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Abstract

Since introduced by Berenger [1], the Perfectly Matched Layers method (PML) has become a popular approach for non reflecting Absorbing Boundary Conditions (ABC) in the numerical solution of the Maxwell equations on unbounded domains. However, most of the formulations only concern parallelepiped or simply convex domains. The goal of this paper is to present the theory for non necessarily convex PML on systems like Maxwell's.

Keywords

Perfectly Matched Layers, Pseudo-riemannian manifolds, Waves propagation equations, Fictive absorbing media, Friedrichs systems

AMS Subject Classification

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Introduction

Whenever one solves a Partial Differential Equations numerically (by a volume discretization), one has to truncate the domain in some way. A key question is how to terminate the mesh without creating excessive echoes from the artificial truncation surface that may spoil the quality of the solution. In 1994, Berenger changed the question : instead of finding an absorbing boundary condition, he found an absorbing boundary layer. That is an artificial material independent of the boundary condition. When a wave enters the absorbing layer, it is attenuated by absorption and decays exponentially. Even if it reflects off the boundary, the returning wave after one round trip through the absorbing layer is exponentially small. Moreover, waves do not reflect at the interface. Although PML were originally derived from electromagnetism and Maxwell equations, the same ideas can immediately be applied to other waves equations.

Chew and Wheedon [2] introduced the notion of complex coordinates stretching, based on analytic continuation of Maxwell's equations into spatial complex coordinates where the fields are exponentially decaying. In this paper, the stretching is interpreted as writing the same equations in a complex tangent bundle of a flat manifold in $\mathbb{C}^n$. For a general system, two cases appear. The first one concerns equations with an intrinsic form given by exterior derivatives and Hodge operators. In that case, we can directly study their integral inverses and thus obtain a general PML theory. This allows to reduce the non necessarily convex PML domain to the closest of the real field region. This leads to a dispersive formulation : the spatial operator is local and the time one is pseudo-differential but a localization can be performed thanks to extra Ordinary Differential Equations. On the contrary, in a second case, like for example aeroacoustics' equations with convection, such a formulation is impossible to obtain. This brings difficulties to prove the existence and uniqueness of the solution, but it is possible to prove that an exponentially absorbing and unreflected solution exists in the entire domain.

This paper is about to generalize the PML formulation for non necessarily convex domains throught a diffeomorphism φ, defined thanks to an affinity normal to the convexified boundary, and a strictly incresing function f, whose properties will be discussed. Such a formulation will also included "classical" PML, such as cartesian and convex ones.

1 Flat complex manifolds

In 2001, Lassas *et al.* [4] showed that all PML can be obtained through a complexification of coordinates that corresponds to flat complex manifolds. Instead of stretching the coordinates, they changed the metric defined on $\mathbb{R}^3$. This method presents several advantages. First, when Maxwell's equations are written in terms of 1-forms, the differential operators take form of exterior derivatives. Second, the stretching of the metric allows to treat more general scattering geometries than before. Finally, this formulation is completely invariant as it is done without a reference to specific coordinate systems.

The manifolds used in this paper are called pseudo-riemannian in $\mathbb{C}^3$, meaning real of dimension 3, with complex tangent and cotangent bundles, with a symmetric complex metric $g(.,.)$ such as its determinant $g = \sqrt{\mathbf{det}(g(.,.))}$ can always be defined with the same determination. The purpose of this section is to geometrically describe the complexification of coordinates that defines the PML.

Let $(\mathbf{M}, g)$ be a real manifold with a complex tangent bundle TM and a riemannian metric g_{jl}. The matrix $\boldsymbol{G} = (g_{ij})$ is associated to the metric $g(.,.)$. The tangent bundle set $T_{\mathbf{x}}M$ for all $\mathbf{x}$ in $\mathbf{M}$, is the sum $U + iV$, with U and V real vectors tangent to $\mathbf{x}$ and its complex dual is the cotangent bundle $T^*_{\mathbf{x}}M$. Their bases are the partial derivatives $\dfrac{\partial}{\partial \mathbf{x}_1}, \ldots, \dfrac{\partial}{\partial \mathbf{x}_n}$ for $T_{\mathbf{x}}M$ and the exterior differentials $d\mathbf{x}_1, \ldots, d\mathbf{x}_n$ for $T^*_{\mathbf{x}}M$. An element of $T_{\mathbf{x}}M$ (a vector) is a complex linear combinaison on this basis $\sum_j X_j(\mathbf{x}) \dfrac{\partial}{\partial \mathbf{x}_j}$. The elements of $T^*_{\mathbf{x}}M$ are covectors on the form $\sum_j X_j(\mathbf{x}) d\mathbf{x}_j$, with $X_j(\mathbf{x}) \in \mathbb{C}$. The bilinear application $TM \times TM \to TM : (\mathbf{u}, \mathbf{v}) \to \nabla_{\mathbf{u}}\mathbf{v}$ with the properties $\nabla_{\mathbf{u}}(\mathbf{v} + \mathbf{w}) = \nabla_{\mathbf{u}}\mathbf{v} + \nabla_{\mathbf{u}}\mathbf{w}$ $\nabla_{f\mathbf{u}}\mathbf{v} = f\nabla_{\mathbf{u}}\mathbf{v}$ and $\nabla_{\mathbf{u}}(f\mathbf{v}) = \mathbf{u}(f)\ \mathbf{v} + f\ \nabla_{\mathbf{u}}\mathbf{v}$ is a linear connection. In particular, the Levi Civita connection is the only linear connection that conserves the metric $g(.,.)$ and can be expressed through the Lie brackets $[\mathbf{u}, \mathbf{v}]$ of $\mathbf{u}$ and $\mathbf{v}$. A connection is torsion free if $\nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u}, \mathbf{v}] = 0$ and $\nabla_i g = 0$. The complex curvature is $R(\mathbf{u}, \mathbf{v})\mathbf{w} = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w} - \nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}$. On a basis of TM, as $\left[\partial_{\mathbf{x}_i}, \partial_{\mathbf{x}_j}\right] = 0$, the nullity of the curvature means the commutation of the covariant derivatives. If η and τ are two p-forms, and if $\eta \neg \tau = \eta_J \tau^J$ is their scalar product (J stands for the set of the ordonate indices p), the Hodge star operator $\star$ is given by the relationship : $\eta \wedge \star\tau = \tau \wedge \star\eta = (\eta \neg \tau)\sqrt{(g)}\ d\mathbf{x}_1 \wedge \cdots \wedge d\mathbf{x}_n$. Moreover, if n is the dimension of the manifold, then $\star^{-1}\tau = (-1)^{p(n-p)}\tau$. The canonic euclidian complex metric $g_C(.,.)$ in $\mathbb{C}^3$ will be considered as the real submanifold of dimension 3 of $\mathbb{R}^6$ with complex tangent and cotangent bundles, and used in cartesian coordinates $\forall(\mathbf{u}, \mathbf{v}) \in \mathbb{C}^3,\quad g_C(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{3} \mathbf{u}_j \mathbf{v}_j$.

If $\mathbf{x} \mapsto \widetilde{\mathbf{x}}(\mathbf{x})$ is a change of variables, a complex metric on $\mathbb{R}^3$, denoted $g_{\mathbf{x}} : T_{\mathbf{x}}M\mathbb{R}^3 \times T_{\mathbf{x}}M\mathbb{R}^3 \to \mathbb{C}^3$, can be defined by $g_{\mathbf{x}}(\mathbf{u}, \mathbf{v}) = g_C\left(d\widetilde{\mathbf{x}}(\mathbf{u}), d\widetilde{\mathbf{x}}(\mathbf{v})\right)$, where $d\widetilde{\mathbf{x}}$ is the differential of $\widetilde{\mathbf{x}}$.

In the natural frame of reference, the tangent vectors always commute but the covariant derivates only commute if the manifold is flat, which provides an intrinsic way of writing the determinant and the symbol of the operator. A flat manifold therefore defines a parallelism structure that generalizes the notion of convolution.

1.1 Formalism

The spatial domains involved are : a bounded smooth obstacle $\mathbf{O}$, non necessarily convex that can be empty, a connex domain Ω, non necessarily simply connex that represents the domain of interests, $\delta\Omega_0$ is the boundary between Ω and $\mathbf{O}$, $\delta\Omega_\infty$ is the external boundary. Both are C^1, and $\delta\Omega_\infty$ can be rejected to infinity if Ω is not bounded.

Consider the real manifolds with values in $\mathbb{C}^3$ described by the change of variables, with $\mathbf{p} = \varepsilon + i\omega \in \mathbb{C}$,

$$\widetilde{\mathbf{x}} = \varphi(\mathbf{x}) + \frac{1}{\mathbf{p}} f(\varphi(\mathbf{x})). \tag{1}$$

Hypotheses 1. *The couple (φ, f) belongs to $C^1(\mathbb{R}^3, \mathbb{R}^3)$. $\forall x \in \Omega$, we have $\varphi(\mathbf{x}) = \mathbf{x}$ and $f(\mathbf{x}) = 0$. Moreover, $|\varphi(\mathbf{x})| = \underset{|\mathbf{x}| \to \infty}{\mathcal{O}}(|\mathbf{x}|)$, f is strictly increasing with a linear growth at infinity and it exists S a convex function such as $f = \underset{\varphi}{\mathbf{grad}} S$.*

These hypotheses will be justified later.

Remark : the regularity $C^1_{pcw}(\mathbb{R}^3, \mathbb{R}^3) \cap C^0(\mathbb{R}^3, \mathbb{R}^3)$ for (φ, f) could be enough if the edges are lipschitzian (no jump discontinuities) and it could be possible to write the weak exterior differentials thanks to the unit partition theorem.

The interest of such a formulation is that the complexification of each component is made through a single function given on all $\mathbb{R}^3$ and does not dependent anymore of the coordinates. Moreover, it does not take into account the shape of the PML or their absorbing direction. As $f_{|\Omega} = \mathbf{0}_\Omega$ and $\varphi_{|\Omega} = \mathbf{id}_\Omega$, the system of equations is unchanged inside the studied domain. If the functions are C^k, then the manifold $\mathbf{M}$ defined by (1) is a submanifold of class C^k and dimension 3 in the $\mathbb{R}$ vectorial space $\mathbb{C}^3$ of dimension 6.

Remark : as announced, this formulation is very general : chosing $\varphi = \mathbf{x}$ and $f(\mathbf{x}) = (f_i(\mathbf{x}))$ with $f_i(\mathbf{x}) = \int_0^{\mathbf{x}} \sigma_i(\tau) d\tau$ and σ_i a positive function that tends to a constant at infinity, means a cartesian formulation, while $f_i(\mathbf{x}) = \partial_i h(\mathbf{dist}(\mathbf{x}, \partial\Omega))$, with h an increasing convex function asymptotically linear is a convex formulation. In both cases, f is the gradient of a convex function and respects Hyp. (1)

2 Helmholtz problem

Let $(\mathbf{M}, g)$ be an pseudo-riemannian manifold. We define the Hodge star operator $\star$ corresponding to the complex metric g, with $U \wedge \star V = g(U, V)\mathtt{d}\mathbf{vol}_g$ and $\mathtt{d}\mathbf{vol}_g = d\widetilde{\mathbf{x}}_1 \wedge d\widetilde{\mathbf{x}}_2 \wedge d\widetilde{\mathbf{x}}_3$. The functions $\widetilde{\mathbf{x}}_j$ are the components of the embedding $\sim : \mathbf{M} \to \mathbb{C}^3$ that defines the immersion. The generalized Laplacian Δ^r_g for r-forms on the manifold $\mathbf{M}$ can be generalized for to the metric $g(.,.)$ through $\Delta^r_g = (-1)^r\left(\star\mathbf{d}\star\mathbf{d} - \mathbf{d}\star\mathbf{d}\star\right)$. If the manifold is flat, $[\nabla_i, \nabla_j] = 0$, the components of 1-forms are given by the metric and Δ^0_g corresponds to the opposite of the Laplace-Beltrami operator $\Delta : \varphi \mapsto g^{ij}\nabla_i\nabla_j\varphi$.

Definition 1. $(\mathbf{M}, g)$ *is an absorbing pseudo-riemannian manifold if*

1. *the manifold* $(\mathbf{M}, g)$ *is flat and* $\mathbf{M}$ *is diffeomorphic to* $\mathbb{R}^3$ *(with* ϕ *the diffeomorphism),*
2. *it exists* $\Omega \subset \mathbf{M}$ *a relatively compact open set where the metric is real and euclidian and given by* $g = \phi^\star g^e$*, with* $\phi^\star$ *the pull-back of* ϕ *and* g^e *the euclidian metric,*
3. $\forall \mathbf{v} \in T^{\mathbb{R}}_{\mathbf{x}}$, $\mathbf{v} \neq 0, \qquad g_{\mathbf{x}}(\mathbf{v}, \mathbf{v}) \neq 0$,
4. *the immersion* $\sim : \mathbf{M} \to \mathbb{C}^3$ *guarantees that the imaginary part of* $\widetilde{\mathbf{x}}(\mathbf{x}_1) - \widetilde{\mathbf{x}}(\mathbf{x}_2)$ *is signed and of constant sign for all* $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbf{M}^2$.

For now on, the manifold considered $(\mathbf{M}, g)$ will be an absorbing pseudo-riemannian manifold. The fourth point of Def. (1) uniquely determines the square root

$$\{\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}}\} = \left(\sum_{j=1}^{3} (\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{y}}_j)^2 \right)^{1/2} .$$

If $\{\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}}\} \geq 0$ is positive, $\mathbf{M}$ is called an outgoing absorbing pseudo-riemannian manifold.

