case where D is a box. We denote by $(\mu_n)_{n\geq 1}$ the sequence of the eigenvalues sorted in a decreasing order and by $(c_n)_{n\geq 1}$ the corresponding eigenfunctions. Since the b_n are distinct, for any $n \geq 1$, there exists a unique $(i,j) \in (\mathbb{N}^*)^2$ such that $c_n(x) = b_i(x_1)b_j(x_2)$ and we have then $\mu_n = \lambda_i\lambda_j$. Indeed the eigenvectors in dimension 2 are obtained as the tensor product of the mono-dimensional eigenvectors, since we choose the norm $\|.\|_1$. We can then define the following bijective function :

$$\begin{aligned} g \quad : \quad & (\mathbb{N}^*)^2 \to \mathbb{N}^* \\ & (i,j) \mapsto n \text{ such that } c_n(x) = b_i(x_1)b_j(x_2) \end{aligned}$$

First we notice that $\left(n - \frac{1}{2}\right)\pi \leq w_n \leq \left(n + \frac{1}{2}\right)\pi$, therefore, for any $n \geq 1$

$$\lambda_n \leq \frac{2\ell\sigma^2}{\ell^2\left(n-\frac{1}{2}\right)^2\pi^2} \leq \frac{8\sigma^2}{\ell\pi^2}\frac{1}{n^2}$$

and

$$\lambda_n \geq \frac{2\ell\sigma^2}{\ell^2\left(n+\frac{1}{2}\right)^2\pi^2 + 1} \geq \frac{2\ell\sigma^2}{1+4\pi^2\ell^2}\frac{1}{n^2}.$$

We recall that the mono-dimensional eigenfunctions b_n are continuously differentiable and that there exists a constant C such that $\|b_n\|_\infty \leq C$ and $\|b'_n\|_\infty \leq Cn$ for all $n \geq 1$. Therefore, for any $(i,j) \in (\mathbb{N}^*)^2$, $c_{g(i,j)}(x) = b_i(x_1)b_j(x_2)$ is continuously differentiable, $\|c_{g(i,j)}\|_\infty \leq \|b_i\|_\infty \|b_j\|_\infty \leq C^2$, and

$$\|\nabla c_{g(i,j)}\|_\infty = \left\| \begin{pmatrix} b'_i(x_1)b_j(x_2) \\ b_i(x_1)b'_j(x_2) \end{pmatrix} \right\|_\infty \leq C^2(i+j) \leq 2C^2 ij.$$

$$\begin{aligned} g(i,j) &\geq Card\,\{(p,q)|\lambda_p\lambda_q > \lambda_i\lambda_j\} \\ &\geq Card\left\{(p,q)|pq \leq \frac{\ell^2\pi^2}{4(1+4\pi^2\ell^2)} ij\right\} \\ &\geq C_\ell\; ij, \end{aligned}$$

where C_ℓ is a constant which depends only on ℓ, because

$$\begin{aligned} Card\{(i,j)|ij \leq n\} &\geq n + E[\frac{n}{2}] + E[\frac{n}{3}] + ... + 1 \\ &\geq n(1+\log(n)). \end{aligned}$$

We now have the required bound on the first derivatives of the eigenfunctions: for any $i,j \geq 1$

$$\|c'_{g(i,j)}\|_\infty \leq 2C^2 ij \leq \frac{2C^2}{C_l} g(i,j),$$

which means that for any $n \geq 1$, $\|c'_n\|_\infty \leq \frac{2C^2}{C_l} n$.
We now prove that the assumptions on the decay of the eigenvalues are fulfilled. Take (i,j) such that $ij = n$, then if (p,q) is such that $\lambda_p\lambda_q \geq \lambda_i\lambda_j$ we have:

$$\left(\frac{8\ell\sigma^2}{\ell^2\pi^2}\right)^2 \frac{1}{p^2q^2} \geq \lambda_p\lambda_q \geq \lambda_i\lambda_j \geq \left(\frac{2\ell\sigma^2}{1+4\pi^2\ell^2}\right)^2 \frac{1}{n^2}$$

