

# Choosability of the square of planar subcubic graphs with large girth

Frédéric Havet

► **To cite this version:**

Frédéric Havet. Choosability of the square of planar subcubic graphs with large girth. *Discrete Mathematics*, Elsevier, 2009, 309, pp.3553–3563. <inria-00496423>

**HAL Id: inria-00496423**

**<https://hal.inria.fr/inria-00496423>**

Submitted on 23 Oct 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Choosability of the square of planar subcubic graphs with large girth

F. Havet\*

Projet Mascotte, CNRS/UNSA/INRIA  
INRIA Sophia-Antipolis,  
2004 route des Lucioles BP 93,  
06902 Sophia-Antipolis Cedex  
France  
e-mail:fhavet@sophia.inria.fr

April 1, 2011

## Abstract

We show that the choice number of the square of a subcubic graph with maximum average degree less than  $18/7$  is at most 6. As a corollary, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6. We then show that the choice number of the square of a subcubic planar graph with girth at least 13 is at most 5.

## 1 Introduction

Let  $G$  be a (simple) graph. The *neighbourhood* of a vertex  $v$  of  $G$ , denoted  $N_G(v)$ , is the set of its *neighbours*, i.e. is the set of vertices  $y$  such that  $xy$  is an edge. The *degree* of a vertex  $v$  in  $G$ , denoted  $d_G(v)$ , is its number of neighbours. Often, when the graph  $G$  is clearly understood from the context, we omit the subscript  $G$ . A graph is *subcubic* if every vertex has degree at most 3.

Let  $p : V(G) \rightarrow \mathbb{N}$ . A  *$p$ -list-assignment* is a list-assignment  $L$  such that  $|L(v)| = p(v)$  for any  $v \in V(G)$ .  $G$  is  *$p$ -choosable* if it is  $L$ -colourable for any  $p$ -list-assignment. By extension, if  $k$  is an integer, we say that  $G$  is  *$k$ -choosable* if it is  $p$ -choosable when  $p$  is the constant function with value  $k$  (i. e.  $p(v) = k$  for all  $v \in V$ ). The *choice number* of  $G$ , denoted  $ch(G)$ , is the smallest integer  $k$  such that  $G$  is  $k$ -choosable. Clearly the choice number of  $G$  is at least as large as  $\chi(G)$ , the *chromatic number* of  $G$ .

The *square* of  $G$  is the graph  $G^2$  with vertex set  $V(G)$  such that two vertices are linked by an edge of  $G^2$  if and only if  $x$  and  $y$  are at distance at most 2 in  $G$ . A graph is called *planar* if it can be embedded in the plane. Wegner [?] proved that the square of a subcubic planar graph is 8-colourable. He also conjectured it is 7-colourable. Recently, this conjecture was proved by Thomassen [?].

**Theorem 1 (Thomassen [?])** *Let  $G$  be a subcubic planar graph. Then  $\chi(G^2) \leq 7$ .*

Kostochka and Woodall [?] conjectured that, for every square of a graph, the chromatic number equals the choice number.

**Conjecture 2 (Kostochka and Woodall [?])** *For all  $G$ ,  $ch(G^2) = \chi(G^2)$ .*

---

\*Partially supported by the European Project IST *AEOLUS*

If true, this conjecture together with Theorem ?? implies that every subcubic planar graph is 7-choosable. Very recently, Cranston and Kim [?] showed that the square of every subcubic graph (non necessarily planar) other than the Petersen graph is 8-choosable.

The *average degree* of  $G$ , denoted  $Ad(G)$  is  $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$ . The *maximum average degree* of  $G$ , denoted  $Mad(G)$ , is  $\max\{Ad(H), H \text{ subgraph of } G\}$ . In [?], Dvořák, Škrekovski and Tancer proved that the choice number of the square of a subcubic graph  $G$  is at most 4 if  $Mad(G) < 24/11$  and  $G$  has no 5-cycle, at most 5 if  $Mad(G) < 7/3$  and at most 6 if  $Mad(G) < 5/2$ .

The *girth* of a graph is the smallest length of a cycle in  $G$ . Planar graphs with prescribed girth have bounded maximum average degree:

**Proposition 3** *Every planar graph with girth at least  $g$  has maximum average degree less than  $2 + \frac{4}{g-2}$ .*

Hence the results of Dvořák, Škrekovski and Tancer imply that the choice number of the square of a planar graph with girth  $g$  is at most 6 if  $g \geq 10$ , at most 5 if  $g \geq 14$  and at most 4 if  $g \geq 24$ . The two latter results had been previously proved by Montassier and Raspaud [?].

In this paper, we improve some of these results. We first show (Theorem ??) that the choice number of the square of a subcubic graph with maximum average degree less than  $18/7$  is at most 6. As a corollary, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6. Note that this corollary has been proved later and independently by Cranston and Kim [?]. We then show (Theorem ??) that the choice number of the square of a subcubic planar graph with girth at least 13 is at most 5.

## 2 The main results

The general frame of the proofs is classical. We consider a  $k$ -*minimal graph*, that is a subcubic graph such that its square is not  $k$ -choosable but the square of every proper subgraph is  $k$ -choosable. We prove that some *configurations* (i.e. induced subgraphs) are forbidden in such a graph and then deduce a contradiction. To do so, we will need the following definitions:

An  $i$ -*vertex* is a vertex of degree  $i$ . We denote by  $V_i$  the set of  $i$ -vertices of  $G$  and by  $v_i$  its cardinality. Let  $v$  be a vertex. An  $i$ -*neighbour* of  $v$  is a neighbour of  $v$  with degree  $i$ . The  $i$ -*neighbourhood* of  $v$  is  $N_i(v) = N(v) \cap V_i$  and its  $i$ -*degree* is  $d_i(v) = |N_i(v)|$ .

Some properties of 6- and 5-minimal graphs have already been proved in [?]. The easy first one is that  $V_0 \cup V_1 = \emptyset$ , so  $G$  has minimum degree 2. This will allow us to use the following definitions for 6- and 5-minimal graphs.

Let  $G$  be a subcubic graph with minimum degree 2. A *thread* of  $G$  is a path whose endvertices are 3-vertices and whose internal vertices are 2-vertices. The *kernel* of  $G$  is the weighted graph  $K_G$  such that  $V(K_G) = V_3(G)$  and  $xy$  is an edge in  $K_G$  with weight  $l$  if and only if  $x$  and  $y$  are connected by a thread of length  $l$  in  $G$ . An edge of weight  $l$  is also called  $l$ -*edge*. Let  $x$  be a 3-vertex of  $G$ . The *type* of  $x$  is the triple  $(l_1, l_2, l_3)$  such that  $l_1 \leq l_2 \leq l_3$  and the three edges (a loop being counted twice) incident to  $x$  have weight  $l_1, l_2$  and  $l_3$  in  $K_G$ . We denote by  $Y_{l_1, l_2, l_3}$  the set of 3-vertices of type  $(l_1, l_2, l_3)$  and  $y_{l_1, l_2, l_3}$  its cardinality. Moreover, for every integer  $i$ , we define  $Z_i := \bigcup_{l_1+l_2+l_3=i} Y_{l_1, l_2, l_3}$  and  $z_i = |Z_i|$ . The number of vertices and edges and thus the average degree of  $G$  may be easily expressed in terms of the  $z_i$ :

$$\begin{aligned} |V(G)| &= \sum_{i \geq 3} \frac{i-1}{2} z_i \\ 2|E(G)| &= \sum_{i \geq 3} i \cdot z_i \end{aligned}$$

$$Ad(G) = \frac{\sum_{i \geq 3} i \cdot z_i}{\sum_{i \geq 3} \frac{i-1}{2} z_i} \quad (1)$$

## 2.1 6-choosability

The aim of this subsection is to prove the following result.

