On the identification of hidden pointwise Hölder exponents

Antoine Ayache, Qidi Peng

On the identification of hidden pointwise Hölder exponents

Antoine Ayache, Qidi Peng

U.M.R. CNRS 8524, Laboratory Paul Painlevé,
University Lille 1, Villeneuve d'Ascq 59655
E-mail: Antoine.Ayache@math.univ-lille1.fr, Qidi.Peng@math.univ-lille1.fr

Un court résumé en français

Mots clés: *Processus, Statistique mathématique.*

L'exposant de Hölder ponctuel (EHP) d'un processus stochastique X permet de mesurer la régularité locale de X. Nombre d'auteurs se sont déjà intéressés au problème de l'estimation de cet exposant à partir de l'observation d'une trajectoire discrétisée de X (voir par exemple [**1, 4, 5, 9**]). *Cependant, il ne semble pas toujours réaliste de supposer qu'une telle observation est directement accessible mais seulement une version corrompue de celle-ci. Est-il alors possible d'effectuer l'estimation ?*

A notre connaissance, cette question a été assez peu étudiée dans la littérature et les articles qui l'abordent se placent dans un cadre qui est essentiellement celui de processus Gaussiens à accoissements stationnaires [**8, 11**]; l'EHP a alors une structure relativement simple puisqu'il reste contant au cours du temps. L'objectif de notre exposé est d'etudier cette question dans un cadre nouveau, celui du mouvement Brownien multifractionnaire; l'EHP de ce processus a généralement une structure assez complexe parce qu'il varie d'un instant à l'autre (voir par [**3, 10, 2**]).

A long summary in English

Let $X = \{X(t)\}_{t\in[0,1]}$ be a stochastic process with continuous and nowhere differentiable trajectories, the pointwise Hölder exponent (PHE) of X, is the stochastic process denoted by $\alpha_X = \{\alpha_X(t)\}_{t\in[0,1]}$ and defined as

$$\alpha_X(t) := \sup\Big\{a : \limsup_{t+h\in[0,1],\, h\to 0} \frac{|X(t+h)-X(t)|}{|h|^a} = 0\Big\}. \tag{0.1}$$

It measures the local smoothness of X: the larger is $\alpha_X(t)$ the more regular is the process X in a neighborhood of the point t. Since several years, a number of authors have been interested in the statistical problem of the estimation of $\alpha_X(t)$ starting from the observation of a discretized trajectory of the process X (see for example [**1, 4, 5, 9**]). *However, it does not always seem to be realistic to assume that such an observation is available but only a corrupted version of it and a natural question one can address is that whether it is still possible to estimate $\alpha_X(t)$.* To our knowledge, only a few number

of articles in the literature deal with this problem and it has been studied only in a setting which basically remains to be that of Gaussian stationary increments processes; typically when the hidden process X is a fractional Brownian motion (fBm for short) [**8, 11**]; in such a setting the hidden PHE of X *has a rather simple structure since it can not evolve with time.* Recall that the fBm of Hurst parameter $H \in (0,1)$ is usually denoted by $B_H = \{B_H(t)\}_{t\in[0,1]}$ and can be defined as the stochastic integral (harmonizable representation):

$$B_H(t) := \int_{\mathbb{R}} \frac{(e^{it\xi}-1)}{|\xi|^{H+1/2}}\, \mathrm{d}\widehat{W}(\xi), \tag{0.2}$$

where $\mathrm{d}\widehat{W}$ is a complex-valued Weiner measure, with adapted real and imaginary parts such that the integral in (0.2) be real-valued.

- When $H = 1/2$, fBm reduces to Brownian motion (Bm).
- Even though fBm constitutes a quite natural extension of Bm, there is a considerable difference between these two processes when $H \neq 1/2$; fBm is non Markovian and its increments even display long-range dependence if $H > 1/2$.
- $\{\alpha_{B_H}(t)\}_{t\in[0,1]}$ the PHE of the fBm $\{B_H(t)\}_{t\in[0,1]}$ can not change with time (see the graph below), since it satisfies almost surely (a.s.) for all $t \in [0,1]$

$$\alpha_{B_H}(t) = H. \tag{0.3}$$

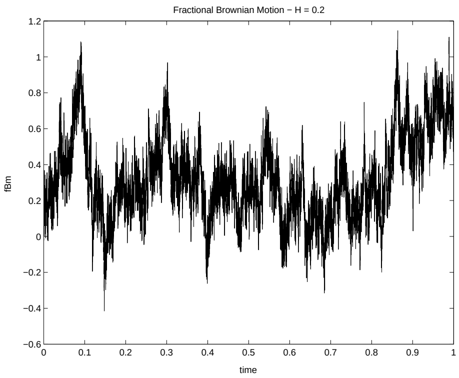
The goal of our talk is to study the statistical problem of the estimation of a hidden PHE in a new setting where this exponent has a rather complex structure since it is allowed to evolve with time. More precisely we assume that the corresponding hidden process X is a multifractional Brownian motion (mBm). Recall that mBm can be obtained by replacing in the harmonizable representation of fBm (see (0.2)) the constant Hurst parameter H by a functional parameter $H(t)$ depending on the time variable t. Therefore, mBm can be represented as:

