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Abstract: We present a new high-order method in space and time for solving the wave equation, based on a new interpretation of the "Modified Equation" technique. Indeed, contrary to most of the works, we consider the time discretization before the space discretization. After the time discretization, an additional bilaplacian operator appears, which can not be discretized by classical finite elements. We propose a new Discontinuous Galerkin method for the discretization of this operator and we present a proof of the convergence of the new scheme. Numerical results illustrate the efficiency of the method.

Key-words: high-order schemes, discontinuous Galerkin method, acoustic wave equation, modified equation technique, convergence

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Nouveaux schémas d'ordre élevés basés sur l'équation modifiée pour résoudre l'équation des ondes

Résumé : Dans ce rapport, nous proposons une nouvelle méthode d'ordre élevé en espace et en temps pour résoudre l'équation des ondes basée sur la technique de "l'Equation Modifiée". En effet, contrairement à la démarche classique, nous allons d'abord considérer la discrétisation en temps avant la discrétisation en espace. Après la discrétisation en temps, un opérateur bilaplacien apparaît et celui-ci ne peut être discrétisé par une méthode classique d'éléments finis. Nous proposons une nouvelle méthode de Galerkine Discontinue pour la discrétisation de cet opérateur et nous présentons la preuve de la convergence de ce nouveau schéma. Des résultats numériques que nous présentons mettent en évidence l'efficacité de la méthode.

Mots-clés : schémas d'ordre élevé, méthode de Galerkine discontinue, équation des ondes accoustiques, technique de l'équation modifiée, convergence

1 Introduction

Highly accurate solution of the full wave equation implies very high computational burdens. Indeed, to improve the accuracy of the numerical solution, one must considerably reduce the space step, which is the distance between two points of the mesh representing the computational domain. Obviously this will result in increasing the number of unknowns of the discrete problem. Besides, the time step, whose value fixes the number of required iterations for solving the evolution problem, is linked to the space step through the CFL (Courant-Friedrichs-Lewy) condition. The CFL number defines an upper bound for the time step in such a way that the smaller the space step is, the higher the number of iterations will be. In the three-dimensional case the problem can have more than ten million unknowns, which must be evaluated at each time-iteration. However, high-order numerical methods can be used for computing accurate solutions with larger space and time steps. Recently, Joly and Gilbert (cf. [8]) have optimized the Modified Equation Technique (MET), which was proposed by Shubin and Bell (cf. [13]) for solving the wave equation, and it seems to be very promising given some improvements. In this work, we apply this technique in a new way. Many works in the litterature (see for instance [5, 6, 13, 2]) consider first the space discretization of the system before addressing the question of the time discretization. We intend here to invert the discretization process by applying first the time discretization using the MET and then to consider the space discretization. The time discretization causes high-order operators to appear (such as p-harmonic operators) and we have therefore to consider appropriate methods to discretize them. The Discontinuous Galerkin Methods are well adapted to this discretization, since they allow to consider piecewise discontinuous functions. In particular, using the Interior Penalty Discontinuous Galerkin (IPDG) method (see for instance [3, 1, 9] for the discretization of the Laplacian and [11] for the discretization of the biharmonic operator), one can enforce through the elements high-order transmission conditions, which are adapted to the high order operators to be discretized. The outline of this paper is as follows. In section 1, we describe the classical application of the MET to the semi-discretized wave equation and we recall its properties. In section 2, we obtain high-order schemes by applying this technique directly to the continuous wave equation and we present the numerical method we have chosen for the space discretization of the high order operators. In section 3, we present numerical results to compare the performances of the new technique with the ones of the classical MET.

2 The Modified Equation Technique

In this section, we recall the principle of the modified equation technique which allow us to obtain even order approximation in time and we refer to [6, 13, 8] for more details on this approach.
We consider here the acoustic wave equation in an heterogeneous bounded media $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$. For the sake of simplicity, we impose homogeneous Dirichlet boundary conditions on the Boundary $\Gamma := \partial\Omega$ but this study can be extended

to Neumann boundary conditions without major difficulties.

$$\begin{cases} \text{Find } u : \Omega \times [0,T] \mapsto \mathbb{R} \text{ such that :} \\ \dfrac{1}{\mu(x)} \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(\dfrac{1}{\rho(x)} \nabla u\right) = f & \text{in } \Omega \times]0,T], \\ u(x,0) = u_0, \quad \dfrac{\partial u}{\partial t}(x,0) = u_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where u stands for the displacement, μ is the compressibility modulus, ρ is the density and f is the source term. We assume that μ and ρ satisfy regularity conditions that we describe later in the report. T denotes the final time, u_0 and u_1 are initial data and n is the unit outward normal vector to Ω.
Applying to (1) a classical space discretization method such as finite difference, finite element or discontinuous Galerkin method, we have to solve the linear system,

$$M \frac{\partial^2 U}{\partial t^2} + KU = F, \tag{2}$$

where M is the mass matrix , K is the stiffness matrix, U is the vector of unknown and F the source vector. In the following, we assume that the space discretization method is such that M is easily invertible (sparse or block-diagonal). This is the case if we consider finite difference, spectral element methods or discontinuous Galerkin methods.
Eq. (2) can be easily discretized by a $2p^{\text{th}}$-order scheme, using a $2p^{\text{th}}$-order Taylor expansion:

$$\frac{U(t+\Delta t) - 2U(t) + U(t-\Delta t)}{\Delta t^2} = \sum_{i=1}^{p} c_i \frac{\partial^{2i} U}{\partial t^{2i}}(t) \tag{3}$$

where $c_i = \dfrac{2\Delta t^{2(i-1)}}{(2i)!}$ and Δt is the time step.
Now, using (2) to rewrite all the partial derivative of U with respect to the time in (3), we can obtain an arbitrary $2p^{\text{th}}$-order modified equation scheme (MES-2p) (assuming that U is at least $C^{2(p+1)}$ in time),

$$\frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} = \sum_{i=1}^{p} c_i (-1)^i \left(M^{-1}K\right)^i + \mathcal{F}_{2p} \tag{4}$$

with $\mathcal{F}_{2p}$ a modified source term such that:

$$\begin{cases} \mathcal{F}_2 = & M^{-1} F(t^n) \\ \mathcal{F}_{2p} = & \mathcal{F}_{2(p-1)} + M^{-1} \displaystyle\sum_{i=1}^{p} \left(KM^{-1}\right)^{p-i} \frac{\partial^{2(i-1)} F}{\partial t^{2(i-1)}}(t^n), p \geq 2. \end{cases}$$

In the following, for the sake of simplicity, we will just consider the cases where $1 \leq p \leq 3$ that is to say the so-called Leap-Frog scheme ($p = 1$), the MES-4 and the MES-6 but we can extend the study to higher orders without any difficulty.

Remark 2.1. *MES-4 requires two matricial multiplications by $M^{-1}K$ whereas there are three matricial multiplications by $M^{-1}K$ for the MES-6 so that the computational burden of one iteration is respectively multiplied by two and three compared to the Leap-Frog scheme.*

For the Leap-Frog scheme the time step has to satisfy a CFL (Courant-Friedrichs-Lewy) condition to ensure the stability of the scheme,

$$\Delta t \leq \Delta t_{LF} := \alpha h,$$

where h is the caracteristic space step of the mesh and α is a constant depending only on the space discretization method and on the physical coefficients.
For the MES-4, this CFL condition is multiplied by $\sqrt{3}$ (cf. [8]),

$$\Delta t \leq \Delta t_{MES-4} := \sqrt{3}\alpha h,$$

whereas the CFL condition of MES-6 is multiplied by 1.38,

$$\Delta t \leq \Delta t_{MES-6} := 1.38\alpha h.$$

Since the MES-4 and the MES-6 require respectively two and three matricial multiplications at each iteration, the computational cost is respectively multiplied by $2/\sqrt{3} = 1.15$ and 3/1.38=2.17, compared to the cost of the Leap-Frog scheme. The additional computational burden of the MES-4 is rather small in comparison of the prohibitive MES-6 costs. Recently, Gilbert and Joly in [8] have shown that it is possible to increase the CFL condition of these schemes, but their technique requires additional multiplications by the matrix $M^{-1}K$ at each time-step.

Instead of trying to increase the CFL condition, the object of our work is to decrease the number of matricial multiplications by adapting this technique in an original way that we present in the next section.

3 Schemes with p-harmonic operators

In this section we detail the construction of the fourth-order scheme and we briefly present the sixth-order scheme. A similar technique can be applied to obtain higher order schemes.

3.1 The scheme with biharmonic operator

The idea of the method is to invert the classical discretization process by applying first the time discretization, using the modified equation technique, before addressing the question of the space discretization. In § 3.1.1, we show that the time discretization causes the apparition of a biharmonic operator which can be discretized with a Discontinuous Galerkin Method presented in § 3.1.2. The stability of the new scheme is discussed in § 3.1.3 and its cost is studied in § 3.1.4.

3.1.1 The time discretization

To perform the time discretization of (1), we consider now a fourth-order Taylor expansion of the continuous quantity

$$\frac{u(t+\Delta t) - 2u(t) + u(t-\Delta t)}{\Delta t^2} = \frac{\partial^2 u(t)}{\partial t^2} + \frac{\Delta t^2}{12}\frac{\partial^4 u(t)}{\partial t^4} + O\left(\Delta t^4\right).$$

Since u is solution to the wave equation (1), we can rewrite the fourth order partial derivative of u with respect to the time

$$\frac{\partial^4 u}{\partial t^4} = \mu \mathrm{div}\left(\frac{1}{\rho}\nabla\left[\mu\mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right]\right) + \mu\frac{\partial^2 f}{\partial t^2} + \mu\mathrm{div}\left(\frac{1}{\rho}\nabla(\mu f)\right).$$

Finally, we obtain the semi-discretized scheme

$$\frac{1}{\mu}\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} = \mathrm{div}\left(\frac{1}{\rho}\nabla u^n\right) + \frac{\Delta t^2}{12}\mathrm{div}\left(\frac{1}{\rho}\nabla\left[\mu\mathrm{div}\left(\frac{1}{\rho}\nabla u^n\right)\right]\right) + f_4, \tag{5}$$

with $f_4 = f + \dfrac{\Delta t^2}{12}\left(\dfrac{\partial^2 f}{\partial t^2} + \mathrm{div}\left(\dfrac{1}{\rho}\nabla(\mu f)\right)\right)$.

Remark 3.1. *In the homogeneous case ($c^2 = \mu/\rho$), the scheme reads as:*

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} = c^2\Delta u^n + \frac{\Delta t^2}{12}c^4\Delta^2 u^n + \mu\left(f + \frac{\Delta t^2}{12}\left(\frac{\partial^2 f}{\partial t^2} + c^2\Delta f\right)\right). \tag{6}$$

This scheme will be called scheme with biharmonic operator. In the case of a $2p^{th}$-order scheme, this latter will be called scheme with p-harmonic operator.

Remark 3.2. *Systems (5) and (6) are actually ill-posed. Indeed, for any Δt it is possible to find initial conditions such that $|u^n| \geq Ce^{\alpha n\Delta t}$ with a constant $C > 0$ and $\alpha > 0$. However, an appropriate space discretization will lead to a stable scheme under a CFL condition.*

3.1.2 The Space Discretization

At this point, we have to choose an appropriate method to discretize the fourth order operator. In [2], Anné, Joly and Tran considered a discretization by Finite Difference Methods. Here, we propose to use a Finite Element Method, which is more flexible to handle complex geometry.

If ρ and μ (and the source term and the initial conditions) are regular enough, it is sufficient to consider discretization methods which can take into account H^2 quantities, such as for instance the Hermite's finite element method (HFEM). If ρ or μ are discontinuous, the solution is no longer H^2 and this method is not appropriate. Furthermore, the HFEM is not adapted to the mass lumping technique and is quite complicate to handle numerically. Therefore, we propose to use an Interior Penalty Discontinuous Galerkin (IPDG) method [3, 1, 9] which is suitable to consider strongly heterogeneous media, due to the discontinuities of the basis functions. Moreover, contrary to HFEM this technique can be easily extended to discretize higher-order operators. Of course, the use of this method requires an appropriate choice of the transmission conditions between each element in order to ensure the consistency of the discretization. We will detail these transmission conditions further in this section.

Let us first of all introduce a triangulation $\mathcal{T}_h$ of Ω by segments (in 1D); triangles (in 2D); or tetrahedra (in 3D). We denote by h_K the diameter of the element $K \in \mathcal{T}_h$. The set of the mesh faces is denoted by $\mathcal{F}_h$ which is partitioned into two subsets $\mathcal{F}_h^i$ and $\mathcal{F}_h^b$ corresponding respectively to the interior faces and those located on the boundary. For $F \in \mathcal{F}_h^i$, we denote arbitrarily by K^+ and K^- the two elements sharing F and $\boldsymbol{\nu}$ the unit outward normal vector pointing from K^+ to K^-.
Moreover, denoting v^+ (resp. v^-) the restriction of a function v to the element K^+ (resp. K^-), we define the jump and the average of v on a face $F \in \mathcal{F}_h^i$ by

$$[\![v]\!] = v^+ - v^-, \qquad\qquad \{\!\{v\}\!\} = \frac{v^+ + v^-}{2}. \tag{7}$$

For an exterior face $F \in \mathcal{F}_h^b$, we define $[\![v]\!] = v$ and $\{\!\{v\}\!\} = v$ and $\boldsymbol{\nu}$ denotes the unit outward normal vector from the element K to which F belongs.
Let us now introduce the space of approximation

$$V_h := \left\{ v \in L^2(\Omega) \ : \ v_{|K} \in P_p(K), \forall K \in \mathcal{T}_h, \ p \geq 3 \right\}.$$

Remark 3.3. *Now, we assume that ρ and μ belong to C^4 on each element of the triangulation $\mathcal{T}_h$ and we need to take into account the transmission conditions*

$$\forall F \in \mathcal{F}_h^i, \ \begin{cases} [\![u]\!] &= 0 \quad on\ F, \\ \left[\!\left[\frac{1}{\rho}\nabla u \cdot \boldsymbol{\nu}\right]\!\right] &= 0 \quad on\ F, \end{cases} \tag{8}$$

which are satisfied provided that $u \in H_0^1(\Omega)$ and $div\left(\frac{1}{\rho}\nabla u\right) \in L^2(\Omega)$.