Theorem 1. *The fundamental solution* G_H *for the g-Helmholtz operator for* 0*-forms on* $(\mathbf{M}, g)$

$$\forall \mathbf{y} \in \mathbf{M}, \quad (\Delta_g - \mathbf{p}^2)\Phi(., \mathbf{y}) = -\delta_{\mathbf{y}} \qquad \textit{is} \qquad G_H(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{exp}(-i\,\omega\ \{\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}}\})}{4\pi\,\{\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}}\}}$$

with G_H *the usual solution for the euclidian metric,* $\delta_{\mathbf{y}}$ *a* 0*-current.*
When $|\mathbf{x} - \mathbf{y}| \to \infty$*,* G_H *and* ∇G_H *have an exponential decay in* $\mathbf{exp}\left(-\mathcal{O}(|\mathbf{x} - \mathbf{y}|)\right)$ *(with* $\mathbf{p} = i\omega + 0$*)*

Remark : the Dirac delta is to be interpreted with respect to the volume form defined by the metric g : if ψ is a C^∞ 0-form on $\mathbf{M}$, $\displaystyle\int_{\mathbf{M}} \psi(\mathbf{x})\delta_{\mathbf{y}}(\mathbf{x})\mathbf{dvol}_g(\mathbf{x}) = \psi(\mathbf{y})$.

The proof of this theorem can be found in Lassas [4]. For the fundamental solution of the Helmholtz equation to be defined, the fourth condition of Def. (1) must be satisfied. The change of variables is given by Eq. (1), and by considering the limit case $\mathbf{p} = i\omega$, we have

$$\{\widetilde{\mathbf{x}} - \widetilde{\mathbf{y}}\}^2 \ = \ [\varphi(\mathbf{x}) - \varphi(\mathbf{y})]^2 \ - \ \frac{1}{\omega^2}\,[f(\varphi(\mathbf{x})) - f(\varphi(\mathbf{y}))]^2 \ - \ \frac{2i}{\omega}\,[\varphi(\mathbf{x}) - \varphi(\mathbf{y})]\,.\,[f(\varphi(\mathbf{x})) - f(\varphi(\mathbf{y}))]\,.$$

Let us denote $[f] = [f(\varphi(\mathbf{x})) - f(\varphi(\mathbf{y}))]$ and $[\varphi] = [\varphi(\mathbf{x}) - \varphi(\mathbf{y})]$. The real and imaginary parts must not be null simultaneously. The condition $[f] = 0$ is straight forward deduced from the previous equation. But

$$[f] = \int_0^1 (1-t)\,\left(\frac{\mathsf{D}f}{\mathsf{D}\varphi}\right)\,\Big(\varphi_1 + t\,(\varphi_2 - \varphi_1)\Big)dt.$$

If $\boldsymbol{M}(\varphi_1, \varphi_2)$ is the previous matrix, a condition for $[f] = 0$ is $(\boldsymbol{M}[\varphi]\,, [\varphi]) = 0$, meaning that $[\varphi]$ is a eigenvector of the matrix $\boldsymbol{M} + \boldsymbol{M}^T$ associated to a null eigenvalue. A sufficient condition is $\boldsymbol{M}$ positive or null. For a continuous determination of the square root, the imaginary part must be of constant sign. A sufficient condition is $\boldsymbol{M}$ symmetric. In that case, the matrix $\left(\dfrac{\mathsf{D}f}{\mathsf{D}\varphi}\right)$ is also symmetric and there exists a convex function S such as $f = \underset{\varphi}{\mathbf{grad}}\, S$. This justifies Hyp. (1).

3 Resolution of the Maxwell equations

The study of these equations will first be made on the whole manifold $\mathbf{M}$ to guarantee the existence and uniqueness of the solution. Then, the problem on unbounded domain will be restrained to a bounded one with adapted boundary conditions.

If the hypotheses from the Rauch's theorem [7] are satisfied, every strong solution is a weak solution, and strong solutions are smooth with second members of the same regularity. If the system has a smooth strong solution, then it has a weak solution. As a consequence, in this paper, the electric and magnetic fields e and h are exclusively smooth 1-forms. Given a metric g, there is a well-known one-to-one correspondence between vector fields and 1-forms. Let j and m be 2-forms standing for the second members (or Right Hand Side (RHS) of the equation) at coefficients in C^∞ with a compact support. The harmonic Maxwell's equation in vacuum are

$$\begin{cases} \mathbf{p} \star e - \mathbf{d}\, h = -j \\ \mathbf{p} \star h + \mathbf{d}\, e = -m, \end{cases} \tag{2}$$

with $\mathbf{p} = \varepsilon + ik \in \mathbb{C}$, $k = \omega/c = \omega\sqrt{\varepsilon_0\,\mu_0}$, and the Hodge star operator $\star$ defined by the Euclidean metric to convert 1-forms to 2-forms. The fields are rescaled for symmetry, meaning $e \to \sqrt{\varepsilon_0}e$ and $h \to \sqrt{\mu_0}h$. The Maxwell operator is

$$\mathcal{M} = \begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix} = -\mathcal{M}^T,$$

and $\star(e,h)$ is the Hodge transformation of (e,h) defined by $(\star e, \star h)$. The Maxwell equations on intrinsic form are

$$(\mathbf{p}\star + \mathcal{M}) \begin{pmatrix} e \\ h \end{pmatrix} = \begin{pmatrix} -j \\ -m \end{pmatrix}. \tag{3}$$

Lemma 1. *If φ is a 0-form and A a 1-form with $\nabla A = 0$ then $\Delta^1(\varphi A) = (\Delta^0\varphi)A$.*

Proof. If A is a 1-form then A is a linear combinaison of $d\widetilde{\mathbf{x}}_i$ whose Hodge transformations are

$$\star d\widetilde{\mathbf{x}}_1 = d\widetilde{\mathbf{x}}_2 \wedge d\widetilde{\mathbf{x}}_3 \qquad \star\, d\widetilde{\mathbf{x}}_2 = d\widetilde{\mathbf{x}}_3 \wedge d\widetilde{\mathbf{x}}_1 \qquad \star\, d\widetilde{\mathbf{x}}_3 = d\widetilde{\mathbf{x}}_1 \wedge d\widetilde{\mathbf{x}}_2.$$

Without a loss of generality, we assume that $A = d\widetilde{\mathbf{x}}_1$ and $\nabla_i := \nabla_{\partial\widetilde{\mathbf{x}}_i}$, then

$$\begin{aligned} \Delta^1(\varphi A) &= -(\star\mathbf{d}\star\mathbf{d} - \mathbf{d}\star\mathbf{d}\star)(\varphi d\widetilde{\mathbf{x}}_1) \\ &= (-\partial_{11}\varphi - \partial_{22}\varphi - \partial_{33}\varphi)d\widetilde{\mathbf{x}}_1 + (\partial_{12}\varphi - \partial_{12}\varphi)d\widetilde{\mathbf{x}}_2 + (\partial_{13}\varphi - \partial_{13}\varphi)d\widetilde{\mathbf{x}}_3 \\ &= -(\Delta^0\varphi)d\widetilde{\mathbf{x}}_1. \end{aligned}$$

Similar results are obtained with $A = d\widetilde{\mathbf{x}}_2$ and $A = d\widetilde{\mathbf{x}}_3$. $\square$

Theorem 2. *If A and B are 2-forms such as $\nabla A = \nabla B = 0$, the application $G_{A,B}(\mathbf{x},\mathbf{y})$ (defined thanks to the Green function $G_H(\mathbf{x},\mathbf{y})$ provided by Thm. 1)*

$$G_{A,B}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \mathbf{p} - \mathbf{p}^{-1}\mathbf{d}\star\mathbf{d}\star & -\star\mathbf{d} \\ \star\mathbf{d} & \mathbf{p} - \mathbf{p}^{-1}\mathbf{d}\star\mathbf{d}\star \end{pmatrix} \begin{pmatrix} G_H(\mathbf{x},\mathbf{y})A \\ G_H(\mathbf{x},\mathbf{y})B \end{pmatrix}.$$

satisfies the properties of the Green function for Maxwell's equations on $(\mathbf{M}, g)$ and

$$\forall \mathbf{y} \in \mathbf{M}, \qquad \left(\mathbf{p} + \star\mathcal{M}^T\right) G_{A,B}(.,\mathbf{y}) = \begin{pmatrix} A\delta_{\mathbf{y}} \\ B\delta_{\mathbf{y}} \end{pmatrix}. \tag{4}$$

Proof. As $\left(\mathbf{p}+\star\mathcal{M}^T\right)=\begin{pmatrix}\mathbf{p} & \star\mathbf{d}\\ -\star\mathbf{d} & \mathbf{p}\end{pmatrix}$, it comes

$$\begin{aligned}\left(\mathbf{p}+\star\mathcal{M}^T\right)G_{A,B}(.,\mathbf{y}) &= \begin{pmatrix}\mathbf{p} & \star\mathbf{d}\\ -\star\mathbf{d} & \mathbf{p}\end{pmatrix}\begin{pmatrix}\mathbf{p}-\mathbf{p}^{-1}\mathbf{d}\star\mathbf{d}\star & -\star\mathbf{d}\\ \star\mathbf{d} & \mathbf{p}-\mathbf{p}^{-1}\mathbf{d}\star\mathbf{d}\star\end{pmatrix}\begin{pmatrix}G_H(.,\mathbf{y})A\\ G_H(.,\mathbf{y})B\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{p}^2-\mathbf{d}\star\mathbf{d}\star+\star\mathbf{d}\star\mathbf{d} & 0\\ 0 & \mathbf{p}^2-\mathbf{d}\star\mathbf{d}\star+\star\mathbf{d}\star\mathbf{d}\end{pmatrix}\begin{pmatrix}G_H(.,\mathbf{y})A\\ G_H(.,\mathbf{y})B\end{pmatrix}\\ &= \begin{pmatrix}(\mathbf{p}^2-\Delta_g)G_H(.,\mathbf{y})A\\ (\mathbf{p}^2-\Delta_g)G_H(.,\mathbf{y})B\end{pmatrix}=\begin{pmatrix}A\delta_{\mathbf{y}}\\ B\delta_{\mathbf{y}}\end{pmatrix} \qquad \text{(lemma (1))}\end{aligned}$$

□

3.1 Case of the Maxwell equations with RHS $\neq 0$ and $\mathbf{O}=\emptyset$

For the rest of this paper, we assume that solutions are smooth enough without loss of generality thanks to the equivalence provided by the Rauch Theorem [7] between strong and weak solutions and the Friedrichs systems finally obtained. The currents will be written as differential forms with distributional coefficients since equations are intrinsic.

Theorem 3. *If* $(\mathbf{M},g)$ *is an outgoing absorbing pseudo-riemannian manifold, then the problem given by* (2) *without any scattering object has a unique solution*

$$e_i=\int_{\mathbf{M}}G_{d\widetilde{\mathbf{x}}_i,0}\wedge\begin{pmatrix}-j\\-m\end{pmatrix}\qquad h_i=\int_{\mathbf{M}}G_{0,d\widetilde{\mathbf{x}}_i}\wedge\begin{pmatrix}-j\\-m\end{pmatrix}.$$

Proof. With X,Y 1-forms, we denote $\star\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}\star X\\ \star Y\end{pmatrix}$. Their exterior product, with P,Q 1-forms, is $\begin{pmatrix}X\\Y\end{pmatrix}\wedge\begin{pmatrix}P\\Q\end{pmatrix}=X\wedge P+Y\wedge Q$. Moreover, $P^*\wedge Q=-P\wedge Q^*$. The weak formulation of Maxwell's equations, with A and B 1-forms, is

$$(\mathbf{p}\star+\mathcal{M})\begin{pmatrix}e\\h\end{pmatrix}\wedge\varphi(A,B)=\begin{pmatrix}e\\h\end{pmatrix}\wedge\left(-\mathbf{p}\star\varphi(A,B)+\mathcal{M}\varphi(A,B)\right)$$

By applying $\star$ to Eq. (4), we have $\left(\mathbf{p}\star+\mathcal{M}^T\right)G_{A,B}(.,\mathbf{y})=\begin{pmatrix}\star A\delta_{\mathbf{y}}\\ \star B\delta_{\mathbf{y}}\end{pmatrix}$. Then,

$$\begin{aligned}e\wedge\star A+h\wedge\star B &= \left\{\int_{\mathbf{M}}\begin{pmatrix}e\\h\end{pmatrix}\wedge\left(\mathbf{p}\star+\mathcal{M}^T\right)G_{A,B}(.,\mathbf{y})\right\}\mathsf{dvol}_g(\mathbf{y})\\ &= \left\{\int_{\mathbf{M}}\begin{pmatrix}e\\h\end{pmatrix}\wedge\left(\mathbf{p}\star G_{A,B}(.,\mathbf{y})\right)\right\}\mathsf{dvol}_g(\mathbf{y})\\ &\quad+\left\{\int_{\mathbf{M}}\begin{pmatrix}e\\h\end{pmatrix}\wedge\left(\mathcal{M}^T G_{A,B}(.,\mathbf{y})\right)\right\}\mathsf{dvol}_g(\mathbf{y})\end{aligned}$$

By commutation of $\star$, $\int_{\mathbf{M}}\begin{pmatrix}e\\h\end{pmatrix}\wedge\left(\mathbf{p}\star G_{A,B}(.,\mathbf{y})\right)=\int_{\mathbf{M}}G_{A,B}(.,\mathbf{y})\wedge\left(\mathbf{p}\star\begin{pmatrix}e\\h\end{pmatrix}\right)$.
If α and β 1-forms, $\mathbf{d}\,(\alpha\wedge\beta)=\mathbf{d}\,\alpha\wedge\beta-\alpha\wedge\mathbf{d}\,\beta$, therefore

$$0=\int_{\mathbf{M}}\mathbf{d}\,(\alpha\wedge\beta)=\int_{\mathbf{M}}\mathbf{d}\,\alpha\wedge\beta-\int_{\mathbf{M}}\alpha\wedge\mathbf{d}\,\beta,$$

and then $\int_{\mathbf{M}}\mathbf{d}\,\alpha\wedge\beta=\int_{\mathbf{M}}\alpha\wedge\mathbf{d}\,\beta$. With the properties of the Green function,

$$\int_{\mathbf{M}} \begin{pmatrix} e \\ h \end{pmatrix} \wedge (\mathcal{M}^T G_{A,B}(.,\mathbf{y})) = \int_{\mathbf{M}} \mathcal{M}^T \begin{pmatrix} e \\ h \end{pmatrix} \wedge G_{A,B}(.,\mathbf{y}) = \int_{\mathbf{M}} G_{A,B}(.,\mathbf{y}) \wedge \mathcal{M} \begin{pmatrix} e \\ h \end{pmatrix}.$$

