# Choosability of the square of planar subcubic graphs with large girth

Frédéric Havet

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F. Havet*
Projet Mascotte, CNRS/UNSA/INRIA
INRIA Sophia-Antipolis,
2004 route des Lucioles BP 93,
06902 Sophia-Antipolis Cedex
France
e-mail:fhavet@sophia.inria.fr

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**Abstract**

We show that the choice number of the square of a subcubic graph with maximum average degree less than 18/7 is at most 6. As a corollary, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6. We then show that the choice number of the square of a subcubic planar graph with girth at least 13 is at most 5.

## 1 Introduction

Let $G$ be a (simple) graph. The *neighbourhood* of a vertex $v$ of $G$, denoted $N_G(v)$, is the set of its *neighbours*, i.e. is the set of vertices $y$ such that $xy$ is an edge. The *degree* of a vertex $v$ in $G$, denoted $d_G(v)$, is its number of neighbours. Often, when the graph $G$ is clearly understood from the context, we omit the subscript $G$. A graph is *subcubic* if every vertex has degree at most 3.

Let $p : V(G) \rightarrow \mathbb{N}$. A *p-list-assignment* is a list-assignment $L$ such that $|L(v)| = p(v)$ for any $v \in V(G)$. $G$ is *p-choosable* if it is $L$-colourable for any $p$-list-assignment. By extension, if $k$ is an integer, we say that $G$ is *k-choosable* if it is $p$-choosable when $p$ is the constant function with value $k$ (i. e. $p(v) = k$ for all $v \in V$). The *choice number* of $G$, denoted $ch(G)$, is the smallest integer $k$ such that $G$ is $k$-choosable. Clearly the choice number of $G$ is at least as large as $\chi(G)$, the *chromatic number* of $G$.

The *square* of $G$ is the graph $G^2$ with vertex set $V(G)$ such that two vertices are linked by an edge of $G^2$ if and only if $x$ and $y$ are at distance at most 2 in $G$. A graph is called *planar* if it can be embedded in the plane. Wegner [?] proved that the square of a subcubic planar graph is 8-colourable. He also conjectured it is 7-colourable. Recently, this conjecture was proved by Thomassen [?].

**Theorem 1 (Thomassen [?])** *Let $G$ be a subcubic planar graph. Then $\chi(G^2) \leq 7$.*

Kostochka and Woodall [?] conjectured that, for every square of a graph, the chromatic number equals the choice number.

**Conjecture 2 (Kostochka and Woodall [?])** *For all $G$, $ch(G^2) = \chi(G^2)$.*

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If true, this conjecture together with Theorem **??** implies that every subcubic planar graph is 7-choosable. Very recently, Cranston and Kim [**?**] showed that the square of every subcubic graph (non necessarily planar) other than the Petersen graph is 8-choosable.

The *average degree* of $G$, denoted $Ad(G)$ is $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of $G$, denoted $Mad(G)$, is $\max\{Ad(H), H \text{ subgraph of } G\}$. In [**?**], Dvořák, Škrekovski and Tancer proved that the choice number of the square of a subcubic graph $G$ is at most 4 if $Mad(G) < 24/11$ and $G$ has no 5-cycle, at most 5 if $Mad(G) < 7/3$ and at most 6 if $Mad(G) < 5/2$.

The *girth* of a graph is the smallest length of a cycle in $G$. Planar graphs with prescribed girth have bounded maximum average degree:

**Proposition 3** *Every planar graph with girth at least $g$ has maximum average degree less than $2 + \frac{4}{g-2}$.*

Hence the results of Dvořák, Škrekovski and Tancer imply that the choice number of the square of a planar graph with girth $g$ is at most 6 if $g \geq 10$, at most 5 if $g \geq 14$ and at most 4 if $g \geq 24$. The two latter results had been previously proved by Montassier and Raspaud [**?**].

In this paper, we improve some of these results. We first show (Theorem **??**) that the choice number of the square of a subcubic graph with maximum average degree less than 18/7 is at most 6. As a corollary, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6. Note that this corollary has been proved later and independently by Cranston and Kim [**?**]. We then show (Theorem **??**) that the choice number of the square of a subcubic planar graph with girth at least 13 is at most 5.

## 2 The main results

The general frame of the proofs is classical. We consider a *k-minimal graph*, that is a subcubic graph such that its square is not $k$-choosable but the square of every proper subgraph is $k$-choosable. We prove that some *configurations* (i.e. induced subgraphs) are forbidden in such a graph and then deduce a contradiction. To do so, we will need the following definitions:

An *i-vertex* is a vertex of degree $i$. We denote by $V_i$ the set of $i$-vertices of $G$ and by $v_i$ its cardinality. Let $v$ be a vertex. An *i-neighbour* of $v$ is a neighbour of $v$ with degree $i$. The *i-neighbourhood* of $v$ is $N_i(v) = N(v) \cap V_i$ and its *i-degree* is $d_i(v) = |N_i(v)|$.

Some properties of 6- and 5-minimal graphs have already been proved in [**?**]. The easy first one is that $V_0 \cup V_1 = \emptyset$, so $G$ has minimum degree 2. This will allow us to use the following definitions for 6- and 5-minimal graphs.

Let $G$ be a subcubic graph with minimum degree 2. A *thread* of $G$ is a path whose endvertices are 3-vertices and whose internal vertices are 2-vertices. The *kernel* of $G$ is the weighted graph $K_G$ such that $V(K_G) = V_3(G)$ and $xy$ is an edge in $K_G$ with weight $l$ if and only if $x$ and $y$ are connected by a thread of length $l$ in $G$. An edge of weight $l$ is also called *l-edge*. Let $x$ be a 3-vertex of $G$. The *type* of $x$ is the triple $(l_1, l_2, l_3)$ such that $l_1 \leq l_2 \leq l_3$ and the three edges (a loop being counted twice) incident to $x$ have weight $l_1$, $l_2$ and $l_3$ in $K_G$. We denote by $Y_{l_1,l_2,l_3}$ the set of 3-vertices of type $(l_1, l_2, l_3)$ and $y_{l_1,l_2,l_3}$ its cardinality. Moreover, for every integer $i$, we define $Z_i := \bigcup_{l_1+l_2+l_3=i} Y_{l_1,l_2,l_3}$ and $z_i = |Z_i|$. The number of vertices and edges and thus the average degree of $G$ may be easily expressed in terms of the $z_i$:

$$
\begin{aligned}
|V(G)| &= \sum_{i \geq 3} \frac{i-1}{2} z_i \\
2|E(G)| &= \sum_{i \geq 3} i.z_i
\end{aligned}
$$

$$Ad(G) = \frac{\sum_{i\geq 3} i.z_i}{\sum_{i\geq 3} \frac{i-1}{2} z_i} \qquad (1)$$

### 2.1 6-choosability

The aim of this subsection is to prove the following result.

**Theorem 4** *Let $G$ be a subcubic graph of maximum average degree $d < 18/7$. Then $G^2$ is 6-choosable.*

**Remark 5** Theorem **??** is tight. Indeed, the graph $J_7$ depicted in Figure **??** has average degree 18/7 and its square is the complete graph on seven vertices $K_7$ which is not 6-choosable (nor 6-colourable).