**Theorem 4** *Let  $G$  be a subcubic graph of maximum average degree  $d < 18/7$ . Then  $G^2$  is 6-choosable.*

**Remark 5** Theorem ?? is tight. Indeed, the graph  $J_7$  depicted in Figure ?? has average degree  $18/7$  and its square is the complete graph on seven vertices  $K_7$  which is not 6-choosable (nor 6-colourable).

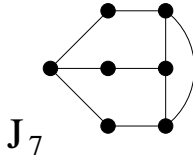


Figure 1: The graph  $J_7$

Theorem ?? and Proposition ?? yield that the square of a subcubic planar graph with girth at least 9 is 6-choosable.

**Corollary 6** *The square of a subcubic planar graph with girth at least 9 is 6-choosable.*

In order to prove Theorem ??, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [?].

**Lemma 7 (Dvořák, Škrekovski and Tancer [?])** *Let  $G$  be a 6-minimal graph. Then the following hold:*

- (i) *all the edges of  $K_G$  have weight at most 2;*
- (ii) *every 3-cycle of  $G$  has its vertices in  $V_3$ ;*
- (iii) *every 4-cycle of  $G$  has at least three vertices in  $V_3$ ;*
- (iv) *a vertex of  $Y_{2,2,2}$  is not adjacent in  $K_G$  to a vertex of  $Y_{1,2,2} \cup Y_{2,2,2}$ .*

We will prove in Subsection ?? some new properties.

**Lemma 8** *Let  $G$  be a 6-minimal graph. Then the following hold:*

- (i) *if  $(v_1, v_2, v_3, v_4, v_1)$  is a 4-cycle with  $v_2 \in V_2$  then  $v_1$  or  $v_3$  is not in  $Y_{1,2,2}$ ;*
- (ii) *a vertex of  $Y_{1,2,2}$  is adjacent in  $K_G$  to at most one vertex of  $Y_{1,2,2}$  by 2-edges.*

**Proof of Theorem ??.** Let  $G$  be a 6-minimal planar graph.  $G$  has minimum degree 2, so its kernel  $K_G$  is defined. Moreover by Lemma ?? (i),  $Z_i$  is empty for  $i \geq 7$  and  $Z_6 = Y_{2,2,2}$  and  $Z_5 = Y_{1,2,2}$ .

Let us consider a vertex of  $Z_4 = Y_{1,1,2}$ . Its neighbour in  $K_G$  via the 2-edge is in  $Z_4 \cup Z_5 \cup Z_6$  because a vertex of  $Z_3 = Y_{1,1,1}$  is incident to no edge of weight 2. For  $i = 4, 5, 6$ , let  $Z_4^i$  be the set of vertices of

$Z_4$  which are adjacent to a vertex of  $Z_i$  by their unique 2-edge and  $z_4^i$  its cardinality.  $(Z_4^4, Z_4^5, Z_4^6)$  is a partition of  $Z_4$  so  $z_4 = z_4^4 + z_4^5 + z_4^6$ . Hence Equation (??) becomes

$$Ad(G) = \frac{6z_6 + 5z_5 + 4z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{\frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4^6 + \frac{3}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma ?? (iv), the three neighbours in  $K_G$  of a vertex of  $Z_6$  are not in  $Z_6 \cup Z_5$ . So they must be in  $Z_4^6$ . It follows that  $3z_6 = z_4^6$ . So

$$Ad(G) = \frac{5z_5 + 6z_4^6 + 4z_4^5 + 4z_4^4 + 3z_3}{2z_5 + \frac{7}{3}z_4^6 + \frac{3}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3}.$$

By Lemma ?? (ii), a vertex of  $Z_5$  is adjacent to at least one vertex of  $Z_4^5$ . Thus  $z_5 \leq z_4^5$ . But  $Ad(G)$  is decreasing as a function of  $z_5$  since  $z_4^6, z_4^5, z_4^4$  and  $z_3$  are non-negative. It follows that

$$Ad(G) \geq \frac{6z_4^6 + 9z_4^5 + 4z_4^4 + 3z_3}{\frac{7}{3}z_4^6 + \frac{7}{2}z_4^5 + \frac{3}{2}z_4^4 + z_3} \geq \frac{18}{7}.$$

□

## 2.2 5-choosability

Dvořák, Škrekovski and Tancer [?] proved that the square of a subcubic graph  $G$  with maximum average degree less than  $7/3$  is 5-choosable. This result is tight since the graph  $J_6$  depicted in Figure ?? has average degree  $7/3$  and its square is the complete graph on six vertices  $K_6$  which is not 5-choosable (nor 5-colourable). However, we will prove that the square of a subcubic planar graph with girth at least 13

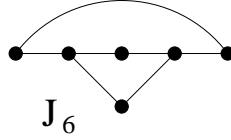


Figure 2: The graph  $J_6$

is 5-choosable, which improves the result of Montassier and Raspaud.

**Theorem 9** *The square of a subcubic planar graph with girth at least 13 is 5-choosable.*

In order to prove this theorem, we need to establish some properties of 5-minimal graphs. Some of them have been proved in [?].

**Lemma 10 (Dvořák, Škrekovski and Tancer [?])** *Let  $G$  be a 5-minimal graph. Then the following hold:*

- (i) *all the edges of  $K_G$  have weight at most 3;*
- (ii) *if  $i \geq 8$ ,  $Z_i$  is empty.*

We will prove in Subsection ?? some new properties.

**Lemma 11** *Let  $G$  be a 5-minimal graph of girth at least 13. Then in  $K_G$  the following hold:*

- (i) a vertex of  $Y_{2,2,3}$  and a vertex of  $Y_{1,2,3} \cup Y_{2,2,3}$  are not linked by a 2-edge;
- (ii) a vertex of  $Y_{1,3,3}$  and a vertex of  $Y_{1,2,3} \cup Y_{1,3,3}$  are not linked by a 1-edge;
- (iii) a vertex of  $Y_{2,2,2}$  is not adjacent in  $K_G$  to three vertices of  $Y_{2,2,3}$  (by 2-edges).

**Proof of Theorem ??.** Let  $G$  be a 5-minimal planar graph with girth at least 13.  $G$  has minimum degree 2, so its kernel  $K_G$  is defined. Moreover, by Lemma ?? (i),  $Z_7 = Y_{2,2,3} \cup Y_{1,3,3}$ , so

$$z_7 = y_{2,2,3} + y_{1,3,3}. \quad (2)$$

Let us count the number  $e_2$  of 2-edges incident to vertices of  $Y_{2,2,3}$ . Recall that  $Z_4 = Y_{1,1,2}$  and  $Z_3 = Y_{1,1,1}$ . Since 2-edges may not link two vertices of type  $(2, 2, 3)$  according to Lemma ?? (i), we have  $e_2 = 2y_{2,2,3}$ . Moreover, the ends of such edges which are not in  $Y_{2,2,3}$  have to be in  $Y_{2,2,2} \cup Y_{1,2,2} \cup Z_4$  by Lemmas ?? and ?? (i). Furthermore, a vertex of  $Y_{2,2,2}$  is incident to at most two edges of  $e_2$  according to Lemma ?? (iii) and a vertex of  $Y_{1,2,2}$  (resp.  $Z_4$ ) is incident to at most two (resp. one) 2-edges. Therefore  $e_2 \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4$ . So,

$$2y_{2,2,3} \leq 2y_{2,2,2} + 2y_{1,2,2} + z_4. \quad (3)$$

Let us now count the number  $e_1$  of 1-edges incident to vertices of  $Y_{1,3,3}$ . Since 1-edges may not link two vertices of type  $(1, 3, 3)$  according to Lemma ?? (ii), we have  $e_1 = y_{1,3,3}$ . Moreover, the ends of such edges which are not in  $Y_{1,1,3}$  have to be in  $Y_{1,2,2} \cup Y_{1,1,3} \cup Z_4 \cup Z_3$  by Lemmas ?? and ?? (ii). Furthermore, vertices of  $Y_{1,2,2}$  (resp.  $Y_{1,1,3} \cup Z_4, Z_3$ ) are incident to at most one (resp. two, three) 1-edges. Thus  $e_1 \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3$ . So,

$$y_{1,3,3} \leq y_{1,2,2} + 2y_{1,1,3} + 2z_4 + 3z_3. \quad (4)$$

$2 \times (??) + (??)$  yields  $2y_{2,2,3} + 2y_{1,3,3} \leq 2y_{2,2,2} + 4y_{1,2,2} + 4y_{1,1,3} + 5z_4 + 6z_3$ . Hence, by Equation (??),  $2z_7 \leq 2z_6 + 4z_5 + 5z_4 + 6z_3$ , so

$$z_7 \leq z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3.$$

Now by Equation (??) the average degree of  $G$  is

$$Ad(G) = \frac{7z_7 + 6z_6 + 5z_5 + 4z_4 + 3z_3}{3z_7 + \frac{5}{2}z_6 + 2z_5 + \frac{3}{2}z_4 + z_3}.$$