Finally

$$\begin{aligned} e \wedge \star A + h \wedge \star B &= \left\{ \int_{\mathbf{M}} G_{A,B}(.,\mathbf{y}) \wedge (\mathbf{p}\star + \mathcal{M}) \begin{pmatrix} e \\ h \end{pmatrix} \right\} \mathsf{d}\mathbf{vol}_g(\mathbf{y}) \\ &\underset{Eq.(3)}{=} \left\{ \int_{\mathbf{M}} G_{A,B}(.,\mathbf{y}) \wedge \begin{pmatrix} -j \\ -m \end{pmatrix} \right\} \mathsf{d}\mathbf{vol}_g(\mathbf{y}). \end{aligned}$$

This formula is established for any 1-forms A and B. As $e = \sum e_i d\widetilde{\mathbf{x}}_i$ and $h = \sum h_i d\widetilde{\mathbf{x}}_i$, by choosing respectively $A = d\widetilde{\mathbf{x}}_i, B = 0$ and $A = 0, B = d\widetilde{\mathbf{x}}_i$, the components e_i and h_i of the electric and magnetic fields are determined. $\square$

3.2 Case of Maxwell's equations with RHS$= 0$ and $\mathbf{O} \neq \emptyset$

Theorem 4. *The Green function $G_{A,B}(.,\mathbf{y})$ can be decomposed as $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$, with A and B 1-forms. If $(\mathbf{M}, g)$ is an outgoing absorbing pseudo-riemannian manifold, we have a Stratton-Chu formula with the solution (e,h) of Problem (2)*

$$e \wedge \star A + h \wedge \star B = \left\{ \int_{\partial\Omega_\infty} e \wedge G_2 - h \wedge G_1 \right\} \mathsf{d}\mathbf{vol}_g(\mathbf{y}).$$

Proof. With A and B 1-forms,

$$\begin{aligned} e \wedge \star A + h \wedge \star B &= \left\{ \int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} e \\ h \end{pmatrix} \wedge \left(\mathbf{p}\star + \mathcal{M}^T\right) G_{A,B}(.,\mathbf{y}) \right\} \mathsf{d}\mathbf{vol}_g(\mathbf{y}) \\ &= \left\{ \int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} e \\ h \end{pmatrix} \wedge (\mathbf{p}\star G_{A,B}(.,\mathbf{y})) \right\} \mathsf{d}\mathbf{vol}_g(\mathbf{y}) \\ &\quad + \left\{ \int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} e \\ h \end{pmatrix} \wedge (\mathcal{M}^T G_{A,B}(.,\mathbf{y})) \right\} \mathsf{d}\mathbf{vol}_g(\mathbf{y}) \end{aligned}$$

Eq. (3) with RHS$= 0$ becomes $\mathbf{p}\star \begin{pmatrix} e \\ h \end{pmatrix} = -\mathcal{M} \begin{pmatrix} e \\ h \end{pmatrix}$. But $-\mathcal{M} = \mathcal{M}^T$, so

$$\begin{aligned} \int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} e \\ h \end{pmatrix} \wedge (\mathbf{p}\star G_{A,B}(.,\mathbf{y})) &= \int_{\mathbb{R}^3/\mathbf{0}} G_{A,B}(.,\mathbf{y}) \wedge \left(\mathbf{p}\star \begin{pmatrix} e \\ h \end{pmatrix}\right) \\ &= \int_{\mathbb{R}^3/\mathbf{0}} G_{A,B}(.,\mathbf{y}) \wedge \begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix} \begin{pmatrix} e \\ h \end{pmatrix}. \end{aligned}$$

Moreover, $\displaystyle\int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} e \\ h \end{pmatrix} \wedge (\mathcal{M}^T G_{A,B}(.,\mathbf{y})) = \int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} e \\ h \end{pmatrix} \wedge \left(\begin{pmatrix} 0 & \mathbf{d} \\ -\mathbf{d} & 0 \end{pmatrix} G_{A,B}(.,\mathbf{y}) \right)$.

With $G_{A,B}(.,\mathbf{y}) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$, we have

$$
\begin{aligned}
e \wedge \star A + h \wedge \star B &= \left\{ \int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} e \wedge \mathbf{d}G_2 + G_2 \wedge \mathbf{d}e \\ -h \wedge \mathbf{d}G_1 + G_1 \wedge (-\mathbf{d}h) \end{pmatrix} \right\} \mathbf{dvol}_g(\mathbf{y}) \\
&= \left\{ \int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} e \wedge \mathbf{d}G_2 - \mathbf{d}e \wedge G_2 \\ -h \wedge \mathbf{d}G_1 + \mathbf{d}h \wedge G_1 \end{pmatrix} \right\} \mathbf{dvol}_g(\mathbf{y}) \\
&= \left\{ \int_{\mathbb{R}^3/\mathbf{0}} \begin{pmatrix} \mathbf{d}(e \wedge G_2) \\ -\mathbf{d}(h \wedge G_1) \end{pmatrix} \right\} \mathbf{dvol}_g(\mathbf{y}).
\end{aligned}
$$

With Stokes formula, $e \wedge \star A + h \wedge \star B = \left\{ \int_{\partial\Omega_\infty} e \wedge G_2 - h \wedge G_1 \right\} \mathbf{dvol}_g(\mathbf{y})$. □

Remark : this theorem gives the solution for a scattering problem for the Maxwell equations, but also an estimation of the error committed by bounding the domain. In all the space, the restriction of the solution coincides with the real solution. If the problem is well-posed, there is a correspondance between the inhomogeous problem and the trace of the solution. If the solution is homogeous, this formula gives the error. Moreover, if the errors created on the articifial boundary are small, it is useless to set an artificial boundary far away for the studied domain $\mathbf{D}$ because of the exponential decay of the Green function : the traces of the PML solution are small and exponentially decay while they return inside $\mathbf{D}$. Finally, this formula allows the superposition of solutions : for a perfect scatterer, the solution of Maxwell's equations depends on the trace of the fields on this object's boundary, which leads to the following corollary.

Corollary 5. *On* $(\mathbf{M}, g)$*, Problem* (2) *has a unique solution*

$$
\begin{aligned}
E &= \left\{ \int_{\partial\Omega_0} \left(-\eta \wedge \left(G_{d\widetilde{\mathbf{x}}_j,0}\right)_2 + \gamma \wedge \left(G_{d\widetilde{\mathbf{x}}_j,0}\right)_1 \right) \right\} d\widetilde{\mathbf{x}}_j, \\
H &= \left\{ \int_{\partial\Omega_0} \left(-\eta \wedge \left(G_{0,d\widetilde{\mathbf{x}}_j}\right)_2 + \gamma \wedge \left(G_{0,d\widetilde{\mathbf{x}}_j}\right)_1 \right) \right\} d\widetilde{\mathbf{x}}_j.
\end{aligned}
$$

with η *and* γ *smooth enough currents.*

4 Harmonic Problem

In this section, we prove that the PML problem for Maxwell's and waves equations can be written on the form

$$
\boldsymbol{K}(\mathbf{p}, \mathbf{x}) \begin{pmatrix} e \\ h \end{pmatrix} + \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix} \begin{pmatrix} e \\ h \end{pmatrix} = \begin{pmatrix} -j \\ -m \end{pmatrix}
$$

with (j, m) smooth enough, as well as the following theorem.

Theorem 6. *With the following hypotheses :*

- $\boldsymbol{K}(\mathbf{p}, \mathbf{x})$ L^∞ *in* $\mathbf{x}$
- $\boldsymbol{K}(\mathbf{p}, \mathbf{x})$ *holomorphic in* $\mathbf{p} \in \mathbf{H} \supset i\mathbb{R}_*^+$
- $\exists \mathbf{p}_0 \in \mathbf{H}$ *such as* $\mathbf{Re}(\boldsymbol{K}(\mathbf{p}_0, \mathbf{x}))$ *is coercive*
- $\exists \mathbf{p}_1 \in \mathbf{H}$ *such as* $\boldsymbol{K}(\mathbf{p}_1, \mathbf{x}) + \boldsymbol{A}_i^c \partial_i$ *with a compact resolvent,*

for all $\mathbf{p} \in \mathbf{H}/\mathbf{S}$ *where* $\mathbf{S}$ *is a locally finite set, eventually empty, and* Ω^∞ *bounded,* $(\boldsymbol{K}(\mathbf{p}, \mathbf{x}) + \boldsymbol{A}_i^c \partial_i)^{-1}$ *is bounded in* $L^2(\Omega)^m$ *with maximal monotone boundary conditions.*

4.1 Intrinsic form of equations

The contravariant coordinates of a random point $\mathbf{x}$ are denoted by $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, the basis vector of the cotangent bundle by $d\mathbf{x}_i$ and $d\check{\mathbf{x}}_i$ stands for the elementary 2-form given by the exterior product $d\mathbf{x}_j \wedge d\mathbf{x}_k$ with $j \neq i$, $k \neq i$ and $j < k$. With E and H 1-forms, $(\mathbf{M}, g)$ an absorbing pseudo-riemannian manifold of metric $g(.,.)$, the Maxwell equations are the following equalities of 2-forms

$$\begin{cases} \mathbf{p} \star E - \mathbf{d}H = 0, \\ \mathbf{p} \star H + \mathbf{d}E = 0. \end{cases}$$

With $\boldsymbol{G}$ matrix of the metric and $g = \mathbf{det}(\boldsymbol{G})$, by the Hodge star operator $\star$ on $\mathbf{M}$, a 1-form $E = \sum e_i d\mathbf{x}_i$ is changed onto a 2-form whose component in $d\check{\mathbf{x}}_i$ is $\sqrt{g} g^{ij} e_j$ (g^{ij} components of the inverse of $g(.,)$, $\sqrt{g}$ complex square root of the determinant g). Therefore $\star E = \sqrt{g}\boldsymbol{G}^{-1}e$. The intrinsic form of Maxwell equations is

$$\begin{cases} \mathbf{p}\sqrt{g}\boldsymbol{G}^{-1}E - \mathbf{d}H = 0, \\ \mathbf{p}\sqrt{g}\boldsymbol{G}^{-1}H + \mathbf{d}E = 0. \end{cases}$$

The purpose of this paper is to restrain the solution of Maxwell's equations to a bounded domain Ω of $\mathbb{R}^3$. The first step is to write the equations in a more appropriated system of coordinates. Let us remind that the PML media is defined by a stretching of coordinates decribed by (1). If $\boldsymbol{J}(\mathbf{p}, \mathbf{x})$ is the jacobian matrix of this transformation, then

$$\boldsymbol{G}(\mathbf{p}, \mathbf{x}) = \boldsymbol{J}(\mathbf{p}, \mathbf{x})^T \boldsymbol{J}(\mathbf{p}, \mathbf{x}).$$

The associated Hodge transformation is

$$\star = \sqrt{g}\boldsymbol{G}^{-1}(\mathbf{p}, \mathbf{x}) = \boldsymbol{K}(\mathbf{p}, \mathbf{x}) = \mathbf{det}(\boldsymbol{J}(\mathbf{p}, \mathbf{x}))\boldsymbol{J}(\mathbf{p}, \mathbf{x})^{-1} \left(\boldsymbol{J}(\mathbf{p}, \mathbf{x})^T\right)^{-1}. \tag{5}$$

The domain of the family of operator $(\boldsymbol{K}(\mathbf{p}, \mathbf{x}) + \boldsymbol{A})_{\mathbf{p} \in \mathbf{D}_0}$ is holomorphic and independent of the frequency ω. As solving a PDE means to inverse its operator, it seems natural to look precisely the kind of operator we have to deal with. The book of Kato [3] gives various tools to prove existence and uniqueness of solution as long as the family of operator is of type A, which is the case here.

4.2 Main results on operators of type A

Definition 2. *Let $\mathbf{X}$,$\mathbf{Y}$ be two Banach spaces. A family $\mathcal{T}(u) \in C(\mathbf{X}, \mathbf{Y})$ defined for u in a domain $\mathbf{D}_0$ of the complex plane is said to be holomorphic of type A if $\mathcal{D}(\mathcal{T}(u)) = \mathcal{D}$ is independent of u, and $\forall (u, v) \in \mathbf{D}_0 \times \mathcal{D}$, $\mathcal{T}(u)v$ is holomorphic.*

Theorem 7. *Let $\mathcal{T}(z) \in C(\mathbf{X})$ be a holomorphic family of type A on a domain $\mathbf{D}_0$ of $\mathbb{C}$. We suppose that the resolvent set of $\mathcal{T}(z)$ is not empty $\forall z \in \mathbf{D}_0$ and that there exists a point z_0 in $\mathbf{D}_0$ such as the resolvent of $\mathcal{T}(z_0)$ is compact. Then, for all z in $\mathbf{D}_0$, the resolvant of $\mathcal{T}(z)$ is compact.*

4.3 Harmonic Maxwell equations in the PML

In the previous section, we have established that harmonic Maxwell equations in the PML media can be written as $(\boldsymbol{K}(\mathbf{p}, \mathbf{x}) + \boldsymbol{A}(\mathbf{x}))\mathbf{u}(\mathbf{x}) = f(\mathbf{x})$, with $\boldsymbol{K} : (\mathbf{p}, \mathbf{x}) \in \mathbf{D}_0 \times \Omega \longrightarrow \mathbf{Hom}(\mathbb{C}^6)$ a holomorphic function in $\mathbf{p} = \varepsilon + i\omega \in \mathbf{D}_0 \subset \mathbb{C}$, in $L^\infty(\Omega)$ for $\mathbf{x}$, and $\boldsymbol{A} = \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}$. We suppose that $\mathbf{D}_0$ is a domain of $\mathbb{C}$ including a segment like $[i\omega_{min}, i\omega_{max}]$, $f \in L^2(\Omega)$ and $\boldsymbol{A}$ is Maxwell's operator whose domain $\mathcal{D}(\boldsymbol{A})$ is independent of $\mathbf{p} \in \mathbb{C}$.