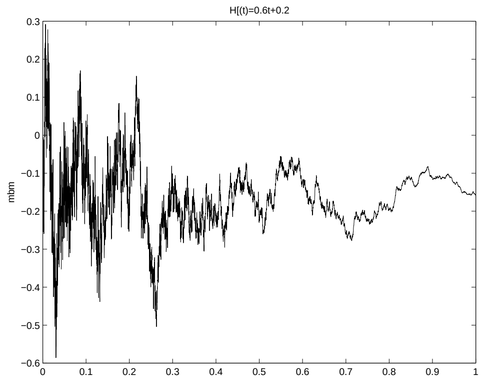
$$X(t) := \int_{\mathbb{R}} \frac{(e^{it\xi}-1)}{|\xi|^{H(t)+1/2}}\, \mathrm{d}\widehat{W}(\xi). \tag{0.4}$$

Throughout this talk we suppose that $H(\cdot)$ is a two times continuously differentiable function taking it values in $[H_*, H^*] \subset (1/2, 1)$, where $H_* := \min_{t\in[0,1]} H(t)$ and $H^* := \max_{t\in[0,1]} H(t)$ are assumed to be known.

- mBm reduces to fBm when the function $H(\cdot)$ is constant.
- mBm is more flexible than fBm since its local Hölder regularity can change from one point to another (see the graph below). More precisely it has been shown in [**3, 10, 2**] that a.s. for all $t \in [0,1]$

$$\alpha_X(t) = H(t). \tag{0.5}$$





Also we assume that we observe the high frequency discrete data $Z(j/(2n))$, $j = 0, \dots, 2n$; the non Gaussian process $Z = \{Z(t)\}_{t\in[0,1]}$ is the "corrupted version" of the mBm X, it is defined as

$$Z(t) := z_0 + \int_0^t \Phi(X(s))\, \mathrm{d}W(s), \tag{0.6}$$

where:

- z_0 is a deterministic real number.
- $\{W(s)\}_{s\in[0,1]}$ is a standard Brownian motion.
- Φ is a unknown deterministic function defined on the real line which vanishes. We assume that Φ vanishes only on a Lebesgue negligible set. We also assume that the function f, defined for every $x \in \mathbb{R}$, as

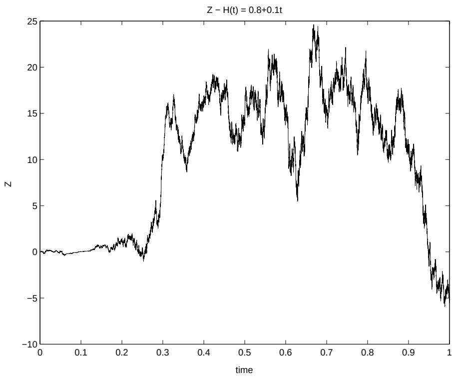
  $$f(x) := (\Phi(x))^2, \tag{0.7}$$

  is two times continuously differentiable and has a at most a slow increase at infinity as well as its two derivatives i.e. there exist two constants $c > 0$ and $L > 0$ such that we have for each $x \in \mathbb{R}$,

  $$\sum_{l=0}^{2} |f^{(l)}(x)| \le c(1 + |x|^L). \tag{0.8}$$

- $\{X(s)\}_{s\in[0,1]}$ is the hidden mBm which is supposed to be independent of the Bm $\{W(s)\}_{s\in[0,1]}$.

It is worth noticing that there is a considerable loss of information when we observe Z instead of X, in fact contrarily to X the PHE of Z is at each point a.s. equal to 1/2 (see the graph below).



Yet we will show that *in spite of this considerable loss of information it is still possible, starting from the observation of a discretized trajectory of Z, to estimate the PHE of X at each point.*

Also it is worth noticing that models of the type (0.6) can be viewed as stochastic volatility models where $Z(t)$ is the logarithm of the price of an underlying asset at time t. Such models have been already considered by Gloter and Hoffmann [**7**] in the case where X is fBm and by Rosenbaum [**10**] in a more general case where X can nicely be expressed in terms of a Gaussian stationary increment process. Let us mention in passing that in [**7**], Gloter and Hoffmann were not interested in the problem of the estimation of PHE (which they suppose to be known) but in that of a parametric estimation of the hidden volatility. In [**10**], Rosenbaum was interested in the same problem of us and he constructed a wavelet estimator of the hidden PHE which converges at the optimal rate in a minimax sense. We preferred to adopt a rather different estimation strategy in order to be able to obtain a Central Limit Theorem for our estimator.

Let us now state our main result, to this end first we need to introduce some notations.