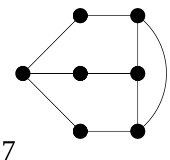


Figure 1: The graph $J_7$

Theorem **??** and Proposition **??** yield that the square of a subcubic planar graph with girth at least 9 is 6-choosable.

**Corollary 6** *The square of a subcubic planar graph with girth at least* 9 *is* 6*-choosable.*

In order to prove Theorem **??**, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [**?**].

**Lemma 7 (Dvořák, Škrekovski and Tancer [?])** *Let $G$ be a* 6*-minimal graph. Then the following hold:*

*(i) all the edges of $K_G$ have weight at most* 2*;*

*(ii) every* 3*-cycle of $G$ has its vertices in $V_3$;*

*(iii) every* 4*-cycle of $G$ has at least three vertices in $V_3$;*

*(iv) a vertex of $Y_{2,2,2}$ is not adjacent in $K_G$ to a vertex of $Y_{1,2,2} \cup Y_{2,2,2}$.*

We will prove in Subsection **??** some new properties.

**Lemma 8** *Let $G$ be a* 6*-minimal graph. Then the following hold:*

*(i) if $(v_1, v_2, v_3, v_4, v_1)$ is a* 4*-cycle with $v_2 \in V_2$ then $v_1$ or $v_3$ is not in $Y_{1,2,2}$;*

*(ii) a vertex of $Y_{1,2,2}$ is adjacent in $K_G$ to at most one vertex of $Y_{1,2,2}$ by* 2*-edges.*

**Proof of Theorem ??**. Let $G$ be a 6-minimal planar graph. $G$ has minimum degree 2, so its kernel $K_G$ is defined. Moreover by Lemma **??** (i), $Z_i$ is empty for $i \geq 7$ and $Z_6 = Y_{2,2,2}$ and $Z_5 = Y_{1,2,2}$.

Let us consider a vertex of $Z_4 = Y_{1,1,2}$. Its neighbour in $K_G$ via the 2-edge is in $Z_4 \cup Z_5 \cup Z_6$ because a vertex of $Z_3 = Y_{1,1,1}$ is incident to no edge of weight 2. For $i = 4, 5, 6$, let $Z_4^i$ be the set of vertices of

$Z_4$ which are adjacent to a vertex of $Z_i$ by their unique 2-edge and $z_4^i$ its cardinality. $(Z_4^4, Z_4^5, Z_4^6)$ is a partition of $Z_4$ so $z_4 = z_4^4 + z_4^5 + z_4^6$. Hence Equation (**??**) becomes

$$Ad(G) = \frac{6z_6 + 5z_5 + 4z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{\frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4^6 + \frac{3}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma **??** (iv), the three neighbours in $K_G$ of a vertex of $Z_6$ are not in $Z_6 \cup Z_5$. So they must be in $Z_4^6$. It follows that $3z_6 = z_4^6$. So

$$Ad(G) = \frac{5z_5 + 6z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{2z_5 + \frac{7}{3}z_4^6 + \frac{3}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma **??** (ii), a vertex of $Z_5$ is adjacent to at least one vertex of $Z_4^5$. Thus $z_5 \leq z_4^5$. But $Ad(G)$ is decreasing as a function of $z_5$ since $z_4^6$, $z_4^5$, $z_4^4$ and $z_3$ are non-negative. It follows that

$$Ad(G) \geq \frac{6z_4^6 + 9z_4^5 + 4z_4^4 + 3z_3}{\frac{7}{3}z_4^6 + \frac{7}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3} \geq \frac{18}{7}.$$

□

## 2.2 5-choosability

Dvořák, Škrekovski and Tancer [?] proved that the square of a subcubic graph $G$ with maximum average degree less than 7/3 is 5-choosable. This result is tight since the graph $J_6$ depicted in Figure **??** has average degree 7/3 and its square is the complete graph on six vertices $K_6$ which is not 5-choosable (nor 5-colourable). However, we will prove that the square of a subcubic planar graph with girth at least 13

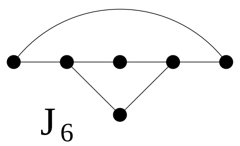


Figure 2: The graph $J_6$

is 5-choosable, which improves the result of Montassier and Raspaud.

**Theorem 9** *The square of a subcubic planar graph with girth at least* 13 *is* 5*-choosable.*

In order to prove this theorem, we need to establish some properties of 5-minimal graphs. Some of them have been proved in [?].

**Lemma 10 (Dvořák, Škrekovski and Tancer [?])** *Let $G$ be a* 5*-minimal graph. Then the following hold:*

*(i) all the edges of $K_G$ have weight at most* 3*;*

*(ii) if $i \geq 8$, $Z_i$ is empty.*

We will prove in Subsection **??** some new properties.

**Lemma 11** *Let $G$ be a* 5*-minimal graph of girth at least* 13*. Then in $K_G$ the following hold:*

*(i) a vertex of $Y_{2,2,3}$ and a vertex of $Y_{1,2,3} \cup Y_{2,2,3}$ are not linked by a 2-edge;*

*(ii) a vertex of $Y_{1,3,3}$ and a vertex of $Y_{1,2,3} \cup Y_{1,3,3}$ are not linked by a 1-edge;*

*(iii) s vertex of $Y_{2,2,2}$ is not adjacent in $K_G$ to three vertices of $Y_{2,2,3}$ (by 2-edges).*

**Proof of Theorem ??**. Let $G$ be a 5-minimal planar graph with girth at least 13. $G$ has minimum degree 2, so its kernel $K_G$ is defined. Moreover, by Lemma **??** (i), $Z_7 = Y_{2,2,3} \cup Y_{1,3,3}$, so

$$z_7 = y_{2,2,3} + y_{1,3,3}. \tag{2}$$

Let us count the number $e_2$ of 2-edges incident to vertices of $Y_{2,2,3}$. Recall that $Z_4 = Y_{1,1,2}$ and $Z_3 = Y_{1,1,1}$. Since 2-edges may not link two vertices of type $(2,2,3)$ according to Lemma **??** (i), we have $e_2 = 2y_{2,2,3}$. Moreover, the ends of such edges which are not in $Y_{2,2,3}$ have to be in $Y_{2,2,2} \cup Y_{1,2,2} \cup Z_4$ by Lemmas **??** and **??** (i). Furthermore, a vertex of $Y_{2,2,2}$ is incident to at most two edges of $e_2$ according to Lemma **??** (iii) and a vertex of $Y_{1,2,2}$ (resp. $Z_4$) is incident to at most two (resp. one) 2-edges. Therefore $e_2 \le 2y_{2,2,2} + 2y_{1,2,2} + z_4$. So,

$$2y_{2,2,3} \le 2y_{2,2,2} + 2y_{1,2,2} + z_4. \tag{3}$$

Let us now count the number $e_1$ of 1-edges incident to vertices of $Y_{1,3,3}$. Since 1-edges may not link two vertices of type $(1,3,3)$ according to Lemma **??** (ii), we have $e_1 = y_{1,3,3}$. Moreover, the ends of such edges which are not in $Y_{1,1,3}$ have to be in $Y_{1,2,2} \cup Y_{1,1,3} \cup Z_4 \cup Z_3$ by Lemmas **??** and **??** (ii). Furthermore, vertices of $Y_{1,2,2}$ (resp. $Y_{1,1,3} \cup Z_4$, $Z_3$) are incident to at most one (resp. two, three) 1-edges. Thus $e_1 \le y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3$. So,

$$y_{1,3,3} \le y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3. \tag{4}$$

