Wavelet Coefficients of Levy Process

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***Abstract*—Cet article presente une expression de la fonction caracteristique des coefficients de decomposition en ondelettes d'un processus de Levy. Le cas particulier de l'ondelette de Haar et d'un processus entrelace est traite plus en detail.**

***Abstract*—The main object of the paper is to study the wavelet decomposition of Levy processes by wavelets with compact support. The general result was applied to the interlacing process with finite different jumps by Haar wavelets.**

Keywords : Ondelettes, Processus

I. Introduction

Estimation of wavelet coefficients of Levy process is a research subject of both theoretical and practical interest as well as its use in a large number of practical application. The statement of problem came from exploring of aircraft trajectories. Fig. 1 and Fig. 2 provide a prototype for the type of altitude of aircraft landing trajectories and vertical velocity. Exploring these different trajectories we can conclude that the velocity of the altitude is an interlacing process. Wavelet decomposition of the altitude velocity involved us to get more general result, i.e. wavelet composition of Levy process by wavelets with compact support.

The theory of stochastic processes was one of the important mathematical developments of the twentieth century. The tools with which this is made precise were provided by A. N. Kolmogorov in the 1930s. In this article we will study wavelet analysis of the class of stochastic processes called Levy processes. Paul Levy first studied them in the 1930s.

Many scientists worked on getting wavelet coefficients of Levy process particular case and achieved definite results [5], [6]. J.C. Simon de Miranda worked on probability density function of the empirical wavelet coefficients of a Poisson process. J. Istas calculated wavelet coefficients of a Gaussian process. The idea of this article is to find wavelet coefficients of general form of the Levy process.

II. Some basics and notations

A. Levy process

Levy processes, i.e. processes in continuous time with stationary and independent increments, are named after Paul Levy: he made the connection with infinitely divisible distributions (The Levy-Khintchine Formulae) and described their structure. Let us give a definition of Levy process:

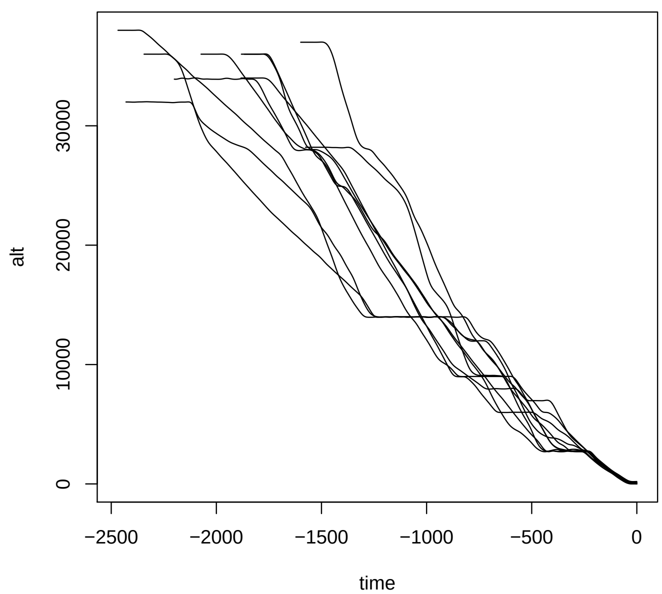

Figure 1. The altitude of 10 aircraft trajectories.

Definition 1. *A Levy process $X = (X_t)_{t\geq 0} = (X(t,\omega))_{t\geq 0}$ is a stochastic process satisfying the following:*

1) *Each $X_0 = 0$ (with probability one),*
2) *X has independent and stationary increments,*
3) *X is stochastically continuous, i.e. for all $\varepsilon > 0$ and for all $s \geq 0$*

$$\lim_{t\to s} \mathbb{P}(|X_t - X_s| > \varepsilon) = 0.$$

Here and in the future we will suppose that we have are called Levy measures.