which yields the following bound for $g(i,j)$:

$$\begin{aligned} g(i,j) &\leq Card\,\{(p,q)|\lambda_p\lambda_q \geq \lambda_i\lambda_j\} \\ &\leq Card\left\{(p,q)|pq \leq \frac{4(1+4\pi^2\ell^2)}{\ell^2\pi^2} n\right\} \\ &\leq C'_\ell\; n\log(n), \end{aligned}$$

where C'_ℓ is a constant which depends only on ℓ, because

$$\begin{aligned} Card\{(i,j)|ij \leq n\} &\leq n + \frac{n}{2} + \frac{n}{3} + ... + 1 \\ &\leq n(1+\log(n+1)). \end{aligned}$$

We can now prove that the last assumption made in section 3 is fulfilled.
Take $b < 1$

$$\sum_{ij=n} \lambda_i\lambda_j g(i,j)^b \quad \leq \quad d(n)C'_\ell \left(\frac{8\ell\sigma^2}{\ell^2\pi^2}\right) (n\log(n))^b \frac{1}{n^2}$$

where $d(n)$ is the number of divisors of n. For any $\varepsilon > 0$ such that $b + \varepsilon < 1$, there exists a constant $k_{\ell,\varepsilon}$ which depends only on ℓ and ε such that:

$$\sum_{ij=n} \lambda_i \lambda_j g(i,j)^b \quad \leq \quad k_{\ell,\varepsilon} n^{b+\varepsilon-2}.$$

Therefore the series $\sum_{n\geq 1} \mu_n n^b$ is convergent for any $b < 1$. We can then apply Theorem 4.2 which gives a strong convergence result

Proposition 7.3. *For every $p > 0$ and $0 < \beta < 1$, there exists a constant $C_{\beta,p}$ such that for any $N \in \mathbb{N}$,*

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq C_{\beta,p} N^{\frac{\beta-1}{2}}.$$

Proof. For any $\varepsilon > 0$ there exists a constant $C_{\ell,\varepsilon}$ such that for all $i, j \geq 1$

$$g(i,j) \leq C_{\ell,\varepsilon}(ij)^{1+\varepsilon}.$$

Therefore $g(i,j) > N$ implies $ij > p_{N,\varepsilon} = \left(\frac{N}{C_{\ell,\varepsilon}}\right)^{\frac{1}{1+\varepsilon}}$ for any $\varepsilon > 0$.
Take $0 < \alpha < 1, \varepsilon > 0$, with $\alpha + \varepsilon < 1$, we have

$$\begin{aligned} R_N^\alpha &= \sum_{n>N} \mu_n n^\alpha \leq \sum_{n>p_{N,\varepsilon}} \sum_{ij=n} \lambda_i \lambda_j g(i,j)^\alpha \\ &\leq k_{\ell,\varepsilon} \sum_{n>p_{N,\varepsilon}} n^{\alpha+\varepsilon-2} \\ &\leq k_{\ell,\varepsilon} \frac{(p_{N,\varepsilon})^{\alpha+\varepsilon-1}}{1-\alpha-\varepsilon} \\ &\leq \frac{k_{\ell,\varepsilon}}{(1-\alpha-\varepsilon) C_{\ell,\varepsilon}^{\frac{\alpha+\varepsilon-1}{1+\varepsilon}}} N^{\frac{\alpha+\varepsilon-1}{1+\varepsilon}} \end{aligned}$$

□

On the other hand, Theorem 5.3 yields the following weak convergence result:

Proposition 7.4. *For any $\varepsilon > 0$, there exists a constant C_ε such that for any $N \in \mathbb{N}$ and $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ, we have:*

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq C_\varepsilon C_\varphi N^{-1+\varepsilon}.$$