As a function of  $z_7$ , this is a decreasing function (on  $\mathbb{R}^+$ ); so it is minimum when  $z_7$  is maximum that is equal to  $z_6 + 2z_5 + \frac{5}{2}z_4 + 3z_3$ . So,

$$Ad(G) \geq \frac{13z_6 + 19z_5 + \frac{43}{2}z_4 + 24z_3}{\frac{11}{2}z_6 + 8z_5 + 9z_4 + 10z_3} \geq \frac{26}{11}.$$

This contradicts the fact that  $G$  has girth 13 by Proposition ??. □

**Remark 12** It is very likely that using the method below, one can prove that a graph  $G$  with maximum average degree less than  $\frac{26}{11}$  is 5-choosable unless it contains  $J_6$  as an induced subgraph. However, this will require the tedious study of a large number of configurations.

### 3 Proofs of Lemmas ?? and ??

In order to prove Lemmas ?? and ??, we need the following lemma proved in [?]. Let  $S$  be a set of vertices of a  $k$ -minimal graph  $G$ . The function  $p_S : S \rightarrow \mathbb{N}$  is defined by  $p_S(v) = k - |N_{G^2}(v) \setminus S|$ . Then  $p_S(v)$  represents the minimum number of available colours at a vertex  $v \in S$  once we have precoloured the square of  $G - S$ . Hence if  $(G - S)^2$  is  $k$ -choosable,  $(G - S)^2 = G^2 - S$  and  $G^2[S]$  is  $p_S$ -choosable, one can extend any  $k$ -list-colouring of  $G - S$  into a  $k$ -list-colouring of  $G$ , which is a contradiction.

**Lemma 13 (Dvořák, Škrekovski and Tancer [?])** *Let  $S$  be a set of vertices of a  $k$ -minimal graph  $G$ . If  $(G - S)^2 = G^2 - S$ , then  $G^2[S]$  is not  $p_S$ -choosable.*

In order to use Lemma ??, we need some results on the choosability of some graphs.

#### 3.1 Some choosability tools

**Definition 14** Let  $x$  and  $y$  be two vertices of a graph  $G$ . An  $(x - y)$ -ordering of  $G$  is an ordering of the vertices such that  $x$  is the minimum and  $y$  the maximum. An  $(x, y - z)$ -ordering is an ordering of the vertices such that  $x$  is minimum,  $y$  is the second minimum and  $z$  is maximum.

Let  $\sigma = (v_1 < v_2 < \dots < v_n)$  be an ordering of the vertices of  $G$  and  $p$  a function  $V(G) \rightarrow \mathbb{N}$ .  $\sigma$  is  $p$ -greedy if, for every  $i$ ,  $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| < p(v_i)$ . It is  $p$ -nice if, for every  $i$  except  $n$ ,  $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| < p(v_i)$  and  $d(v_n) = p(v_n)$ . It is  $p$ -good if, for every  $3 \leq i \leq n$ ,  $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| - \epsilon(v_i) < p(v_i)$  with  $\epsilon(v_i) = 1$  if  $v_i$  is adjacent to both  $v_1$  and  $v_2$  and  $\epsilon(v_i) = 0$  otherwise. By extension, if  $k$  is an integer, we say that  $\sigma$  is  $k$ -greedy (resp.  $k$ -nice,  $k$ -good) if it is  $p$ -greedy (resp.  $p$ -nice,  $p$ -good) when  $p$  is the constant function with value  $k$  (i. e.  $p(v_i) = k$  for every  $1 \leq i \leq n$ ).

The greedy algorithm according to greedy, nice and good orderings yields the following three lemmas.

**Lemma 15** *If  $G$  has a  $p$ -greedy ordering then  $G$  is  $p$ -choosable.*

**Proof.** Applying the greedy algorithm according to the  $p$ -greedy ordering gives the desired colouring.  $\square$

**Lemma 16** *Let  $xy$  be an edge of graph  $G$  and  $L$  be a  $p$ -list-assignment of  $G$ . If  $L(x) \not\subseteq L(y)$  and  $G$  has a  $p$ -nice  $(x - y)$ -ordering, then  $G$  is  $L$ -colourable.*

**Proof.** Let  $a$  be a colour in  $L(x) \setminus L(y)$ . Assign  $a$  to  $x$  and proceed the greedy algorithm according to the  $p$ -nice  $(x - y)$ -ordering. The only vertex which has not more colour in its list than previously coloured neighbours is  $y$  for which  $|L(y)| = d(y)$ . But since  $a \notin L(y)$ , at most  $d(y) - 1$  colours of  $L(y)$  are assigned to the neighbours of  $y$ . Hence one can colour  $y$ .  $\square$

**Lemma 17** *Let  $x, y$  and  $z$  be three vertices of a graph  $G = (V, E)$  such that  $xy \notin E$ . If  $L(x) \cap L(y) \neq \emptyset$  and  $G$  has a  $p$ -good  $(x, y - z)$ -ordering, then  $G$  is  $L$ -colourable.*

**Proof.** Let  $a$  be a colour in  $L(x) \cap L(y)$  and  $\sigma = (v_1 < v_2 < \dots < v_n)$  be a  $p$ -good  $(x, y - z)$ -ordering. (In particular,  $v_1 = x$ ,  $v_2 = y$  and  $v_n = z$ .) Assign  $a$  to  $x$  and  $y$  and proceed the greedy algorithm according to  $\sigma$ . For every  $3 \leq i \leq n$ , the number of colours assigned to already coloured neighbours of  $v_i$  is at most  $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| - \epsilon(v_i)$  since  $v_1$  and  $v_2$  are coloured the same. Hence the greedy algorithm gives an  $L$ -colouring.  $\square$

**Remark 18** Note that if  $xz, yz \in E$ , a  $p$ -nice  $(x, y - z)$ -ordering is also  $p$ -good.

**Definition 19** The *blocks* of a graph are its maximal 2-connected components. A connected graph is said to be a *Gallai tree* if each of its blocks is either a complete graph or an odd cycle.