$2 \times$ (**??**) + (**??**) yields $2y_{2,2,3} + 2y_{1,3,3} \le 2y_{2,2,2} + 4y_{1,2,2} + 4y_{1,1,3} + 5z_4 + 6z_3$. Hence, by Equation (**??**), $2z_7 \le 2z_6 + 4z_5 + 5z_4 + 6z_3$, so

$$z_7 \le z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3.$$

Now by Equation (**??**) the average degree of $G$ is

$$Ad(G) = \frac{7z_7 + 6z_6 + 5z_5 + 4z_4 + 3z_3}{3z_7 + \frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4 + z_3}.$$

As a function of $z_7$, this is a decreasing function (on $\mathbb{R}^+$); so it is minimum when $z_7$ is maximum that is equal to $z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3$. So,

$$Ad(G) \ge \frac{13z_6 + 19z_5 + \frac{43}{2}z_4 + 24z_3}{\frac{11}{2}z_6 + 8z_5 + 9z_4 + 10z_3} \ge \frac{26}{11}.$$

This contradicts the fact that $G$ has girth 13 by Proposition **??**. □

**Remark 12** It is very likely that using the method below, one can prove that a graph $G$ with maximum average degree less than $\frac{26}{11}$ is 5-choosable unless it contains $J_6$ as an induced subgraph. However, this will require the tedious study of a large number of configurations.

## 3 Proofs of Lemmas ?? and ??

In order to prove Lemmas **??** and **??**, we need the following lemma proved in [**?**]. Let $S$ be a set of vertices of a $k$-minimal graph $G$. The function $p_S : S \to \mathbb{N}$ is defined by $p_S(v) = k - |N_{G^2}(v) \setminus S|$. Then $p_S(v)$ represents the minimum number of available colours at a vertex $v \in S$ once we have precoloured the square of $G - S$. Hence if $(G-S)^2$ is $k$-choosable, $(G-S)^2 = G^2 - S$ and $G^2[S]$ is $p_S$-choosable, one can extend any $k$-list-colouring of $G - S$ into a $k$-list-colouring of $G$, which is a contradiction.

**Lemma 13 (Dvořák, Škrekovski and Tancer [?])** *Let $S$ be a set of vertices of a $k$-minimal graph $G$. If $(G-S)^2 = G^2 - S$, then $G^2[S]$ is not $p_S$-choosable.*

In order to use Lemma **??**, we need some results on the choosability of some graphs.

### 3.1 Some choosability tools

**Definition 14** Let $x$ and $y$ be two vertices of a graph $G$. An $(x-y)$*-ordering* of $G$ is an ordering of the vertices such that $x$ is the minimum and $y$ the maximum. An $(x, y-z)$*-ordering* is an ordering of the vertices such that $x$ is minimum, $y$ is the second minimum and $z$ is maximum.

Let $\sigma = (v_1 < v_2 < \ldots < v_n)$ be an ordering of the vertices of $G$ and $p$ a function $V(G) \to \mathbb{N}$. $\sigma$ is *$p$-greedy* if, for every $i$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| < p(v_i)$. It is *$p$-nice* if, for every $i$ except $n$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| < p(v_i)$ and $d(v_n) = p(v_n)$. It is *$p$-good* if, for every $3 \le i \le n$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| - \epsilon(v_i) < p(v_i)$ with $\epsilon(v_i) = 1$ if $v_i$ is adjacent to both $v_1$ and $v_2$ and $\epsilon(v_i) = 0$ otherwise. By extension, if $k$ is an integer, we say that $\sigma$ is *$k$-greedy* (resp. *$k$-nice*, *$k$-good*) if it is $p$-greedy (resp. $p$-nice, $p$-good) when $p$ is the constant function with value $k$ (i. e. $p(v_i) = k$ for every $1 \le i \le n$).

The greedy algorithm according to greedy, nice and good orderings yields the following three lemmas.

**Lemma 15** *If $G$ has a $p$-greedy ordering then $G$ is $p$-choosable.*

**Proof**. Applying the greedy algorithm according to the $p$-greedy ordering gives the desired colouring. □

**Lemma 16** *Let $xy$ be an edge of graph $G$ and $L$ be a $p$-list-assignment of $G$. If $L(x) \not\subset L(y)$ and $G$ has a $p$-nice $(x-y)$-ordering, then $G$ is $L$-colourable.*

**Proof**. Let $a$ be a colour in $L(x) \setminus L(y)$. Assign $a$ to $x$ and proceed the greedy algorithm according to the $p$-nice $(x-y)$-ordering.. The only vertex which has not more colour in its list than previously coloured neighbours is $y$ for which $|L(y)| = d(y)$. But since $a \notin L(y)$, at most $d(y) - 1$ colours of $L(y)$ are assigned to the neighbours of $y$. Hence one can colour $y$. □

**Lemma 17** *Let $x$, $y$ and $z$ be three vertices of a graph $G = (V, E)$ such that $xy \notin E$. If $L(x) \cap L(y) \neq \emptyset$ and $G$ has a $p$-good $(x, y-z)$-ordering, then $G$ is $L$-colourable.*

**Proof**. Let $a$ be a colour in $L(x) \cap L(y)$ and $\sigma = (v_1 < v_2 < \ldots < v_n)$ be a $p$-good $(x, y-z)$-ordering. (In particular, $v_1 = x$, $v_2 = y$ and $v_n = z$.) Assign $a$ to $x$ and $y$ and proceed the greedy algorithm according to $\sigma$. For every $3 \le i \le n$, the number of colours assigned to already coloured neighbours of $v_i$ is at most $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| - \epsilon(v_i)$ since $v_1$ and $v_2$ are coloured the same. Hence the greedy algorithm gives an $L$-colouring. □

**Remark 18** Note that if $xz, yz \in E$, a $p$-nice $(x, y-z)$-ordering is also $p$-good.

**Definition 19** The *blocks* of a graph are its maximal 2-connected components. A connected graph is said to be a *Gallai tree* if each of its blocks is either a complete graph or an odd cycle.