Solving this equation means inversing the operator $(\boldsymbol{K}(\mathbf{p},.)+\boldsymbol{A})$ of domain $\mathcal{D}(\boldsymbol{A})$ at $\mathbf{p}=i\omega$, which is an operator of type A. Moreover, $(\boldsymbol{A},\mathcal{D}(\boldsymbol{A}))$ is maximal monotone and its resolvent set is not empty. As $\boldsymbol{K}(\mathbf{p},.)$ is $L^\infty(\Omega)$, the operator $(\boldsymbol{K}(\mathbf{p},.)+\boldsymbol{A},\mathcal{D}(\boldsymbol{A}))$ has a non empty resolvent set for all $\mathbf{p}$ in $\mathbf{D}_0$.

Thm. (7), the main point of this section, will prove that studying the $(\boldsymbol{K}(\mathbf{p},.)+\boldsymbol{A}$ is an operator of type A with a compact resolvent set, and the study of this set's compactness can be reduced to the study in a single point, and therefore ensures the existence and unicity of the solution.

Theorem 8 (Petkov[6]). *If* $Q=\begin{pmatrix}\mathbf{div} & 0\\ 0 & \mathbf{div}\end{pmatrix}$, *for all* u *in* $\mathcal{D}(\boldsymbol{A})\cap(H(\mathbf{div},\Omega)^2)$, *then*

$$\| u \|_{(H^1(\Omega))^6}\leq C\left\{\| u \|_{(L^2(\Omega))^6}+\| \boldsymbol{A}u \|_{(L^2(\Omega))^6}+\| Qu \|_{(L^2(\Omega))^6}\right\}.$$

Definition 3. *If* $P_0: L^2(\Omega)^3\to H(\mathbf{div}_0,\Omega)$ *and* $P_\perp: L^2(\Omega)^3\to(\mathbf{grad}(H_0^1(\Omega))^2)$ *are the projections associated to the Hodge decomposition, for all* $g\in L^2(\Omega)^3$, $g_0=P_0g$, $g_\perp=P_\perp g$. *The space* $H(\mathbf{div}_0,\Omega)^2$ *is stable under the action of* $\boldsymbol{A}$.

Lemma 2. *The operator* $P_\perp\boldsymbol{K}(\mathbf{p},.)P_\perp$ *is invertible on* $(\mathbf{grad}(H_0^1(\Omega)))^2$ *for all* $\mathbf{p}$ *in* $\mathbf{D}_0$ *except for a finite subset (eventually empty) of* $\mathbf{D}_0$, *denoted by* $\mathbf{S}$. *Moreover, its inverse is holomorphic on* $\mathbf{D}_0\backslash\mathbf{S}$.

Proof. Let h be in $H^{-1}(\Omega)^2$. Inversing $P_\perp\boldsymbol{K}(\mathbf{p},.)P_\perp$ on $(\mathbf{grad}(H_0^1(\Omega)))^2$ requiere to find ω in $H_0^1(\Omega)^2$ such as $\forall v\in H_0^1(\Omega)^2$,

$$\int_\Omega\left\langle\boldsymbol{K}(\mathbf{p},.)\nabla\omega,\overline{\nabla v}\right\rangle\,dx=<h,v>_{H^{-1}(\Omega)^2\times H_0^1(\Omega)^2},\tag{6}$$

with $\nabla\omega=\begin{pmatrix}\nabla\omega_1\\ \nabla\omega_2\end{pmatrix}$.

If $a(\omega,v)=\displaystyle\int_\Omega\left\langle\boldsymbol{K}(\mathbf{p},.)\nabla\omega,\overline{\nabla v}\right\rangle dx$, $L(v)=<h,v>_{H^{-1}(\Omega)^2\times H_0^1(\Omega)^2}$, the applications $a(.,.)$ and $L(.)$ are respectively bilinear and linear. A Cauchy-Schwartz inequality proves that $a(.,.)$ is continuous on $H_0^1(\Omega)^2\times H_0^1(\Omega)^2$ and $L(.)$ is continuous on $H_0^1(\Omega)^2$. The previous problem is in fact, to find ω in $H_0^1(\Omega)^2$ such as $\forall v\in H_0^1(\Omega)^2$,

$$a(\omega,v)=L(v).\tag{7}$$

The Lax-Milgram theorem for $\mathbf{p}=\mathbf{p}_0$ guarantees that this equation is well-posed. The coercivity of $\boldsymbol{K}(\mathbf{p}_0,.)$ and the continuity of $L(.)$ on $H_0^1(\Omega)^2$ leads to the inequality satisfied by the solution $\beta\|\omega\|_{H_0^1(\Omega)^2}\leq\|h\|_{H^{-1}(\Omega)^2}$, with $\beta>0$. Therefore, the resolvent of $P_\perp\boldsymbol{K}(\mathbf{p}_0,.)P_\perp$ is compact because of the compactness of the injection $H_0^1(\Omega)$ in $H^{-1}(\Omega)$. By Theorem (7), the resolvent of $P_\perp\boldsymbol{K}(\mathbf{p},.)P_\perp$ is compact for all $\mathbf{p}$ in $\mathbf{D}_0$. As a consequence, this operator is either singular on $\mathbf{D}_0$ either invertible for all $\mathbf{p}$ in $\mathbf{D}_0$ except for a finite subset (eventually empty) of $\mathbf{D}_0$ denoted $\mathbf{S}$. As Problem (7) is well-posed for $\mathbf{p}_0$, this operator is invertible $\forall\mathbf{p}$ in $\mathbf{D}_0\backslash\mathbf{S}$.

Let us show that operator $(P_\perp\boldsymbol{K}(\mathbf{p},.)P_\perp)^{-1}$ is holomorphic on $\mathbf{D}_0\backslash\mathbf{S}$. By application of the closed graph theorem to the closed set $\mathbf{grad}(H_0^1(\Omega))^2$ of $L^2(\Omega)^6$, we have $(P_\perp\boldsymbol{K}(\mathbf{p},.)P_\perp)^{-1}$ in $\mathcal{L}((\mathbf{grad}(H_0^1(\Omega)))^2)$. As $P_\perp\boldsymbol{K}(\mathbf{p},.)P_\perp$ is holomorphic on $\mathbf{D}_0$, its inverse is holomorphic on $\mathbf{D}_0\backslash\mathbf{S}$. □

Theorem 9 (Kato[3]). *If it exists* $\mathbf{p}_0\in\mathbf{D}_0$ *such as* $\boldsymbol{K}(\mathbf{p}_0,.)$ *coercive, then harmonic Maxwell system in the PML media is well-posed for all real frequency except for a finite discrete subset (eventually empty) of* $\mathbb{R}$.

Proof. By projection of Eq. (6) with the Hodge decomposition, the problem is to find $u=u_0+u_\perp\in\mathcal{D}(\boldsymbol{A})$ such as

$$\begin{cases}P_0\boldsymbol{K}(\mathbf{p},.)\,u+\boldsymbol{A}\,u_0=f_0,\\ P_\perp\boldsymbol{K}(\mathbf{p},.)u=f_\perp.\end{cases}$$

As $u = P_0\,u + P_\perp\,u$, the previous system becomes

$$\begin{cases} P_0\boldsymbol{K}(\mathbf{p},.)P_0u + P_0\boldsymbol{K}(\mathbf{p},.)P_\perp u + \boldsymbol{A}u = f_0, \\ P_\perp\boldsymbol{K}(\mathbf{p},.)P_\perp u + P_\perp\boldsymbol{K}(\mathbf{p},.)P_0u = f_\perp. \end{cases}$$

With Lem. (2), if $\widetilde{\boldsymbol{A}}$ is the operator defined by the restriction of the operator $\boldsymbol{A}$ to the set $(H(\mathbf{div}_0,\Omega))^2$, then $\mathcal{D}(\widetilde{\boldsymbol{A}}) = \mathcal{D}(\boldsymbol{A}) \cap (H(\mathbf{div}_0,\Omega))^2$, so the problem is to inverse in $(H(\mathbf{div}_0,\Omega))^2$ the closed operator

$$\mathcal{R}(\mathbf{p}) = P_0\boldsymbol{K}_0(\mathbf{p},.)P_0 - P_0\boldsymbol{K}_0(\mathbf{p},.)P_\perp(P_\perp\boldsymbol{K}_0(\mathbf{p},.)P_\perp)^{-1}P_\perp\boldsymbol{K}_0(\mathbf{p},.)P_0 + P_0\boldsymbol{K}_1(\mathbf{p},.)\widetilde{\boldsymbol{A}}$$

The operator $\mathcal{R}(\mathbf{p})$ is holomorphic on $\mathbf{D}_0\backslash\mathbf{S} = \mathbf{D}_1$. By use of Thm. (8) and (7) to the family of holomorphic operator $\mathcal{R}(\mathbf{p}) + \widetilde{\boldsymbol{A}}$ of type A, it appears that $\mathcal{R}(\mathbf{p}) + \widetilde{\boldsymbol{A}}$ is a holomorphic family of closed operators with compact resolvents for $\mathbf{p} \in \mathbf{D}_1$. As a consequence, this operator is either singular on $\mathbf{D}_1$ either invertible for all $\mathbf{p}$ in $\mathbf{D}_1$ except for a locally finite discrete subset (eventually empty) of $\mathbf{D}_1$. Operator $P_\perp\boldsymbol{K}(\mathbf{p}_0,.)P_\perp$ defines a sesquilinear coercive form in $(\mathbf{grad}(H_0^1(\Omega)))^2$ because $\boldsymbol{K}(\mathbf{p}_0)$ is coercive in $L^2(\Omega)^6$ which implies $P_\perp\boldsymbol{K}(\mathbf{p}_0,.)P_\perp$ invertible. $\boldsymbol{A}$ is maximal monotone and $\boldsymbol{K}(\mathbf{p}_0,.)$ is monotone coercive and bounded in $L^2(\Omega)^6$, thus the operator $\boldsymbol{A} + \boldsymbol{K}(\mathbf{p}_0,.)$ is invertible. Finally, the operator $\mathcal{R}(\mathbf{p}_0,.) + \widetilde{\boldsymbol{A}}$ is invertible on $(H(\mathbf{div}_0,\Omega))^2$. So $\mathcal{R}(\mathbf{p}) + \widetilde{\boldsymbol{A}}$ is invertible for all p in $\mathbf{D}_1$ except for a locally finite discrete subset (eventually empty) of $\mathbf{D}_1$. This proves that the operator $\boldsymbol{K}(\mathbf{p},.)+\boldsymbol{A}$ is invertible on $L^2(\Omega)^6$ for all $\mathbf{p}$ in $\mathbf{D}_1$ except for a locally finite discrete subset (eventually empty) of $\mathbf{D}_1$. □

4.4 Study of the coercivity of the PML matrix

According to Thm. (9), to have existence and uniqueness of the solution, it must exists a $\mathbf{p}_0$ in $\mathbb{C}$ such as $\boldsymbol{K}(\mathbf{p}_0,\mathbf{x})$ is coercive. If we exhibit such a point, the resolvent is not empty and there is no more alternative.

Property 10. *The matrix $\boldsymbol{K}(1,\mathbf{x})$ is symmetric definite and positive for all x in $\mathbb{R}^3$.*

Proof. As $\boldsymbol{K}(\mathbf{p},\mathbf{x})$ is defined by Eq. (5), we have

$$\begin{aligned} \boldsymbol{K}(\mathbf{p},\mathbf{x})^T &= \left(\mathbf{det}(\boldsymbol{J}(\mathbf{p},\mathbf{x}))\boldsymbol{J}(\mathbf{p},\mathbf{x})^{-1}\left(\boldsymbol{J}(\mathbf{p},\mathbf{x})^T\right)^{-1}\right)^T \\ &= \mathbf{det}(\boldsymbol{J}(\mathbf{p},\mathbf{x}))\boldsymbol{J}(\mathbf{p},\mathbf{x})^{-1}\left(\boldsymbol{J}(\mathbf{p},\mathbf{x})^T\right)^{-1} = \boldsymbol{K}(\mathbf{p},\mathbf{x}), \end{aligned}$$

with $\boldsymbol{J}(\mathbf{p},\mathbf{x}) = \dfrac{\mathsf{D}(\widetilde{\mathbf{x}})}{\mathsf{D}(\varphi(\mathbf{x}))}\dfrac{\mathsf{D}(\varphi(\mathbf{x}))}{\mathsf{D}(\mathbf{x})}$. So $\boldsymbol{K}(\mathbf{p},\mathbf{x})$ is symmetric for all $\mathbf{p} \in \mathbb{C}$, and a fortiori $\boldsymbol{K}(1,\mathbf{x})$ is symmetric. If $\mathbf{X} \in \mathbb{R}^3$ with $\mathbf{X} \neq \mathbf{0}$, then

$$\begin{aligned} \mathbf{X}^T\boldsymbol{K}(1,\mathbf{x})\mathbf{X} &= \mathbf{X}^T\left(\mathbf{det}(\boldsymbol{J}(1,\mathbf{x}))\boldsymbol{J}(1,\mathbf{x})^{-1}\left(\boldsymbol{J}(1,\mathbf{x})^T\right)^{-1}\right)\mathbf{X} \\ &= \mathbf{det}(\boldsymbol{J}(1,\mathbf{x}))\mathbf{X}^T\boldsymbol{J}(1,\mathbf{x})^{-1}\left(\boldsymbol{J}(1,\mathbf{x})^T\right)^{-1}\mathbf{X} \\ &= \mathbf{det}(\boldsymbol{J}(1,\mathbf{x}))\mathbf{X}^T\boldsymbol{J}(1,\mathbf{x})^{-1}\left(\mathbf{X}^T\boldsymbol{J}(1,\mathbf{x})^{-1}\right)^T. \end{aligned}$$

If $\boldsymbol{A} = \mathbf{X}^T\boldsymbol{J}(1,\mathbf{x})^{-1}$, then $\mathbf{X}^T\boldsymbol{K}(1,\mathbf{x})\mathbf{X} = \mathbf{det}(\boldsymbol{J}(1,\mathbf{x}))\boldsymbol{A}\boldsymbol{A}^T$. As $\boldsymbol{A}\boldsymbol{A}^T$ is definite positive for all matrix $\boldsymbol{A}$, we deduced that $\boldsymbol{K}(1,\mathbf{x})$ is definite positive if $\mathbf{det}(\boldsymbol{J}(1,\mathbf{x})) > 0$. Let $\boldsymbol{F}(\mathbf{x}) = \dfrac{\mathsf{D}(\varphi(\mathbf{x}))}{\mathsf{D}(\mathbf{x})}$ be the jacobian matrix of φ. As φ is an increasing function that represents the embedding in a manifold $\mathbf{M}$ of $\mathbb{C}^3$, we have $\mathbf{det}(\boldsymbol{F}(\mathbf{x})) > 0$. Let us denote $\boldsymbol{F}_0(\mathbf{x}) = \dfrac{\mathsf{D}(\widetilde{\mathbf{x}})}{\mathsf{D}(\varphi(\mathbf{x}))}$. For all $(i,j) \in [\![1,n]\!]$, $\boldsymbol{F}_0(\mathbf{x})_{ij} = \delta_{ij} + f'(\varphi(\mathbf{x}))$. As f is strictly increasing, $\mathbf{det}(\boldsymbol{I}_3 + \boldsymbol{F}_0(\mathbf{x})) > 0$. □

5 Explicit formulations for Maxwell's Equations

Lemma 3. *The matrix of the harmonic PML formulation is*

$$\boldsymbol{K}(\mathbf{p},\mathbf{x}) = \mathbf{det}\,\boldsymbol{T}(\mathbf{x})\left(\boldsymbol{T}(\mathbf{x})^{-1}\boldsymbol{P}^{*}\boldsymbol{M}\boldsymbol{P}\boldsymbol{T}(\mathbf{x})^{*^{-1}}\right)$$

with $\boldsymbol{T}(\mathbf{x}) = \dfrac{\mathsf{D}\varphi}{\mathsf{D}\mathbf{x}}$, $\boldsymbol{M} = \mathbf{diag}\left(\prod_{k}^{\overset{i}{\vee}}(\mathbf{p}+\lambda_k)\right)$ *and* $\boldsymbol{P}^{*} = \boldsymbol{P}^{-1}$.