The following theorem was proved independently by Borodin [?] and Erdős, Rubin and Taylor [?]:

**Theorem 20 (Borodin [?], Erdős, Rubin and Taylor [?])** *Let  $G$  be a connected graph and  $d_G$  the degree function in  $G$ . Then  $G$  is  $d_G$ -choosable if and only if  $G$  is not a Gallai tree.*

**Lemma 21** *Let  $G = (V, E)$  be a graph and  $p : V(G) \rightarrow \mathbb{N}$ . Let  $S$  be a set of vertices such that  $p(v) \geq d(v)$  for all  $v \in S$ . If  $G[S]$  is not a Gallai tree and  $G - S$  is  $p$ -choosable then  $G$  is  $p$ -choosable.*

**Proof.** Let  $L$  be a  $p$ -list-assignment of  $G$ . Since  $G - S$  is  $p$ -choosable, it admits an  $L$ -colouring  $c$ . Let us now extend it to  $S$ . The list  $I(v) = L(v) \setminus \{c(w), w \in N(v) \setminus S\}$  of available colours of a vertex  $v \in S$  is of size at least  $p'(v) = p(v) - |N(v) \setminus S| \geq d_{G[S]}(v)$ . Since  $G[S]$  is not a Gallai tree, by Theorem ??,  $G[S]$  is  $p'$ -choosable and thus  $I$ -colourable. So,  $G$  is  $L$ -colourable.  $\square$

A 4-regular graph  $G$  is *cycle+triangles* if it is the edge union of a Hamiltonian cycle  $C$  and a 2-factor consisting of triangles. In other words, the graph induced by the edges of  $E(G) \setminus E(C)$  is the disjoint union of 3-cycles.

**Theorem 22 (Fleischner and Stiebitz [?])** *Every cycle+triangles graph is 3-choosable.*

### 3.2 Proof of Lemma ??

**Lemma 23** *Let  $q \geq 2$  and  $C_{4q} = (v_1, \dots, v_{4q}, v_1)$  be the  $4q$ -cycle and  $p$  defined by  $p(v_i) = 4$  if  $i$  is odd and  $p(v_i) = 2$  otherwise. Then  $C_{4q}^2$  is  $p$ -choosable.*

**Proof.** The set  $S$  of vertices  $v$  for which  $p(v) \geq d_{C_{4q}^2}(v)$  is the set of  $v_i$  with odd indices.  $C_{4q}^2[S]$  is a  $2q$ -cycle and thus is not a Gallai tree. Moreover  $C_{4q}^2 - S$  is also a  $2q$ -cycle and is 2-choosable. Hence Lemma ?? gives the result.  $\square$

**Proposition 24** *Let  $P_7 = (v_1, \dots, v_7)$  be a path and  $p$  the function defined by  $p(v_1) = p(v_2) = p(v_6) = p(v_7) = 2$ ,  $p(v_3) = p(v_5) = 4$  and  $p(v_4) = 3$ . Then  $P_7^2$  is  $p$ -choosable.*

**Proof.** Let  $L$  be a  $p$ -list-assignment of  $P_7^2$ . Since  $(v_2 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$  is a  $p$ -nice ordering of  $P_7^2$ , by Lemma ??, we may assume that  $L(v_1) = L(v_2)$ , and by symmetry of  $P_7$  and  $p$  that  $L(v_6) = L(v_7)$ .

Since  $(v_1 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$  is  $p$ -good, by Lemma ??, we may assume that  $L(v_1) \cap L(v_4) = \emptyset$ , and by symmetry  $L(v_7) \cap L(v_4) = \emptyset$ .

Now one can find  $c(v_1) \in L(v_1)$ ,  $c(v_2) \in L(v_2) \setminus \{c(v_1)\}$ ,  $c(v_6) \in L(v_6)$ ,  $c(v_7) \in L(v_7) \setminus \{c(v_6)\}$ ,  $c(v_3) \in L(v_3) \setminus \{c(v_1), c(v_2)\}$ , and  $c(v_5) \in L(v_5) \setminus \{c(v_3), c(v_6), c(v_7)\}$ . Now since  $L(v_1) \cap L(v_4) = \emptyset$  and  $L(v_1) = L(v_2)$ ,  $c(v_2) \notin L(v_4)$ . Analogously,  $c(v_6) \notin L(v_4)$ . Hence,  $L(v_4) \setminus \{c(v_2), c(v_3), c(v_5), c(v_6)\} = L(v_4) \setminus \{c(v_3), c(v_5)\} \neq \emptyset$ . So, one can choose  $c(v_4)$  in this set to get an  $L$ -colouring  $c$  of  $P_7^2$ .  $\square$

**Lemma 25** *For  $1 \leq i \leq 17$ , let  $F_i$  be the graphs and  $p_i$  be the functions depicted in Figure ??.*

- (i)  $F_1^2 \cup \{v_5v_6\}$  is  $p_1$ -choosable.
- (ii)  $F_2^2 \cup \{v_1v_4\}$  and  $F_2^2 \cup \{v_4v_7\}$  are  $p_2$ -choosable.
- (iii)  $F_3^2 \cup \{v_4v_8\}$  is  $p_3$ -choosable.



- (iv)  $F_4^2$  is 6-choosable.
- (v)  $F_5^2 \cup \{v_1v_4, v_1v_6\}$  is  $p_5$ -choosable.
- (vi)  $F_6^2$  is  $p_6$ -choosable.
- (vii)  $F_7^2 \cup \{v_9v_{10}\}$  is  $p_7$ -choosable.
- (viii)  $F_8^2$  is  $p_8$ -choosable.
- (ix)  $F_9^2 \cup \{v_2v_9\}$  and  $F_9^2 \cup \{v_6v_9\}$  are  $p_9$ -choosable.
- (x)  $F_{10}^2 \cup \{v_4v_8\}$  is  $p_{10}$ -choosable.
- (xi)  $F_{11}^2 \cup \{v_4v_8, v_8v_9\}$ ,  $F_{11}^2 \cup \{v_4v_8, v_9v_4\}$  and  $F_{11}^2 \cup \{v_8v_9, v_9v_4\}$  are  $p_{11}$ -choosable and  $F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$  is 5-choosable.
- (xii)  $F_{12}^2 \cup \{v_4v_8\}$  is  $p_{12}$ -choosable.
- (xiii)  $F_{13}^2$  is 6-choosable.

**Proof.**

- (i) In  $F_1^2 \cup \{v_5v_6\}$ ,  $(v_6 < v_5 < v_4 < v_3 < v_1 < v_2)$  is  $p_1$ -greedy. So, by Lemma ??,  $F_1^2 \cup \{v_5v_6\}$  is  $p_1$ -choosable.
- (ii) In  $F_2^2 \cup \{v_4v_7\}$ ,  $(v_2 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$  is  $p_2$ -nice and  $p_2(v_2) > p_2(v_1)$ . So, by Lemma ??,  $F_2^2 \cup \{v_4v_7\}$  is  $p_2$ -choosable.  
By symmetry, one shows that  $F_2^2 \cup \{v_1v_4\}$  is  $p_2$ -choosable.
- (iii) In  $F_3^2 \cup \{v_4v_8\}$ ,  $(v_2 < v_8 < v_4 < v_7 < v_6 < v_5 < v_3 < v_1)$  is  $p_3$ -nice and  $p_3(v_2) > p_3(v_1)$ . So, by Lemma ??,  $F_3^2 \cup \{v_4v_8\}$  is  $p_3$ -choosable.
- (iv) Let  $L$  be a 6-list-assignment of  $F_4^2$ . Every ordering with maximum  $v_1$  and second maximum  $v_7$  is 6-nice. Thus, by Lemma ??, we may assume that  $L(v_j) = L(v_1)$  for  $j \in \{2, 3, 4, 5, 6, 8\}$ . Analogously, we may assume that  $L(v_j) = L(v_7)$  for  $j \in \{2, 3, 4, 5, 6, 8\}$ . Hence all the lists are the same, say  $\{1, 2, 3, 4, 5, 6\}$ . Now  $c(v_1) = c(v_5) = 1$ ,  $c(v_2) = 2$ ,  $c(v_3) = c(v_7) = 3$ ,  $c(v_4) = 4$ ,  $c(v_6) = 5$  and  $c(v_8) = 6$  is an  $L$ -colouring of  $F_4^2$ .
- (v) In  $F_5^2 \cup \{v_1v_4, v_1v_6\}$ ,  $(v_7 < v_6 < v_1 < v_4 < v_2 < v_3 < v_5)$  is  $p_5$ -nice and  $p_5(v_7) > p_5(v_5)$ . So, by Lemma ??,  $F_5^2 \cup \{v_1v_5\}$  is  $p_5$ -choosable.
- (vi) In  $F_6^2$ ,  $(v_4 < v_2 < v_8 < v_1 < v_3 < v_5)$  is  $p_6$ -nice and  $p_6(v_4) > p_6(v_5)$ . So, by Lemma ??,  $F_6^2$  is  $p_6$ -choosable.
- (vii) Let  $L$  be a  $p_7$ -list-assignment of  $F_7^2 \cup \{v_9v_{10}\}$ .  $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$ ,  $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_5 < v_7 < v_1 < v_3)$  and  $(v_4 < v_9 < v_{10} < v_8 < v_6 < v_2 < v_5 < v_7 < v_1 < v_3)$  are  $p_7$ -nice. Thus, by Lemma ??, we may assume that  $L(v_2) \subset L(v_1)$ ,  $L(v_2) \subset L(v_3)$  and  $L(v_4) \subset L(v_3)$ . It follows that  $L(v_1) \cap L(v_4) \neq \emptyset$ . Because  $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$  is  $p_7$ -good, by Lemma ??,  $F_7^2 \cup \{v_9v_{10}\}$  is  $L$ -colourable.
- (viii) In  $F_8^2$ ,  $(v_6 < v_5 < v_7 < v_9 < v_8 < v_4 < v_3 < v_2 < v_1)$  is  $p_8$ -greedy. So, by Lemma ??,  $F_8^2$  is  $p_8$ -choosable.