The following theorem was proved independently by Borodin [?] and Erdős, Rubin and Taylor [?]:

**Theorem 20 (Borodin [?], Erdős, Rubin and Taylor [?])** *Let $G$ be a connected graph and $d_G$ the degree function in $G$. Then $G$ is $d_G$-choosable if and only if $G$ is not a Gallai tree.*

**Lemma 21** *Let $G = (V, E)$ be a graph and $p : V(G) \to I\!N$. Let $S$ be a set of vertices such that $p(v) \geq d(v)$ for all $v \in S$. If $G[S]$ is not a Gallai tree and $G - S$ is $p$-choosable then $G$ is $p$-choosable.*

**Proof.** Let $L$ be a $p$-list-assignment of $G$. Since $G - S$ is $p$-choosable, its admits an $L$-colouring $c$. Let us now extend it to $S$. The list $I(v) = L(v) \setminus \{c(w), w \in N(v) \setminus S\}$ of available colours of a vertex $v \in S$ is of size at least $p'(v) = p(v) - |N(v) \setminus S| \geq d_{G[S]}(v)$. Since $G[S]$ is not a Gallai tree, by Theorem **??**, $G[S]$ is $p'$-choosable and thus $I$-colourable. So, $G$ is $L$-colourable. □

A 4-regular graph $G$ is *cycle+triangles* if it is the edge union of a Hamiltonian cycle $C$ and a 2-factor consisting of triangles. In other words, the graph induced by the edges of $E(G) \setminus E(C)$ is the disjoint union of 3-cycles.

**Theorem 22 (Fleischner and Stiebitz [?])** *Every cycle+triangles graph is 3-choosable.*

## 3.2 Proof of Lemma ??

**Lemma 23** *Let $q \geq 2$ and $C_{4q} = (v_1, \ldots, v_{4q}, v_1)$ be the $4q$-cycle and $p$ defined by $p(v_i) = 4$ if $i$ is odd and $p(v_i) = 2$ otherwise. Then $C_{4q}^2$ is $p$-choosable.*

**Proof.** The set $S$ of vertices $v$ for which $p(v) \geq d_{C_{4q}^2}(v)$ is the set of $v_i$ with odd indices. $C_{4q}^2[S]$ is a $2q$-cycle and thus is not a Gallai tree. Moreover $C_{4q}^2 - S$ is also a $2q$-cycle and is 2-choosable. Hence Lemma **??** gives the result. □

**Proposition 24** *Let $P_7 = (v_1, \ldots, v_7)$ be a path and $p$ the function defined by $p(v_1) = p(v_2) = p(v_6) = p(v_7) = 2$, $p(v_3) = p(v_5) = 4$ and $p(v_4) = 3$. Then $P_7^2$ is $p$-choosable.*

**Proof.** Let $L$ be a $p$-list-assignment of $P_7^2$. Since $(v_2 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$ is a $p$-nice ordering of $P_7^2$, by Lemma **??**, we may assume that $L(v_1) = L(v_2)$, and by symmetry of $P_7$ and $p$ that $L(v_6) = L(v_7)$.

Since $(v_1 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$ is $p$-good, by Lemma **??**, we may assume that $L(v_1) \cap L(v_4) = \emptyset$, and by symmetry $L(v_7) \cap L(v_4) = \emptyset$.

Now one can find $c(v_1) \in L(v_1)$, $c(v_2)$ in $L(v_2) \setminus \{c(v_1)\}$, $c(v_6)$ in $L(v_6)$, $c(v_7)$ in $L(v_7) \setminus \{c(v_6)\}$, $c(v_3)$ in $L(v_3) \setminus \{c(v_1), c(v_2)\}$, and $c(v_5)$ in $L(v_5) \setminus \{c(v_3), c(v_6), c(v_7)\}$. Now since $L(v_1) \cap L(v_4) = \emptyset$ and $L(v_1) = L(v_2)$, $c(v_2) \notin L(v_4)$. Analogously, $c(v_6) \notin L(v_4)$. Hence, $L(v_4) \setminus \{c(v_2), c(v_3), c(v_5), c(v_6)\} = L(v_4) \setminus \{c(v_3), c(v_5)\} \neq \emptyset$. So, one can choose $c(v_4)$ in this set to get an $L$-colouring $c$ of $P_7^2$. □

**Lemma 25** *For $1 \leq i \leq 17$, let $F_i$ be the graphs and $p_i$ be the functions depicted in Figure* **??**.

*(i)* $F_1^2 \cup \{v_5v_6\}$ *is $p_1$-choosable.*

*(ii)* $F_2^2 \cup \{v_1v_4\}$ *and* $F_2^2 \cup \{v_4v_7\}$ *are $p_2$-choosable.*

*(iii)* $F_3^2 \cup \{v_4v_8\}$ *is $p_3$-choosable.*

*(iv)* $F_4^2$ *is* 6*-choosable.*

*(v)* $F_5^2 \cup \{v_1v_4, v_1v_6\}$ *is* $p_5$*-choosable.*

*(vi)* $F_6^2$ *is* $p_6$*-choosable.*

*(vii)* $F_7^2 \cup \{v_9v_{10}\}$ *is* $p_7$*-choosable.*

*(viii)* $F_8^2$ *is* $p_8$*-choosable.*

*(ix)* $F_9^2 \cup \{v_2v_9\}$ *and* $F_9^2 \cup \{v_6v_9\}$ *are* $p_9$*-choosable.*

*(x)* $F_{10}^2 \cup \{v_4v_8\}$ *is* $p_{10}$*-choosable.*

*(xi)* $F_{11}^2 \cup \{v_4v_8, v_8v_9\}$*,* $F_{11}^2 \cup \{v_4v_8, v_9v_4\}$ *and* $F_{11}^2 \cup \{v_8v_9, v_9v_4\}$ *are* $p_{11}$*-choosable and* $F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ *is* 5*-choosable.*

*(xii)* $F_{12}^2 \cup \{v_4v_8\}$ *is* $p_{12}$*-choosable.*

*(xiii)* $F_{13}^2$ *is* 6*-choosable.*

**Proof.**

(i) In $F_1^2 \cup \{v_5v_6\}$, $(v_6 < v_5 < v_4 < v_3 < v_1 < v_2)$ is $p_1$-greedy. So, by Lemma **??**, $F_1^2 \cup \{v_5v_6\}$ is $p_1$-choosable.

(ii) In $F_2^2 \cup \{v_4v_7\}$, $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is $p_2$-nice and $p_2(v_2) > p_2(v_1)$. So, by Lemma **??**, $F_2^2 \cup \{v_4v_7\}$ is $p_2$-choosable.

By symmetry, one shows that $F_2^2 \cup \{v_1v_4\}$ is $p_2$-choosable.

(iii) In $F_3^2 \cup \{v_4v_8\}$, $(v_2 < v_8 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$ is $p_3$-nice and $p_3(v_2) > p_3(v_1)$. So, by Lemma **??**, $F_3^2 \cup \{v_4v_8\}$ is $p_3$-choosable.

(iv) Let $L$ be a 6-list-assignment of $F_4^2$. Every ordering with maximum $v_1$ and second maximum $v_7$ is 6-nice. Thus, by Lemma **??**, we may assume that $L(v_j) = L(v_1)$ for $j \in \{2, 3, 4, 5, 6, 8\}$. Analogously, we may assume that $L(v_j) = L(v_7)$ for $j \in \{2, 3, 4, 5, 6, 8\}$. Hence all the lists are the same, say $\{1, 2, 3, 4, 5, 6\}$. Now $c(v_1) = c(v_5) = 1$, $c(v_2) = 2$, $c(v_3) = c(v_7) = 3$, $c(v_4) = 4$, $c(v_6) = 5$ and $c(v_8) = 6$ is an $L$-colouring of $F_4^2$.

