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Un court résumé en français

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L'exposant de Hölder ponctuel (EHP) d'un processus stochastique X permet de mesurer la régularité locale de X. Nombre d'auteurs se sont déjà intéressés au problème de l'estimation de cet exposant à partir de l'observation d'une trajectoire discrétisée de X (voir par exemple [**1, 4, 5, 9**]). *Cependant, il ne semble pas toujours réaliste de supposer qu'une telle observation est directement accessible mais seulement une version corrompue de celle-ci. Est-il alors possible d'effectuer l'estimation ?*

A notre connaissance, cette question a été assez peu étudiée dans la littérature et les articles qui l'abordent se placent dans un cadre qui est essentiellement celui de processus Gaussiens à accoissements stationnaires [**8, 11**]; l'EHP a alors une structure relativement simple puisqu'il reste contant au cours du temps. L'objectif de notre exposé est d'etudier cette question dans un cadre nouveau, celui du mouvement Brownien multifractionnaire; l'EHP de ce processus a généralement une structure assez complexe parce qu'il varie d'un instant à l'autre (voir par [**3, 10, 2**]).

A long summary in English

Let $X = \{X(t)\}_{t\in[0,1]}$ be a stochastic process with continuous and nowhere differentiable trajectories, the pointwise Hölder exponent (PHE) of X, is the stochastic process denoted by $\alpha_X = \{\alpha_X(t)\}_{t\in[0,1]}$ and defined as

$$\alpha_X(t) := \sup\Big\{a \,:\, \limsup_{t+h\in[0,1],\, h\to 0} \frac{|X(t+h)-X(t)|}{|h|^a} = 0\Big\}. \tag{0.1}$$

It measures the local smoothness of X: the larger is $\alpha_X(t)$ the more regular is the process X in a neighborhood of the point t. Since several years, a number of authors have been interested in the statistical problem of the estimation of $\alpha_X(t)$ starting from the observation of a discretized trajectory of the process X (see for example [**1, 4, 5, 9**]). *However, it does not always seem to be realistic to assume that such an observation is available but only a corrupted version of it and a natural question one can address is that whether it is still possible to estimate $\alpha_X(t)$.* To our knowledge, only a few number

of articles in the literature deal with this problem and it has been studied only in a setting which basically remains to be that of Gaussian stationary increments processes; typically when the hidden process X is a fractional Brownian motion (fBm for short) **[8, 11]**; in such a setting the hidden PHE of X *has a rather simple structure since it can not evolve with time.* Recall that the fBm of Hurst parameter $H \in (0,1)$ is usually denoted by $B_H = \{B_H(t)\}_{t\in[0,1]}$ and can be defined as the stochastic integral (harmonizable representation):

$$B_H(t) := \int_{\mathbb{R}} \frac{(e^{it\xi}-1)}{|\xi|^{H+1/2}}\,\mathrm{d}\hat{W}(\xi), \tag{0.2}$$

where $\mathrm{d}\hat{W}$ is a complex-valued Weiner measure, with adapted real and imaginary parts such that the integral in (0.2) be real-valued.

- When $H = 1/2$, fBm reduces to Brownian motion (Bm).
- Even though fBm constitutes a quite natural extension of Bm, there is a considerable difference between these two processes when $H \neq 1/2$; fBm is non Markovian and its increments even display long-range dependence if $H > 1/2$.
- $\{\alpha_{B_H}(t)\}_{t\in[0,1]}$ the PHE of the fBm $\{B_H(t)\}_{t\in[0,1]}$ can not change with time (see the graph below), since it satisfies almost surely (a.s.) for all $t \in [0,1]$

$$\alpha_{B_H}(t) = H. \tag{0.3}$$

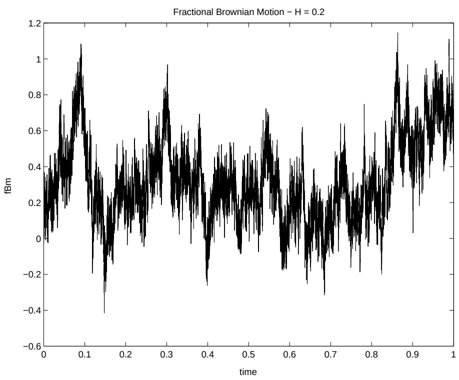
The goal of our talk is to study the statistical problem of the estimation of a hidden PHE in a new setting where this exponent has a rather complex structure since it is allowed to evolve with time. More precisely we assume that the corresponding hidden process X is a multifractional Brownian motion (mBm). Recall that mBm can be obtained by replacing in the harmonizable representation of fBm (see (0.2)) the constant Hurst parameter H by a functional parameter $H(t)$ depending on the time variable t. Therefore, mBm can be represented as:

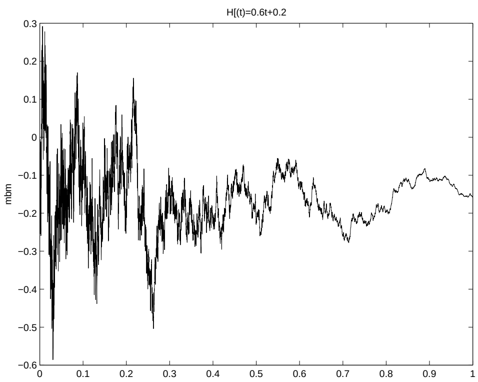
$$X(t) := \int_{\mathbb{R}} \frac{(e^{it\xi}-1)}{|\xi|^{H(t)+1/2}}\,\mathrm{d}\hat{W}(\xi). \tag{0.4}$$

Throughout this talk we suppose that $H(\cdot)$ is a two times continuously differentiable function taking it values in $[H_*, H^*] \subset (1/2, 1)$, where $H_* := \min_{t\in[0,1]} H(t)$ and $H^* := \max_{t\in[0,1]} H(t)$ are assumed to be known.

- mBm reduces to fBm when the function $H(\cdot)$ is constant.
- mBm is more flexible than fBm since its local Hölder regularity can change from one point to another (see the graph below). More precisely it has been shown in **[3, 10, 2]** that a.s. for all $t \in [0,1]$

$$\alpha_X(t) = H(t). \tag{0.5}$$





Also we assume that we observe the high frequency discrete data $Z(j/(2n))$, $j = 0, \ldots, 2n$; the non Gaussian process $Z = \{Z(t)\}_{t\in[0,1]}$ is the "corrupted version" of the mBm X, it is defined as

$$Z(t) := z_0 + \int_0^t \Phi(X(s))\, \mathrm{d}W(s), \tag{0.6}$$

where:

- z_0 is a deterministic real number.
- $\{W(s)\}_{s\in[0,1]}$ is a standard Brownian motion.
- Φ is a unknown deterministic function defined on the real line which vanishes. We assume that Φ vanishes only on a Lebesgue negligible set. We also assume that the function f, defined for every $x \in \mathbb{R}$, as

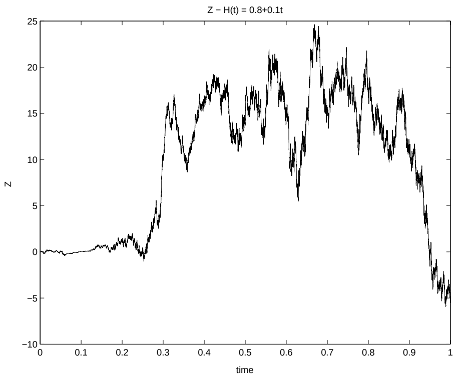
  $$f(x) := (\Phi(x))^2, \tag{0.7}$$

  is two times continuously differentiable and has a at most a slow increase at infinity as well as its two derivatives i.e. there exist two constants $c > 0$ and $L > 0$ such that we have for each $x \in \mathbb{R}$,

  $$\sum_{l=0}^{2} |f^{(l)}(x)| \le c(1 + |x|^L). \tag{0.8}$$

- $\{X(s)\}_{s\in[0,1]}$ is the hidden mBm which is supposed to be independent of the Bm $\{W(s)\}_{s\in[0,1]}$.

It is worth noticing that there is a considerable loss of information when we observe Z instead of X, in fact contrarily to X the PHE of Z is at each point a.s. equal to 1/2 (see the graph below).