Proof. Given the embedding $\widetilde{\mathbf{x}} = \varphi(\mathbf{x}) + \dfrac{1}{\mathbf{p}}f(\varphi(\mathbf{x}))$, it is possible to have a general formulation for the PML matrix. The jacobian matrix $\boldsymbol{J}(\mathbf{p},\mathbf{x})$ is $\dfrac{\mathsf{D}\widetilde{\mathbf{x}}}{\mathsf{D}\mathbf{x}} = \left(\boldsymbol{I} + \dfrac{1}{\mathbf{p}}\dfrac{\mathsf{D}f}{\mathsf{D}\varphi}\right)\dfrac{\mathsf{D}\varphi}{\mathsf{D}\mathbf{x}}$. As $f = \underset{\varphi}{\mathbf{grad}}S$, is exists $\boldsymbol{P}$ a unitary matrix $\boldsymbol{P}^{*} = \boldsymbol{P}^{-1}$ such as $\dfrac{\mathsf{D}f}{\mathsf{D}\varphi} = \boldsymbol{P}^{*}\left(\boldsymbol{I} + \dfrac{1}{\mathbf{p}}\boldsymbol{D}\right)\boldsymbol{P}$ with $\boldsymbol{D}$ a diagonal matrix and λ_i, $i = 1,2,3$ its eigenvalues. With $\boldsymbol{T}(\mathbf{x}) = \dfrac{\mathsf{D}\varphi}{\mathsf{D}\mathbf{x}}$, we have $\boldsymbol{J}(\mathbf{p},\mathbf{x}) = \boldsymbol{P}^{*}\left(\boldsymbol{I} + \dfrac{1}{\mathbf{p}}\boldsymbol{D}\right)\boldsymbol{P}\boldsymbol{T}(\mathbf{x})$. The Hodge transformation associated is $\star = \mathbf{det}(\boldsymbol{J}(\mathbf{p},\mathbf{x}))\boldsymbol{J}(\mathbf{p},\mathbf{x})^{-1}\left(\boldsymbol{J}(\mathbf{p},\mathbf{x})^{T}\right)^{-1}$. The proof is ended by writing $\boldsymbol{K}(\mathbf{p},\mathbf{x}) = \mathbf{p}\star$. □

Theorem 11. *The unsteady Maxwell equations in the PML media are*

$$\bar{\boldsymbol{I}}(\mathbf{x})\frac{\partial}{\partial t}\begin{pmatrix} e \\ h \end{pmatrix} + \bar{\boldsymbol{A}}(\mathbf{x})\begin{pmatrix} e \\ h \end{pmatrix} + \bar{\boldsymbol{B}}(\mathbf{x})\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}\begin{pmatrix} e \\ h \end{pmatrix} = \begin{pmatrix} -j \\ -m \end{pmatrix},$$

where (u,v) *are solutions of the ODE*

$$\begin{pmatrix} \boldsymbol{T}(\mathbf{x})^{*}\boldsymbol{P}^{-1} & 0 \\ 0 & \boldsymbol{T}(\mathbf{x})^{*}\boldsymbol{P}^{-1} \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \boldsymbol{T}(\mathbf{x})^{*}\boldsymbol{P}^{-1}\boldsymbol{F}(\mathbf{x}) & 0 \\ 0 & \boldsymbol{T}(\mathbf{x})^{*}\boldsymbol{P}^{-1}\boldsymbol{F}(\mathbf{x}) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e \\ h \end{pmatrix},$$

with $\boldsymbol{F}(\mathbf{x}) = \mathbf{diag}\,(\lambda_i)\,, \boldsymbol{A}(\mathbf{x}) = \mathbf{diag}\left(\sum_{k}^{\overset{i}{\vee}}\lambda_k - \lambda_i\right), \boldsymbol{B}(\mathbf{x}) = \mathbf{diag}\left(\prod_{k}^{\overset{i}{\vee}}(\lambda_i - \lambda_k)\right)$ *and* $\bar{\boldsymbol{I}}(\mathbf{x}) = \boldsymbol{M}(\mathbf{x})\Big(\boldsymbol{M}(\mathbf{x})\Big)^{*^{-1}}$, $\bar{\boldsymbol{A}}(\mathbf{x}) = \boldsymbol{M}(\mathbf{x})\begin{pmatrix} \boldsymbol{A}(\mathbf{x}) & 0 \\ 0 & \boldsymbol{A}(\mathbf{x}) \end{pmatrix}\Big(\boldsymbol{M}(\mathbf{x})\Big)^{*^{-1}}$ $\bar{\boldsymbol{B}}(\mathbf{x}) = \boldsymbol{M}(\mathbf{x})\begin{pmatrix} \boldsymbol{B}(\mathbf{x}) & 0 \\ 0 & \boldsymbol{B}(\mathbf{x}) \end{pmatrix}\Big(\boldsymbol{M}(\mathbf{x})\Big)^{*^{-1}}$ *with* $\boldsymbol{M}(\mathbf{x}) = \mathbf{det}(\boldsymbol{T}(\mathbf{x}))\begin{pmatrix} \boldsymbol{T}(\mathbf{x})^{-1}\boldsymbol{P}^{*} & 0 \\ 0 & \boldsymbol{T}(\mathbf{x})^{-1}\boldsymbol{P}^{*} \end{pmatrix}$.

Proof. Let (i,j,k) be three different indices.

$$\frac{(\mathbf{p}+\lambda_i)(\mathbf{p}+\lambda_j)}{\mathbf{p}+\lambda_k} = \mathbf{p} + (\lambda_i + \lambda_j - \lambda_k) - \frac{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}{\mathbf{p}+\lambda_k},$$

The diagonal matrix of Lem. (3) can be decomposed as $\mathbf{p}\boldsymbol{I}_3 + \boldsymbol{A}(\mathbf{x}) + \boldsymbol{B}(\mathbf{x})\,(\mathbf{p}\boldsymbol{I}_3 + \boldsymbol{F}(\mathbf{x}))^{-1}$ with

$$\boldsymbol{F}(\mathbf{x}) = \mathbf{diag}\,(\lambda_i) \quad \boldsymbol{A}(\mathbf{x}) = \mathbf{diag}\left(\sum_{k}^{\overset{i}{\vee}}\lambda_k - \lambda_i\right) \quad \boldsymbol{B}(\mathbf{x}) = \mathbf{diag}\left(\prod_{k}^{\overset{i}{\vee}}(\lambda_i - \lambda_k)\right).$$

With $\boldsymbol{T}(\mathbf{x}) = \left(\dfrac{\mathsf{D}\varphi}{\mathsf{D}\mathbf{x}}\right)$, the PML matrix can be decomposed as

$$\mathbf{det}(\boldsymbol{T}(\mathbf{x}))\left(\boldsymbol{T}(\mathbf{x})^{-1}\boldsymbol{P}^{*}\left(\mathbf{p}\boldsymbol{I}_3 + \boldsymbol{A}(\mathbf{x}) + \boldsymbol{B}(\mathbf{x})\,(\mathbf{p}\boldsymbol{I}_3 + \boldsymbol{F}(\mathbf{x}))^{-1}\right)\boldsymbol{P}\boldsymbol{T}(\mathbf{x})^{*^{-1}}\right). \tag{8}$$

We notice that $(\mathbf{p}\boldsymbol{I}_3 + \boldsymbol{F}(\mathbf{x}))^{-1}\,\boldsymbol{P}\boldsymbol{T}(\mathbf{x})^{*^{-1}} = \Big(\boldsymbol{T}(\mathbf{x})^{*}\boldsymbol{P}^{-1}\,(\mathbf{p}\boldsymbol{I}_3 + \boldsymbol{F}(\mathbf{x}))\Big)^{-1}$. By choosing $\bar{\boldsymbol{I}}(\mathbf{x})$, $\bar{\boldsymbol{A}}(\mathbf{x})$, $\bar{\boldsymbol{B}}(\mathbf{x})$ and $\widetilde{\boldsymbol{F}}(\mathbf{x})$ as written above, the Maxwell equations in the PML become

$$\left(\mathbf{p}\bar{\boldsymbol{I}}(\mathbf{x}) + \bar{\boldsymbol{A}}(\mathbf{x}) + \bar{\boldsymbol{B}}(\mathbf{x})\left(\mathbf{p}\widetilde{\boldsymbol{I}}(\mathbf{x}) + \widetilde{\boldsymbol{F}}(\mathbf{x})\right)^{-1}\right)\begin{pmatrix} e \\ h \end{pmatrix} + \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}\begin{pmatrix} e \\ h \end{pmatrix} = \begin{pmatrix} -j \\ -m \end{pmatrix}.$$

If $\begin{pmatrix} u \\ v \end{pmatrix} = \left(\mathbf{p}\widetilde{\boldsymbol{I}}(\mathbf{x}) + \widetilde{\boldsymbol{F}}(\mathbf{x})\right)^{-1}\begin{pmatrix} e \\ h \end{pmatrix}$, it is a solution of an ODE that can be introduced in Maxwell's equations on $\mathbb{R}^6$ to have

$$\begin{cases} \mathbf{p}\bar{\boldsymbol{I}}(\mathbf{x})\begin{pmatrix} e \\ h \end{pmatrix} + \bar{\boldsymbol{A}}(\mathbf{x})\begin{pmatrix} e \\ h \end{pmatrix} + \bar{\boldsymbol{B}}(\mathbf{x})\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}\begin{pmatrix} e \\ h \end{pmatrix} = \begin{pmatrix} -j \\ -m \end{pmatrix} \\ \mathbf{p}\widetilde{\boldsymbol{I}}(\mathbf{x})\begin{pmatrix} u \\ v \end{pmatrix} + \widetilde{\boldsymbol{F}}(\mathbf{x})\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e \\ h \end{pmatrix}. \end{cases}$$

Using the Laplace inverse transformation ends the proof. □

Important remark : this theorem shows an important property of the PML. The spatial part of the operator $\begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}$ is not modified in the PML media. This guarantees the possibility to choose without constrain the numerical approximation method (Finite Elements, Finites Volumes, Discontinuous Galerkin, ...). Moreover, a Gedney formulation can be obtained with an other simple element decomposition (for example for an FTDT approximation with leap frog schemes).

The previous formula has been established for Maxwell's equations in 3D. For bidimensional equations, the formula is deduced for Thm. (11) through an Hadamard's method of descent.