Herein, we do not detail the use of the IPDG method to obtain the bilinear form corresponding to the 2^{nd}-order operator, we just refer to [3, 1, 9] for more details on its properties, but we present in a second part the technique to discretize the 4^{th}-order operator.
First, applying an IPDG discretization to (5), we obtain the following scheme

$$\begin{cases} \text{Find } u_h^{n+1} \in V_h \text{ such that,} \quad \forall v \in V_h, \\ \displaystyle\sum_{K \in \mathcal{T}_h} \int_K \frac{1}{\mu} \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} v = \ -a_{1h}(u_h^n, v) + \frac{\Delta t^2}{12} a_{2h}(u_h^n, v) + \sum_{K \in \mathcal{T}_h} \int_K f_4(., n\Delta t) v. \end{cases}$$

where a_{1h} is a symmetric and coercive bilinear form defined by

$$a_{1h}(u_h^n, v) = B_{\mathcal{T}_{h_1}}(u_h^n, v) - \mathcal{I}_1(u_h^n, v) - \mathcal{I}_1(v, u_h^n) + B_{S1}(u_h^n, v),$$

with

$$B_{\mathcal{T}_{h_1}}(u_h^n, v) = \sum_{K \in \mathcal{T}_h} \int_K \frac{1}{\rho} \nabla u_h^n \cdot \nabla v, \quad \mathcal{I}_1(u_h^n, v) = \sum_{F \in \mathcal{F}_h} \int_F [\![u_h^n]\!] \left\{\!\!\left\{ \frac{1}{\rho} \nabla v \cdot \boldsymbol{\nu} \right\}\!\!\right\},$$

and

$$B_{S_1}(u_h^n, v) = \sum_{F \in \mathcal{F}_h} \int_F \alpha_1 [\![u_h^n]\!] [\![v]\!].$$

The penalization function α_1 is introduced to ensure the stability of the bilinear form a_{1h}. We recall that a bilinear form is stable if it satisfies the stability condition (cf. [3]) $a_{1h}(v,v) \geq C\|v\|^2$, $\forall v \in V_h$ with $C > 0$. α_1 is defined on each interior face F by

$$\alpha_1 = \frac{\gamma_1}{\min(h_{K^+}, h_{K^-}) \min(\rho_{K^+}, \rho_{K^-})},$$

where γ_1 is a positive parameter depending only on the choice of the basis functions of V_h and on each exterior face by

$$\alpha_1 = \frac{\gamma_1}{h_K \rho_K}.$$

If we consider the classical Lagrange basis functions of degree p, it has been shown in [1] that it is sufficient to choose $\gamma_1 > \gamma_1^0 = p(p+1)/2$.

Now, considering the 4^{th} order operator, we perform a double integration by part on each element of the equation (5) and we obtain the bilinear form a_{2h}:

$$\begin{aligned} a_{2h}(u,v) = & \sum_{K \in \mathcal{T}_h} \int_K \mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right) \mathrm{div}\left(\frac{1}{\rho}\nabla v\right) - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\left(\frac{1}{\rho}\nabla v \cdot \nu\right) \\ + & \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{1}{\rho}\left(\nabla\left(\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right) \cdot \nu\right) v \end{aligned}$$

We denote by Q_2 the second term of this expression and by Q_3 the third one. Q_2 reads as

$$Q_2 = -\sum_{F \in \mathcal{F}_h} \int_F \left[\!\!\left[\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\left(\frac{1}{\rho}\nabla v \cdot \boldsymbol{\nu}\right)\right]\!\!\right].$$

Using the equality $[\![u\, v]\!] = \{\!\{u\}\!\}\,[\![v]\!] + \{\!\{v\}\!\}\,[\![u]\!]$ on the interior faces and the fact that $[\![u]\!] = \{\!\{u\}\!\} = u$ on the exterior faces it holds

$$\begin{aligned} Q_2 = & -\sum_{F \in \mathcal{F}_h^i} \int_F \left(\left\{\!\!\left\{\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right\}\!\!\right\}\left[\!\!\left[\frac{1}{\rho}\nabla v \cdot \boldsymbol{\nu}\right]\!\!\right] - \left\{\!\!\left\{\frac{1}{\rho}\nabla v \cdot \boldsymbol{\nu}\right\}\!\!\right\}\left[\!\!\left[\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right]\!\!\right]\right) \\ & -\sum_{F \in \mathcal{F}_h^b} \int_F \left\{\!\!\left\{\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right\}\!\!\right\}\left[\!\!\left[\frac{1}{\rho}\nabla v \cdot \boldsymbol{\nu}\right]\!\!\right]. \end{aligned}$$

To rewrite this expression, we need additional transmission conditions on u. Let us remark that, if u is regular enough in time, the transmission conditions (8) imply

$$\forall F \in \mathcal{F}_h^i, \left\{ \begin{aligned} \left[\!\!\left[\frac{\partial^2 u}{\partial t^2}\right]\!\!\right] &= 0 \quad \text{on } F, \\ \left[\!\!\left[\frac{1}{\rho}\nabla\frac{\partial^2 u}{\partial t^2} \cdot \boldsymbol{\nu}\right]\!\!\right] &= 0 \quad \text{on } F. \end{aligned} \right. \tag{9}$$

Using the wave equation (1) we obtain

$$\forall F \in \mathcal{F}_h^i, \left\{ \begin{aligned} \left[\!\!\left[\mu \mathrm{div}\left(\tfrac{1}{\rho}\nabla u\right)\right]\!\!\right] &= 0 \quad \text{on } F, \\ \left[\!\!\left[\tfrac{1}{\rho}\nabla\left(\mu \mathrm{div}\left(\tfrac{1}{\rho}\nabla u\right)\right) \cdot \boldsymbol{\nu}\right]\!\!\right] &= 0 \quad \text{on } F. \end{aligned} \right. \tag{10}$$

Let us now remark that if we derive two times with respect to the time the Dirichlet boundary condition and we use the wave equation (1), we obtain the additional boundary condition

$$\mu \text{div}\left(\frac{1}{\rho}\nabla u\right) = 0 \text{ on } F \in \mathcal{F}_h^b, \tag{11}$$

so that, thanks to the first condition of (10), that:

$$Q_2 = -\sum_{F\in\mathcal{F}_h^i}\int_F \left\{\!\!\left\{\mu \text{div}\left(\frac{1}{\rho}\nabla u\right)\right\}\!\!\right\}\left[\!\!\left[\frac{1}{\rho}\nabla v\cdot\boldsymbol{\nu}\right]\!\!\right].$$

In the same way, with the second condition of (10), we have:

$$Q_3 = \sum_{F\in\mathcal{F}_h}\int_F \left\{\!\!\left\{\frac{1}{\rho}\nabla\left(\mu \text{div}\left(\frac{1}{\rho}\nabla u\right)\right)\cdot\boldsymbol{\nu}\right\}\!\!\right\}[\![v]\!].$$

Finally, we obtain:

$$a_{2h}(u,v) = \quad B_{\mathcal{T}_{h_2}}(u,v) + \mathcal{I}_2(u,v)$$

where:

$$\begin{cases} B_{\mathcal{T}_{h_2}}(u,v) = & \displaystyle\sum_{K\in\mathcal{T}_h}\int_K \mu \text{div}\left(\frac{1}{\rho}\nabla u\right)\text{div}\left(\frac{1}{\rho}\nabla v\right), \\ \mathcal{I}_2(u,v) = & -\mathcal{I}_{2,1}(u,v) + \mathcal{I}_{2,2}(u,v), \\ \mathcal{I}_{2,1}(u,v) = & \displaystyle\sum_{F\in\mathcal{F}_h^i}\int_F \left\{\!\!\left\{\mu \text{div}\left(\frac{1}{\rho}\nabla u\right)\right\}\!\!\right\}\left[\!\!\left[\frac{1}{\rho}\nabla v\cdot\boldsymbol{\nu}\right]\!\!\right], \\ \mathcal{I}_{2,2}(u,v) = & \displaystyle\sum_{F\in\mathcal{F}_h}\int_F \left\{\!\!\left\{\frac{1}{\rho}\nabla\left(\mu \text{div}\left(\frac{1}{\rho}\nabla u\right)\right)\cdot\boldsymbol{\nu}\right\}\!\!\right\}[\![v]\!]. \end{cases}$$

The bilinear form $a_{2h} : (u_h, v) \in V_h^2 \mapsto B_{\mathcal{T}_{h_2}}(u_h, v) + \mathcal{I}_2(u_h, v)$ is clearly not symmetric, so we add the term $\mathcal{I}_2(v, u_h)$ which does not hamper the consistency of the approximation since $\mathcal{I}_2(v,u) = 0$ by the second transmission condition of (10). To enforce the stability we have to add the two forms $B_{S,2,1}(u_h, v)$ and $B_{S,2,2}(u_h, v)$ defined by

$$\begin{cases} B_{S,2,1}(u_h,v) = & \displaystyle\sum_{F\in\mathcal{F}_h^i}\int_F \alpha_{2,1}\left[\!\!\left[\frac{1}{\rho}\nabla u_h\cdot\boldsymbol{\nu}\right]\!\!\right]\left[\!\!\left[\frac{1}{\rho}\nabla v\cdot\boldsymbol{\nu}\right]\!\!\right], \\ B_{S,2,2}(u_h,v) = & \displaystyle\sum_{F\in\mathcal{F}_h}\int_F \alpha_{2,2}[\![u_h]\!][\![v]\!]. \end{cases}$$

The penalization functions $\alpha_{2,1}$ and $\alpha_{2,2}$ are defined on each interior face F by

$$\alpha_{2,1} = \gamma_{2,1}\frac{\max(\mu_{K^+},\mu_{K^-})}{\min(h_{K^+},h_{K^-})} \text{ and } \alpha_{2,2} = \frac{\gamma_{2,2}}{\min(h_{K^+}^3,h_{K^-}^3)}\max\left(\frac{\mu_{K^+}}{\rho_{K^+}^2},\frac{\mu_{K^-}}{\rho_{K^-}^2}\right).$$

and $\alpha_{2,2}$ is defined on an exterior face F by $\alpha_{2,2} = \frac{\gamma_{2,2}}{h_K^3}\frac{\mu_K}{\rho_K^2}$, where K is the element to which F belongs. The parameters $\gamma_{2,1}$ and $\gamma_{2,2}$ are positive and

depend only on the choice of the basis functions of V_h. It can be proved, by using inverse inequalities that $\gamma_{2,1} > \gamma_{2,1}^0 \approx c_1 p^2$ and $\gamma_{2,2} > \gamma_{2,2}^0 \approx c_2 p^6$, where p denotes the degree of the basis functions. We refer to [11] for the proof of these relations in the case of an homogeneous medium.
Finally, a_{2h} is also a symmetric and stable form defined by

$$a_{2h}(u,v) = B_{\mathcal{T}_{h_2}}(u,v) + \mathcal{I}_2(u,v) + \mathcal{I}_2(v,u) + B_{S,2,1}(u,v) + B_{S,2,2}(u,v). \quad (12)$$

In an homogeneous medium (i.e. ρ and μ constant), a_{2h} is similar to the form proposed by [11] for the solution of the biharmonic equation.

Remark 3.4. *In all the numerical experiments we have carried out, we have set $\gamma_{2,2} = 0$ and we did not observed any instability. This is due to the fact that the form $B_{S,2,2}$ is similar to $B_{S,1}$, so that the term $B_{S,1}$ is sufficient to ensure the stability of both a_{1h} and a_{2h}. Indeed, if we assume that $\gamma_{2,2} = 0$, $\gamma_{2,1} > \gamma_{2,1}^0$ and $\gamma_{1,1} > \gamma_{1,1}^0$ then, $\forall \gamma > \gamma_{2,2}^0$*

$$\begin{aligned} a_{1h}(u,v) - \frac{\Delta t^2}{12} a_{2h}(u,v) = & \quad a_{1h}(u,v) + \frac{\Delta t^2}{12}\frac{\gamma}{h^3}\sum_{F\in\mathcal{F}_h}\int_F [\![u]\!]\,[\![v]\!] \\ & - \quad \frac{\Delta t^2}{12}\left(a_{2h}(u,v) + \frac{\gamma}{h^3}\sum_{F\in\mathcal{F}_h}\int_F [\![u]\!]\,[\![v]\!]\right) \\ = & \quad +\widetilde{a}_{1h}(u,v) - \frac{\Delta t^2}{12}\widetilde{a}_{2h}(u,v) \end{aligned}$$

where $\widetilde{a}_{1h}$ (resp. $\widetilde{a}_{2h}$) is a bilinear form whose coefficient of penalization is (resp. are) $\widetilde{\gamma}_{1,1} = \gamma_{1,1} + \frac{\Delta t^2}{12} h \gamma_{2,2}^0 > \gamma_{1,1}^0$ (resp. $\widetilde{\gamma}_{2,1} = \gamma_{2,1} > \gamma_{2,1}^0$ and $\widetilde{\gamma}_{2,2} = \gamma > \gamma_{2,2}^0$). Consequently, the stability of $\widetilde{a}_{1h}$ and $\widetilde{a}_{2h}$ is ensured even if $\gamma_{2,2} = 0$.