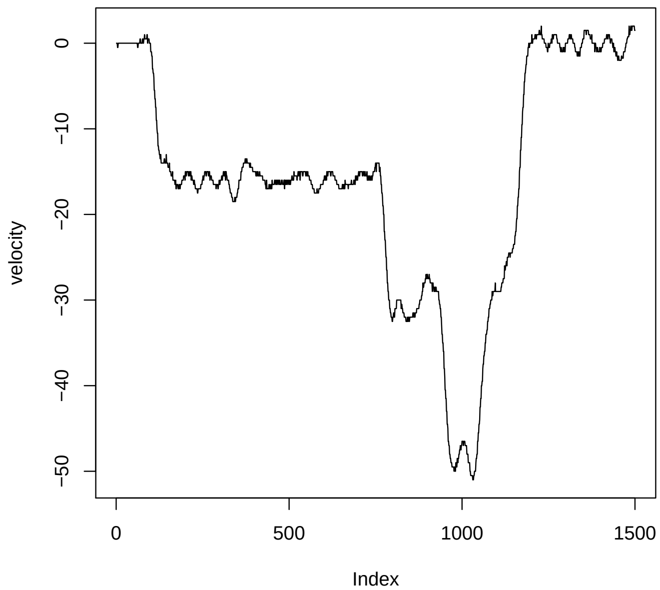

Figure 2. The velocity of aircraft trajectory altitude.

Levy process modification $X' = (X'(t))_{t\geq 0}$ of process $X = (X(t))_{t\geq 0}$, such that $X' = X'(t,\omega)$ is right-continuous with left limit, see [4].

To understand the structure of a generic Levy process, we employ Fourier analysis. The characteristic functions of Levy process were completely characterized by Levy and Khintchine in the 1930s.

Theorem 1. The Levy-Khintchine Formula. *If* $X = \{X(t)\}_{t\geq 0}$ *is a Levy process, then* $\phi_t(\theta) = e^{t\eta(\theta)}$*, for each* $t \geq 0, \theta \in \mathbb{R}$*, where*

$$\eta(\theta) = ia\theta - \frac{1}{2}\sigma^2\theta^2 + \int\limits_{\mathbb{R}/\{0\}} \left(e^{i\theta y} - 1 - i\theta y I_{||y||<1}(y)\right)\nu(\mathrm{d}y),$$

for some $a \in \mathbb{R}, \sigma \geq 0$ *and a Borel measure* ν *on* $\mathbb{R}/\{0\}$ *for which*

$$\int\limits_{\mathbb{R}/\{0\}} (|y|^2 \wedge 1)\nu(\mathrm{d}y) < \infty.$$

Conversely, given a mapping of the form (1) we can always construct a Levy process for which $\phi_t(\theta) = e^{t\eta(\theta)}$.

The triple (a, σ^2, ν) is called the characteristics of X. It determines the law p_t . The measures ν that can appear in (1) are called Levy measures.

B. Wavelets

For many years, the sine, cosine and imaginary exponential functions have been the basic functions of analysis. The sequence $(2\pi)^{1/2}e^{ikx}, k = 0, \pm 1, \pm 2, \ldots$ forms an orthonormal basis of the standard space $L_2[0, 2\pi]$; Fourier series are the linear combinations $\sum a_k e^{ikx}$. Their study has been and remains, an unquenchable source of problems and discoveries in mathematical analysis. The problem arise from the absence of a good dictionary for translating the properties of a function into those of its Fourier coefficients.

At the beginning of 1980s, many scientists were already using ”wavelets” as an alternative to traditional Fourier analysis. This alternative gave grounds for hoping for simpler numerical analysis and more robust synthesis of certain transitory phenomena. The theory of wavelets was developed by Y.Meyer, I.Daubechies, S.Mallat and others in the end of 1980s (see [1], [3], [4]).

The word ”wavelet” is used in mathematics to denote a kind of orthonormal bases in L_2 with remarkable approximation properties. This means that any $f \in L_2(\mathbb{R})$ can be represented as a series (convergent in $L_2(\mathbb{R})$):

$$f(x) = \sum_k \alpha_k \varphi_{0k}(x) + \sum_{j=0}^{\infty} \sum_k \beta_{jk}\psi_{jk}(x) \tag{2.1}$$

where α_k, β_{jk} are some coefficients, and $\psi_{jk}, k \in \mathbb{Z}$, is a basis for W_j which is called resolution level of multiresolution analysis. The wavelet expansion needs to justify the use of

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k) \tag{2.2}$$

in (2.1), i.e. the existence of such a function called mother wavelet.