(v) In $F_5^2 \cup \{v_1v_4, v_1v_6\}$, $(v_7 < v_6 < v_1 < v_4 < v_2 < v_3 < v_5)$ is $p_5$-nice and $p_5(v_7) > p_5(v_5)$. So, by Lemma **??**, $F_5^2 \cup \{v_1v_5\}$ is $p_5$-choosable.

(vi) In $F_6^2$, $(v_4 < v_2 < v_8 < v_1 < v_3 < v_5)$ is $p_6$-nice and $p_6(v_4) > p_6(v_5)$. So, by Lemma **??**, $F_6^2$ is $p_6$-choosable.

(vii) Let $L$ be a $p_7$-list-assignment of $F_7^2 \cup \{v_9v_{10}\}$. $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$, $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_5 < v_7 < v_1 < v_3)$ and $(v_4 < v_9 < v_{10} < v_8 < v_6 < v_2 < v_5 < v_7 < v_1 < v_3)$ are $p_7$-nice. Thus, by Lemma **??**, we may assume that $L(v_2) \subset L(v_1)$, $L(v_2) \subset L(v_3)$ and $L(v_4) \subset L(v_3)$. It follows that $L(v_1) \cap L(v_4) \neq \emptyset$. Because $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$ is $p_7$-good, by Lemma **??**, $F_7^2 \cup \{v_9v_{10}\}$ is $L$-colourable.

(viii) In $F_8^2$, $(v_6 < v_5 < v_7 < v_9 < v_8 < v_4 < v_3 < v_2 < v_1)$ is $p_8$-greedy. So, by Lemma **??**, $F_8^2$ is $p_8$-choosable.

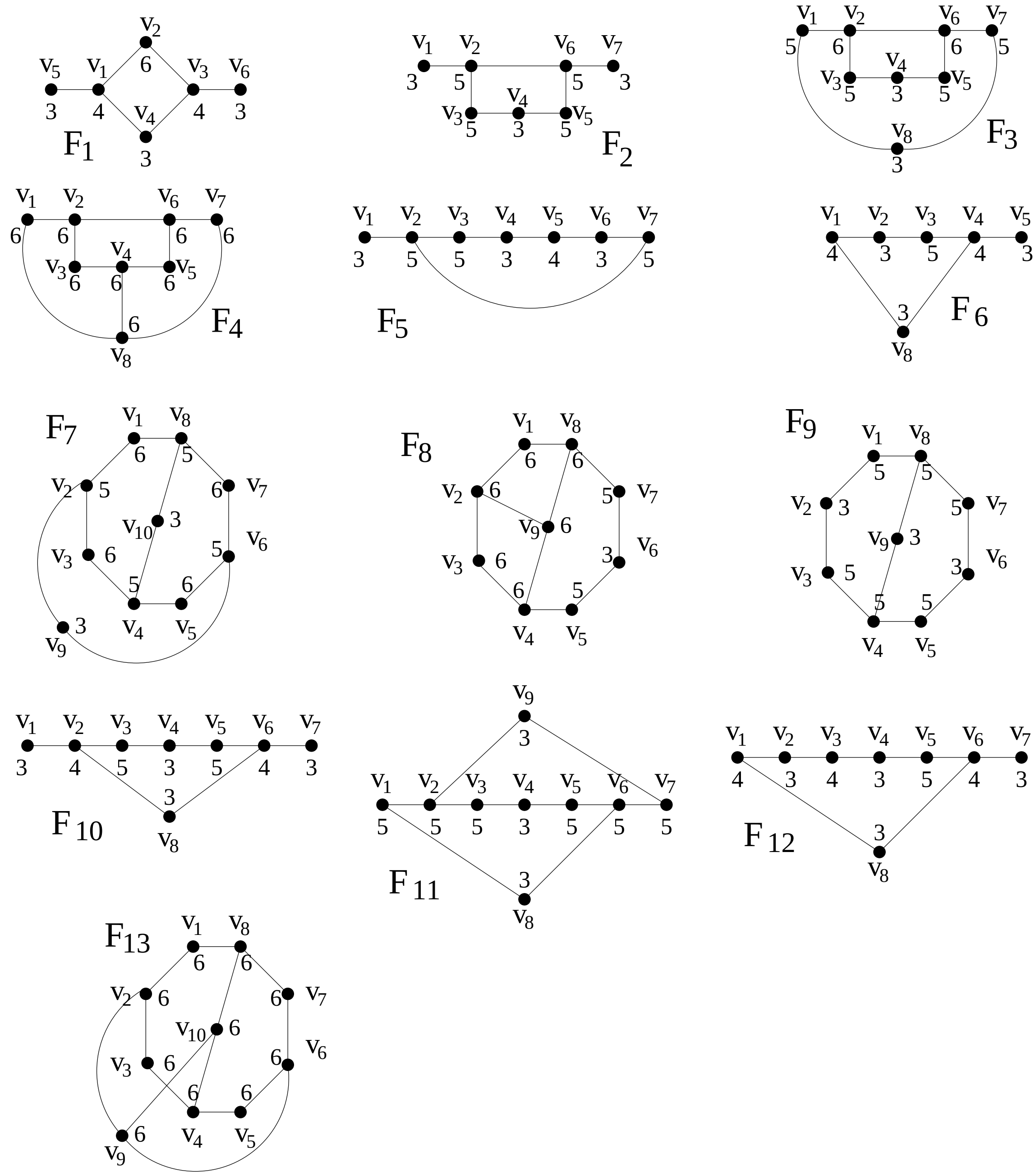


Figure 3: The graphs $F_i$ and functions $p_i$ for $1 \leq i \leq 13$

(ix) Let $L$ be a $p_9$-list-assignment of $F_9^2 \cup \{v_2v_9\}$. Then $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_3 < v_1)$ and $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_1 < v_3)$ are $p_9$-nice so by Lemma **??**, we may assume that $L(v_2) \subset L(v_3) \cap L(v_1)$. Moreover, $(v_4 < v_2 < v_9 < v_6 < v_8 < v_7 < v_5 < v_1 < v_3)$ is $p_9$-nice so by Lemma **??**, we may assume that $L(v_4) = L(v_3)$. It follows that $L(v_1) \cap L(v_4) \neq \emptyset$. Thus, by Lemma **??**, since $(v_1 < v_4 < v_2 < v_9 < v_8 < v_6 < v_7 < v_5 < v_3)$ is $p_9$-good, $F_9^2 \cup \{v_2v_9\}$ is $L$-colourable.

By symmetry, one shows that $F_9^2 \cup \{v_6v_9\}$ is $p_9$-choosable.

(x) In $F_{10}^2 \cup \{v_4v_8\}$, $(v_2 < v_8 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$ is $p_{10}$-nice and $p_{10}(v_2) > p_{10}(v_1)$. So, by Lemma **??**, $F_{10}^2 \cup \{v_4v_8\}$ is $p_{10}$-colourable.