Remark 3.5. *The consistency of the bilinear form a_{1h} is well known [3] and the consistency of a_{2h} can be easily derived using Green's formula then it is clear that the form $a_{1h} - \frac{\Delta t^2}{12} a_{2h}$ is consistent.*

Now, we consider $\{\varphi_i\}_{i=1,\dots,m}$, the classical discontinuous Lagrange basis functions of degree p of V_h, where m denotes the number of degrees of freedom of the problem, and we obtain the linear system

$$\frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} + M^{-1}\left(K_1 - \frac{\Delta t^2}{12}K_2\right)U^n = M^{-1}F^n, \quad (13)$$

where

$$(M)_{i,j} = \sum_{K\in\mathcal{T}_h}\int_K \varphi_i\varphi_j,\ (K_1)_{i,j} = a_{1h}(\varphi_i,\varphi_j),\ (K_2)_{i,j} = a_{2h}(\varphi_i,\varphi_j),$$

$$(F^n)_i = \sum_{K\in\mathcal{T}_h}\int_K f_4(\cdot, n\Delta t)\,\varphi_i.$$

The mass matrix M is block-diagonal by construction and therefore easily invertible.
The initial conditions U^0, $U^1 \in V_h$ are given by

$$\begin{cases} U^0 = P_h(u_0), \quad V^0 = P_h(v_0), \\ U^1 = U^0 + \Delta t V_0 + \frac{\Delta t^2}{2}\widetilde{U}_0 + \frac{\Delta t^3}{6}\widetilde{V}_0 + \frac{\Delta t^4}{24}\widehat{U}_0 \end{cases}$$

where $P_h(u)$ is the L^2 projection of $u \in H^4(\Omega)$ on V_h. $\widetilde{U}_0$, $\widetilde{V}_0$, $\widehat{U}_0 \in V_h$ are such that $\forall v \in V_h$:

$$\begin{aligned}
\left(\widetilde{U}_0, v\right) &= \left(\frac{d^2u}{dt^2}(\cdot,0), v\right) = a_{1h}(u_0, v) + \left(f^0, v\right), \\
\left(\widetilde{V}_0, v\right) &= \left(\frac{d^3u}{dt^3}(\cdot,0), v\right) = a_{1h}(v_0, v) + \left(\partial_t f^0, v\right), \\
\left(\widehat{U}_0, v\right) &= \left(\frac{d^4u}{dt^4}(\cdot,0), v\right) = a_{2h}(u_0, v) + \left(\partial_t^2 f^0, v\right) + \left(\Delta f^0, v\right).
\end{aligned} \tag{14}$$

Remark 3.6. *For the sake of simplicity, we denote $\partial_t^k u(\cdot, t^n)$ by $\partial_t^k u^n$.*

3.1.3 Stability

The following theorem and its corollary guarantee the stability of the scheme under a CFL condition.

Theorem 3.7. *The scheme with biharmonic operator is stable if the matrices $A = M - \frac{\Delta t^2}{4} K_1$ and $K^* = K_1 - \frac{\Delta t^2}{12} K_2$ are positive matrices.*

Proof. For the sake of simplicity, we will consider the formulation (13) without source term that is to say

$$M \frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} + K^* U^n = 0, \tag{15}$$

with $K^* = K_1 - \frac{\Delta t^2}{12} K_2$. Using classical techniques, we prove the conservation of the quantity

$$E^{n+\frac{1}{2}} = \left(\left(M - \frac{\Delta t^2}{4} K^*\right) \frac{U^{n+1} - U^n}{\Delta t}, \frac{U^{n+1} - U^n}{\Delta t}\right) + \left(K^* \frac{U^{n+1} + U^n}{2}, \frac{U^{n+1} + U^n}{2}\right)$$

which defines a discrete energy if $M - \frac{\Delta t^2}{4} K^*$ and K^* are two positive matrices. If these conditions are satisfied, the stability of the scheme will be guaranteed. Since $K^* = K_1 - \frac{\Delta t^2}{12} K_2$, we have to ensure the positivity of the matrices $A + \frac{\Delta t^4}{48} K_2$ and K^*. Moreover, as K_2 is positive, the positivity of A implies the positivity of $A + \frac{\Delta t^4}{48} K_2$. □

Corollary 3.8. *The scheme with biharmonic operator is stable under a CFL condition.*

Proof. Since M, K_1 and K_2 are positive matrices, it is clear that there exist $(\Delta t_1, \Delta t_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that A_1 is positive $\forall \Delta t < \Delta t_1$ and A_2 is positive $\forall \Delta t < \Delta t_2$. Therefore, the scheme with biharmonic operator is stable for all $\Delta t < \min(\Delta t_1, \Delta t_2)$. □

Remark 3.9. *The parameter Δt_1 is also the CFL condition of the Leap-Frog scheme. It is well known that it is a decreasing function of γ_1 (see for instance [10]). However, there is no analytical expression of this parameter and we have to evaluate it numerically. We observed numerically that the parameter Δt_2 is a decreasing function of $\gamma_{2,1}$ and $\gamma_{2,2}$. In all the numerical experiments we performed, Δt_2 was larger than Δt_1 that is to say that the stability of the Leap-Frog scheme seems to be a sufficient condition of the stability of the scheme with biharmonic operator.*

3.1.4 Numerical cost of the scheme

Let us now compare the cost of this scheme (that we denote by Δ^2-scheme) to the cost of the Leap-Frog scheme and of the MES-4. We suppose here that the matrix K in (2) has been obtained by using an IPDG method of order p, so that $(K)_{ij} = a_{1h}(\varphi_i, \varphi_j) = (K_1)_{ij}$.

In practice we compute $K^* := K_1 - \frac{\Delta t^2}{12} K_2$, so that we have only one matricial multiplication by $M^{-1}K^*$ to perform at each iteration. Moreover, it is clear that $a_{1h}(\varphi_i, \varphi_j) = a_{2h}(\varphi_i, \varphi_j) = 0$, as soon as the degrees of freedom i and j are respectively associated to two elements which do not share a common edge. This means that $M^{-1}K_1$ and $M^{-1}K_2$ have the same number of non-zero elements and that the cost of one multiplication by $M^{-1}K^*$ is the same as the cost of one multiplication by $M^{-1}K = M^{-1}K_1$. It is therefore clear that the cost of one iteration of the Δ^2-scheme is the same as the cost of one iteration of the Leap-Frog scheme and is the half of the cost of one iteration of MES-4.

The global cost of these schemes is the cost of one iteration multiplied by the number of iterations, which is imposed by the CFL condition. We did not obtain an explicit CFL condition for the Δ^2-scheme, but the numerical experiments we have carried out (see section 5) show that this condition is a little bit higher than the condition of the Leap-Frog scheme, so that the global cost of the Δ^2-scheme is equivalent to the one of the Leap-Frog scheme. Moreover, since the CFL condition of MES-4 is about 1.73 times the condition of the Leap-Frog scheme, we can deduce that the global cost of the Δ^2-scheme is smaller than the one of MES-4.

3.2 Scheme with triharmonic operator

We do not detail here the construction of the scheme with triharmonic operator and only give its expression. The problem to be solved is

$$\left\{\begin{array}{l} \text{Find } u_h^{n+1} \in V_h \text{ such that,} \quad \forall v \in V_h : \\ \displaystyle\sum_{K\in\mathcal{T}_h} \int_K \frac{1}{\mu} \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} v = \quad -a_{1h}(u_h^n, v) + \frac{\Delta t^2}{12} a_{2h}(u_h^n, v) \\ \qquad\qquad\qquad\qquad\qquad\qquad \displaystyle -\frac{\Delta t^4}{360} a_{3h}(u_h^n, v) + \sum_{K\in\mathcal{T}_h} \int_K f_6(., n\Delta t) v, \end{array}\right.$$

where

$$f_6 = f_4 + \frac{\Delta t^4}{360}\left(\frac{\partial^4 f}{\partial t^4} + \mathrm{div}\left(\frac{1}{\rho}\nabla\left(\mu\frac{\partial^2 f}{\partial t^2}\right)\right) + \mathrm{div}\left(\frac{1}{\rho}\nabla\left(\mu \mathrm{div}\left(\frac{1}{\rho}\nabla(\mu f)\right)\right)\right)\right)$$

and

$$a_{3h}(u_h, v_h) = B_{\mathcal{T}_{h_3}}(u_h, v_h) + \mathcal{I}_3(u_h, v_h) + \mathcal{I}_3(v_h, u_h) + B_{S_{3,1}}(u_h, v_h) + B_{S_{3,2}}(u_h, v_h) + B_{S_{3,3}}(u_h, v_h),$$

with

$$\begin{cases} B_{\mathcal{T}_{h_3}}(u,v) = & \displaystyle\sum_{K\in\mathcal{T}_h}\int_K \frac{1}{\rho}\nabla\left(\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right)\nabla\left(\mu \mathrm{div}\left(\frac{1}{\rho}\nabla v\right)\right), \\ \mathcal{I}_3(u,v) = & \mathcal{I}_{3,1}(u,v) - \mathcal{I}_{3,2}(u,v) + \mathcal{I}_{3,3}(u,v), \\ \mathcal{I}_{3,1}(u,v) = & \displaystyle\sum_{F\in\mathcal{F}_h^i}\int_F \left\{\!\!\left\{\frac{1}{\rho}\nabla\left(\mu \mathrm{div}\left(\frac{1}{\rho}\nabla\left(\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right)\right)\right)\right\}\!\!\right\} [\![v]\!], \\ \mathcal{I}_{3,2}(u,v) = & \displaystyle\sum_{F\in\mathcal{F}_h}\int_F \left\{\!\!\left\{\mu \mathrm{div}\left(\frac{1}{\rho}\nabla\left(\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right)\right)\right\}\!\!\right\} \left[\!\!\left[\frac{1}{\rho}\nabla v\cdot\boldsymbol{\nu}\right]\!\!\right], \\ \mathcal{I}_{3,3}(u,v) = & \displaystyle\sum_{F\in\mathcal{F}_h^i}\int_F \left\{\!\!\left\{\frac{1}{\rho}\nabla\left(\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right)\right\}\!\!\right\} \left[\!\!\left[\mu \mathrm{div}\left(\frac{1}{\rho}\nabla v\right)\right]\!\!\right], \\ B_{S_{3,1}}(u,v) = & \displaystyle\sum_{F\in\mathcal{F}_h^i}\int_F \alpha_{3,1}\left[\!\!\left[\mu \mathrm{div}\left(\frac{1}{\rho}\nabla u\right)\right]\!\!\right]\left[\!\!\left[\mu \mathrm{div}\left(\frac{1}{\rho}\nabla v\right)\right]\!\!\right]. \\ B_{S_{3,2}}(u,v) = & \displaystyle\sum_{F\in\mathcal{F}_h}\int_F \alpha_{3,2}\left[\!\!\left[\frac{1}{\rho}\nabla u\cdot\boldsymbol{\nu}\right]\!\!\right]\left[\!\!\left[\frac{1}{\rho}\nabla v\cdot\boldsymbol{\nu}\right]\!\!\right], \\ B_{S_{3,3}}(u,v) = & \displaystyle\sum_{F\in\mathcal{F}_h^i}\int_F \alpha_{3,3}[\![u]\!][\![v]\!], \end{cases}$$

The penalization functions $\alpha_{3,1}$, $\alpha_{3,2}$ and $\alpha_{3,3}$ are defined on each interior face F by

$$\alpha_{3,1} = \frac{\gamma_{3,1}}{\min(\rho_{K^+},\rho_{K^-})\min(h_{K^+},h_{K^-})}, \quad \alpha_{3,2} = \frac{\gamma_{3,2}}{\min(h_{K^+}^3,h_{K^-}^3)}\max\left(\frac{\mu_{K^+}^2}{\rho_{K^+}},\frac{\mu_{K^-}^2}{\rho_{K^-}}\right),$$

$$\text{and } \alpha_{3,3} = \frac{\gamma_{3,3}}{\min(h_{K^+}^5,h_{K^-}^5)}\max\left(\frac{\mu_{K^+}^2}{\rho_{K^+}^3},\frac{\mu_{K^-}^2}{\rho_{K^-}^3}\right),$$

and $\alpha_{3,2}$ is respectively defined on an exterior face F by

$$\alpha_{3,2} = \frac{\gamma_{3,2}}{h_K^3}\frac{\mu_K^2}{\rho_K},$$

where K is the element to which F belongs. The parameters $\gamma_{3,1}$, $\gamma_{3,2}$ and $\gamma_{3,3}$ are positive and depend only on the choice of the basis functions of V_h. It can be proved, by using inverse inequalities that $\gamma_{3,1} \geq \gamma_{3,1}^0 \approx c_1p^2$, $\gamma_{3,2} \geq \gamma_{3,2}^0 \approx c_2p^6$ and $\gamma_{3,3} \geq \gamma_{3,3}^0 \approx c_3p^{10}$, where p denotes the degree of the basis functions. In practice, as for the Δ^2-scheme, the parameters $\gamma_{3,2}$ and $\gamma_{3,3}$ can be set to zero. Indeed, the forms $B_{S,3,3}$ and $B_{S,3,2}$ are respectively similar to $B_{S,1}$ and $B_{S,2,2}$, so that $B_{S,1}$ and $B_{S,2,2}$ are sufficient to ensure the stability of a_{1h}, a_{2h} and a_{3h}.