III. Wavelet composition of Levy process by wavelets with compact support

Let X be a Levy process satisfying $E[X(1)] = a_0$, $E[X(1)^2] < \infty$. Let $\psi(t)$ be a mother wavelet with compact support, such that supp $\psi(t) \in \mathbb{R}^+$. We can define a function

$$\xi : \omega \mapsto \xi(\omega) = \int\limits_{-\infty}^{+\infty} X(t,\omega)\psi(t)\mathrm{d}t, \tag{3.1}$$

where we assume, that $X(t,\omega) = 0,\ \forall t < 0$. The existence of that integral follows from existence of cadlag Levy process modification.

Theorem 2. *The function* ξ *is a random variable, such that* $E[\xi] = a_0 \int\limits_{\mathbb{R}} t\psi(t)\mathrm{d}t$.

Proof. Let supp $\psi(t) \subset [0, b]$, $T = \{0 = x_0 = t_0 < t_1 < \cdots < t_n = x_n = b\}$ be a target partition. We can define a random process

$$\xi_T = \sum_{i=1}^{n} X(t_i, \omega)\psi(t_i)\Delta x_i. \tag{3.2}$$

Obviously, $\forall\ \omega \in \Omega \Rightarrow \lim_{\Delta \to 0} \xi_T(\omega) = \xi(\omega)$. It means, that ξ is measurable function or random variable.

Then

$$E\xi = \int_{\Omega} \xi(\omega)\mathrm{d}\mathbb{P} = \int_{\Omega} \left(\int_{\mathbb{R}} X(t,\omega)\psi(t)\mathrm{d}t \right) \mathrm{d}\mathbb{P} = \int_{\mathbb{R}} \left(\int_{\Omega} X(t,\omega)\psi(t)\mathrm{d}\mathbb{P} \right) \mathrm{d}t = \int_{\mathbb{R}} \psi(t) \left(\int_{\Omega} X(t,\omega)\mathrm{d}\mathbb{P} \right) \mathrm{d}t = a_0 \int_{\mathbb{R}} t\psi(t)\mathrm{d}t.$$

Let us calculate the characteristic function of the random variable ξ:

Theorem 3. *Let* $\Psi(t) = \int_t^{+\infty} \psi(s)ds$. *The characteristic function of* ξ *is* $\phi(\theta) = e^{\eta(\theta)}$, *where*

$$\eta(\theta) = ia_0\theta \int_{\mathbb{R}} \Psi(t)dt - \frac{1}{2}\sigma^2\theta^2 \int_{\mathbb{R}} \Psi(t)^2 dt + \int_{\mathbb{R}} \int_{\mathbb{R}} (e^{i\theta\Psi(t)x} - 1 - i\theta\Psi(t)x)\ dt\ \nu(dx)$$

Proof of the theorem 3 see in the appendix.

Let $\psi_{ij}(x) = 2^{i/2}\psi(2^i x - j)$ be a wavelet, where $\psi(x)$ a mother wavelet. Let us define ξ_{ij} as:

$$\xi_{ij} = \xi_{ij}(\omega) = \int_{-\infty}^{+\infty} X(t,\omega)\psi_{ij}(t)dt. \tag{3.3}$$

Theorem 4. *In our conditions*

1) ξ_{ij} *are identically distributed random variables* $\forall j \in \mathbb{Z}$ *and fixed* i;
2) *if* $\forall j_1, j_2$ *that supp* $\psi_{ij_1} \bigcap$ *supp* $\psi_{ij_2} = \varnothing$ *then* ξ_{ij_1}, ξ_{ij_2} *are independent.*

Proof. These follow from definition of Levy process and equation

$$\xi_{ij}(\omega) = \int_{-\infty}^{+\infty} X(t,\omega)\psi_{ij}(t)dt = \int_{-\infty}^{+\infty} X(t + \frac{j}{2^i}, \omega)\psi_{i0}(t)dt = \int_{-\infty}^{+\infty} \left(X(t + \frac{j}{2^i}, \omega) - X(\frac{j}{2^i}, \omega) \right) \psi_{i0}(t)dt \sim \int_{-\infty}^{+\infty} X(t,\omega)\psi_{i0}(t)dt.$$

IV. The interlacing process composition by Haar wavelet

Let $C_{a,\sigma}(t) = at + B_\sigma(t)$ be a Brownian motion with drift. Let $Z_\lambda(t) = \sum_{j=1}^{N_\lambda(t)} Y_j$ be a Compound Poisson process, where $(Y_n, n \in N)$ is a sequence of independent identically distributed random variables with common law q and N_λ is an independent Poisson process.