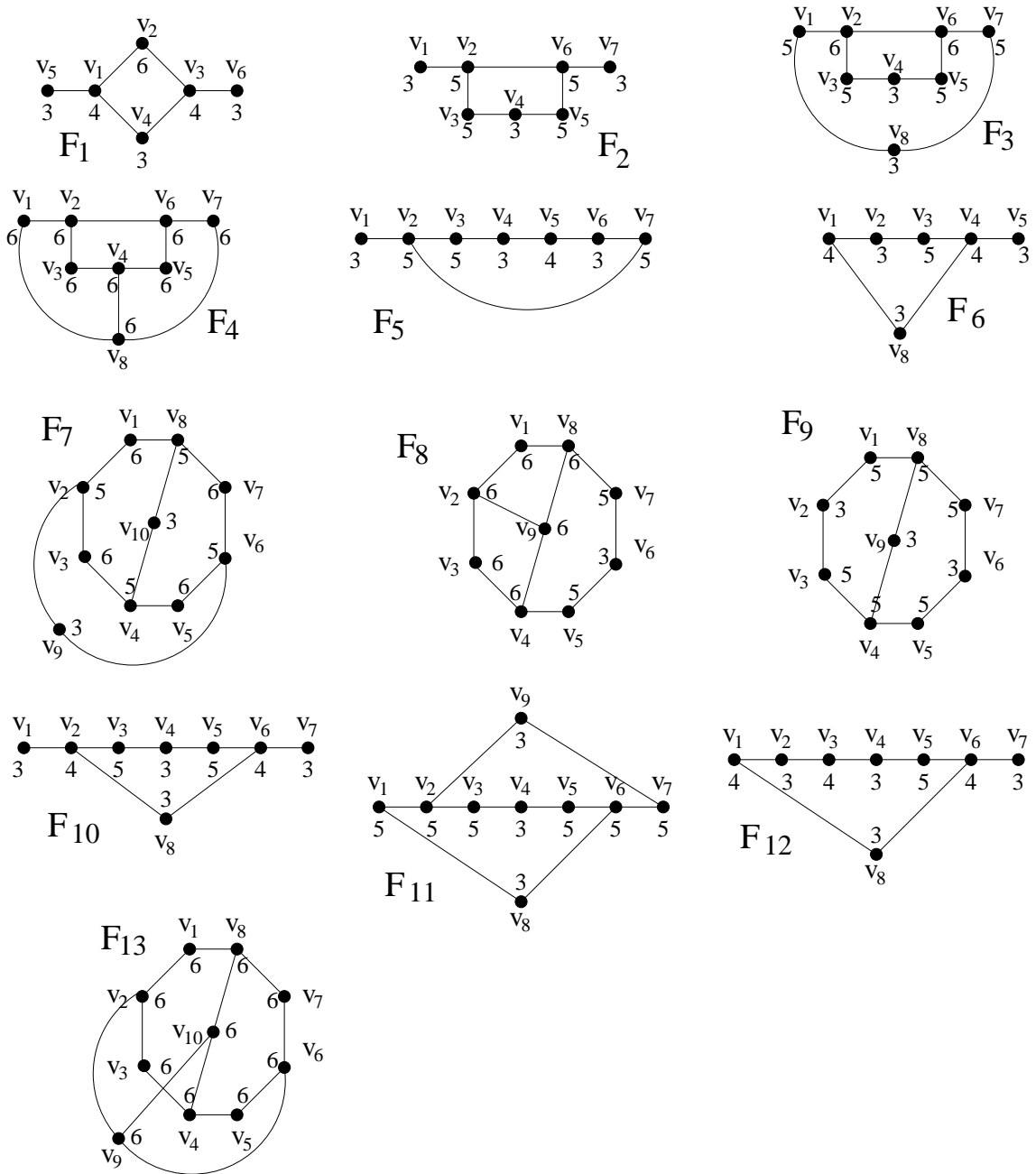


Figure 3: The graphs  $F_i$  and functions  $p_i$  for  $1 \leq i \leq 13$

(ix) Let  $L$  be a  $p_9$ -list-assignment of  $F_9^2 \cup \{v_2v_9\}$ . Then  $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_3 < v_1)$  and  $(v_2 < v_9 < v_6 < v_4 < v_8 < v_7 < v_5 < v_1 < v_3)$  are  $p_9$ -nice so by Lemma ??, we may assume that  $L(v_2) \subset L(v_3) \cap L(v_1)$ . Moreover,  $(v_4 < v_2 < v_9 < v_6 < v_8 < v_7 < v_5 < v_1 < v_3)$  is  $p_9$ -nice so by Lemma ??, we may assume that  $L(v_4) = L(v_3)$ . It follows that  $L(v_1) \cap L(v_4) \neq \emptyset$ . Thus, by Lemma ??, since  $(v_1 < v_4 < v_2 < v_9 < v_8 < v_6 < v_7 < v_5 < v_3)$  is  $p_9$ -good,  $F_9^2 \cup \{v_2v_9\}$  is  $L$ -colourable.

By symmetry, one shows that  $F_9^2 \cup \{v_6v_9\}$  is  $p_9$ -choosable.

(x) In  $F_{10}^2 \cup \{v_4v_8\}$ ,  $(v_2 < v_8 < v_4 < v_6 < v_7 < v_5 < v_3 < v_1)$  is  $p_{10}$ -nice and  $p_{10}(v_2) > p_{10}(v_1)$ . So, by Lemma ??,  $F_{10}^2 \cup \{v_4v_8\}$  is  $p_{10}$ -colourable.

(xi) Let  $F \in \{F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}, F_{11}^2 \cup \{v_4v_8, v_8v_9\}, F_{11}^2 \cup \{v_4v_8, v_9v_4\}, F_{11}^2 \cup \{v_8v_9, v_9v_4\}\}$  and  $L$  be a 5-list-assignment if  $F = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$  and a  $p_{11}$ -list-assignment of  $F$  otherwise.

Then  $(v_1 < v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3)$ ,  $(v_7 < v_9 < v_8 < v_4 < v_6 < v_2 < v_1 < v_5 < v_3)$  and  $(v_7 < v_9 < v_8 < v_4 < v_6 < v_2 < v_1 < v_3 < v_5)$  are  $p$ -nice in  $F$ . So by Lemma ??, we may assume that  $L(v_1) = L(v_3) = L(v_5) = L(v_7)$ .