Using the same arguments as for the Δ^2-scheme, it can be shown that the cost of one iteration of the Δ^3-scheme is the same as the one of the Leap-Frog scheme and is three times smaller than the one of the MES-6. The numerical experiments we have performed show that the CFL condition of the Δ^3-scheme is slightly higher than the one of Leap-Frog scheme, so that the global cost of both schemes is equivalent. Moreover the global cost of the Δ^3-scheme is much smaller than the one of MES-6.

4 A Convergence Result for the scheme with biharmonic operator

In this part, we propose a result which proves the convergence between the solution of the totally discretized system and the continuous solution at the instant t^n. The idea of the proof is based on the work of Grote and al. in [10] in the case of the second order approximation in time for the wave equation. We assume here that ρ and μ are constant.

Theorem 4.1. *Let u be the solution of the wave equation (1) satisfying the regularity assumptions*

$$u \in C^2\left(\overline{J}; H^{p+1}(\Omega)\right), \quad \partial_t^5 u \in C\left(\overline{J}; L^2(\Omega)\right), \quad \partial_t^6 u \in L^1\left(J; L^2(\Omega)\right) \tag{16}$$

with $J = (0,T)$. Let $(U^n)_{n=0}^N$ the discrete solution defined by (13)-(14). If Δt satisfies

$$\Delta t \leq \beta h \tag{17}$$

with $\beta \in \mathbb{R}^+$ small enough, and if ${}^{12}/_{\Delta t^2}$ is not an eigenvalue of the operator $-\Delta$ then there exists a constant $C > 0$ independent of h and Δt such that:

$$\max_{n=0}^{N} \|u^n - U^n\|_0 \leq C\left(h^{p+1} + \Delta t^4\right).$$

The following subsections present the main steps in the proof of the above theorem. First, we have to look at the properties of the bilinear form a_h.

4.1 Properties of the bilinear form a_h

First, we present the three norms defined on the spaces $V_h + H^4(\Omega)$

$$\|u\|_{\mathrm{DG}_2}^2 = |u|_{1,h}^2 + \sum_K h_K^2 |u|_{2,K}^2 + |\alpha_1^{1/2} u|_*^2$$

$$\|u\|_{\mathrm{DG}_4}^2 = |u|_{2,h}^2 + |\alpha_{2,2}^{1/2} u|_*^2 + |\alpha_{2,1}^{1/2} \nabla u|_*^2 + \|\alpha_{2,2}^{-1/2} \{\!\{\nabla(\Delta u)\cdot \boldsymbol{\nu}\}\!\}\|_{L^2(\Gamma)}^2 + \|\alpha_{2,1}^{-1/2} \{\!\{\Delta u\}\!\}\|_{L^2(\Gamma)}^2$$

$$|\!|\!|u|\!|\!|^2 = \|u\|_{\mathrm{DG}_2}^2 + \frac{\Delta t^2}{12}\|u\|_{\mathrm{DG}_4}^2$$

where $|u|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} (\nabla u, \nabla u)_{L^2(K)}$, $|u|_{2,h}^2 = \sum_{K\in\mathcal{T}_h} (\Delta u, \Delta u)_{L^2(K)}$ and $|u|_*^2 = \int_\Gamma [\![u]\!]^2\, ds$.

Lemma 4.2. *If $\gamma_1 > 0$, $\gamma_{2,1} > 0$ and $\gamma_{2,2} > 0$ then $|\!|\!|\cdot|\!|\!|_{DG_2}$, $|\!|\!|\cdot|\!|\!|_{DG_4}$ and $|\!|\!|\cdot|\!|\!|$ are norms on $H^4(\Omega) + V_h$.*

Now, let us look at some properties of a_h into the spaces V_h and $V_h + H^4(\Omega)$. We recap results established in [9]:

Lemma 4.3. *There exists two constants $C_{coer1}, C_{cont1} > 0$ independent of the mesh size such that:*

$$a_{1h}(u,u) \geq C_{coer1}\|u\|_{DG_2}^2, \quad \forall u \in V_h$$

and

$$|a_{1h}(u,v)| \leq C_{cont1}\|u\|_{DG_2}\|v\|_{DG_2}, \quad \forall u,v \in V_h + H^4(\Omega)$$

Moreover, for quasi-uniform mesh:

$$a_{1h}(u,u) \leq C_S c_{max}^2 h^{-2}\|u\|_0^2, \quad u \in V_h + H^4(\Omega).$$

with C_S *independent of the mesh-size, of* c^2 *and of* σ_{α_1}.

Thanks to [11], we have a continuity result for the bilinear form a_{2h}:

Lemma 4.4. *There exists a constant* $C_{cont2} > 0$ *independent of* h *such that:*

$$a_{2h}(u,v) \leq C_{cont2}\|u\|_{DG_4}\|v\|_{DG_4}, \quad \forall u,v \in V_h + H^4(\Omega).$$

We introduce the bilinear form

$$a_h(u,v) = a_{1h}(u,v) - \frac{\Delta t^2}{12} a_{2h}(u,v).$$

Finally, we need the following result in order to establish the convergence theorem:

Theorem 4.5. *The bilinear form* a_h *satisfies a stability condition on* V_h *i.e.* $\exists C > 0$ *such that :*

$$a_h(u,u) \geq C\|u\|_{DG_2}^2, \quad \forall u \in V_h$$

and a_h *is continuous on* V_h *i.e.* $\exists C > 0$ *such that :*

$$|a_h(u,v)| \leq C\|u\|_{DG_2}\|v\|_{DG_2}, \quad \forall u,v \in V_h.$$

The proof of this theorem is presented in subsection A.1.

4.2 Error bound

First of all, we have to introduce the Galerkin projection:

Definition 4.6. *Let* $u \in H^4(\Omega)$*, we consider the Galerkin projection* $p_h : u \to p_h(u) = u_h(\cdot,t)$ *the projection of* u *on* V_h *such that:*

$$a_h(p_h(u(\cdot,t)),v) = a_h(u(\cdot,t),v), \quad \forall v \in V_h.$$

Remark 4.7. *The existence and the uniqueness of* $p_h(u)$ *are guaranteed by the stability of* a_h *in* V_h.

The Galerkin projection satisfies the following estimates

Lemma 4.8. *If* $u \in H^{p+1}(\Omega)$ *with* $p \geq 1$*, then there exists* $C > 0$ *independent of* h *such that:*

$$\begin{aligned}
\|u - p_h(u)\|_{DG_2} &\leq Ch^p\|u\|_{p+1}, \\
\|u - p_h(u)\|_{DG_4} &\leq Ch^{p-1}\|u\|_{p+1}, \\
\|u - p_h(u)\|_0 &\leq Ch^{p+1}\|u\|_{p+1}.
\end{aligned}$$

We refer to section A.3 for the proof of this lemma.

Now, we want to obtain a bound of the error between the continuous solution at the instant t^n and the totally discretized solution at the same time. We set, at time t^n, $e^n = u^n - U^n$ which we rewrite as:

$$e^n = \eta^n + \phi^n, \quad n = 0, \ldots, N$$

where

$$\begin{cases} \eta^n = u^n - w^n \\ \phi^n = w^n - U^n \\ w^n = p_h\left(u^n\right) \end{cases}$$

Thanks to the triangle inequality, we have:

$$\max_{n=0}^{N} \|e^n\|_0 \leq \max_{n=0}^{N} \|\phi^n\|_0 + \max_{n=0}^{N} \|\eta^n\|_0. \tag{18}$$

We introduce:

$$r^n = \begin{cases} \delta^2 w^n - \partial_t^2 u^n - \frac{\Delta t^4}{12} \partial_t^4 u^n, & n = 1, \ldots, N-1, \\ \dfrac{\phi^1 - \phi^0}{\Delta t^2}, & n = 0 \end{cases}$$

with $\delta^2 w^n = \frac{w^{n+1} - 2w^n + w^{n-1}}{\Delta t^2}$ and we denote $R^n = \Delta t \sum\limits_{m=0}^{n} r^m$.

Proposition 4.9. *Under the CFL condition (17), there exists a constant $C > 0$ such that:*

$$\underset{n=0}{\overset{N}{max}} \|\phi^n\|_0 \leq C\left(\|e^0\|_0 + \|\eta^0\|_0\right) + C^2 \Delta t \sum_{n=0}^{N-1} \|R^n\|_0.$$

where N is the number of time steps and $C > 0$ is a constant independent of h, Δt and T.

We refer to the subsection A.4 for the proof of this proposition.

Combining proposition 4.9 with (18), we have:

$$\max_{n=0}^{N} \|e^n\|_0 \leq C\left(\|e^0\|_0 + \max_{n=0}^{N} \|\eta^n\|_0 + \Delta t \sum_{n=0}^{N-1} \|R^n\|_0\right) \tag{19}$$

Moreover, it is obvious that

$$\Delta t \sum_{n=1}^{N-1} \|R^n\|_0 \leq T \max_{n=1}^{N-1} \|R^n\|_0,$$

so that (19) can be rewritten as

$$\max_{n=0}^{N} \|e^n\|_0 \leq C\left(\|e^0\|_0 + \max_{n=0}^{N} \|\eta^n\|_0 + T \max_{n=1}^{N-1} \|R^n\|_0\right).$$

Recalling that: $\max_{n=0}^{N}\|\eta^n\|_0 = \max_{n=0}^{N}\|u^n - p_h\left(u^n\right)\|_0$ and using lemma 4.8 :

$$\max_{n=0}^{N}\|\eta^n\|_0 \leq Ch^{p+1}\|u\|_{C\left(\overline{J};H^{p+1}(\Omega)\right)}.$$

Moreover, thanks to the properties of the L^2 projection and to lemma 4.8, we have:

$$\|e^0\|_0 = \|u^0 - P_h\left(u^0\right)\|_0 \leq Ch^{p+1}\|u_0\|_{p+1} \leq Ch^{p+1}\|u\|_{C\left(\overline{J};H^{p+1}(\Omega)\right)}.$$

Now we have this result, we have to bound $\|R^n\|_0$. We must distinguish two cases: the case $n=0$ and the case the case $n \geq 1$. In the first case, we have the following lemma:

Lemma 4.10. *There exists a constant $C > 0$ independent of h, Δt and T such that:*

$$\|r^0\|_0 \leq C\left(\Delta t^{-1}h^{p+1}\|\partial_t u\|_{C\left(\overline{J},H^{p+1}(\Omega)\right)} + \Delta t^3\|\partial_t^5 u\|_{C\left(\overline{J},L^2(\Omega)\right)}\right)$$

Now, let us look at $\|r^n\|_0$ for $1 \leq n \leq N-1$.

Lemma 4.11. *For $1 \leq n \leq N-1$, there exists $C > 0$ independent of h, Δt and T such that :*

$$\|r^n\|_0 \leq C\left(\frac{h^{p+1}}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\|\partial_t^2 u\left(\cdot,s\right)\|_{p+1}ds + \Delta t^3\int_{t_{n-1}}^{t_{n+1}}\|\partial_t^6 u\left(\cdot,s\right)\|_0 ds\right)$$

These two lemma are proved in the subsections A.5 and A.6.

By definition of R^n and by the triangle inequality, we have:

$$\|R^n\|_0 \leq \Delta t\|r^0\|_0 + \Delta t\sum_{m=1}^{N-1}\|r^m\|_0.$$

Thanks to lemmas 4.10 and 4.11, $\forall n \in \{1\ldots N-1\}$, there exists a constant $C > 0$ independent of h, Δt and T such that:

$$\begin{aligned}\|R^n\|_0 \leq \quad & C\Delta t^4\left(\|\partial_t^5 u\|_{C\left(\overline{J},L^2(\Omega)\right)} + \|\partial_t^6 u\|_{L^1(J,L^2(\Omega))}\right) \\ & + Ch^{p+1}\left(\|\partial_t u\|_{C\left(\overline{J},H^{p+1}(\Omega)\right)} + \|\partial_t^2 u\|_{C\left(\overline{J},H^{p+1}(\Omega)\right)}\right)\end{aligned} \tag{20}$$

Combining proposition 4.9 and (20), we obtain:

$$\max_{n=0}^{N}\|e^n\|_0 \leq Ch^{p+1}\|u\|_{C^2\left(\overline{J};H^{p+1}(\Omega)\right)} + C\Delta t^4\left(\|\partial_t^5 u\|_{C\left(\overline{J};L^2(\Omega)\right)} + \|\partial_t^6 u\|_{L^1(J;L^2(\Omega))}\right).$$

This proves that:

$$\max_{n=0}^{N}\|u^n - U^n\| \leq C\left(h^{p+1} + \Delta t^4\right)$$

5 Numerical results

In this section, we present numerical results in the one dimensional and in the two dimensional cases in order to compare the performances of the Δ^2- and Δ^3-schemes to the ones of MES-4 and MES-6. In particular, we compare the accuracy and the computational costs of both techniques.