(xi) Let $F \in \{F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}, F_{11}^2 \cup \{v_4v_8, v_8v_9\}, F_{11}^2 \cup \{v_4v_8, v_9v_4\}, F_{11}^2 \cup \{v_8v_9, v_9v_4\}\}$ and $L$ be a 5-list-assignment if $F = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ and a $p_{11}$-list-assignment of $F$ otherwise.

Then $(v_1 < v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$, $(v_7 < v_9 < v_8 < v_4 < v_6 < v_2 < v_1 < v_5 < v_3)$ and $(v_7 < v_9 < v_8 < v_4 < v_6 < v_2 < v_1 < v_3 < v_5)$ are $p$-nice in $F$. So by Lemma **??**, we may assume that $L(v_1) = L(v_3) = L(v_5) = L(v_7)$.

If $L(v_8) \not\subset L(v_2)$, let us colour $v_8$ with $c_8 \in L(v_8) \setminus L(v_2)$, $v_4$ with $c_4 \in L(v_4) \setminus \{c_8\}$, $v_9$ with $c_8 \in L(v_8) \setminus \{c_4, c_8\}$, $v_1$ and $v_5$ with the same colour $c_1 \in L(v_1) \setminus \{c_4, c_8, c_9\}$, $v_3$ and $v_7$ with the same colour $c_3 \in L(v_1) \setminus \{c_1, c_4, c_8, c_9\}$, $v_6$ with $c_6 \in L(v_6) \setminus \{c_1, c_3, c_8, c_9\}$ and finally $v_2$ with $c_2 \in L(v_2) \setminus \{c_1, c_3, c_6, c_8, c_9\} = L(v_2) \setminus \{c_1, c_3, c_6, c_9\}$. This gives an $L$-colouring of $F$. So we may assume that $L(v_8) \subset L(v_2)$. Exchanging the role of $c_4$ in $c_8$ in the preceding argument, we may assume that $L(v_4) \subset L(v_2)$. Moreover by symmetry, we may assume that $L(v_9) \cup L(v_4) \subset L(v_6)$. In particular, this implies that the sets $L(v_8) \cap L(v_9)$, $L(v_8) \cap L(v_4)$, $L(v_9) \cap L(v_4)$ and $L(v_2) \cap L(v_6)$ are non empty.

If $F = F_{11}^2 \cup \{v_4v_8, v_9v_4\}$ then $v_8v_9 \notin F$. Hence $(v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3 < v_1)$ is $p_{11}$-good, so by Lemma **??**, $F$ is $L$-colourable.

If $F = F_{11}^2 \cup \{v_8v_9, v_9v_4\}$ then $v_8v_4 \notin F$. Hence $(v_4 < v_8 < v_9 < v_1 < v_2 < v_6 < v_7 < v_3 < v_5)$ is $p_{11}$-good, so by Lemma **??**, $F$ is $L$-colourable.

If $F = F_{11}^2 \cup \{v_4v_8, v_8v_9\}$ then $v_9v_4 \notin F$. Hence $(v_4 < v_9 < v_8 < v_1 < v_2 < v_6 < v_7 < v_3 < v_5)$ is $p_{11}$-good, so by Lemma **??**, $F$ is $L$-colourable.

If $F = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$, then $(v_2 < v_6 < v_4 < v_8 < v_9 < v_7 < v_5 < v_3 < v_1)$ is 5-good. So, by Lemma **??**, $F$ is $L$-colourable.

(xii) In $F_{12}^2 \cup \{v_4v_8\}$, $(v_6 < v_8 < v_4 < v_2 < v_1 < v_3 < v_5 < v_7)$ is $p_{12}$-nice and $p_{12}(v_6) > p_{12}(v_7)$. So by Lemma **??**, $F_{12}^2 \cup \{v_4v_8\}$ is $p_{12}$-choosable.

(xiii) Let $L$ be a 6-list-assignment of $F_{13}^2$. $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$, $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_5 < v_7 < v_1 < v_3)$ and $(v_4 < v_9 < v_{10} < v_8 < v_6 < v_2 < v_5 < v_7 < v_1 < v_3)$ are 6-nice. Thus, by Lemma **??**, we may assume that $L(v_1) = L(v_2) = L(v_3) = L(v_4)$. Because $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$ is 6-good, by Lemma **??**, $F_{13}^2$ is $L$-colourable.

$\square$

**Proof of Lemma ??.**

To prove this lemma, we will suppose for a contradiction that it does not hold. Then we will find a set $X$ of vertices contradicting Lemma **??**. Indeed Lemma **??** will show that $G^2[X]$ is $p_X$-choosable and for each set $X$ we consider, every vertex of $X$ has at most one neighbour in $G - X$, so $(G - X)^2 = G^2 - X$. Lemma **??** completes the proof.

(i) Suppose for a contradiction that $v_1$ and $v_3$ are in $Y_{1,2,2}$. Let $v_5$ (resp. $v_6$) be the neighbour of $v_1$ (resp. $v_3$) distinct from $v_2$ and $v_4$. By Lemma **??** (iv), $v_4$ is in $V_3$ and $v_5 \neq v_6$. Set $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then $G[S] = F_1$, $p_S \geq p_1$ and $G^2[S] \subset F_1^2 \cup \{v_5v_6\}$. So Lemma **??** contradicts Lemma **??**.

(ii) Suppose for a contradiction that, in $K_G$, a vertex $v_4$ of $Y_{1,2,2}$ is adjacent to two vertices of $Y_{1,2,2}$ $v_2$ and $v_6$ by 2-edges. According to Lemma **??** (iii), $v_2 \neq v_6$. Let $v_3$ and $v_5$ be the 2-neighbours of $v_4$ common with $v_2$ and $v_6$ respectively, and $v_1$ (resp. $v_7$) be the 2-neighbour of $v_2$ (resp. $v_6$) not adjacent to $v_4$. Set $S = \{v_1, \ldots, v_7\}$.

We first claim that $v_1 \neq v_7$. Suppose not. Then $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ is a cycle $C$. It has no chord by Lemma **??** (ii), so $C^2 = G^2[S]$. Moreover, $p_S(v_i) \geq 4$ if $i$ is even and $p_S(v_i) \geq 3$ otherwise. $C^2$ is a cycle+triangle graph, thus, by Theorem **??**, it is 3-choosable and so $p_S$-choosable. This contradicts Lemma **??**.

Let $w_1$ (resp. $w_7$) be the neighbour of $v_1$ (resp. $w_7$) distinct from $v_2$ (resp. $v_6$) and for $i \in \{2, 4, 6\}$, let $w_i$ be the neighbour of $v_i$ not in $\{v_{i-1}, v_{i+1}\}$. Let $W = \{w_1, w_2, w_4, w_6, w_7\}$.