If  $L(v_8) \not\subset L(v_2)$ , let us colour  $v_8$  with  $c_8 \in L(v_8) \setminus L(v_2)$ ,  $v_4$  with  $c_4 \in L(v_4) \setminus \{c_8\}$ ,  $v_9$  with  $c_8 \in L(v_8) \setminus \{c_4, c_8\}$ ,  $v_1$  and  $v_5$  with the same colour  $c_1 \in L(v_1) \setminus \{c_4, c_8, c_9\}$ ,  $v_3$  and  $v_7$  with the same colour  $c_3 \in L(v_1) \setminus \{c_1, c_4, c_8, c_9\}$ ,  $v_6$  with  $c_6 \in L(v_6) \setminus \{c_1, c_3, c_8, c_9\}$  and finally  $v_2$  with  $c_2 \in L(v_2) \setminus \{c_1, c_3, c_6, c_8, c_9\} = L(v_2) \setminus \{c_1, c_3, c_6, c_9\}$ . This gives an  $L$ -colouring of  $F$ . So we may assume that  $L(v_8) \subset L(v_2)$ . Exchanging the role of  $c_4$  in  $c_8$  in the preceding argument, we may assume that  $L(v_4) \subset L(v_2)$ . Moreover by symmetry, we may assume that  $L(v_9) \cup L(v_4) \subset L(v_6)$ . In particular, this implies that the sets  $L(v_8) \cap L(v_9)$ ,  $L(v_8) \cap L(v_4)$ ,  $L(v_9) \cap L(v_4)$  and  $L(v_2) \cap L(v_6)$  are non empty.

If  $F = F_{11}^2 \cup \{v_4v_8, v_9v_4\}$  then  $v_8v_9 \notin F$ . Hence  $(v_8 < v_9 < v_4 < v_2 < v_6 < v_7 < v_5 < v_3 < v_1)$  is  $p_{11}$ -good, so by Lemma ??,  $F$  is  $L$ -colourable.

If  $F = F_{11}^2 \cup \{v_8v_9, v_9v_4\}$  then  $v_8v_4 \notin F$ . Hence  $(v_4 < v_8 < v_9 < v_1 < v_2 < v_6 < v_7 < v_3 < v_5)$  is  $p_{11}$ -good, so by Lemma ??,  $F$  is  $L$ -colourable.

If  $F = F_{11}^2 \cup \{v_4v_8, v_8v_9\}$  then  $v_9v_4 \notin F$ . Hence  $(v_4 < v_9 < v_8 < v_1 < v_2 < v_6 < v_7 < v_3 < v_5)$  is  $p_{11}$ -good, so by Lemma ??,  $F$  is  $L$ -colourable.

If  $F = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ , then  $(v_2 < v_6 < v_4 < v_8 < v_9 < v_7 < v_5 < v_3 < v_1)$  is 5-good. So, by Lemma ??,  $F$  is  $L$ -colourable.

(xii) In  $F_{12}^2 \cup \{v_4v_8\}$ ,  $(v_6 < v_8 < v_4 < v_2 < v_1 < v_3 < v_5 < v_7)$  is  $p_{12}$ -nice and  $p_{12}(v_6) > p_{12}(v_7)$ . So by Lemma ??,  $F_{12}^2 \cup \{v_4v_8\}$  is  $p_{12}$ -choosable.

(xiii) Let  $L$  be a 6-list-assignment of  $F_{13}^2$ .  $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_7 < v_5 < v_3 < v_1)$ ,  $(v_2 < v_9 < v_{10} < v_8 < v_6 < v_4 < v_5 < v_7 < v_1 < v_3)$  and  $(v_4 < v_9 < v_{10} < v_8 < v_6 < v_2 < v_5 < v_7 < v_1 < v_3)$  are 6-nice. Thus, by Lemma ??, we may assume that  $L(v_1) = L(v_2) = L(v_3) = L(v_4)$ . Because  $(v_1 < v_4 < v_{10} < v_9 < v_2 < v_8 < v_6 < v_7 < v_5 < v_3)$  is 6-good, by Lemma ??,  $F_{13}^2$  is  $L$ -colourable.

□

### Proof of Lemma ??.

To prove this lemma, we will suppose for a contradiction that it does not hold. Then we will find a set  $X$  of vertices contradicting Lemma ?. Indeed Lemma ?? will show that  $G^2[X]$  is  $p_X$ -choosable and for each set  $X$  we consider, every vertex of  $X$  has at most one neighbour in  $G - X$ , so  $(G - X)^2 = G^2 - X$ . Lemma ?? completes the proof.

(i) Suppose for a contradiction that  $v_1$  and  $v_3$  are in  $Y_{1,2,2}$ . Let  $v_5$  (resp.  $v_6$ ) be the neighbour of  $v_1$  (resp.  $v_3$ ) distinct from  $v_2$  and  $v_4$ . By Lemma ?? (iv),  $v_4$  is in  $V_3$  and  $v_5 \neq v_6$ . Set  $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ . Then  $G[S] = F_1$ ,  $p_S \geq p_1$  and  $G^2[S] \subset F_1^2 \cup \{v_5v_6\}$ . So Lemma ?? contradicts Lemma ??.

(ii) Suppose for a contradiction that, in  $K_G$ , a vertex  $v_4$  of  $Y_{1,2,2}$  is adjacent to two vertices of  $Y_{1,2,2}$   $v_2$  and  $v_6$  by 2-edges. According to Lemma ?? (iii),  $v_2 \neq v_6$ . Let  $v_3$  and  $v_5$  be the 2-neighbours of  $v_4$  common with  $v_2$  and  $v_6$  respectively, and  $v_1$  (resp.  $v_7$ ) be the 2-neighbour of  $v_2$  (resp.  $v_6$ ) not adjacent to  $v_4$ . Set  $S = \{v_1, \dots, v_7\}$ .

We first claim that  $v_1 \neq v_7$ . Suppose not. Then  $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$  is a cycle  $C$ . It has no chord by Lemma ?? (ii), so  $C^2 = G^2[S]$ . Moreover,  $p_S(v_i) \geq 4$  if  $i$  is even and  $p_S(v_i) \geq 3$  otherwise.  $C^2$  is a cycle+triangle graph, thus, by Theorem ??, it is 3-choosable and so  $p_S$ -choosable. This contradicts Lemma ??.

Let  $w_1$  (resp.  $w_7$ ) be the neighbour of  $v_1$  (resp.  $w_7$ ) distinct from  $v_2$  (resp.  $v_6$ ) and for  $i \in \{2, 4, 6\}$ , let  $w_i$  be the neighbour of  $v_i$  not in  $\{v_{i-1}, v_{i+1}\}$ . Let  $W = \{w_1, w_2, w_4, w_6, w_7\}$ .

We claim that  $W \cap S \neq \emptyset$ . Indeed, suppose for a contradiction that  $W \cap S = \emptyset$ . Since  $G$  is simple,  $w_1 \neq v_2$  and  $w_7 \neq v_6$ . Moreover by Lemma ?? (i),  $w_1$  and  $w_7$  are in  $V_3$ , so  $w_1 \neq v_7$  and  $w_7 \neq v_1$ . Furthermore, by Lemma ?? (ii),  $w_2 \neq v_4$  and  $w_6 \neq v_4$  and by Lemma ?? (iii),  $w_1 \neq v_4$  and  $w_7 \neq v_4$ . Last, we may not have  $w_1 = v_6$  and  $w_2 = v_7$  otherwise the 4-cycle  $(v_1, v_6, v_7, v_2, v_1)$  would contradict Lemma ?? (iii). Then, by symmetry, we only need to consider the cases  $w_2 = v_6$ ,  $w_2 = v_7$ .