5.1 One-dimensional results

In all this section, we consider the simulation of wave propagation in an homogeneous 1D domain $\Omega = [0, 10]$ with a velocity $c = \left(\mu/\rho\right)^{1/2} = 1 \text{ ms}^{-1}$. We impose also periodic boundary conditions at the both ends of the domain. The source term are set to 0 and the initial data are

$$u_0(x) = \begin{cases} (x - x_0)\, e^{-\left(\dfrac{2\pi(x - x_0)}{r_0}\right)^2} & \text{if } |x - x_0| \leq r_0 \\ 0 & \text{else,} \end{cases}$$

and

$$u_1(x) = \begin{cases} \left(8\left(\frac{(x - x_0)\pi}{r_0}\right)^2 - 1\right) e^{-\left(\dfrac{2\pi(x - x_0)}{r_0}\right)^2} & \text{if } |x - x_0| \leq r_0 \\ 0 & \text{else.} \end{cases}$$

such that the exact solution u^{ex} can be easily computed. In the following, we set $x_0 = 3$ and $r_0 = 4$.
To discretize the wave equation (1), we considered

1. MES-4, based on a space discretization with P^3-Lagrange polynomials and a penalization parameter of $\gamma_1 = 8$. With these basis functions and this parameter the CFL condition of the Leap-Frog scheme is (experimentally) $\Delta t_{LF_4} = 0.1533h$ so that the CFL condition of MES-4 is $\Delta t_{MES-4} = 0.1533\sqrt{3}h = 0.2655h$.

2. MES-6, based on a space discretization with P^5-Lagrange polynomials and a penalization parameter of $\gamma_1 = 20$. With these basis functions and this parameter the CFL condition of the Leap-Frog scheme is (experimentally) $\Delta t_{LF_6} = 0.073h$ so that the CFL condition of MES-6 is $\Delta t_{MES_6} = 1.38 \times 0.073h = 0.101h$.

3. The Δ^2-scheme, with P^3-Lagrange basis functions and with the penalization parameters $\gamma_1 = 8$, $\gamma_{2,1} = 10$ and $\gamma_{2,2} = 0$. The CFL condition of this scheme is (experimentally) $\Delta t_{\Delta^2} = 0.1821h$.

4. The Δ^3-scheme, with P^5-Lagrange basis functions and with the penalization parameters $\gamma_1 = 20$, $\gamma_{2,1} = 20$, $\gamma_{2,2} = 0$, $\gamma_{3,1} = 20$, $\gamma_{3,2} = 0$ and $\gamma_{3,3} = 0$. With these parameters, the CFL condition is (experimentally) $\Delta t_{\Delta^3} = 0.077h$.

Remark 5.1. *Following [1], we chose $\alpha_1 > \alpha_1^0 = p(p+1)/2$. Since we do not have an explicit expression of the other penalization coefficients, we evaluated them numerically in order to obtain a stable solution.*

Let us remark that the CFL condition of the Δ^2-scheme and the Δ^3-scheme are respectively slightly higher than the CFL condition of the classical Leap-Frog scheme Δt_{LF_4} and Δt_{LF_6}. Since the Δ^p-schemes only require one multiplication by iteration, this means that their computational costs is even smaller than the classical Leap-Frog scheme (at least for p=2 and 3).

We compute the relative $L^2\left([0,T],\Omega\right)$ error, given by $\left(\int_0^T \left(\int_\Omega (u-u_h)^2\,dx\right)dt\right)^{1/2}$ where u and u_h represent respectively the exact solution and the approximation, for $T=100$ and for various space steps: $h = 0.25, 0.125, 0.0625, 0.03125$ for the fourth order schemes and $h = 1, 0.5, 0.25, 0.125$ for the sixth order schemes. In Tab. 1 (resp. Tab. 2), we present the $L^2\left([0,T],\Omega\right)$ error of each scheme and in Fig. 1 (resp. in Fig. 2) we represent the relative L^2 error as a function of the mesh size for the MES-4 (resp. MES-6) (cyan line with diamonds) and the Δ^2-scheme (resp. Δ^2-scheme) (green line with squares) in log-log scale. All the schemes converges with the expected order and the Δ^p-schemes perform as well as the corresponding MES-p. Since the CFL condition of the Δ^p-schemes is slightly higher than the CFL condition of the Leap-Frog schemes and only require one matricial multiplication at each iteration, that means that they allow for high-order accuracy with a smaller cost than the Leap-Frog scheme. In comparison, we recall that the computational costs of the MES-4 and of the MES-6 are respectively 1.15 and 2.17 times higher than the cost of the Leap-Frog schemes.

h	MES-4	Δ^2 scheme
0.25	$1.1e-3$	$2.0e-3$
0.125	$7.4e-5$	$4.5e-5$
0.0625	$5.6e-6$	$2.1e-6$
0.03125	$3.7e-7$	$1.2e-7$

Table 1: Relative $L^2\left([0,T],\Omega\right)$ error at time $T=100$s for the fourth order schemes

h	MES-6	Δ^3 scheme
1	$2.4e-1$	$2.8e-1$
0.5	$2.3e-4$	$4.1e-4$
0.25	$2.1e-6$	$2.5e-6$
0.125	$4.6e-8$	$4.2e-8$

Table 2: Relative $L^2\left([0,T],\Omega\right)$ error at time $T=100$s for the sixth order schemes.

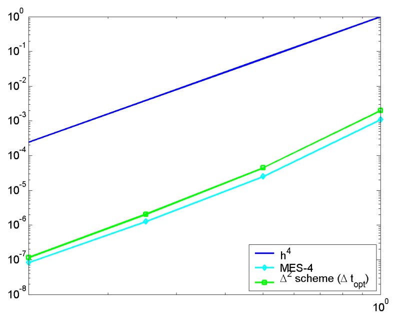

Figure 1: Convergence curves for the 4^{th}-order schemes in 1D

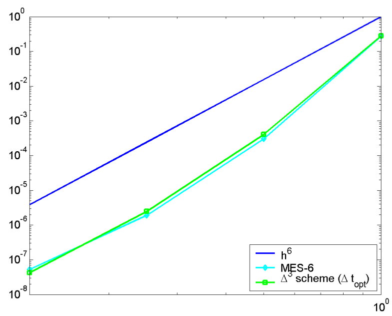

Figure 2: Convergence curves for the 6^{th}-order schemes in 1D

5.2 Two-dimensional results

In this section, we consider the simulation of wave propagation in a 2D two-layered media $\Omega = [-1,1]^2 = \Omega_t \cap \Omega_b$ where $\Omega_t = [-1,1]\times[0,1]$ and $\Omega_b =$

$[-1,1] \times [-1,0]$ are two homogeneous layers respectively charaterized by $\mu = 2$, $\rho = 2$ and $\mu = 8$, $\rho = 4$. We consider zero-initial conditions and a source which is a second derivative of a Gaussian in time and a point source in space:

$$f = \delta_{x_0} 2\lambda \left(\lambda \left(t - t_0\right)^2 - 1 \right) e^{-\lambda(t-t_0)^2},$$

with $x_0 = (0, 0.5)$, $\lambda = \pi^2 f_0^2$, $f_0 = 5$ and $t_0 = 1/f_0$.
To discretize the wave equation (1) we used the two following methods:

1. MES-4, based on a space discretization with P^3-Lagrange polynomials and a penalization parameter of $\gamma_1 = 10$. With these basis functions and this parameter the CFL condition of the Leap-Frog scheme is (experimentally) $\Delta t_{LF_4} = 0.058h$ so that the CFL condition of MES-4 is $\Delta t_{MES-4} = 0.058\sqrt{3}h = 0.100h$.

2. The Δ^2-scheme, with P^3-Lagrange basis functions and with the penalization parameters $\gamma_1 = 10$, $\gamma_{2,1} = 10$ and $\gamma_{2,2} = 0$. The CFL condition of this scheme is (experimentally) $\Delta t_{\Delta^2} = 0.061h$.

As for the one dimensional case, the CFL condition of the Δ^2-scheme is slightly higher than the CFL condition of the Leap-Frog scheme Δt_{LF_4}.
To compare the performances of the different methods, we compute the solution on a receiver at point $x_1 = (0.25, 0.25)$ and we calculate the relative $L^2\left([0,T], x_1\right)$ error for different mean space steps $h = 3e-3, 1.5e-3, 7.5e-4$ and a final time approximatively equal to 2. The analytical solution is computed thanks to a Cagnard-De Hoop solution. The results are presented in Tab. 3 and the convergence curves in log-log scale are plotted in Fig. 3.

h	MES-4	Δ^2-scheme
$3.0e-3$	$2.5e-2$	$2.3e-2$
$1.5e-3$	$1.2e-3$	$1.1e-3$
$7.5e-4$	$6.6e-5$	$6.5e-5$

Table 3: Relative L^2 error in time at the receiver.

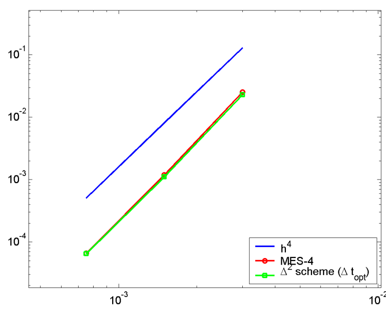

Figure 3: Convergence curves for the 4^{th}-order schemes in 2D

As for the 1D case, the two methods are fourth order approximations and give similar results. Once again, we conclude that the cost of the Δ^2-scheme is smaller than the MES-4.

6 Conclusion

In this paper, we have constructed new high-order schemes both in time and space to solve the acoustic wave equation and we have proved the convergence of the method. The numerical results we have presented illustrate the fact that the computational cost of these schemes is the same as the one of the Leap-Frog scheme and is therefore smaller than the ones of the MES-4 and MES-6. However, the CFL of the new schemes are only computed numerically and we are now trying to determine them analytically. It is worth noting that the CFL condition of the classical Leap-Frog scheme combined with IPDG method is neither known analytically. The first step is then to compute this CFL condition before addressing the issue of the Δ^2 and Δ^3-schemes.

Another very interesting property of these schemes is the fact that they seem to be very well-adapted to p-adaptivity. Indeed, if we combine for instance the Δ^2-scheme with a mesh composed of P^1 and P^3 cells, it is clear that $a_2h(\phi_i, \phi_j)$ vanish if the degrees of freedom i and j belong to a P^1-cells. Therefore, we infer that the scheme will be of second order on the P^1-cells and of fourth order on the P^3-cells. This will be the object of a forthcoming work.

A Proofs of auxiliary theorems and lemma

A.1 Proof of Theorem 4.5

Proof. From lemma 4.3 and 4.4,we have, $\forall u \in V_h$,

$$a_h\left(u,u\right) \geq C_{coer1}\|u\|^2_{DG_2} - \frac{\Delta t^2}{12} C_{cont2}\|u\|^2_{DG_4}.$$

Moreover, we have the next result which is proved in subsection A.2:

Lemma A.1. *There exists $\gamma > 0$ such that for all $u \in V_h$:*

$$\|u\|^2_{DG_4} \leq \gamma h^{-2}\|u\|^2_{DG_2}$$

Consequently,

$$a_h\left(u,u\right) \geq C_{coer1}\|u\|^2_{DG_2} - \frac{\Delta t^2}{12} C_{cont2}\|u\|^2_{DG_4} \geq \left(C_{coer1} - \gamma\frac{\Delta t^2}{12}h^{-2}C_{cont2}\right)\|u\|^2_{DG_2}.$$

Using the CFL condition (17), it implies:

$$\left(C_{coer1} - \gamma\frac{\Delta t^2}{12}h^{-2}C_{cont2}\right)\|u\|^2_{DG_2} \geq \left(C_{coer1} - \gamma C_{cont2}\frac{\beta^2}{12}\right)\|u\|^2_{DG_2}.$$

Then, for $\beta > \sqrt{\frac{12}{C\gamma C_{cont2}}}$, there exists a constant $C > 0$ such that

$$a_h\left(u,u\right) \geq C\|u\|^2_{DG_2}, \quad \forall u \in V_h,$$

which is the wanted result.
Concerning the second part of the theorem, lemma 4.3 and 4.4 imply that, $\forall u, v \in V_h$,

$$|a_h(u,v)| \leq C_{\text{cont1}} \|u\|_{DG_2} \|v\|_{DG_2} + \frac{\Delta t^2}{12} C_{\text{cont2}} \|u\|_{DG_4} \|v\|_{DG_4}.$$

Using lemma A.1, we have

$$|a_h(u,v)| \leq \left(C_{\text{cont1}} + \gamma \frac{\Delta t^2}{12h^2} C_{\text{cont2}} \right) \|u\|_{DG_2} \|v\|_{DG_2}.$$

Then thanks to the CFL condition (17),

$$|a_h(u,v)| \leq C \|u\|_{DG_2} \|v\|_{DG_2}$$

with $C = C_{\text{cont1}} + \gamma \frac{\beta^2}{12} C_{\text{cont2}}$. □

A.2 Proof of lemma A.1

Proof. First of all, let us recap some trace inequalities (cf. [15], [12]) : $\forall v \in P^p(K)$,

$$\|v\|^2_{L^2(\partial K)} \leq \frac{(p+1)(p+2)}{2h} \|v\|^2_{L^2(K)}, \quad \|\nabla v\|^2_{L^2(\partial K)} \leq \frac{\lambda(p)}{h^3} \|v\|^2_{L^2(K)}$$

with $\lambda(p) \sim p^6$ and an inverse inequality (cf. [4]), for $n \in \mathbb{N}$ and $j \geq 1$:

$$|v|_{n+j,K} \leq \alpha h^{-n} |v|_{j,K} \quad \forall v \in V_h$$

We now have to bound each term contained in the norm $\|\cdot\|_{DG_4}$ by terms contained in $\|\cdot\|_{DG_2}$. Let us study the term $|u|_{2,h}$ for all $u \in V_h$.
Let $u \in V_h$, thanks to the inverse inequalities, we have:

$$|u|^2_{2,h} = \sum_{K \in \mathcal{T}_h} |u|^2_{2,K} \leq \alpha^2 \sum_{K \in \mathcal{T}_h} h_K^{-2} |u|^2_{1,K} \leq \alpha^2 h^{-2} |u|^2_{1,h}$$