We can define a Levy process by the prescription $X(t) = C_{a,\sigma}(t) + Z_\lambda(t)$, provided the two summands are assumed to be independent. We call this an interlacing process since its paths have the form of continuous motion interlaced with random jumps of size $||Y_n||$ occurring at the random times τ_n. X has characteristic exponent

$$\eta(\theta) = ia\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (e^{i\theta y} - 1)\lambda q(dy), \tag{4.1}$$

which is quite close to the general form in theorem. Indeed was proposed as the form of the most general η by the Italian mathematician Bruno de Finetti in the 1920s. His error was in failing to appreciate that the finite measure λq can be replaced by a σ-finite Levy measure ν. But if we do this, $(e^{i\theta y} - 1)$ may not be ν-integrable and hence we must adjust the integrand.

Let us see a simple example: the Interlacing Process $(X_t)_{t \geq 0}$ composition by Haar wavelet $H(t)$, where Y_i discrete random variables with possible values $\{a_i\}_{i=1}^n$ and the probability $\mathbb{P}(Y = a_i) = p_i$.

Definition 2. *The Haar wavelet is the function defined on the real line* $\mathbb{R}$ *as*

$$H(t) = \begin{cases} -1, & t \in [-\frac{1}{2}, 0]; \\ 1, & t \in (0, \frac{1}{2}]; \\ 0, & \textit{otherwise}. \end{cases} \tag{4.2}$$

Therefore,

$$\Psi(t)=\int_{t}^{+\infty}H(s)ds=\begin{cases}\frac{1}{2}+t, & t\in[-\frac{1}{2},0];\\ \frac{1}{2}-t, & t\in(0,\frac{1}{2}];\\ 0, & \text{otherwise.}\end{cases} \tag{4.3}$$

and

$$\eta_{\xi}(\theta)=ia_0\theta\int_{\mathbb{R}}\Psi(t)dt-\frac{1}{2}\sigma^2\theta^2\int_{\mathbb{R}}\Psi(t)^2dt+$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}}(e^{i\theta\Psi(t)x}-1-i\theta\Psi(t)x)dt\nu(dx)=$$
$$i\left(\frac{a_0}{4}\right)\theta-\frac{1}{2}\left(\frac{\sigma^2}{12}\right)\theta^2+$$
$$\lambda\sum_{i=1}^{n}p_i\int_{\mathbb{R}}(e^{i\theta\Psi(t)a_i}-1)dt=$$
$$i\left(\frac{a_0}{4}\right)\theta-\frac{1}{2}\left(\frac{\sigma^2}{12}\right)\theta^2+$$
$$\lambda\int_{\mathbb{R}}(e^{i\theta y}-1)d\left(\sum_{i=1}^{n}p_iF_{sign(a_i)}\left(\frac{y}{|a_i|}\right)\right),$$

where

$$F_{+}(s)=\begin{cases}0, & s<0;\\ 2s, & s\in[0,\frac{1}{2}];\\ 1, & s>\frac{1}{2}.\end{cases}$$

and

$$F_{-}(s)=\begin{cases}0, & s<-\frac{1}{2};\\ 1+2s, & s\in[-\frac{1}{2},0];\\ 1, & s>0.\end{cases}$$

It means that ξ has the same distribution as $C_{a/4,\sigma/\sqrt{12}}(1)+\sum_{i=1}^{N_{\lambda}(1)}T_i$ where T_i have common distribution function $\sum_{i=1}^{n}p_iF_{sign(a_k)}(\frac{s}{|a_k|})$.