- Assume that  $w_2 = v_6$ . Then  $G[S] = F_2$ ,  $p_S \geq p_2$  and  $G^2[S] \subset F_2^2 \cup \{v_1v_4, v_4v_7, v_1v_7\}$ . Thus, by Lemmas ?? and ??,  $F_2^2 \cup \{v_1v_7\} \subset G^2[S]$ , so  $w_1 = w_7 = v_8$ . Let  $T = S \cup \{v_8\}$ . If  $v_8 \neq w_4$ , then  $G[T] = F_3$  and  $p_T \geq p_3$  and  $G^2[T] \subset F_3^2 \cup \{v_4v_8\}$ . So Lemma ?? contradicts Lemma ?? . If not then  $G[T] = G = F_4$ , so  $G$  is 6-choosable, by Lemma ?? . This is a contradiction.
- Suppose that  $w_2 = v_7$ . Then  $G[S] = F_5$ ,  $p_S \geq p_5$  and  $G^2[S] \subset F_5^2 \cup \{v_1v_4, v_1v_6\}$ . Thus Lemma ?? contradicts Lemma ?? .

This proves the claim.

Note that by Lemma ?? (ii),  $w_1 \neq w_2$  and  $w_6 \neq w_7$ .

Suppose  $w_1 = w_4 = v_8$ . Then let  $R = \{v_1, v_2, v_3, v_4, v_5, v_8\}$  and  $w_8$  the neighbour of  $v_8$ . Then  $(G[R], p_R) = (F_6, p_6)$  and  $G^2[R] = F_6^2$ . Thus Lemma ?? contradicts Lemma ?? . Therefore,  $w_1 \neq w_4$  and, by symmetry,  $w_4 \neq w_7$ .

Suppose  $w_1 = w_7 = v_8$ . Let  $T = S \cup \{v_8\}$ . Then  $G[T]$  is the cycle  $C_8$  and  $p_T$  is greater or equal to the function  $p$  defined in Lemma ?? . So, by Lemmas ?? and ??,  $G^2[T] \neq C_8^2$ . It follows that either  $w_2 = w_6$  or  $w_4 = w_8$  with  $w_8$  be the neighbour of  $v_8$  not in  $S$ .

- Suppose  $w_2 = w_6 = v_9$ , and  $w_4 = w_8 = v_{10}$ . Set  $W = \{v_1, \dots, v_{10}\}$ . If  $v_9v_{10} \notin E(G)$  then  $G[W] = F_7$ ,  $p_W \geq p_7$  and  $G^2[W] \subset F_7^2 \cup \{v_9v_{10}\}$ ; so Lemma ?? contradicts Lemma ?? . If not,  $G = G[W] = F_{13}$ , so  $G^2$  is 6-choosable, according to Lemma ??, a contradiction.
- Suppose  $w_2 = w_4 = w_6 = v_9$ . Setting  $U = \{v_1, \dots, v_9\}$ , we have  $(G[U], p_U) = (F_8, p_8)$  and  $G^2[U] = F_8^2$ . Hence Lemma ?? contradicts Lemma ?? .

By symmetry, we get a contradiction if  $w_2 = w_6 = w_8$ ,  $w_2 = w_4 = w_8$  or  $w_4 = w_6 = w_8$ .

- Suppose  $w_4 = w_8 = v_9$ ,  $w_2 \neq v_9$ ,  $w_6 \neq v_9$  and  $w_2 \neq w_6$ . Setting  $U = \{v_1, \dots, v_9\}$ , we have  $G[U] = F_9$ ,  $p_U \geq p_9$  and  $G^2[U] \subset F_9^2 \cup \{v_2v_9\}$  or  $G^2[U] \subset F_9^2 \cup \{v_6v_9\}$ . Hence Lemma ?? contradicts Lemma ?? .

By symmetry, we get a contradiction if  $w_2 = w_6 = v_9$ ,  $w_4 \neq v_9$ ,  $w_8 \neq v_9$  and  $w_4 \neq w_8$ .

Therefore,  $w_1 \neq w_7$ .

Suppose that  $w_2 = w_6 = v_8$ . Let  $T = S \cup \{v_8\}$ . Then  $G[T] = F_{10}$ ,  $p_T \geq p_{10}$ , and  $G^2[T] \subset F_{10}^2 \cup \{v_4v_8\}$ , since  $w_1, w_4$  and  $w_7$  are distinct vertices. Hence Lemma ?? contradicts Lemma ??.

Therefore,  $w_2 \neq w_6$ .

Suppose that  $w_1 = w_6 = v_8$  and  $w_2 = w_7 = v_9$ . Let  $U = S \cup \{v_8, v_9\}$ . Then  $G[U] = F_{11}$  and  $G^2[U]$  is a subgraph of  $F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ . Moreover  $p_U \geq p_{11}$  and, if  $G^2[U] = F_{11}^2 \cup \{v_4v_8, v_8v_9, v_9v_4\}$ ,  $p_U(v_i) = 5$  for  $1 \leq i \leq 9$ . . Hence Lemma ?? contradicts Lemma ??.

Therefore,  $w_1 \neq w_6$  or  $w_2 \neq w_7$ . By symmetry,  $w_2 \neq w_7$ .

Suppose  $w_1 = w_6 = v_8$ . Let  $T = S \cup \{v_8\}$  and let  $w_8$  be the neighbour of  $v_8$  not in  $S$ . Then  $G[T] = F_{12}$ ,  $p_T \geq p_{12}$  and  $G^2[T] \subset F_{12}^2 \cup \{v_4v_8\}$ . Hence Lemma ?? contradicts Lemma ??.

Therefore,  $w_1 \neq w_6$ .

Hence all the  $w_i$  are distinct, so  $G[S]^2 = G^2[S]$ . Thus Proposition ?? contradicts Lemma ??.

□

**Remark 26** The proof of Lemma ?? in the case of planar graphs of girth at least 9 is simpler and shorter because all the configurations considered in the above proof (except the path  $P_7$ ) have girth less than 9. Thus Corollary ?? has a short direct proof which requires only Proposition ??.