Now, we focus on the term $\|\alpha_{2,1}^{-1/2} \{\!\{\nabla(\Delta u) \cdot \boldsymbol{\nu}\}\!\} \|^2_{L^2(\Gamma)}$ and we have to consider the restriction of the norm to an unique edge $F \in \mathcal{F}_h$.
Let $F \in \mathcal{F}_h$, using the algebraic inequality $(a+b)^2 \leq 2(a^2+b^2)$ and the trace inequalities, we have:

$$\begin{aligned}
\|\alpha_{2,2}^{-1/2} \{\!\{\nabla(\Delta u) \cdot \boldsymbol{\nu}\}\!\} \|^2_{L^2(F)} &\leq \tfrac{h^3}{p^6 \gamma_{2,2}} \left(\left\| \tfrac{1}{2} \left(\nabla(\Delta u^+) \cdot \boldsymbol{\nu} + \nabla(\Delta u^-) \cdot \boldsymbol{\nu} \right) \right\|^2_{L^2(F)} \right) \\
&\leq \tfrac{h^3}{2p^6 \gamma_{2,2}} \left(\|\nabla(\Delta u^+) \cdot \boldsymbol{\nu}\|^2_{L^2(F)} + \|\nabla(\Delta u^-) \cdot \boldsymbol{\nu}\|^2_{L^2(F)} \right) \\
&\leq \tfrac{\lambda(p)}{2p^6 \gamma_{2,2}} \left(\|\Delta u\|^2_{L^2(K^+)} + \|\Delta u\|^2_{L^2(K^-)} \right).
\end{aligned}$$

Consequently, we obtain

$$\|\alpha_{2,2}^{-1/2} \{\!\{\nabla(\Delta u) \cdot \boldsymbol{\nu}\}\!\} \|^2_{L^2(\Gamma)} \leq \frac{\lambda(p)}{p^6 \gamma_{2,2}} |u|^2_{2,h}.$$

By the inverse inequalities, we have

$$\|\alpha_{2,2}^{-1/2} \{\!\{\nabla(\Delta u) \cdot \boldsymbol{\nu}\}\!\} \|^2_{L^2(\Gamma)} \leq \frac{\alpha^2 \lambda(p) h^{-2}}{p^6 \gamma_{2,2}} |u|^2_{1,h}.$$

With a same reasoning on the other term involving a mean term:

$$\|\alpha_{2,1}^{-1/2}\,\{\!\{\Delta u\}\!\}\,\|_{L^2(\Gamma)}^2 \leq \frac{\alpha^2\,(p+1)\,(p+2)\,h^{-2}}{2p^2\gamma_{2,1}}|u|_{1,h}^2.$$

Now, we consider $\|\alpha_{2,1}^{1/2}\,[\![\nabla u\cdot\boldsymbol{\nu}]\!]\,\|_{L^2(\Gamma)}^2$.
Let $F\in\mathcal{F}_h$,

$$\begin{aligned}
\|\alpha_{2,1}^{1/2}\,[\![\nabla u\cdot\boldsymbol{\nu}]\!]\,\|_{L^2(F)}^2 &= \alpha_{2,1}\|\nabla u^+\cdot\boldsymbol{\nu}-\nabla u^-\cdot\boldsymbol{\nu}\|_{L^2(\Gamma)}^2\\
&= \alpha_{2,1}\int_F\left(\nabla v^+\cdot\boldsymbol{\nu}\right)^2+\left(\nabla v^-\cdot\boldsymbol{\nu}\right)^2-2\left(\nabla v^+\cdot\boldsymbol{\nu}\right)\left(\nabla v^-\cdot\boldsymbol{\nu}\right)ds\\
&\leq \alpha_{2,1}\left(\|\nabla v^+\|_{L^2(F)}^2+\|\nabla v^-\|_{L^2(F)}^2+2\int_F|\left(\nabla v^+\cdot\boldsymbol{\nu}\right)\left(\nabla v^-\cdot\boldsymbol{\nu}\right)|ds\right).
\end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\|\alpha_{2,1}^{1/2}\,[\![\nabla u\cdot\boldsymbol{\nu}]\!]\,\|_{L^2(F)}^2 \leq \alpha_{2,1}\left(\|\nabla v^+\|_{L^2(F)}^2+\|\nabla v^-\|_{L^2(F)}^2+2\|\nabla v^+\|_{L^2(F)}\|\nabla v^-\|_{L^2(F)}\right)$$

then, by the algebraic inequality $2ab\leq a^2+b^2$, we have

$$\|\alpha_{2,1}^{1/2}\,[\![\nabla u\cdot\boldsymbol{\nu}]\!]\,\|_{L^2(F)}^2 \leq \alpha_{2,1}\left(\|\nabla v^+\|_{L^2(F)}^2+\|\nabla v^-\|_{L^2(F)}^2+\|\nabla v^+\|_{L^2(F)}^2+\|\nabla v^-\|_{L^2(F)}^2\right)$$

which leads to

$$\|\alpha_{2,1}^{1/2}\,[\![\nabla u\cdot\boldsymbol{\nu}]\!]\,\|_{L^2(F)}^2 \leq 2\alpha_{2,1}\left(\|\nabla v^+\|_{L^2(F)}^2+\|\nabla v^-\|_{L^2(F)}^2\right).$$

Moreover, thanks to the trace inequalities, we have

$$\|\alpha_{2,1}^{1/2}\,[\![\nabla u\cdot\boldsymbol{\nu}]\!]\,\|_{L^2(\Gamma)}^2 \leq \sum_{F\in\mathcal{F}_h}\frac{2\gamma_{2,1}p^2}{h}\frac{(p+1)\,(p+2)}{2h}\left(\|\nabla v^+\|_{L^2(K^+)}^2+\|\nabla v^-\|_{L^2(K^-)}^2\right).$$

so that,

$$\|\alpha_{2,1}^{1/2}\,[\![\nabla u\cdot\boldsymbol{\nu}]\!]\,\|_{L^2(\Gamma)}^2 \leq 2h^{-2}\gamma_{2,1}p^2\,(p+1)\,(p+2)\,|u|_{1,h}^2.$$

Finally, we consider the term $|\alpha_{2,2}^{1/2}u|_*^2$ which can be easily rewritten as

$$|\alpha_{2,2}^{1/2}u|_*^2 = \frac{\gamma_1 p^2}{h}\left(\frac{\gamma_{2,2}p^4}{\gamma_1 h^2}\right)|u|_*^2 = \frac{\gamma_{2,2}p^4}{\gamma_1 h^2}|\alpha_1^{1/2}u|_*^2$$

so that

$$\|u\|_{DG_4}^2 \leq \gamma h^{-2}\left(|u|_{1,h}^2+|\alpha_1^{1/2}u|_*^2\right) \leq \gamma h^{-2}\|u\|_\sim^2,$$

where $\|\cdot\|_\sim$ denotes the norm:

$$\|v\|_\sim^2 := |v|_{1,h}^2+|\alpha_1^{1/2}v|_*^2.$$

In [3], we have equivalence on V_h between the norm $\|\cdot\|_{DG_2}$ and the norm $\|\cdot\|_\sim$.
Consequently,

$$\|u\|_{DG_4}^2 \leq \gamma h^{-2}\|u\|_{DG_2}^2.$$

□

A.3 Proof of lemma 4.8

A.3.1 Proof of the first estimate

Proof. We are seeking a bound of the quantity $\|u - p_h(u)\|_{DG_2}$. Introducing a suitable projector $\pi_p^h u \in V_h$ of the exact solution u, we have

$$\|u - p_h(u)\|_{DG_2} \leq \|u - \pi_p^h u\|_{DG_2} + \|\pi_p^h u - p_h(u)\|_{DG_2}. \tag{21}$$

To define an appropriate projector, we use the following theorem (cf. [14]).

Theorem A.2. *Suppose that a partition $\mathcal{T}_h$ of Ω consists of d-dimensional simplexes or parallelepipeds. Then, for every $u \in H^t(\Omega)$, t and $r \in \mathbb{N}$, there exists a projector*

$$\pi_r^h : H^t(\Omega) \longrightarrow V_h, \qquad \left(\pi_r^h u\right)_{|K} = \pi_r^h\left(u_{|K}\right)$$

such that, for $0 \leq q \leq t$,

$$\|u - \pi_r^h u\|_{q,K} \leq C\frac{h^{s-q}}{r^{t-q}}\|u\|_{t,K} \quad \forall K \in \mathcal{T}_h,$$

and for $0 \leq q \leq t_K - 1$

$$\|D^\alpha\left(u - \pi_r^h u\right)\|_{0,\partial K} \leq C\frac{h^{s-q-\frac{1}{2}}}{r^{t-q-\frac{1}{2}}}\|u\|_{t,K}, \quad |\alpha| = q, \forall K \in \mathcal{T}_h,$$

where $s = min(r+1,t)$ and C is a constant independant of u, h and r, but dependent on t.

From this theorem, choosing $r = p$, we deduce that [3]

$$\begin{cases} \|u - \pi_p^h u\|_{DG_2} \leq Ch^p\|u\|_{p+1} \\ \|u - \pi_p^h u\|_{DG_4} \leq Ch^{p-1}\|u\|_{p+1} \end{cases} \tag{22}$$

Now, we have to find a similar bound on the quantity $\|\pi_p^h u - p_h(u)\|_{DG_2}$. Thanks to the properties of the Galerkin projection and of the bilinear form a_h, we have

$$\begin{aligned} a_h\left(\pi_p^h u - p_h(u), \pi_p^h u - p_h(u)\right) &= a_h\left(\pi_p^h u, \pi_p^h u - p_h(u)\right) - a_h\left(p_h(u), \pi_p^h u - p_h(u)\right) \\ &= a_h\left(\pi_p^h u - u, \pi_p^h u - p_h(u)\right). \end{aligned} \tag{23}$$

Since, $\pi_p^h u - p_h(u) \in V_h$, using theorem 4.5, there exists a constant C such that

$$a_h\left(\pi_p^h u - p_h(u), \pi_p^h u - p_h(u)\right) \geq C\|\pi_p^h u - p_h(u)\|^2_{DG_2}. \tag{24}$$

Furthermore, we want an upper bound over the term $a_h\left(\pi_p^h u - u, \pi_p^h u - p_h(u)\right)$ but since $\pi_p^h u - u \in V_h + H^{p+1}(\Omega)$ we cannot use the continuity of a_h in V_h. Let u, $v \in V_h + H^{p+1}(\Omega)$, it is obvious that

$$|a_h(u,v)| \leq |a_{1h}(u,v)| + \frac{\Delta t^2}{12}|a_{2h}(u,v)|. \tag{25}$$

According to lemma 4.3, there exists a positive constant $C > 0$ such that

$$|a_{1h}(u,v)| \leq C\|u\|_{DG_2}\|v\|_{DG_2}, \qquad \forall u,v \in V_h + H^{p+1}(\Omega).$$

From the definition of the norm $\|\!|\cdot|\!\|$, it is clear that

$$|a_{1h}(u,v)| \leq C|\!|\!|u|\!|\!|\,|\!|\!|v|\!|\!|, \qquad \forall u,v \in V_h + H^{p+1}(\Omega). \tag{26}$$

Similarly, it has been proved in [11] that $\forall u,v \in H^{p+1}(\Omega)$, $\exists C > 0$:

$$|a_{2h}(u,v)| \leq C\|u\|_{DG_4}\|v\|_{DG_4}$$

which implies

$$\frac{\Delta t^2}{12}|a_{2h}(u,v)| \leq C|\!|\!|u|\!|\!|\,|\!|\!|v|\!|\!|, \qquad \forall u,v \in V_h + H^{p+1}(\Omega). \tag{27}$$

Using (26) and (27), we obtain

$$a_h\left(\pi_p^h u - u, \pi_p^h u - p_h(u)\right) \leq C|\!|\!|\pi_p^h u - u|\!|\!|\,|\!|\!|\pi_p^h u - p_h(u)|\!|\!|.$$

Then, combining this inequality with (24) and (23), we have

$$\|\pi_p^h u - p_h(u)\|_{DG_2}^2 \leq C|\!|\!|\pi_p^h u - u|\!|\!|\,|\!|\!|\pi_p^h u - p_h(u)|\!|\!|. \tag{28}$$