V. Conclusion

We determined in this paper the general expression of the characteristic function and therefore probability distribution function of wavelet coefficients of Levy process. We considered the special case of the Haar wavelet and interlacing process for which we calculated probability distribution function.

In the future works estimation of wavelet coefficients of Levy processes will be implemented in different fields such as the case of aircraft landing trajectories, financial data, etc.

VI. Appendix

Proof of **Theorem 3**:

As $E[|X(t)|]<\infty$, then (Levy-Khinchine formulae)

$$Ee^{i\theta X_t}=e^{t\eta(\theta)},$$

where

$$\eta(\theta)=ia_0\theta-\frac{1}{2}\sigma^2\theta^2+\int_{\mathbb{R}}(e^{i\theta x}-1-i\theta x)\nu(dx),$$

and $a_0=E[X_1]$, $\int_{\mathbb{R}}(|x^2|\wedge|x|)\nu(dx)<\infty$.

Let us use the same notations as in theorem 2. Then

$$Ee^{i\theta\xi_T}=Ee^{\sum_{i=1}^{n}ib_i\theta(X(t_i)-X(t_{i-1}))}=$$
$$\prod_{i=1}^{n}Ee^{ib_i\theta(X(t_i)-X(t_{i-1}))}=\prod_{i=1}^{n}e^{(t_i-t_{i-1})\eta(b_i\theta)}=$$
$$e^{\sum_{i=1}^{n}(t_i-t_{i-1})\eta(b_i\theta)},$$

where

$$b_i=\sum_{j=i}^{n}\psi(t_j)\Delta x_j.$$

We have the next:

$$\sum_{i=1}^{n}(t_i-t_{i-1})\eta(b_i\theta)=$$
$$ia_0\theta\sum_{i=1}^{n}(t_i-t_{i-1})b_i-\frac{1}{2}\sigma^2\theta^2\sum_{i=1}^{n}(t_i-t_{i-1})b_i^2+$$
$$\sum_{i=1}^{n}(t_i-t_{i-1})\int_{\mathbb{R}}(e^{ib_i\theta x}-1-ib_i\theta x)\nu(dx).$$

Let us to calculate the

$$\lim_{\Delta_T\to 0}\sum_{i=1}^{n}(t_i-t_{i-1})\int_{\mathbb{R}}(e^{ib_i\theta x}-1-ib_i\theta x)\nu(dx).$$

$\forall\varepsilon>0\ \exists\delta_1=\delta_1(\varepsilon)>0:\forall T\ \Delta_T<\delta_1\Rightarrow$

$$\left|\int_{x_i}^{+\infty}\psi(s)ds-\sum_{j=i}^{n}\psi(t_i)\Delta x_i\right|<\varepsilon.$$

Let θ be fixed. Then $\forall\varepsilon_1>0\ \exists N_{\varepsilon_1}>0:$

$$\int_{\mathbb{R}\setminus[-N_{\varepsilon_1},N_{\varepsilon_1}]}(2+C_1\theta|x|)\nu(dx)<\varepsilon_1,$$

where $C_1=\max_{t\in\mathbb{R}}(\psi(t))b$.

And it is simple to get, that

$$|b_i|\le\max_{t\in\mathbb{R}}(\psi(t))\sum_{j=i}^{n}\Delta x_i\le C_1.$$

Therefore,

$$\left|\sum_{i=1}^{n}(t_i-t_{i-1})\int_{\mathbb{R}\setminus[-N_{\varepsilon_1},N_{\varepsilon_1}]}(e^{i\theta b_ix}-1-i\theta b_ix)\nu(dx)\right|\le$$
$$\sum_{i=1}^{n}(t_i-t_{i-1})\int_{\mathbb{R}\setminus[-N_{\varepsilon_1},N_{\varepsilon_1}]}(2+C_1\theta|x|)\nu(dx)<\varepsilon_1 b.$$