- We denote by $\beta \in (0,1)$ a fixed parameter and for all integer n large enough, we set $N_n := [n^\beta]$ and $m_n := [n/N_n]$, $[\cdot]$ being the integer part function. We also set

$$\widehat{Y}_{i,n} := N_n \sum_{k=0}^{j_{i+1}-j_i-1} \Big(Z((j_i+k+1)/n) - Z((j_i+k)/n) \Big)^2 .$$

  where for all $i = 0, \dots, N_n$, $j_i := [in/N_n]$. It is worth noticing that $\widehat{Y}_{i,n}$ provides an estimation of the following average value of the hidden process $\{Y(s)\}_{s\in[0,1]} := \{f(X(s))\}_{s\in[0,1]}$:

$$\overline{Y}_{i,N_n} := N_n \int_{i/N_n}^{(i+1)/N_n} Y(s)\, ds;$$

  moreover, the smaller is the parameter β the more precise is the estimation.

- The generalized increments of the discrete process $\{\widehat{Y}_{i,n}\}_i$ are defined for all $i \in \{0, \ldots, N_n - p - 1\}$ as,

$$\Delta_a \widehat{Y}_{i,n} := \sum_{k=0}^{p} a_k \widehat{Y}_{i+k,n},$$

  where $a = (a_0, \ldots, a_p) \in \mathbb{R}^{p+1}$ is an arbitrary finite fixed sequence having $M \geq 2$ vanishing moments i.e.:

$$\sum_{k=0}^{p} k^l a_k = 0, \text{for all } l = 0, \ldots, M-1.$$

  and

$$\sum_{k=0}^{p} k^M a_k \neq 0.$$

- Let $t_0 \in (0,1)$ be an arbitrary fixed point, We denote by $\nu_{N_n}(t_0)$ the set of indices around t_0, defined as

$$\nu_{N_n}(t_0) = \Big\{ i \in \{0, \ldots, N_n - p - 1\} \ : \ |i/N_n - t_0| \leq N_n^{-\gamma} \Big\},$$

  where $\gamma \in (0,1)$ is a fixed parameter which allows to control the size of $\nu_{N_n}(t_0)$.

- Finally, for any n big enough, we denote by V_n the localized generalized quadratic variation defined as

$$V_n = \Big(\sum_{i \in \nu_{N_n}(t_0)} (\Delta_a \widehat{Y}_{i,n})^2 \Big)^{1/2}.$$

Our main result is the following theorem:

Theorem 1 *We observe $Z(i/(2n)), i = 0, \ldots, 2n$. Let $t_0 \in (0,1)$ be an arbitrary fixed point. We set*

$$\widehat{H}_{n,t_0} = \frac{1-\gamma}{2} + \frac{\log_2(V_n/V_{2n})}{\beta}.$$

- *When the parameter β satisfies the inequality $\beta < 1/(4H^* + 2)$, then*

$$\widehat{H}_{n,t_0} \xrightarrow[n \to +\infty]{a.s.} H(t_0).$$

- *When the parameters β and γ satisfy the inequalities $\gamma > 1/(2H_* + 1)$ and $\beta < 1/(2H^* + 2 - \gamma)$, then*

$$n^{\beta(1-\gamma)/2}(\widehat{H}_{n,t_0} - H(t_0))$$

  converges in distribution to a centred Gaussian variable.

Recall that $H_* := \min_{t \in [0,1]} H(t)$ and $H^* := \max_{t \in [0,1]} H(t)$.

Bibliography

[1] Ayache, A. and Lévy Véhel, J. (2004) On the identification of the pointwise Hölder exponent of the generalized multifractional Brownian motion, *Stoch. Proc. Appl*, 111, 119–156.

[2] Ayache, A., Jaffard, S. and Taqqu, M. S. (2007) Wavelet construction of generalized multifractional processes, *Rev. Mat. Iberoam*, 23, no. 1, 327–370.

[3] Benassi, A., Jaffard, S. and Roux, D. (1997) Gaussian processes and pseudodifferential elliptic operators, *Rev. Mat. Iberoam*, 13, no. 1, 19–81.

[4] Benassi, A., Cohen, S. and Istas, J. (1998) Identifying the multifractional function of a Gaussian process, *Stat. Prob. lett.*, 39, 31–49.

[5] Benassi, A., Cohen, S., Istas, J. and Jaffard, S. (1998) Identification of filtered white noises, *Stoch. Proc. Appl*, 75, 31–49.

[6] Coeurjolly, J. F. (2008) Hurst exponent estimation of locally self-similar Gaussian processes using sample quantiles, *Ann. Statist.*, 36, no. 3, 1404–1434.

[7] Gloter, A. and Hoffmann, M. (2004) Stochastic volatility and fractional Brownian motion, *Stoch. Proc. Appl*, 113, 143–172.

[8] Gloter, A. and Hoffmann, M. (2007) Estimation of the Hurst parameter from discrete noisy data, *Ann. Statist.*, 35, no. 5, 1947-1974.

[9] Istas, J. and Lang, G. (1997) Quadratic variations and estimation of the local Hölder index of a Gaussian process, *Ann. Inst. Henri Poincaré*, Vol. 33, no 4, 407–436.

[10] Peltier, R. F. and Lévy Véhel, J. (1995) Multifractional Brownian motion: definition and preliminary results, *Rapport de recherche de l'INRIA*, no.2645.

[11] Rosenbaum, M. (2008) Estimation of the volatility persistence in a discretely observed diffusion model, *Stoch. Proc. Appl*, 118, 1434–1462.