However, $\pi_p^h u - p_h(u) \in V_h$ and we know (cf. lemma A.1) that $\forall v \in V_h$, $\exists C > 0$ such that $\|v\|_{DG_4}^2 \leq Ch^{-2}\|v\|_{DG_2}^2$.
Consequently, $\forall v \in V_h$, the CFL condition (17) implies

$$|\!|\!|v|\!|\!|^2 \leq \|v\|_{DG_2}^2 + C\frac{\Delta t^2}{12h^2}\|v\|_{DG_2}^2 \leq \widetilde{C}\|v\|_{DG_2}^2$$

with $\widetilde{C} = 1 + C\dfrac{\beta^2}{12}$, so that

$$\|\pi_p^h u - p_h(u)\|_{DG_2}^2 \leq C|\!|\!|\pi_p^h u - u|\!|\!|\,\|\pi_p^h u - p_h(u)\|_{DG_2}, \tag{29}$$

which obviously reads as

$$\|\pi_p^h u - p_h(u)\|_{DG_2} \leq C|\!|\!|\pi_p^h u - u|\!|\!|. \tag{30}$$

Moreover, using (22), we have, $\forall u \in H^{p+1}(\Omega)$

$$|\!|\!|\pi_p^h u - u|\!|\!|^2 \leq C\left(h^{2p} + \frac{\Delta t^2}{12}h^{2(p-1)}\right)\|u\|_{p+1,\Omega}^2$$

so that, under the CFL condition (17),

$$|\!|\!|\pi_p^h u - u|\!|\!|^2 \leq C_1 h^{2p}\|u\|_{p+1,\Omega}^2 + \beta^2\frac{C_2}{12}h^{2p}\|u\|_{p+1,\Omega}^2$$

which leads to

$$|\!|\!|\pi_p^h u - u|\!|\!| \leq Ch^p\|u\|_{p+1,\Omega}. \tag{31}$$

Finally, considering (22) and (31), we obtain

$$\|u - p_h(u)\|_{DG_2} \leq Ch^p\|u\|_{p+1,\Omega} \tag{32}$$

which proves the first estimate. □

A.3.2 Proof of the second estimate

Proof. As in the proof of the first estimate, we have

$$\|u - p_h(u)\|_{DG_4} \leq \|u - \pi_p^h u\|_{DG_4} + \|\pi_p^h u - p_h(u)\|_{DG_4}. \tag{33}$$

and we have already seen that

$$\|u - \pi_p^h u\|_{DG_4} \leq C h^{p-1} \|u\|_{p+1} \tag{34}$$

To bound the second term of (33), we can remark that $\pi_p^h u - p_h(u) \in V_h$ then applying the lemma A.1, it follows

$$\|\pi_p^h u - p_h(u)\|_{DG_4} \leq \gamma h^{-1} \|\pi_p^h u - p_h(u)\|_{DG_2}.$$

Then, using the first estimate of lemma 4.8, we have

$$\|\pi_p^h u - p_h(u)\|_{DG_4} \leq \gamma h^{p-1} \|u\|_{p+1}. \tag{35}$$

Combining (33), (34) and (35) we have,

$$\|u - p_h(u)\|_{DG_4} \leq C h^{p-1} \|u\|_{p+1}.$$

□

A.3.3 Proof of the third estimate

Proof. Let us first consider the following auxiliary problem to prove the second estimate.

$$\left\{ \begin{array}{rll} \Delta z + \dfrac{\Delta t^2}{12} \Delta^2 z & = u - p_h(u) & \text{in } \Omega, \\ z & = 0 & \text{on } \partial\Omega, \\ \Delta z & = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{36}$$

First of all, we have to prove that there exists an unique solution $z \in H^4(\Omega)$ to this problem and to obtain a bound of the quantities $\|z\|_2$ and $\|z\|_4$.
The problem (36) can be rewritten as two coupled problems:

$$\left\{ \begin{array}{rll} \Delta z_1 & = u - p_h(u) & \text{in } \Omega, \\ z_1 & = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{37}$$

and

$$\left\{ \begin{array}{rll} z + \dfrac{\Delta t^2}{12} \Delta z & = z_1 & \text{in } \Omega, \\ z & = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{38}$$

The domain Ω is assumed to be convex so the elliptic regularity gives that the solution z_1 of the problem (37) belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ and there exists a constant C_1 such that

$$\|z_1\|_2 \leq C_1 \|u - p_h(u)\|_0. \tag{39}$$

Now, we are interested in the problem (38) which is an Helmholtz problem.
We know that, thanks to the Fredholm alternative, the problem

$$\left\{ \begin{array}{rll} Lu & = \lambda u + f & \text{in } \Omega, \\ u & = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{40}$$

has an unique solution $u \in H_0^1(\Omega)$ if $f \in L^2(\Omega)$ and $\lambda \notin Sp(L)$.
Here, $L = -\Delta$, $\lambda = {}^{12}/_{\Delta t^2}$ and $f = ({}^{12}/_{\Delta t^2})\, z_1 \in H^2(\Omega)$ then, if $\lambda \notin Sp(-\Delta)$ there exists an unique $z \in H_0^1(\Omega)$ solution of the problem (40).
Now, applying the Boundary H^m-Regularity theorem (cf. chap. 6 in [7]) it follows that there exists an unique $z \in H^4(\Omega)$ solution of the problem (36) such that:

$$\|z\|_2 \leq C_2 \|z_1\|_0 \quad \text{and} \quad \|z\|_4 \leq C_2 \|z_1\|_2. \tag{41}$$

Consequently, combining (39) and (41), it holds that

$$\|z\|_2 \leq C \|u - p_h(u)\|_0 \tag{42}$$

with a constant $C > 0$.

Now, multiplying the first equation of (36) by $u - p_h(u)$ and integrating it over Ω it follows, using the consistency of a_h,

$$\|u - p_h(u)\|_0^2 = a_h(z, u - p_h(u)). \tag{43}$$

Moreover, $\forall z_h \in V_h$ we have, from the definition of $p_h(u)$

$$a_h(z, u - p_h(u)) = a_h(z - z_h, u - p_h(u))$$

and, thanks to the continuities of a_{1h} and a_{2h} in $H^{p+1}(\Omega) + V_h$, there exists a constant C such that

$$a_h(z, u - p_h(u)) \leq C \left(\|z - z_h\|_{DG_2} \|u - p_h(u)\|_{DG_2} + \frac{\Delta t^2}{12} \|z - z_h\|_{DG_4} \|u - p_h(u)\|_{DG_4} \right). \tag{44}$$

Choosing $z_h = \pi_1^h z$, we have, thanks to theorem A.2,

$$\|z - z_h\|_{DG_2} \leq Ch\|z\|_2$$

and, thanks to the estimate (42), we obtain

$$\|z - z_h\|_{DG_2} \leq Ch\|u - p_h(u)\|_0. \tag{45}$$

Now, in the same way

$$\|z - z_h\|_{DG_4} \leq C\|z\|_4.$$

Using the second inequality of (41) and (39), it holds

$$\|z - z_h\|_{DG_4} \leq C\|u - p_h(u)\|_0. \tag{46}$$

Combining the results (43), (44), (45) and (46), we can conclude that there exists a constant C such that

$$\|u - p_h(u)\|_0 \leq C \left(h\|u - p_h(u)\|_{DG_2} + \frac{\Delta t^2}{12} \|u - p_h(u)\|_{DG_4} \right).$$

Thanks to the CFL condition (17),

$$\|u - p_h(u)\|_0 \leq Ch \left(\|u - p_h(u)\|_{DG_2} + \frac{\beta^2 h}{12} \|u - p_h(u)\|_{DG_4} \right).$$

Finally, we conclude by using the two first estimate of the lemma 4.8. □

A.4 Proof of proposition 4.9

Proof. We recall that we have

$$\max_{n=0}^{N}\|e^n\|_0 \leq \max_{n=0}^{N}\|\phi^n\|_0 + \max_{n=0}^{N}\|\eta^n\|_0.$$

We already have a bound on the second term (cf. lemma 4.8) so we just have to find a bound on the first one.
Thanks to the wave equation, we have:

$$\partial_t^2 u^n + \frac{\Delta t^2}{12}\Delta^2 u^n - \frac{\Delta t^2}{12}\Delta^2 u^n - \Delta u^n = f^n.$$

Since a_h is consistent, we have, $\forall v \in V_h$:

$$\left(\partial_t^2 u^n + \frac{\Delta t^2}{12}\Delta^2 u^n, v\right) + a_h\left(u^n, v\right) = \left(f^n, v\right). \tag{47}$$

Moreover, (13) reads as

$$\left(\delta U^n, v\right) + a_h\left(U^n, v\right) = \left(f^n + \frac{\Delta t^2}{12}\frac{\partial^2 f^n}{\partial t^2} + \Delta f^n, v\right)$$

The difference between this equation and (47) gives:

$$\left(\partial_t^2 u^n + \frac{\Delta t^2}{12}\Delta^2 u^n - \delta^2 U^n, v\right) + a_h\left(u^n - U^n, v\right) = -\frac{\Delta t^2}{12}\left(\frac{\partial^2 f^n}{\partial t^2} + \Delta f^n, v\right)$$

that is to say, $\forall v \in V_h$, $n = 1, \ldots, N-1$:

$$\left(\partial_t^2 u^n - \delta^2 w^n + \delta^2 w^n - \delta^2 U^n + \frac{\Delta t^2}{12}\Delta^2 u^n, v\right) + a_h\left(u^n - w^n + w^n - U^n, v\right) = -\frac{\Delta t^2}{12}\left(\frac{\partial^2 f^n}{\partial t^2} + \Delta f^n, v\right).$$

Since $a_h\left(u^n - w^n, v\right) = 0$ $\forall v \in V_h$, we have:

$$\left(\delta^2 \phi^n, v\right) + a_h\left(\phi^n, v\right) = \left(\delta^2 w^n - \partial_t^2 u^n - \frac{\Delta t^2}{12}\left(\Delta^2 u^n + \frac{\partial^2 f^n}{\partial t^2} + \Delta f^n\right), v\right).$$

Applying a second derivative with respect to the time to the wave equation, we have the following equality:

$$\Delta^2 u^n + \frac{\partial^2 f^n}{\partial t^2} + \Delta f^n = \partial_t^4 u^n$$

i.e. $\forall v \in V_h$, $n = 1, \ldots, N-1$:

$$\left(\delta^2 \phi^n, v\right) + a_h\left(\phi^n, v\right) = \left(r^n, v\right).$$

If we sum this equality from $n = 1$ to $n = m$, $1 \leq m \leq N-1$, we have:

$$\left(\frac{\phi^{m+1} - \phi^m}{\Delta t^2}, v\right) + \left(\frac{\phi^0 - \phi^1}{\Delta t^2}, v\right) + \sum_{n=1}^{m} a_h\left(\phi^n, v\right) = \sum_{n=1}^{m}\left(r^n, v\right)$$

or:

$$\left(\frac{\phi^{m+1} - \phi^m}{\Delta t}, v\right) + \Delta t\sum_{n=1}^{m} a_h\left(\phi^n, v\right) = \Delta t\sum_{n=1}^{m}\left(r^n, v\right) + \Delta t\left(r^0, v\right).$$

We denote $\xi^n = \Delta t \sum\limits_{n=1}^{m} \phi^n$ with $\xi^0 = 0$.
Then, $\forall v \in V_h$, $0 \leq m \leq N-1$:

$$\left(\frac{\phi^{m+1} - \phi^m}{\Delta t}, v\right) + a_h\left(\xi^m, v\right) = \left(R^m, v\right)$$

where $(R^m, v) = \Delta t \sum\limits_{n=0}^{m} (r^n, v)$.
We choose $v = \phi^{m+1} + \phi^m \in V_h$ so that, $\forall m \in \{0 \dots n-1\}$

$$\|\phi^{m+1}\|_0^2 - \|\phi^m\|_0^2 + \Delta t a_h\left(\xi^m, \phi^{m+1} + \phi^m\right) = \Delta t\left(R^m, \phi^{m+1} + \phi^m\right).$$

and we sum from $m = 0$ to $n-1$, $\forall n \in \{1 \dots N\}$ to obtain

$$\|\phi^n\|_0^2 - \|\phi^0\|_0^2 + \Delta t \sum_{m=0}^{n-1} a_h\left(\xi^m, \phi^{m+1} + \phi^m\right) = \Delta t \sum_{m=0}^{n-1}\left(R^m, \phi^{m+1} + \phi^m\right).$$

From the definition of ξ^m, we have, $\forall m \in \{1 \dots N-1\}$: $\xi^{m+1} - \xi^{m-1} = \Delta t\left(\phi^{m+1} + \phi^m\right)$, so that

$$\begin{aligned}
\Delta t \sum_{m=0}^{n-1} a_h\left(\xi^m, \phi^{m+1} + \phi^m\right) &= \sum_{m=1}^{n-1} a_h\left(\xi^m, \xi^{m+1} - \xi^{m-1}\right) \\
&= \sum_{m=1}^{n-1} a_h\left(\xi^m, \xi^{m+1}\right) - \sum_{m=0}^{n-2} a_h\left(\xi^{m+1}, \xi^m\right) \\
&= a_h\left(\xi^{n-1}, \xi^n\right) - a_h\left(\xi^1, \xi^0\right).
\end{aligned}$$

Moreover, thanks to the bilinearity of a_h, we have

$$a_h\left(\xi^{n-1}, \xi^n\right) = a_h\left(\frac{\xi^{n-1} + \xi^n}{2}, \frac{\xi^{n-1} + \xi^n}{2}\right) - a_h\left(\frac{\xi^n - \xi^{n-1}}{2}, \frac{\xi^n - \xi^{n-1}}{2}\right).$$

Recalling that $\xi^0 = 0$, using theorem 4.5 and that $\xi^{n+1} - \xi^n = \Delta t \phi^{n+1}$, $\forall n \in \{0, \dots, N-1\}$ it follows:

$$a_h\left(\xi^{n-1}, \xi^n\right) \geq -\frac{\Delta t^2}{4} a_h\left(\phi^n, \phi^n\right).$$