6 Some numerical PML examples for a L-shaped geometry

6.1 A general non convex formulation

The embedding is given by $\widetilde{\theta} = \theta$,

$$\widetilde{\rho} = \phi(\rho,\theta) + \frac{1}{\mathbf{p}} f\left(\phi(\rho) - R\right),$$

and the diffeomorphism is $\phi(\rho,\theta) = k\rho + (1-k)\rho_0(\theta)$. The eigenvalues of the jacobian matrix are $\alpha(\rho,\theta) = f'\left(\phi(\rho,\theta) - R\right)$, $\beta(\rho,\theta) = \dfrac{f\left(\phi(\rho,\theta) - R\right)}{\phi(\rho,\theta)}$. The PML matrix is

$$\boldsymbol{K}(\mathbf{p},\rho,\theta) \quad = \quad \boldsymbol{Q}(\theta)\boldsymbol{P}(\rho,\theta)\left(\mathbf{p}\,\boldsymbol{I}(\rho,\theta) + \boldsymbol{A}(\rho,\theta) + \boldsymbol{B}(\rho,\theta)\Big(\mathbf{p}\boldsymbol{I}_3 + \boldsymbol{F}(\rho,\theta)\Big)^{-1}\right)\boldsymbol{P}^T(\rho,\theta)\,\boldsymbol{Q}(\theta)^T, \quad (9)$$

with $\boldsymbol{I}(\rho,\theta) = \left(k\dfrac{\phi(\rho,\theta)}{\rho}\right)\boldsymbol{I}_3$, $\boldsymbol{F}(\rho,\theta) = \mathbf{diag}\Big(\alpha(\rho,\theta),\beta(\rho,\theta),0\Big)$,

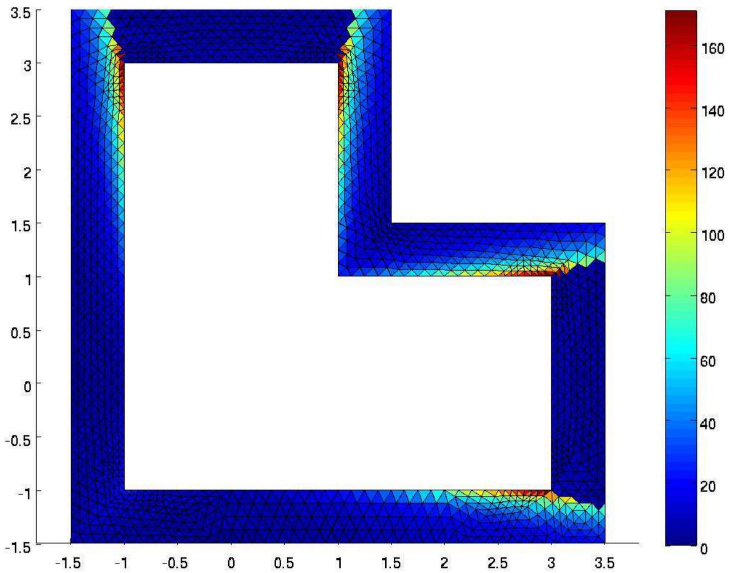
$\boldsymbol{B}(\rho,\theta) = \left(k\dfrac{\phi(\rho,\theta)}{\rho}\right)\mathbf{diag}\Big(\alpha(\rho,\theta)(\alpha(\rho,\theta) - \beta(\rho,\theta)), \beta(\rho,\theta)(\beta(\rho,\theta) - \alpha(\rho,\theta)), \alpha(\rho,\theta)\beta(\rho,\theta)\Big)$,

$\boldsymbol{A}(\rho,\theta) = \left(k\dfrac{\phi(\rho,\theta)}{\rho}\right)\mathbf{diag}\Big(\beta(\rho,\theta) - \alpha(\rho,\theta), \alpha(\rho,\theta) - \beta(\rho,\theta), \alpha(\rho,\theta) + \beta(\rho,\theta)\Big)$,

$\boldsymbol{P}(\rho,\theta) = \begin{pmatrix} \dfrac{\phi(\rho,\theta)}{\rho} & -2\,(1-k)\,\dfrac{\rho_0'(\theta)}{\rho} & 0 \\ 0 & k & 0 \\ 0 & 0 & \left(k\dfrac{\phi(\rho,\theta)}{\rho}\right)^{-1} \end{pmatrix}$, and $\boldsymbol{Q}(\theta)$ the rotation matrix around the z-axis. In that case, the matrix behind $\mathbf{p}$ is no longer the identity matrix. If $[\varepsilon]$ corresponds to this part of the matrix $\mathbf{det}(\boldsymbol{J}(\mathbf{x}))\left(\boldsymbol{J}(\mathbf{x})^T\boldsymbol{J}(\mathbf{x})\right)^{-1}$, meaning $\widetilde{\boldsymbol{A}_0} = \begin{pmatrix} [\varepsilon] & 0 \\ 0 & [\varepsilon] \end{pmatrix}$, the CFL condition associated is defined though λ_0 the smallest eigenvalue of $[\varepsilon]$ by $\dfrac{dt}{d\mathbf{x}} \leq \lambda_0$. As its value depends on the mesh, the CFL condition is the maximal value of

$$\frac{k^2 + 4\left((1-k)\dfrac{\rho_0'(\theta)}{\rho}\right)^2 + \left(k + (1-k)\dfrac{\rho_0(\theta)}{\rho}\right)^2}{2k\left(k + (1-k)\dfrac{\rho_0(\theta)}{\rho}\right)} - \sqrt{\frac{\left(k^2 + 4\left((1-k)\dfrac{\rho_0'(\theta)}{\rho}\right)^2 + \left(k + (1-k)\dfrac{\rho_0(\theta)}{\rho}\right)^2\right)^2 - 1}{4k^2\left(k + (1-k)\dfrac{\rho_0(\theta)}{\rho}\right)^2}}.$$

A numerical visualisation on an unstructured mesh of the previous formula shows that the CFL condition can reach 180, but only for few elements, where the values of $\rho_0'(\theta)$ are important. As the matrix is well-conditioned, this formulation can be used for harmonic problems. The domain is reduced by comparison with a convex formulation, and few more complexity is involved. But for unsteady problems, the penalization induced by the CFL condition may no be balanced by the reduction of the triangles' number. A first solution is to work implicitely.



6.2 A more specific formulation

A part of a L-shaped geometry is convex, so an other idea is to establish a new formula that takes advantage of this convex part. The PML domain is divided in two parts : one with a cartesian formula, and the other one with a non convex formula, that matched continously the cartesian PML at the border of each domain.

Fig. 1 explains the geometry. The domain of study $\mathbf{D}$, the interior L, is represented in white while the cartesian PML are with horizontal stripes. The diffeomorphim ϕ transforms $\mathbf{D}$ by adding the black area. In this specific area, whose thickness is controlled by the parameter κ, the waves do not decrease. The attenuative effect of the PML starts in the vertical striped area, perpendicularly to the external boundary of the black area. The embedding is given by $\widetilde{\mathbf{x}} = \phi(\mathbf{x}) + \frac{1}{\mathbf{p}} \begin{pmatrix} \int \sigma_x(\tau)d\tau \\ \int \sigma_y(\tau)d\tau \\ 0 \end{pmatrix}$ and the diffeomorphim is $\phi(\mathbf{x}) = \begin{pmatrix} x + (\kappa - 1)\,\mathbf{inf}(x, y) \\ y + (\kappa - 1)\,\mathbf{inf}(x, y) \\ 1 \end{pmatrix}$.

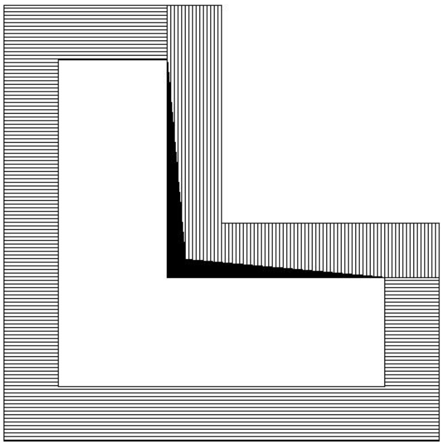


Figure 1: The different domains for a non convex PML formulation

The PML matrix can be decomposed as

$$\boldsymbol{K}(\mathbf{p}, \mathbf{x}) = \boldsymbol{T}(\mathbf{x}) \left(\mathbf{p}\boldsymbol{I} + \boldsymbol{A}(\mathbf{x}) + \boldsymbol{B}(\mathbf{x}) \left(\mathbf{p}\boldsymbol{I}_3 + \boldsymbol{F}(\mathbf{x}) \right)^{-1} \right) \boldsymbol{T}(\mathbf{x})^T,$$

with $\boldsymbol{I} = \frac{1}{\kappa}\boldsymbol{I}_3$, $\boldsymbol{A}(\mathbf{x}) = \frac{1}{\kappa}\mathbf{diag}\Big(\sigma_y(\mathbf{x}) - \sigma_x(\mathbf{x}), \sigma_x(\mathbf{x}) - \sigma_y(\mathbf{x}), \sigma_x(\mathbf{x}) + \sigma_y(\mathbf{x})\Big)$,
$\boldsymbol{B}(\mathbf{x}) = \frac{1}{\kappa}\mathbf{diag}\left(\sigma_x(\mathbf{x})(\sigma_x(\mathbf{x}) - \sigma_y(\mathbf{x})), \sigma_y(\mathbf{x})(\sigma_y(\mathbf{x}) - \sigma_x(\mathbf{x})), \sigma_x(\mathbf{x})\sigma_y(\mathbf{x})\right)$ and $\boldsymbol{F}(\mathbf{x}) = \mathbf{diag}\Big(\sigma_x(\mathbf{x}), \sigma_y(\mathbf{x}), 0\Big)$.
The matrix $\boldsymbol{T}(\mathbf{x})$ is triangular and depends on the non convex part of the domain where it is calculated. If $y \geq x$, $\boldsymbol{T}^+ = \begin{pmatrix} 1 & 1-\kappa & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix}$, otherwise $\boldsymbol{T}^- = \begin{pmatrix} \kappa & 0 & 0 \\ 1-\kappa & 1 & 0 \\ 0 & 0 & \kappa \end{pmatrix}$. The smallest eigenvalue of the matrix $\boldsymbol{T}(\mathbf{x})\,\boldsymbol{T}(\mathbf{x})^T$ is $\frac{1}{\kappa} + \kappa - 1 - (1 - \frac{1}{\kappa})\sqrt{1 + \kappa^2}$. As $\kappa \geq 1$, the maximal CFL condition is 1 and does not depend on the shape of the domain. Even if the L-shaped has an important length by comparison with its width, the CFL condition remains the same, contrarily to the first formulation established. This embedding has a strong analogy with the cartesian PML and therefore presents a dissymetry in the waves absorption.
An other solution, based on the convex PML, is to choose $\widetilde{\mathbf{x}} = \phi(\mathbf{x}) + \frac{1}{\mathbf{p}} f\Big(\phi(\mathbf{x}) - R\boldsymbol{I}_3\Big)$. The PML matrix becomes $\boldsymbol{K}(\mathbf{p}, \mathbf{x}) = \boldsymbol{T}^T(\mathbf{x}) \left(\mathbf{p}\boldsymbol{I} + \boldsymbol{A}(\mathbf{x}) + \frac{1}{\mathbf{p} + 2\sigma(\mathbf{x})}\boldsymbol{B}(\mathbf{x}) \right) \boldsymbol{T}(\mathbf{x})$, where σ stands for the derivative of the function f, the triangular matrix $\boldsymbol{T}(\mathbf{x})$ is unchanged, $\boldsymbol{I} = \frac{1}{\kappa}\boldsymbol{I}_3$, $\boldsymbol{A}(\mathbf{x}) = \frac{2\sigma(\mathbf{x})}{\kappa} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{B}(\mathbf{x}) = \frac{2\sigma^2(\mathbf{x})}{\kappa} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The numerical experiments of this paper are obtained with this formulation.

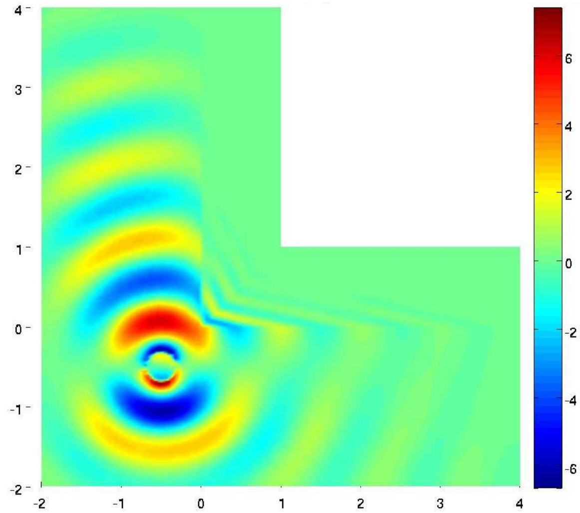
6.3 Numerical Simulations

All the following simulations are made with the Maxwell Bidimensional Equations, with Silver Müller conditions at the boudary of the PML domain, and meshes are unstructured. The origin of the reference frame is the corner of the L, bewteen the domain of study and the non convex PML.

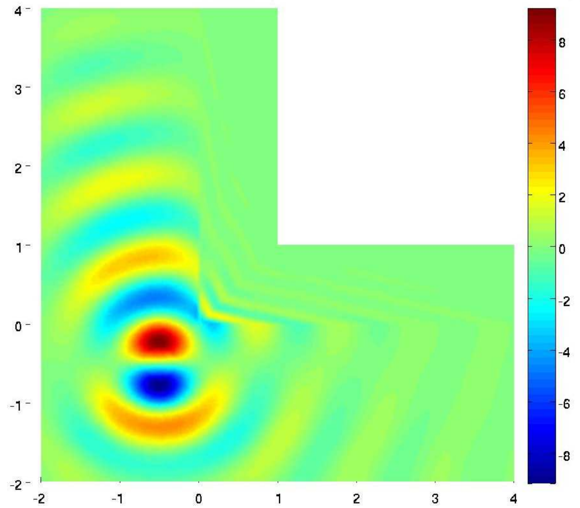
Two kind of problems will be computed, which correspond to the cases studied in Sec. 3. The first simulation is the propagation of a RHS

$$f(\mathbf{x}) = \left(0, 0, \mathbf{exp}\left(\frac{(x-x_c)^2 + (y-y_c)^2}{(x-x_c)^2 + (y-y_c)^2 - R^2}\right) \mathbb{1}_{\mathcal{D}}(\mathbf{x})\right)^T,$$

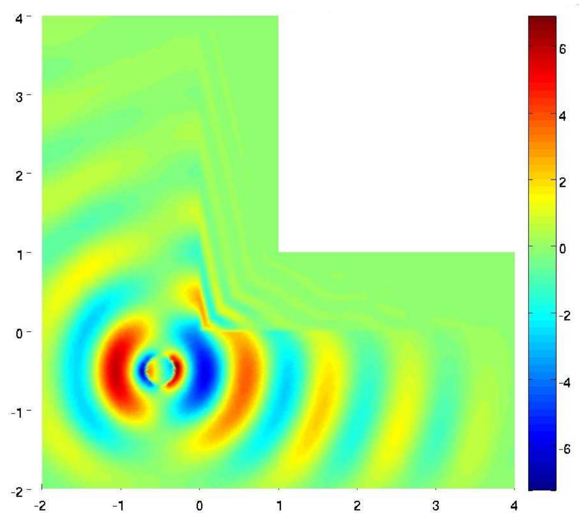
where $\mathcal{D}$ is the disc of center $(x_c, y_c) = (-1, -1)$ and radius $R = 0.5$. The pulsation is $\omega = 2\pi$, the σ function is of order 2, with a coefficient that guarantees a decrease in the PML at 10^{-2}, and $\kappa = 1/3$.