We claim that $W \cap S \neq \emptyset$. Indeed, suppose for a contradiction that $W \cap S \neq \emptyset$. Since $G$ is simple, $w_1 \neq v_2$ and $w_7 \neq v_6$. Moreover by Lemma **??** (i), $w_1$ and $w_7$ are in $V_3$, so $w_1 \neq v_7$ and $w_7 \neq v_1$. Furthermore, by Lemma **??** (ii), $w_2 \neq v_4$ and $w_6 \neq v_4$ and by Lemma **??** (iii), $w_1 \neq v_4$ and $w_7 \neq v_4$. Last, we may not have $w_1 = v_6$ and $w_2 = v_7$ otherwise the 4-cycle $(v_1, v_6, v_7, v_2, v_1)$ would contradict Lemma **??** (iii). Then, by symmetry, we only need to consider the cases $w_2 = v_6$, $w_2 = v_7$.

- Assume that $w_2 = v_6$. Then $G[S] = F_2$, $p_S \geq p_2$ and $G^2[S] \subset F_2^2 \cup \{v_1v_4, v_4v_7, v_1v_7\}$. Thus, by Lemmas **??** and **??**, $F_2^2 \cup \{v_1v_7\} \subset G^2[S]$, so $w_1 = w_7 = v_8$. Let $T = S \cup \{v_8\}$. If $v_8 \neq w_4$, then $G[T] = F_3$ and $p_T \geq p_3$ and $G^2[T] \subset F_3^2 \cup \{v_4v_8\}$. So Lemma **??** contradicts Lemma **??**. If not then $G[T] = G = F_4$, so $G$ is 6-choosable, by Lemma **??**. This is a contradiction.

- Suppose that $w_2 = v_7$. Then $G[S] = F_5$, $p_S \geq p_5$ and $G^2[S] \subset F_5^2 \cup \{v_1v_4, v_1v_6\}$. Thus Lemma **??** contradicts Lemma **??**.

This proves the claim.

Note that by Lemma **??** (ii), $w_1 \neq w_2$ and $w_6 \neq w_7$.

Suppose $w_1 = w_4 = v_8$. Then let $R = \{v_1, v_2, v_3, v_4, v_5, v_8\}$ and $w_8$ the neighbour of $v_8$. Then $(G[R], p_R) = (F_6, p_6)$ and $G^2[R] = F_6^2$. Thus Lemma **??** contradicts Lemma **??**. Therefore, $w_1 \neq w_4$ and, by symmetry, $w_4 \neq w_7$.

Suppose $w_1 = w_7 = v_8$. Let $T = S \cup \{v_8\}$. Then $G[T]$ is the cycle $C_8$ and $p_T$ is greater or equal to the function $p$ defined in Lemma **??**. So, by Lemmas **??** and **??**, $G^2[T] \neq C_8^2$. It follows that either $w_2 = w_6$ or $w_4 = w_8$ with $w_8$ be the neighbour of $v_8$ not in $S$.

- Suppose $w_2 = w_6 = v_9$, and $w_4 = w_8 = v_{10}$. Set $W = \{v_1, \ldots, v_{10}\}$. If $v_9v_{10} \notin E(G)$ then $G[W] = F_7$, $p_W \geq p_7$) and $G^2[W] \subset F_7^2 \cup \{v_9v_{10}\}$; so Lemma **??** contradicts Lemma **??**. If not, $G = G[W] = F_{13}$, so $G^2$ is 6-choosable, according to Lemma **??**, a contradiction.

- Suppose $w_2 = w_4 = w_6 = v_9$. Setting $U = \{v_1, \ldots, v_9\}$, we have $(G[U], p_U) = (F_8, p_8)$ and $G^2[U] = F_8^2$. Hence Lemma **??** contradicts Lemma **??**.

  By symmetry, we get a contradiction if $w_2 = w_6 = w_8$, $w_2 = w_4 = w_8$ or $w_4 = w_6 = w_8$.

- Suppose $w_4 = w_8 = v_9$, $w_2 \neq v_9$, $w_6 \neq v_9$ and $w_2 \neq w_6$. Setting $U = \{v_1, \ldots, v_9\}$, we have $G[U] = F_9$, $p_U \geq p_9$ and $G^2[U] \subset F_9^2 \cup \{v_2v_9\}$ or $G^2[U] \subset F_9^2 \cup \{v_6v_9\}$. Hence Lemma **??** contradicts Lemma **??**.

  By symmetry, we get a contradiction if $w_2 = w_6 = v_9$, $w_4 \neq v_9$, $w_8 \neq v_9$ and $w_4 \neq w_8$.

Therefore, $w_1 \neq w_7$.

Suppose that $w_2 = w_6 = v_8$. Let $T = S \cup \{v_8\}$. Then $G[T] = F_{10}$, $p_T \geq p_{10}$, and $G^2[T] \subset F_{10}^2 \cup \{v_4v_8\}$, since $w_1$, $w_4$ and $w_7$ are distinct vertices. Hence Lemma **??** contradicts Lemma **??**.
Therefore, $w_2 \neq w_6$.

Suppose that $w_1 = w_6 = v_8$ and $w_2 = w_7 = v_9$. Let $U = S \cup \{v_8, v_9\}$. Then $G[U] = F_{11}$ and $G^2[U]$ is a subgraph of $F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$. Moreover $p_U \geq p_{11}$ and, if $G^2[U] = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$, $p_U(v_i) = 5$ for $1 \leq i \leq 9$. . Hence Lemma **??** contradicts Lemma **??**.

Therefore, $w_1 \neq w_6$ or $w_2 \neq w_7$. By symmetry, $w_2 \neq w_7$.

Suppose $w_1 = w_6 = v_8$. Let $T = S \cup \{v_8\}$ and let $w_8$ be the neighbour of $v_8$ not in $S$. Then $G[T] = F_{12}$, $p_T \geq p_{12}$ and $G^2[T] \subset F_{12}^2 \cup \{v_4v_8\}$. Hence Lemma **??** contradicts Lemma **??**.
Therefore, $w_1 \neq w_6$.

Hence all the $w_i$ are distinct, so $G[S]^2 = G^2[S]$. Thus Proposition **??** contradicts Lemma **??**.

□

**Remark 26** The proof of Lemma **??** in the case of planar graphs of girth at least 9 is simpler and shorter because all the configurations considered in the above proof (except the path $P_7$) have girth less than 9. Thus Corollary **??** has a short direct proof which requires only Proposition **??**.