### 3.3 Proof of Lemma ??

**Definition 27** For  $1 \leq j \leq 4$ , let  $I_j$  and  $q_j$  be the graphs and functions depicted in Figure ??.

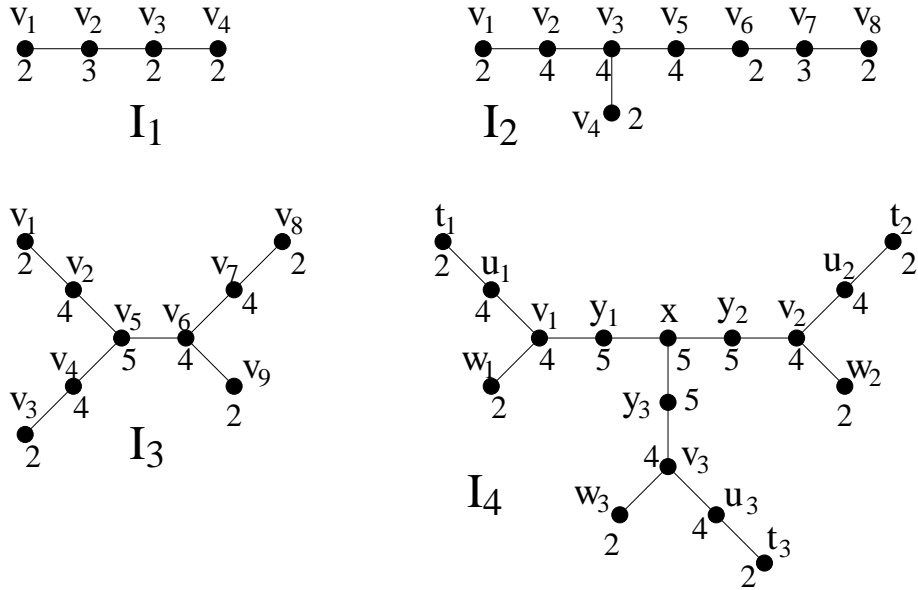


Figure 4: The graphs  $I_j$  and functions  $q_j$ ,  $1 \leq j \leq 4$

**Lemma 28** For  $1 \leq j \leq 4$ ,  $I_j^2$  is  $q_j$ -choosable.

**Proof.**

- Let  $L$  be a  $q_1$ -list-assignment of  $I_1^2$ . The orderings  $(v_4 < v_3 < v_1 < v_2)$  and  $(v_1 < v_3 < v_4 < v_2)$  are  $q_1$ -nice. So, by Lemma ??, we may assume that  $L(v_1) \cup L(v_4) \subset L(v_2)$ . Hence  $L(v_1) \cap L(v_4) \neq \emptyset$ . But  $(v_4 < v_1 < v_3 < v_2)$  is  $q_1$ -good. Thus, by Lemma ??,  $I_1^2$  is  $L$ -colourable.
- Let  $L$  be a  $q_2$ -list-assignment of  $I_2^2$ .  
 Suppose first that  $L(v_3) \not\subset L(v_1) \cup L(v_6)$ . Then choose  $c(v_3)$  in  $L(v_3) \setminus (L(v_1) \cup L(v_6))$  and  $c(v_4) \in L(v_4) \setminus \{c(v_3)\}$ . Since  $I_1^2$  is  $q_1$ -choosable, one can extend  $c$  to  $\{v_5, v_6, v_7, v_8\}$ . Then one can find  $c(v_2) \in L(v_2) \setminus \{c(v_3), c(v_4), c(v_5)\}$  and  $c(v_1) \in L(v_1) \setminus \{c(v_2), c(v_3)\} = L(v_1) \setminus \{c(v_2)\}$ . Hence we may assume that  $L(v_3) \subset L(v_1) \cup L(v_6)$ , so  $L(v_3) = L(v_1) \cup L(v_6)$  and  $L(v_1) \cap L(v_6) = \emptyset$ .  
 Suppose now that  $L(v_4) \cap L(v_6) \neq \emptyset$ . Then colour  $v_4$  and  $v_6$  with the same colour  $c(v_4) = c(v_6) \in L(v_4) \cap L(v_6)$ . Choose  $c(v_8) \in L(v_8) \setminus \{c(v_6)\}$  and  $c(v_7) \in L(v_7) \setminus \{c(v_6), c(v_8)\}$ . Now since  $I_1^2$  is  $q_1$ -choosable, one can extend  $c$  into an  $L$ -colouring of  $I_2^2$ . So we may assume that  $L(v_4) \cap L(v_6) = \emptyset$ . Now  $(v_4 < v_1 < v_6 < v_8 < v_7 < v_5 < v_3 < v_2)$  is  $q_2$ -good so, by Lemma ??, we may assume that  $L(v_4) \cap L(v_1) = \emptyset$ . It follows that  $L(v_4) \cap L(v_3) = \emptyset$  since  $L(v_3) = L(v_1) \cup L(v_6)$ .  
 The ordering  $(v_4 < v_8 < v_6 < v_7 < v_5 < v_3 < v_1 < v_2)$  is  $q_2$ -nice so, by Lemma ??, we may assume that  $L(v_4) \subset L(v_2)$ . Then one may assign  $c(v_4) \in L(v_4)$  and  $c(v_2) \in L(v_4) \setminus \{c(v_4)\}$  to the vertices  $v_4$  and  $v_2$ . Now, because  $L(v_4) \cap L(v_3) = \emptyset$ , one can extend  $c$  into an  $L$ -colouring of  $I_2^2$  by colouring greedily according to the ordering  $(v_1 < v_8 < v_6 < v_7 < v_5 < v_3)$ .
- Let  $L$  be a  $q_3$ -list-assignment of  $I_3^2$ . Assign to  $v_5$  a colour  $c_5$  in  $L(v_5) \setminus (L(v_1) \cup L(v_9))$  and to  $v_6$  a colour in  $L(v_6) \setminus (L(v_8) \cup \{c_5\})$ . Then colour the remaining vertices greedily according to  $(v_3 < v_4 < v_2 < v_1 < v_9 < v_7 < v_8)$  to get an  $L$ -colouring of  $I_3^2$ .
- Let  $L$  be  $q_4$ -list-assignment of  $I_4^2$ . Pick  $c(y_1)$  in  $L(y_1) \setminus L(w_1)$ ,  $c(y_2)$  in  $L(y_2) \setminus (L(w_2) \cup \{c(y_1)\})$ ,  $c(y_3)$  in  $L(y_3) \setminus (L(w_3) \cup \{c(y_1), c(y_2)\})$  and  $c(x)$  in  $L(x) \setminus \{c(y_1), c(y_2), c(y_3)\}$ . Since  $I_1^2$  is  $q_1$ -choosable, one can extend  $c$  to a colouring of  $I_4^2$ .

□

### Proof of Lemma ??.

- Suppose that a vertex  $v_3$  of  $Y_{2,2,3}$  and  $v_6$  of  $Y_{1,2,3} \cup Y_{2,2,3}$  are adjacent via a 2-edge in  $K_G$ . Then the subgraph of  $G$  induced by  $v_3, v_6$  and the 2-vertices of their incident threads contains  $I_2$  as an induced subgraph. (It is  $I_2$  if  $v_6$  is in  $Y_{1,2,3}$  and has one extra vertex otherwise.) Since  $G$  has girth at least 13, then  $G^2[V(I_2)] = I_2^2$ ,  $(G - V(I_2))^2 = G^2 - V(I_2)$  and  $p_{V[I_2]} = q_2$ , so Lemma ?? contradicts Lemma ??.
- Suppose that a vertex  $v_5$  of  $Y_{1,3,3}$  and  $v_6$  of  $Y_{1,2,3} \cup Y_{1,3,3}$  are adjacent via a 1-edge in  $K_G$ . Then the subgraph of  $G$  induced by  $v_5, v_6$  and the 2-vertices of their incident threads contains  $I_3$ . So Lemma ?? contradicts Lemma ??.
- Suppose that a vertex  $x$  of  $Y_{2,2,2}$  is adjacent to three vertices  $v_1, v_2$  and  $v_3$  of  $Y_{2,2,3}$  in  $K_G$ . Then the subgraph of  $G$  induced by  $x, v_1, v_2, v_3$  and the 2-vertices of their incident threads is  $I_4$ . So Lemma ?? contradicts Lemma ??.

□

### Acknowledgement

The author would like to thank André Raspaud and Riste Škrekovski for stimulating discussions. He is also grateful to the two anonymous referees whose comments helped a lot to improve the presentation of this paper.

## References

- [1] O. V. Borodin. Criterion of chromaticity of a degree prescription (in Russian). In *Abstracts of IV All-Union Conf. on Theoretical Cybernetics (Novosibirsk)*, pages 127–128, 1977.
- [2] D. W. Cranston and S.-J. Kim, List-coloring the square of a subcubic graph. *manuscript*, (2006).
- [3] Z. Dvořák, R. Škrekovski and M. Tancer, List-colouring squares of sparse subcubic graphs, IMFM Preprint PS-985, 2005.
- [4] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, Congress. Numer., XXVI, pages 125–157, Winnipeg, Man., 1980. Utilitas Math.
- [5] H. Fleischner and M. Stiebitz, A solution to a Colouring problem of P. Erdős, *Discrete Math.*, **101** (1992), 39–48.
- [6] A. V. Kostochka and D. R. Woodall, Choosability conjectures and multicircuits. *Discrete Math.*, **240**(1-3) (2001), 123–143.
- [7] M. Montassier and A. Raspaud, A note on 2-facial coloring of plane graphs, *Information Processing Letters*, **98**(6) (2006), 211–266.
- [8] C. Thomassen, The square of a planar cubic graph is 7-colorable, *manuscript* (2006).
- [9] G. Wegner, Graphs with given diameter and a coloring problem, *Technical Report*, University of Dortmund, Germany, 1977.