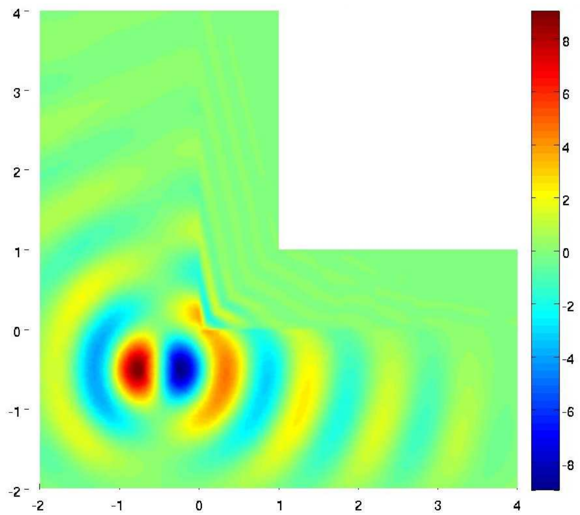
(a) Real part of E_x



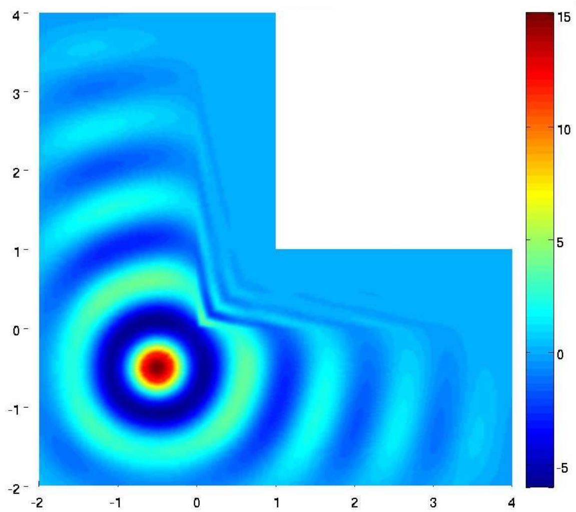
(b) Imaginary part of E_x



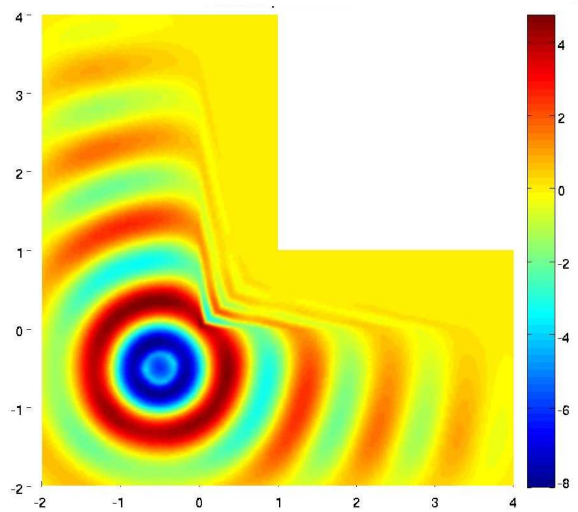
(c) Real part of E_y



(d) Imaginary part of E_y

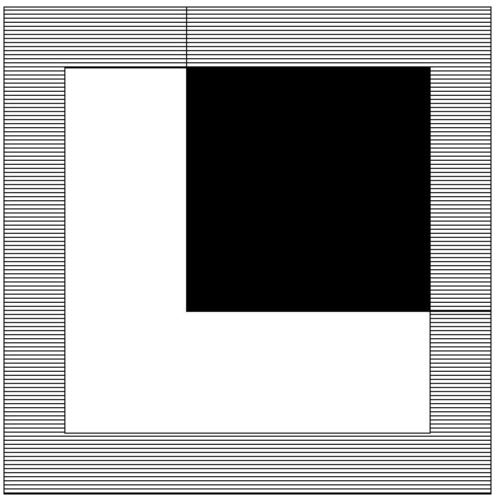


(e) Real part of H_z

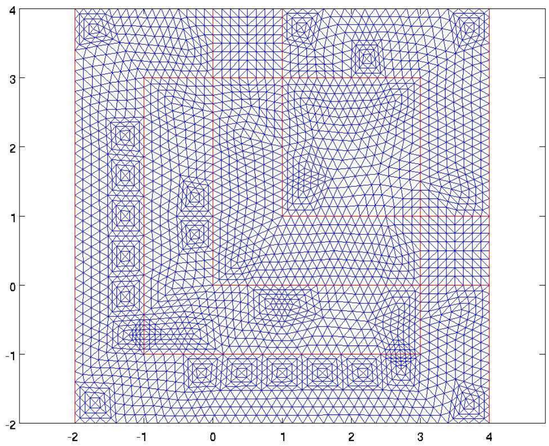


(f) Imaginary part of H_z

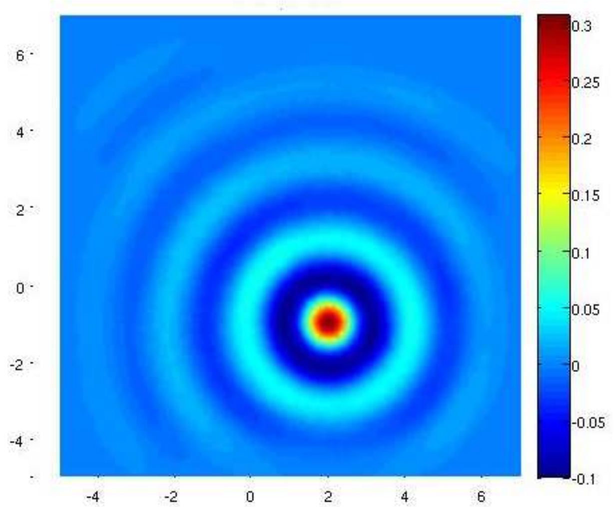
As the cartesian PML are a considered as a reference solution for the problems studied, here are comparison with the non convex PML introduced in this paper. The simulation has the same initial condition as previously, but the center is moved to the right. For the wave to propagate correctly onto the L-shaped domain, it has to go through the non convex PML. Fig. (g) shows the cartesian domain involved, and the black area is the one added in order to apply a cartesian method. Fig. (h) represents the mesh generated by the PDETOOLBOX of MatLab (like all the others used in this paper).



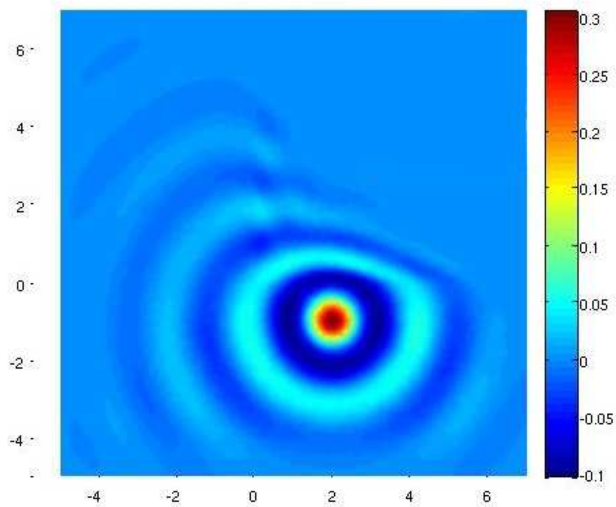
(g) The different domains for a cartesian PML formulation



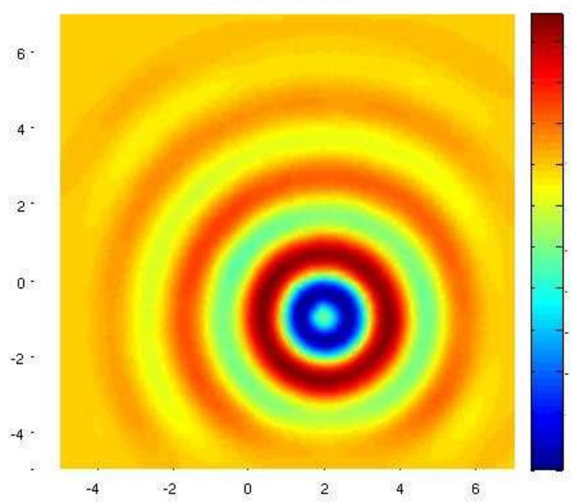
(h) Roughest mesh for the problem with a RHS$\neq 0$



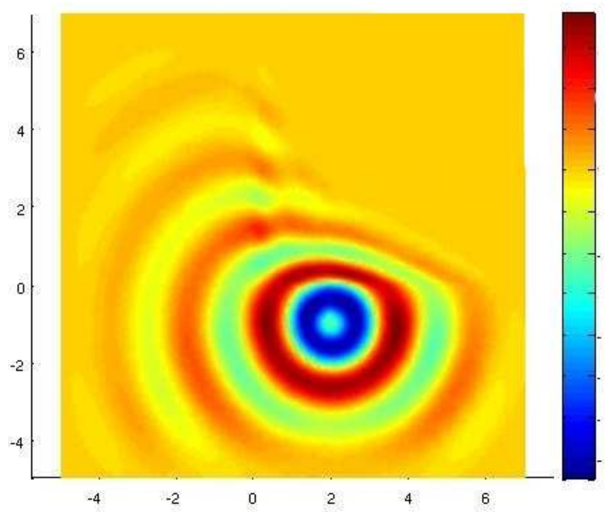
(i) Real part of H_z - cartesian PML



(j) Real part of H_z - non convex PML

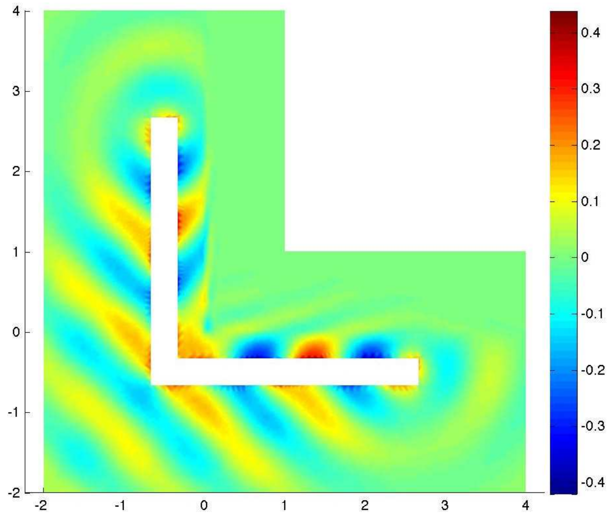


(k) Imaginary part of H_z - cartesian PML

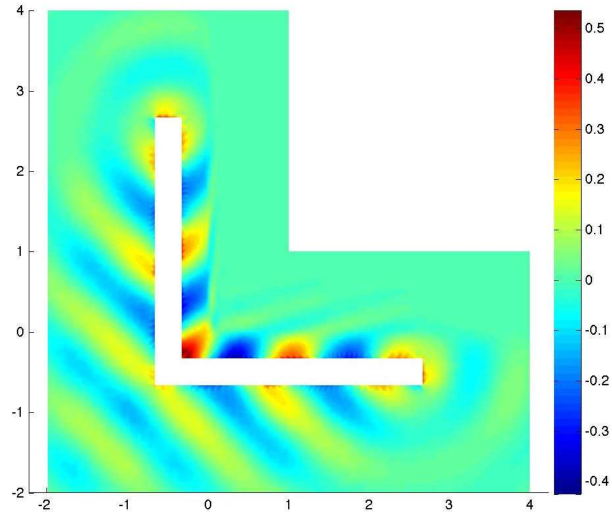


(l) Imaginary part of H_z - non convex PML

For the scattering problem, the study of an incident plane wave φ_{inc} reflected by a L-shaped geometry Γ of length 3 and width 1 is processed, with $\varphi_{inc}(\mathbf{x}) = (-k_y, k_x, 1)^T \mathbf{exp}\,(\omega(-\mathbf{k}.\mathbf{x}))$. $\mathbf{k} = (k_x, k_y)^T$ gives the direction of the incident wave, and it is chosen as $\mathbf{k} = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)^T$. The boundary conditions at the boundary of the scatterer are $\forall \mathbf{x} \in \partial\Gamma,\ f(\mathbf{x}) = \int_{\partial_{\omega_k} \cap \partial\Gamma} -\boldsymbol{M}_1 \varphi_{inc}(\mathbf{x}) \mathbf{exp}\,(-\omega \mathbf{k}.\mathbf{x})$, with $\boldsymbol{M}_1 = \begin{pmatrix} \mathbf{n} \otimes \mathbf{n} & 0 \\ -\mathbf{n}^T & 0 \end{pmatrix}$, $\mathbf{n}$ the outer-pointing unit normal and ω_k the element considered. The PML domain has a thickness of 1/3, the function σ is of order 1. Others parameters remain the same as the previous simulation.

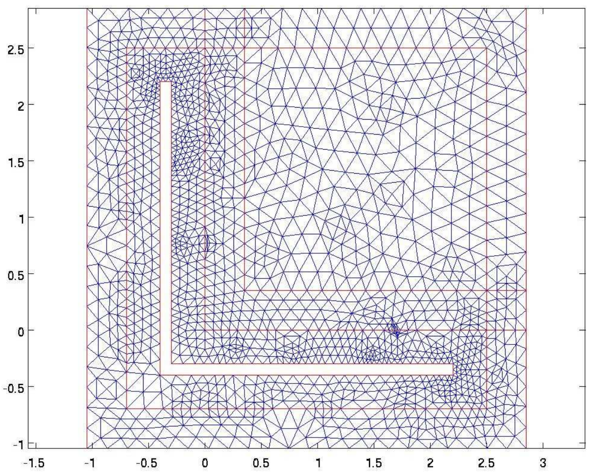


(m) Imaginary part of H_z

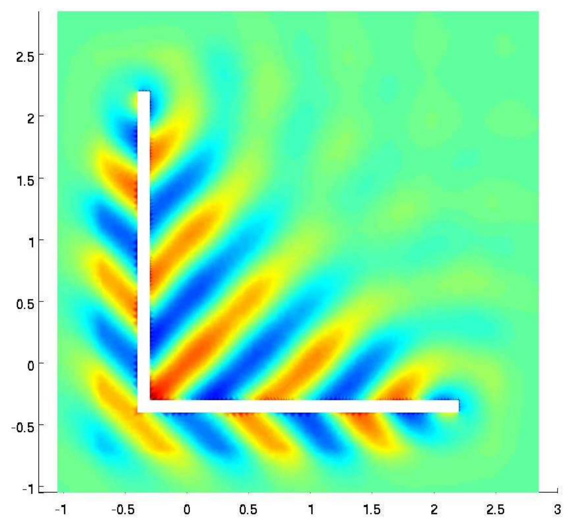


(n) Real part of H_z

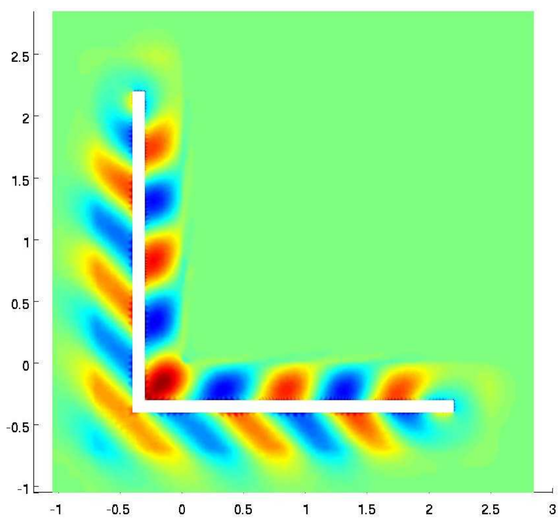
For the comparison with cartesian PML, the pulsation chosen is $\omega = 3\pi$, the order of the sigma function is 2 and the mesh is modified to use a thicker scattering object.