Yet we will show that *in spite of this considerable loss of information it is still possible, starting from the observation of a discretized trajectory of Z, to estimate the PHE of X at each point.*

Also it is worth noticing that models of the type (0.6) can be viewed as stochastic volatility models where $Z(t)$ is the logarithm of the price of an underlying asset at time t. Such models have been already considered by Gloter and Hoffmann [**7**] in the case where X is fBm and by Rosenbaum [**10**] in a more general case where X can nicely be expressed in terms of a Gaussian stationary increment process. Let us mention in passing that in [**7**], Gloter and Hoffmann were not interested in the problem of the estimation of PHE (which they suppose to be known) but in that of a parametric estimation of the hidden volatility. In [**10**], Rosenbaum was interested in the same problem of us and he constructed a wavelet estimator of the hidden PHE which converges at the optimal rate in a minimax sense. We preferred to adopt a rather different estimation strategy in order to be able to obtain a Central Limit Theorem for our estimator.

Let us now state our main result, to this end first we need to introduce some notations.

- We denote by $\beta \in (0,1)$ a fixed parameter and for all integer n large enough, we set $N_n := [n^\beta]$ and $m_n := [n/N_n]$, $[\cdot]$ being the integer part function. We also set

  $$\widehat{Y}_{i,n} := N_n \sum_{k=0}^{j_{i+1}-j_i-1} \Big(Z((j_i+k+1)/n) - Z((j_i+k)/n) \Big)^2 .$$

  where for all $i = 0, \ldots, N_n$, $j_i := [in/N_n]$. It is worth noticing that $\widehat{Y}_{i,n}$ provides an estimation of the following average value of the hidden process $\{Y(s)\}_{s\in[0,1]} := \{f(X(s))\}_{s\in[0,1]}$:

  $$\overline{Y}_{i,N_n} := N_n \int_{i/N_n}^{(i+1)/N_n} Y(s)\, ds;$$

  moreover, the smaller is the parameter β the more precise is the estimation.

- The generalized increments of the discrete process $\{\widehat{Y}_{i,n}\}_i$ are defined for all $i \in \{0, \ldots, N_n - p - 1\}$ as,

$$\Delta_a \widehat{Y}_{i,n} := \sum_{k=0}^{p} a_k \widehat{Y}_{i+k,n},$$

where $a = (a_0, \ldots, a_p) \in \mathbb{R}^{p+1}$ is an arbitrary finite fixed sequence having $M \geq 2$ vanishing moments i.e.:

$$\sum_{k=0}^{p} k^l a_k = 0, \text{for all } l = 0, \ldots, M-1.$$

and

$$\sum_{k=0}^{p} k^M a_k \neq 0.$$

- Let $t_0 \in (0,1)$ be an arbitrary fixed point, We denote by $\nu_{N_n}(t_0)$ the set of indices around t_0, defined as

$$\nu_{N_n}(t_0) = \Big\{ i \in \{0, \ldots, N_n - p - 1\} \ : \ |i/N_n - t_0| \leq N_n^{-\gamma} \Big\},$$

where $\gamma \in (0,1)$ is a fixed parameter which allows to control the size of $\nu_{N_n}(t_0)$.

- Finally, for any n big enough, we denote by V_n the localized generalized quadratic variation defined as

$$V_n = \Big(\sum_{i \in \nu_{N_n}(t_0)} (\Delta_a \widehat{Y}_{i,n})^2 \Big)^{1/2}.$$

Our main result is the following theorem:

Theorem 1 *We observe $Z(i/(2n)), i = 0, \ldots, 2n$. Let $t_0 \in (0,1)$ be an arbitrary fixed point. We set*

$$\widehat{H}_{n,t_0} = \frac{1-\gamma}{2} + \frac{\log_2(V_n/V_{2n})}{\beta}.$$

- *When the parameter β satisfies the inequality $\beta < 1/(4H^* + 2)$, then*

$$\widehat{H}_{n,t_0} \xrightarrow[n \to +\infty]{a.s.} H(t_0).$$

- *When the parameters β and γ satisfy the inequalities $\gamma > 1/(2H_* + 1)$ and $\beta < 1/(2H^* + 2 - \gamma)$, then*

$$n^{\beta(1-\gamma)/2}(\widehat{H}_{n,t_0} - H(t_0))$$

converges in distribution to a centred Gaussian variable.

Recall that $H_* := \min_{t \in [0,1]} H(t)$ and $H^* := \max_{t \in [0,1]} H(t)$.

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