Consequently, we obtain

$$\|\phi^n\|_0^2 - \frac{\Delta t^2}{4} a_h\left(\phi^n, \phi^n\right) \leq \|\phi^0\|_0^2 + \Delta t \sum_{m=0}^{n-1}\left(R^m, \phi^{m+1} + \phi^m\right). \tag{48}$$

Thanks to the continuity of a_{1h} and a_{2h} and also lemma A.1

$$\begin{aligned}
a_h\left(\phi^n, \phi^n\right) &\leq |a_{1h}\left(\phi^n, \phi^n\right)| + \tfrac{\Delta t^2}{12}|a_{2h}\left(\phi^n, \phi^n\right)| \\
&\leq C_{cont1}\|\phi^n\|_{DG_2}^2 + \tfrac{\Delta t^2}{12} C_{cont2}\|\phi^n\|_{DG_4}^2 \\
&\leq \left(C_{cont1} + \tfrac{\Delta t^2}{12} C_{cont2}\gamma h^{-2}\right)\|\phi^n\|_{DG_2}^2
\end{aligned}$$

Then, thanks to the lemma 4.3 :

$$C_{coer1}\|\phi^n\|_{DG_2}^2 \leq C_S c_{\max}^2 h^{-2}\|\phi^n\|_0^2,$$

so that

$$a_h(u,u) \le \left(C_{cont1} + \frac{\Delta t^2}{12} C_{cont2}\gamma h^{-2}\right) \frac{C_S}{C_{coer1}} c_{\max}^2 h^{-2}\|u\|_0^2$$

Under the CFL condition (17), it follows from (48) that

$$C^*\|\phi^n\| \le \|\phi^0\|_0^2 + \Delta t \sum_{m=0}^{n-1} \left(R^m, \phi^{m+1} + \phi^m\right), \quad 0 \le n \le N,$$

with:

$$C^* := 1 - \frac{\beta^2}{4}\left(C_{cont1} + \frac{\beta^2}{12} C_{cont2}\gamma\right) \frac{C_S}{C_{coer1}} c_{\max}^2. \tag{49}$$

C^* is obviously positive for β small enough.
We apply the Cauchy-Schwarz Inequality:

$$C^*\|\phi^n\|_0^2 \le \|\phi^0\|_0^2 + \Delta t \sum_{m=0}^{n-1} \|R^m\|_0 \|\phi^{m+1} + \phi^m\|_0.$$

Then, $\|\phi^m + \phi^{m+1}\|_0 \le 2\max_{m=0}^{N}\|\phi^m\|_0$ implies:

$$C^*\|\phi^n\|_0^2 \le \|\phi^0\|_0^2 + 2\left(\Delta t \sum_{m=0}^{N-1} \|R^m\|_0\right)\left(\max_{m=0}^{N}\|\phi^m\|_0\right)$$

Using the algebraic inequality $2ab \le \varepsilon^{-1}a^2 + \varepsilon b^2$, $\forall \epsilon > 0$, we obtain

$$C^*\|\phi^n\|_0^2 \le \|\phi^0\|_0^2 + \frac{C^*}{2}\max_{m=0}^{N}\|\phi^m\|_0^2 + \frac{2}{C^*}\left(\Delta t \sum_{m=0}^{N-1} \|R^m\|_0\right)^2.$$

Since the right hand side is not dependent of n,

$$C^*\max_{n=0}^{N}\|\phi^n\|_0^2 \le \|\phi^0\|_0^2 + \frac{C^*}{2}\max_{m=0}^{N}\|\phi^m\|_0^2 + \frac{2}{C^*}\left(\Delta t \sum_{m=0}^{N-1} \|R^m\|_0\right)^2$$

which is equivalent to:

$$C^*\max_{n=0}^{N}\|\phi^n\|_0^2 \le 2\|\phi^0\|_0^2 + \frac{4}{C^*}\left(\Delta t \sum_{m=0}^{N-1} \|R^m\|_0\right)^2.$$

But, for $a, b \ge 0$, $(a+b)^2 \ge a^2 + b^2$ then:

$$C^*\max_{n=0}^{N}\|\phi^n\|_0^2 \le \left(\sqrt{2}\|\phi^0\|_0 + \frac{2\Delta t}{\sqrt{C^*}} \sum_{m=0}^{N-1} \|R^m\|_0\right)^2.$$

Consequently,

$$\max_{n=0}^{N}\|\phi^n\|_0 \le \sqrt{\frac{2}{C^*}}\|\phi^0\|_0 + \frac{2\Delta t}{C^*} \sum_{m=0}^{N-1} \|R^m\|_0.$$

Moreover, $\|\phi^0\|_0 \le \|e^0\|_0 + \|\eta^0\|_0$ because $e^0 = \phi^0 + \eta^0$, so:

$$\max_{n=0}^{N}\|\phi^n\|_0 \le \sqrt{\frac{2}{C^*}}\left(\|e^0\|_0 + \|\eta^0\|_0\right) + \frac{2\Delta t}{C^*} \sum_{m=0}^{N-1} \|R^m\|_0.$$

□

A.5 Proof of lemma 4.10

Proof. We recall that $\Delta t^2 r^0 = \phi^1 - \phi^0$. In the following, we want to bound $\|\phi^1 - \phi^0\|_0$. Let $v \in V_h$, since $\left(u^0 - U^0, v\right) = 0$, by definition of U^0, we have:

$$\begin{aligned} \left(\phi^1 - \phi^0, v\right) &= \left(w^1 - U^1, v\right) - \left(w^0 - U^0, v\right) \\ &= \left(w^1 - u^1, v\right) + \left(u^1 - U^1, v\right) - \left(w^0 - u^0, v\right) - \left(u^0 - U^0, v\right) \\ &= \left(\left(p_h - I\right)\left(u^1 - u^0\right), v\right) + \left(u^1 - U^1, v\right) \end{aligned} \tag{50}$$

Using lemma 4.8, a Taylor's expansion with integral remainder and the fact that $\partial_t^i \left(p_h\left(u\right)\right) = p_h\left(\partial_t^i u\right)$ with $i = 0, \ldots, 2$, we obtain, with $t_1 = \Delta t$:

$$\begin{aligned} \left|\left(\left(p_h - I\right)\left(u^1 - u^0\right), v\right)\right| &\leq \int_0^{t_1} \left|\left(\partial_t \left(p_h - I\right) u, v\right)\right| dt \\ &\leq \int_0^{t_1} \left|\left(\left(p_h - I\right) u_t, v\right)\right| dt \\ &\leq C \Delta t h^{p+1} \|u_t\|_{C\left(\overline{J}; H^{p+1}(\Omega)\right)} \|v\|_0 \end{aligned}$$

Now, we have to study the second term. Thanks to Taylor's expansion with integral remainder, we have:

$$u^1 = u_0 + \Delta t v_0 + \frac{\Delta t^2}{2} \partial_t^2 u^0 + \frac{\Delta t^3}{6} \partial_t^3 u^0 + \frac{\Delta t^4}{24} \partial_t^4 u^0 + \frac{1}{24} \int_0^{t_1} \left(\Delta t - s\right)^4 \partial_t^5 u\left(\cdot, s\right) ds.$$

Then,

$$\begin{aligned} \left(u^1 - U^1, v\right) = \quad & \left(u_0 - P_h u_0, v\right) + \Delta t \left(v_0 - P_h v_0, v\right) + \tfrac{\Delta t^2}{2} \left(\partial_t^2 u^0 - \widetilde{U}_0, v\right) + \tfrac{\Delta t^3}{6} \left(\partial_t^3 u^0 - \widetilde{V}_0, v\right) \\ & + \tfrac{\Delta t^4}{24} \left(\partial_t^4 u^0 - \widehat{U_0}, v\right) + \tfrac{1}{24} \int_0^{t_1} \left(\Delta t - s\right)^4 \partial_t^5 u\left(\cdot, s\right) ds \end{aligned}$$

By the definition of the projection P_h, $\forall v \in V_h$:

$$\left(u_0 - P_h\left(u_0\right), v\right) = 0 \quad \left(v_0 - P_h\left(v_0\right), v\right) = 0$$

The consistency of the method and the definitions of $\widetilde{U}_0$, $\widetilde{V}_0$ and $\widehat{U_0}$ in 14 give immediately

$$\begin{cases} \left(\partial_t^2 u^0 - \widetilde{U}_0, v\right) = 0, \\ \left(\partial_t^3 u^0 - \widetilde{V}_0, v\right) = 0, \\ \left(\partial_t^4 u^0 - \widehat{U_0}, v\right) = 0. \end{cases}$$

So,

$$\begin{aligned} \left|\left(u^1 - U^1, v\right)\right| &\leq \frac{1}{24} \int_0^{t_1} \left(\Delta t - s\right)^4 \left|\left(\partial_t^5 u\left(\cdot, s\right), v\right)\right| ds \\ &\leq C \Delta t^5 \|\partial_t^5 u\|_{C\left(\overline{J}; L^2(\Omega)\right)} \|v\|_0. \end{aligned}$$

Since $\phi^1 - \phi^0 \in V_h$ we can choose $v = \phi^1 - \phi^0$ as test function in (50):

$$\|\phi^1 - \phi^0\|_0 \leq C \left(\Delta t h^{p+1} \|u_t\|_{C\left(\overline{J};H^{p+1}(\Omega)\right)} + \Delta t^5 \|\partial_t^5 u\|_{C\left(\overline{J};L^2(\Omega)\right)} \right).$$

Consequently, we obtain:

$$\|r^0\|_0 \leq C \left(\Delta t^{-1} h^{p+1} \|u_t\|_{C\left(\overline{J};H^{p+1}(\Omega)\right)} + \Delta t^3 \|\partial_t^5 u\|_{C\left(\overline{J};L^2(\Omega)\right)} \right).$$

□

A.6 Proof of lemma 4.11

Proof. By the triangle inequality, we have:

$$\begin{aligned} \|r^n\|_0 &= \|\delta^2 w^n - \partial_t^2 u^n - \tfrac{\Delta t^2}{12} \partial_t^4 u^n\|_0 \\ &\leq \|\delta^2 (p_h - I) u^n\|_0 + \|\delta^2 u^n - \partial_t^2 u^n - \tfrac{\Delta t^2}{12} \partial_t^4 u^n\|_0 \end{aligned}$$

In order to bound the first term, we use:

$$v(\cdot, t_{n+1}) - 2v(\cdot, t_n) + v(\cdot, t_{n-1}) = \Delta t \int_{t_{n-1}}^{t_{n+1}} \left(1 - \frac{|s-t|}{\Delta t}\right) \partial_t^2 v(\cdot, s)\, ds.$$

Thanks to lemma 4.8 and that $\partial_t^i (u_h) = \pi_h \left(\partial_t^i u\right)$ with $i = 0, \ldots, 2$, we obtain:

$$\begin{aligned} \|\delta^2 (p_h - I) u^n\|_0 &\leq \tfrac{1}{\Delta t} \int_{t_{n-1}}^{t_{n+1}} \left(1 - \frac{|s-t_n|}{\Delta t}\right) \|\partial_t^2 (p_h - I) u\|_0 ds \\ &\leq C \tfrac{h^{p+1}}{\Delta t} \int_{t_{n-1}}^{t_{n+1}} \|\partial_t^2 u(\cdot, s)\|_{p+1} ds. \end{aligned}$$

For the second term, using a Taylor's expansion with integral remainder, we have:

$$\delta^2 u^n - \partial_t^2 u^n - \frac{\Delta t^2}{12} \partial_t^4 u^n = \frac{\Delta t^{-2}}{120} \left(\int_{t_n}^{t_{n+1}} (t_{n+1} - s)^5 \partial_t^6 u(\cdot, s)\, ds + \int_{t_n}^{t_{n-1}} (t_{n-1} - s)^5 \partial_t^6 u(\cdot, s)\, ds \right).$$

And, $\forall s \in [t_n, t_{n+1}]$, $t_{n+1} - s \leq \Delta t$ then :

$$\int_{t_n}^{t_{n+1}} (t_{n+1} - s)^5 \partial_t^6 u(\cdot, s)\, ds \leq \Delta t^5 \int_{t_n}^{t_{n+1}} \partial_t^6 u(\cdot, s)\, ds.$$

In the same way, $\forall s \in [t_{n-1}, t_n]$, $s - t_{n-1} \leq \Delta t$ and $(t_{n-1} - s)^5 = -(s - t_{n-1})^5$ so:

$$\int_{t_n}^{t_{n-1}} (t_{n-1} - s)^5 \partial_t^6 u(\cdot, s)\, ds \leq \Delta t^5 \int_{t_{n-1}}^{t_n} \partial_t^6 u(\cdot, s)\, ds.$$

Consequently,

$$\|\delta^2 u^n - \partial_t^2 u^n - \frac{\Delta t^2}{12} \partial_t^4 u^n\|_0 \leq \frac{\Delta t^3}{120} \int_{t_{n-1}}^{t_{n+1}} \|\partial_t^6 u(\cdot, s)\|_0 ds.$$

Then it holds:

$$\|r^n\|_0 \leq C \left(\frac{h^{p+1}}{\Delta t} \int_{t_{n-1}}^{t_{n+1}} \|\partial_t^2 u(\cdot, s)\|_{p+1} ds + \Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \|\partial_t^6 u(\cdot, s)\|_0 ds \right).$$

□

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