Proof. Take $1 > \varepsilon > 0$, we have

$$\begin{aligned} R_N^0 &= \sum_{n>N} \mu_n \leq \sum_{n>p_{N,\varepsilon}} \sum_{ij=n} \lambda_i \lambda_j \\ &\leq \frac{k_{\ell,\varepsilon}}{(\varepsilon-1) C_{\ell,\varepsilon}^{\frac{\varepsilon-1}{1+\varepsilon}}} N^{\frac{\varepsilon-1}{1+\varepsilon}} \end{aligned}$$

□

7.2 The analytic covariance kernel case

We suppose here that the covariance kernel $cov[g]$ is analytic on $D \times D$, which is the case of a gaussian covariance kernel $cov[g](x,y) = \sigma^2 e^{-\frac{\|x-y\|^2}{\ell^2|D|^2}}$ in particular. Then we have the following result from Frauenfelder, Schwab and Todor given in [7] about the eigenvalues decay and about the decay of the derivatives of the eigenfunctions:

Proposition 7.5. *There exists two constants $c_1, c_2 > 0$ such that for all $n \geq 1$*

$$\lambda_n \leq c_1 e^{-c_2 n^{1/d}}.$$

For any $s > 0$ there exists a constant c_s such that for any $n \geq 1$,

$$\|b_n\|_\infty \leq c_s |\lambda_n|^{-s} \text{ and } \|b_n'\|_\infty \leq c_s |\lambda_n|^{-s}.$$

The methods developed to prove the results given in the previous sections apply with some slight modifications, leading to similar results. In particular we have the following strong error and weak error results:

Proposition 7.6. *For any $0 < s < \frac{1}{2}$, and $p > 0$, there exists a constant $H_{s,p}$ such that for all $N \in \mathbb{N}$*

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq H_{s,p} \sqrt{\sum_{n>N} \lambda_n^{1-2s}},$$

therefore, for any $0 < s < 1/2$ and $p > 0$ there exists a constant $I_{d,s,p}$ depending on p, s, d, c_1 and c_2 such that for any $N \in \mathbb{N}$

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq I_{d,s,p} N^{\frac{d-1}{2d}} e^{-\frac{c_2(1-2s)}{2} N^{1/d}}.$$

Proposition 7.7. *For any $0 < s < \frac{1}{2}$, there exists a constant J_s such that for all $N \in \mathbb{N}$, for all $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ, we have:*

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq J_s C_\varphi \sum_{n>N} \lambda_n^{1-2s},$$

therefore, for any $0 < s < \frac{1}{2}$, there exists a constant $K_{d,s}$ depending on d, s, c_1 and c_2 such that for all $N \in \mathbb{N}$, for all $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq K_{d,s} C_\varphi N^{\frac{d-1}{d}} e^{-c_2(1-2s)N^{1/d}}.$$

8 Conclusions

In this work we have established estimates of the error on the solution of the elliptic partial differential equation, resulting from the approximation of the lognormal random field a through the truncature of the KL expansion of $\log(a)$. This approximation is indeed the first step of several numerical methods, in particular galerkin stochastic methods and collocation methods. In these methods, since the computational cost increases very fast with the truncature order N, it is crucial to have good estimates of the error commited on the solution u.

We first showed that the strong error decreases like the error on the KL expansion of $\log(a)$ in the natural $L^2(\Omega \times D)$ norm, i.e. is bounded by the squared root of the remainder of

the eigenvalues series. We next showed that this bound can be improved by looking at the weak error, which is a natural quantity of interest since we are interested on the law of the solution. The bound for the weak error is indee d the square of the previous bound for the strong error.
We complete then this work by generalizing the result of [1], which gives an estimate of the collocation error, to the case considered here where the random field a is neither uniformly bounded nor uniformly coercive with respect to ω.
Finally we show that the strong and weak error results apply to two examples which are important on a practical point of view, the case of an exponential covariance and the case of an analytic covariance, which includes the case of a gaussian covariance in particular. We give then explicit bounds for the error in these two cases which are among the most frequently used to model permeability fields in the context of flow computation in porous media.
The analysis of the dependance of the error on the correlation length ℓ and on the multiplicative factor σ in the covariance is the subject of ongoing research.

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