And

$$\left|\sum_{i=1}^{n}(t_i-t_{i-1})\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}(e^{i\theta b_i x}-1-i\theta b_i x)\nu(dx)\right.$$

$$\left.-\sum_{i=1}^{n}(t_i-t_{i-1})\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}(e^{i\theta\Psi(x_i)x}-1-i\theta\Psi(x_i)x)\nu(dx)\right|=$$

$$\left|\sum_{i=1}^{n}(t_i-t_{i-1})\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}\sum_{j=2}^{+\infty}\frac{(i\theta x)^j}{j!}(b_i^j-\Psi(x_i)^j)\nu(dx)\right|\leq$$

$$\sum_{i=1}^{n}(t_i-t_{i-1})\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}\sum_{j=2}^{+\infty}\frac{(|\theta x|)^j}{j!}\varepsilon(2C_1)^j\nu(dx)=$$

$$\varepsilon b\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}(e^{2C_1|\theta x|}-1-2C_1|\theta x|)\nu(dx).$$

Then, $\exists\delta_2=\delta_2(\varepsilon)>0$, that $\forall T:\ \Delta_T<\delta_2\Rightarrow$

$$\left|\int\limits_0^b(e^{i\theta\Psi(t)x}-1-i\theta\Psi(t)x)dt\right.$$

$$\left.-\sum_{i=1}^{n}(t_i-t_{i-1})(e^{i\theta\Psi(x_i)x}-1-i\theta\Psi(x_i)x)\right|<$$

$$\varepsilon(|x^2|\wedge|x|)$$

$\forall x\in[-N_{\varepsilon_1},N_{\varepsilon_1}]$.

Therefore,

$$\left|\sum_{i=1}^{n}(t_i-t_{i-1})\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}(e^{i\theta\Psi(x_i)x}-1-i\theta\Psi(x_i)x)\nu(dx)\right.$$

$$\left.-\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}\left(\int\limits_0^b(e^{i\theta\Psi(t)x}-1-i\theta\Psi(t)x)dt\right)\nu(dx)\right|<$$

$$\varepsilon\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}(|x^2|\wedge|x|)\ \nu(dx).$$

Thus,

$$\left|\sum_{i=1}^{n}(t_i-t_{i-1})\int\limits_{\mathbb{R}}(e^{i\theta b_i x}-1-i\theta b_i x)\nu(dx)\right.$$

$$\left.-\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}\int\limits_0^b(e^{i\theta\Psi(t)x}-1-i\theta\Psi(t)x)dt\nu(dx)\right|<$$

$$\varepsilon_1 b+\varepsilon C(\varepsilon_1).$$

Therefore,

$$\left|\lim_{\Delta_T\to 0}\sum_{i=1}^{n}(t_i-t_{i-1})\int\limits_{\mathbb{R}}(e^{i\theta b_i x}-1-i\theta b_i x)\nu(dx)\right.$$

$$\left.-\int\limits_{-N_{\varepsilon_1}}^{N_{\varepsilon_1}}\int\limits_0^b(e^{i\theta\Psi(t)x}-1-i\theta\Psi(t)x)dt\nu(dx)\right|<$$

$$\varepsilon_1 b,$$

and

$$\lim_{\Delta_T\to 0}\sum_{i=1}^{n}(t_i-t_{i-1})\int\limits_{\mathbb{R}}(e^{i\theta b_i x}-1-i\theta b_i x)\nu(dx)=$$

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}(e^{i\theta\Psi(t)x}-1-i\theta\Psi(t)x)dt\nu(dx).$$

It is similar to get:

$$\lim_{\Delta_T\to 0}(ia_0\theta\sum_{i=1}^{n}(t_i-t_{i-1})b_i-\frac{1}{2}\sigma^2\theta^2\sum_{i=1}^{n}(t_i-t_{i-1})b_i^2)=$$

$$ia_0\theta\int\limits_{\mathbb{R}}\Psi(t)dt-\frac{1}{2}\sigma^2\theta^2\int\limits_{\mathbb{R}}\Psi(t)^2dt.$$

Therefore,

$$\eta(\theta)=ia_0\theta\int\limits_{\mathbb{R}}\Psi(t)dt-\frac{1}{2}\sigma^2\theta^2\int\limits_{\mathbb{R}}\Psi(t)^2dt+$$

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}(e^{i\theta\Psi(t)x}-1-i\theta\Psi(t)x)\ dt\ \nu(dx).$$

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