(o) Roughest mesh for the scattering problem



(p) Real part of H_z - cartesian PML



(q) Real part of H_z - non convex PML

On a visual point of view, the non convex PML and the cartesian ones lead to the same solution in the physical L-shaped domain D. To confirm this, we take into account two errors. Let φ_{NC} be the non convex PML solution and φ_C the cartesian PML one. The first error, ε_1, corresponds to the l^2 norm of the difference on D, and the second one, ε_2, represents the local error on each element :

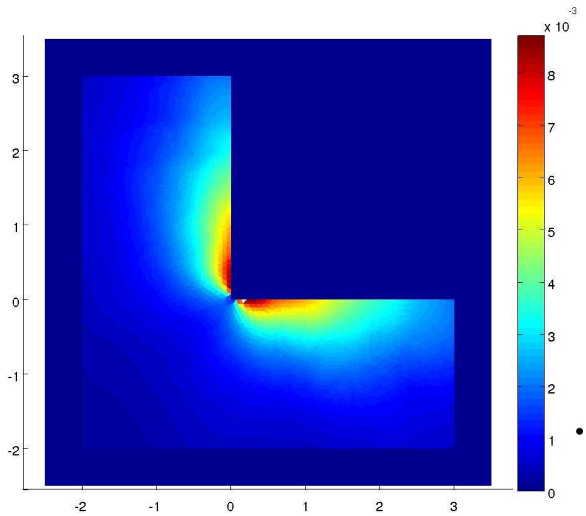
$$\varepsilon_1 = \frac{\sum\limits_{\omega_k \in \mathsf{D}} |\varphi_{NC}(\omega_k) - \varphi_C(\omega_k)|^2 \mathbf{vol}(\omega_k)}{\sum\limits_{\omega_k} |\varphi_C(\omega_k)|^2 \mathbf{vol}(\omega_k)} \qquad \varepsilon_2(\omega_k) = |\varphi_{NC}(\omega_k) - \varphi_C(\omega_k)|^2.$$

Tab. 1 gives the error ε_1 function of the number of elements per wave-length λ. The mesh are rough, with very few layers, to place ourselves in the worst case. The error ε_1 decreases slower with the refinement of the mesh for the scattering problem. The mesh of Fig. 2(o) is very rough, especially in the area used especially for the cartesian simulation. The damping starts outside the convexified of the scatterer, the waves are brutally moved away : there is a lot of dissipation.

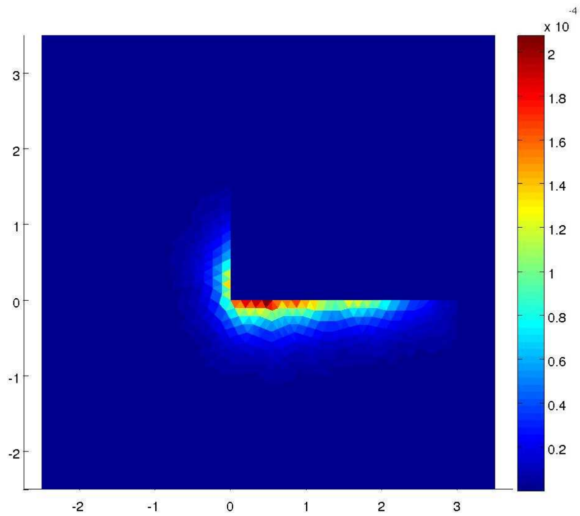
ε_1	$\lambda/8$	$\lambda/16$	$\lambda/32$
Scattering	$\begin{pmatrix} 5.81e-2 \\ 5.65e-2 \\ 6.67e-2 \end{pmatrix}$	$\begin{pmatrix} 1.65e-2 \\ 1.61e-2 \\ 1.76e-2 \end{pmatrix}$	$\begin{pmatrix} 1.031e-2 \\ 9.7e-3 \\ 1.07e-2 \end{pmatrix}$
RHS$\neq 0$	$\begin{pmatrix} 1.28e-2 \\ 5.4e-3 \\ 5e-3 \end{pmatrix}$	$\begin{pmatrix} 5.6e-3 \\ 5.4e-3 \\ 2.3e-3 \end{pmatrix}$	$\begin{pmatrix} 9.72e-4 \\ 9.25e-4 \\ 3e-4 \end{pmatrix}$

Table 1: ε_1 for (E_x, E_y, H_z) depending of the number of elements per wave length

Fig. (r) gives the error ε_2 with a RHS$\neq 0$, whose support's center is "centered", *i.e.* $(-1,-1)$ and a mesh of $\lambda/16$ elements per wave-length and Fig. (s) for a mesh of $\lambda/8$ elements per wave-length, corresponding to the test (j).



(r) ε_2 with $\lambda/16$ elements per wave length with a centered" RHS

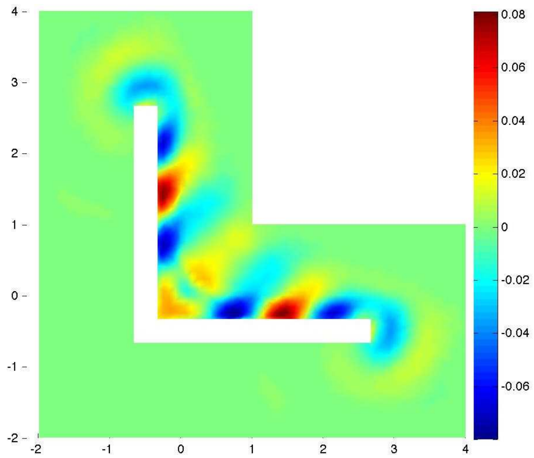


(s) ε_2 with $\lambda/8$ elements per wave length (case test (j))

For unsteady problems, as the PML matrix contains some terms at order -1 in $\mathbf{p}$, an extra equation has to be added. The Maxwell Equations on $\mathbb{R}^3 \times \mathbb{R}^+$ are

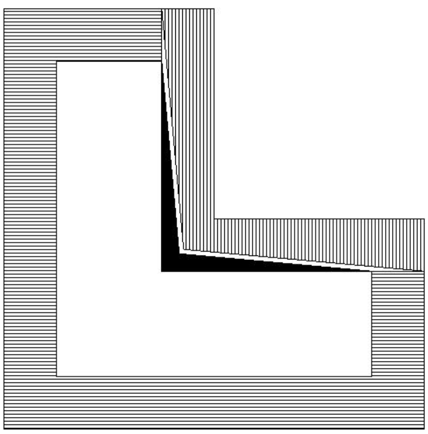
$$\begin{cases} \bar{\boldsymbol{I}}(\mathbf{x})\dfrac{\partial \varphi(\mathbf{x},t)}{\partial t} + \bar{\boldsymbol{A}}(\mathbf{x})\varphi(\mathbf{x},t) + \psi(\mathbf{x},t) + \begin{pmatrix} 0 & 0 & -\partial_y \\ 0 & 0 & \partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \varphi(\mathbf{x},t) = 0 \\ \dfrac{\partial \psi(\mathbf{x},t)}{\partial t} + 2\sigma(\mathbf{x})\psi(\mathbf{x},t) = \bar{\boldsymbol{B}}(\mathbf{x})\varphi(\mathbf{x},t). \end{cases}$$

with $\bar{\boldsymbol{I}}(\mathbf{x}) = \boldsymbol{T}(\mathbf{x})\boldsymbol{I}(\mathbf{x})\boldsymbol{T}(\mathbf{x})^T$, $\bar{\boldsymbol{A}}(\mathbf{x}) = \boldsymbol{T}(\mathbf{x})\boldsymbol{A}(\mathbf{x})\boldsymbol{T}(\mathbf{x})^T$ and $\bar{\boldsymbol{B}}(\mathbf{x}) = \boldsymbol{T}(\mathbf{x})\boldsymbol{B}(\mathbf{x})\boldsymbol{T}(\mathbf{x})^T$, with the same kind of boundary conditions as the harmonic problem :The boundary conditions at the boundary of the scatterer are $\forall \mathbf{x} \in \partial\Gamma$, $\int_{\partial\omega_k \cap \partial\Gamma} -\boldsymbol{M}_1 \varphi_{inc}(\mathbf{x},t)\mathbf{exp}\,(-\omega\mathbf{k}.\mathbf{x})\,\mathbf{exp}(i\omega t)$. For the simulation, an implicit Euler scheme is used. The time step is determined by $\Delta t = \underset{e}{\mathbf{min}} \dfrac{\mathbf{vol}(\omega_e)}{\mathbf{vol}(\partial\omega_e)}$.

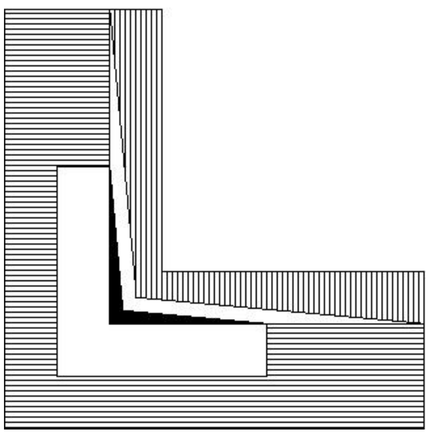


(t) Solution of the unsteady problem : third component $H_z(\mathbf{x}, T)$ with $T = 1.8$

Remark : the previous numerical experiments have an unnecessary domain of calculus inside the non convex PML. At the external boundary of the cartesian PML, the solution is vanishing. This property is verified in the non convex PML on a parallel of the exhaustion. As a consequence, for a L-shaped geometry, the non convex area required is less than a L. On Fig. (u) and (v), the vertical stripes represent the useless domain : the non convex PML domain is very thin. Fig. (u) corresponds to Fig. (1) while Fig. (v) has an elarged PML domain to show with more visibility the effective non convex PML domain, which is very thin by comparison with the thickness of the cartesian area required on this specific geometry.



(u)



(v)

Conclusions

This paper presents a fully proved non convex PML theory and some numerical experiments. The aim is to obtain the most general formulation possible, thanks to the families (φ, f). The hypothesis of convexity is underneath, through the diffeomorphism φ, and absolutely necessary to prevent singularities to appear in the Green function.

The case of the L-shaped geometry can be generalized as long as it is possible to express the diffeomorphism from an affinity normal to the boundary of the convexified domain. It is always possible if the domain is not trapping. For example, a U-shaped geometry where the boundaries are strictly parallel can not give such an affinity, but if the boundaries are "opened", φ can be found by using the normals to the convexified.

In the non convex PML, even if the luminuous rays are twisted (consequence of the complex metric with a different wave speed), the permittivity tensor is more complex -with anisotropic permittivities- but the restriction of the solution to the physical domain are numerically correct. The trace of the solution on the boundary, which almost inverses the problem, will be an excellent preconditionner for integral methods. For harmonic problems, non convex PML present two advantages : the number of cells (and unknowns) is reduced, and there is no need to inverse the matrix on every elements, which is sparse. For unsteady problems, the change of the wave speed implies a drop of the CFL conditions : this saves some memories but no calculus time. As the exageration of the CFL condition depends on a very few elements, an implicit formulation can be used to solve this problem.

The comparison with cartesian PML gives excellent results, even if the choice of the numerical approximation was very poor with a Finite Volums method. The use of methods like Discontinuous Galerkin has to be considered.

In 2001, this approach of the PML theory thanks to pseudo riemannian manifolds with complex tangent and cotangent bundles was already tried by Lassas [4] for harmonic problem. There are four major differences between their work and the present one. For the restriction to a bounded domain, for harmonic problem, some frequencies can be excluded (we remind that Thm. (9) specifies for all $\mathbf{p}$ in $\mathbf{H}$ except for a locally finite eventually empty set $\mathbf{S}$) Lassas *and Co.* do not have that restriction but their theory implies to set the artificial boundary "far enough" of the domain $\mathbf{D}$. In this paper, it can be placed as closed as we want. In fact, Lassas *and Co.* requiere an extra hypothesis to define their absorbing pseudo riemannian manifold : the asymptotic η-euclidianity of the metric, meaning they impose a strict convexity and both the metric and the connection have to be majored asymptotically by an euclidian metric. We do not need that hypothesis. Last point : we study the unsteady problems, while their work was only about harmonic formulations. Moreover, this paper explicitely gives some formulations (diffeomorphims and functions) that leads to the unsteady formulation. As the embedding is defined in a very general way (cartesian and convex PML can be described as well), and the PML matrix obtained is composed of rational fraction that can be decomposed and gives the Friedrichs unsteady system while adding an ODE. The problem of chosing the couples (φ, f) is general for every non trapping domain, but has to be written. The diffeomorphism is deductible from an affinity normal to the boundary of the convexified domain.

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