### 3.3 Proof of Lemma ??

**Definition 27** For $1 \leq j \leq 4$, let $I_j$ and $q_j$ be the graphs and functions depicted in Figure **??**.

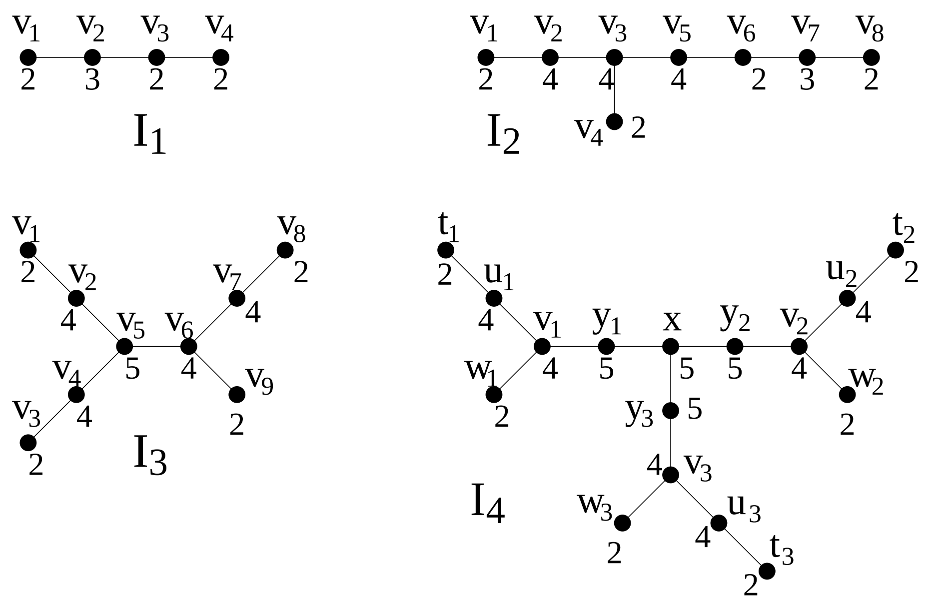


Figure 4: The graphs $I_j$ and functions $q_j$, $1 \leq j \leq 4$

**Lemma 28** *For $1 \leq j \leq 4$, $I_j^2$ is $q_j$-choosable.*

**Proof**.

- Let $L$ be a $q_1$-list-assignment of $I_1^2$. The orderings $(v_4 < v_3 < v_1 < v_2)$ and $(v_1 < v_3 < v_4 < v_2)$ are $q_1$-nice. So, by Lemma **??**, we may assume that $L(v_1) \cup L(v_4) \subset L(v_2)$. Hence $L(v_1) \cap L(v_4) \neq \emptyset$. But $(v_4 < v_1 < v_3 < v_2)$ is $q_1$-good. Thus, by Lemma **??**, $I_1^2$ is $L$-colourable.

- Let $L$ be a $q_2$-list-assignment of $I_2^2$.

  Suppose first that $L(v_3) \not\subset L(v_1) \cup L(v_6)$. Then choose $c(v_3)$ in $L(v_3) \setminus (L(v_1) \cup L(v_6))$ and $c(v_4) \in L(v_4) \setminus \{c(v_3)\}$. Since $I_1^2$ is $q_1$-choosable, one can extend $c$ to $\{v_5, v_6, v_7, v_8\}$. Then one can find $c(v_2) \in L(v_2) \setminus \{c(v_3), c(v_4), c(v_5)\}$ and $c(v_1) \in L(v_1) \setminus \{c(v_2), c(v_3)\} = L(v_1) \setminus \{c(v_2)\}$. Hence we may assume that $L(v_3) \subset L(v_1) \cup L(v_6)$, so $L(v_3) = L(v_1) \cup L(v_6)$ and $L(v_1) \cap L(v_6) = \emptyset$.

  Suppose now that $L(v_4) \cap L(v_6) \neq \emptyset$. Then colour $v_4$ and $v_6$ with the same colour $c(v_4) = c(v_6) \in L(v_4) \cap L(v_6)$. Choose $c(v_8) \in L(v_8) \setminus \{c(v_6)\}$ and $c(v_7) \in L(v_7) \setminus \{c(v_6), c(v_8)\}$. Now since $I_1^2$ is $q_1$-choosable, one can extend $c$ into an $L$-colouring of $I_2^2$. So we may assume that $L(v_4) \cap L(v_6) = \emptyset$. Now $(v_4 < v_1 < v_6 < v_8 < v_7 < v_5 < v_3 < v_2)$ is $q_2$-good so, by Lemma **??**, we may assume that $L(v_4) \cap L(v_1) = \emptyset$. It follows that $L(v_4) \cap L(v_3) = \emptyset$ since $L(v_3) = L(v_1) \cup L(v_6)$.

  The ordering $(v_4 < v_8 < v_6 < v_7 < v_5 < v_3 < v_1 < v_2)$ is $q_2$-nice so, by Lemma **??**, we may assume that $L(v_4) \subset L(v_2)$. Then one may assign $c(v_4) \in L(v_4)$ and $c(v_2) \in L(v_4) \setminus \{c(v_4)\}$ to the vertices $v_4$ and $v_2$. Now, because $L(v_4) \cap L(v_3) = \emptyset$, one can extend $c$ into an $L$-colouring of $I_2^2$ by colouring greedily according to the ordering $(v_1 < v_8 < v_6 < v_7 < v_5 < v_3)$.

- Let $L$ be a $q_3$-list-assignment of $I_3^2$. Assign to $v_5$ a colour $c_5$ in $L(v_5) \setminus (L(v_1) \cup L(v_9))$ and to $v_6$ a colour in $L(v_6) \setminus (L(v_8) \cup \{c_5\})$. Then colour the remaining vertices greedily according to $(v_3 < v_4 < v_2 < v_1 < v_9 < v_7 < v_8)$ to get an $L$-colouring of $I_3^2$.

- Let $L$ be $q_4$-list-assignment of $I_4^2$. Pick $c(y_1)$ in $L(y_1) \setminus L(w_1)$, $c(y_2)$ in $L(y_2) \setminus (L(w_2) \cup \{c(y_1)\})$, $c(y_3)$ in $L(y_3) \setminus (L(w_3) \cup \{c(y_1), c(y_2)\})$ and $c(x)$ in $L(x) \setminus \{c(y_1), c(y_2), c(y_3)\}$. Since $I_1^2$ is $q_1$-choosable, one can extend $c$ to a colouring of $I_4^2$.

□

**Proof of Lemma ??.**

(i) Suppose that a vertex $v_3$ of $Y_{2,2,3}$ and $v_6$ of $Y_{1,2,3} \cup Y_{2,2,3}$ are adjacent via a 2-edge in $K_G$. Then the subgraph of $G$ induced by $v_3$, $v_6$ and the 2-vertices of their incident threads contains $I_2$ as an induced subgraph. (It is $I_2$ if $v_6$ is in $Y_{1,2,3}$ and has one extra vertex otherwise.) Since $G$ has girth at least 13, then $G^2[V(I_2)] = I_2^2$, $(G - V(I_2))^2 = G^2 - V(I_2)$ and $p_{V[I_2]} = q_2$, so Lemma **??** contradicts Lemma **??**.

(ii) Suppose that a vertex $v_5$ of $Y_{1,3,3}$ and $v_6$ of $Y_{1,2,3} \cup Y_{1,3,3}$ are adjacent via a 1-edge in $K_G$. Then the subgraph of $G$ induced by $v_5$, $v_6$ and the 2-vertices of their incident threads contains $I_3$. So Lemma **??** contradicts Lemma **??**.

(iii) Suppose that a vertex $x$ of $Y_{2,2,2}$ is adjacent to three vertices $v_1$, $v_2$ and $v_3$ of $Y_{2,2,3}$ in $K_G$. Then the subgraph of $G$ induced by $x$, $v_1$, $v_2$, $v_3$ and the 2-vertices of their incident threads is $I_4$. So Lemma **??** contradicts Lemma **??**.

□

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