Optimizing Power Allocation in Interference Channels Using D.C. Programming

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***Abstract*—Power allocation is a promising approach for optimizing the performance of mobile radio systems in interference channels. In the present paper, the non-convex objective function of the power allocation problem aiming at maximizing the sum rate with a total power constraint is reformulated as a difference of two concave functions. A global optimum power allocation is found by applying a branch and bound based algorithm to the new formulation. The algorithm basically splits the feasible region consecutively into subregions where for every subregion the objective function is upper and lower bounded. For a certain partition of the feasible region, a power allocation corresponding to the highest lower bound which is upper bounded by the highest upper bound with some insignificant difference is found as the global optimum. A convex maximization formulation of the optimization problem with a piecewise linearly outer approximated feasible region is essentially applied for finding an upper bound which only requires solving a linear program problem. The simulation results show a significant improvement in the sum rate of the proposed algorithm over the conventional suboptimal techniques.**

I. Introduction

Interference is a dominant source of performance degradations in mobile radio systems. If the nodes communicate autonomously through a shared channel, i.e., there is no cooperation among nodes, the interference channel can be considered as a system level model [1]. In the interference channel, it is assumed that a number of transmitter-receiver pairs communicate with each other through a shared medium where any transmission from a transmitter would not just result in a useful signal at its corresponding receiver but also in an interference signal at all other receivers.

Power allocation plays a key role in improving the system performance in interference channels. A smart power allocation resulting in minimizing the received interference and thus maximizing the system sum rate is required. If the interference is treated as noise, the power allocation optimization problem aiming at maximizing the sum rate with a total power constraint is a non-convex problem. Consequently, sub-optimal solutions of the problem and heuristic algorithms are proposed [2].

In the present study, we solve this non-convex problem by rewriting the non-convex objective function of the sum rate as a difference of two concave functions. The new formulation of the problem can be solved using a class of the global optimization methods called difference of two convex functions programming or shortly D.C. programming [3]. Because of the nice properties of the D.C. functions, D.C. programming attained a great attention during the last few decades and several efficient algorithms for a variety of applications are proposed [3], [4].

Our approach uses a branch and bound algorithm. The algorithm initially estimates an upper bound and a lower bound of the D.C. function of the sum rate over the whole feasible region. Then it splits the feasible region recursively into subregions where it estimates the bounds for every subregion. Considering the highest lower bound, many subregions with lower upper bounds are not of interest. Also if this bound reaches the highest upper bound for a certain partition of the feasible region with some insignificant difference, the corresponding power allocation is taken as the optimum.

The fruition of the algorithm is based on a good estimation of the bounds. By introducing a new variable, i.e., adding a new dimension to the problem, the D.C. problem is reformulated as a convex maximization problem over a piecewise linearly outer approximated convex set. An upper bound of a subregion is found by computing the greatest distance with respect to the new added dimension between a point in the convex set corresponding to a local lower bound and a point in the envelope of the convex set. Furthermore, a corner point in a subregion which leads to the highest sum rate is considered as a lower bound.

In [5], the authors find a global optimum power allocation for multiuser DSL networks. Assuming a power constraint per user, the optimization problem for maximizing the sum rate is decoupled across all tones to make it solvable on a per tone basis. So they form a D.C. program out of a dual form of a weighted sum rate for a single tone with weighted powers of all users. These weights are updated iteratively to meet both power and rate constraints. The D.C. problem is solved using a prismatic branch and bound algorithm which is proposed in [3].

The remainder of the paper is organized as follows. The next section describes the system model. Section III introduces the problem statement and the D.C. formulation. In Section IV the proposed algorithm is described. Section V presents some simulation results. The conclusions are drawn in Section VI.

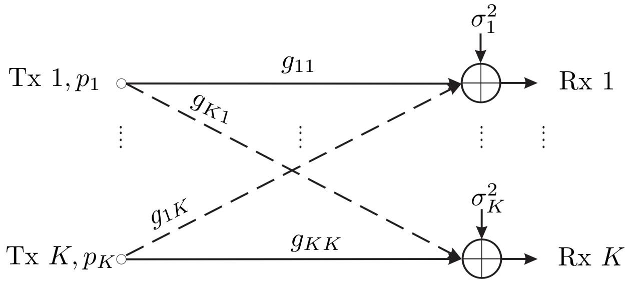


Fig. 1: An interference channel scenario containing K transmitter-receiver pairs

II. System Model

A general memoryless interference channel with perfect channel knowledge at the transmitters is assumed. A scenario consisting of K transmitter-receiver pairs which are coupled by interfering links is considered. Let $\underline{h}_{kl}$ be the channel coefficient of the link between the transmitter l and the receiver k, where $k, l = 1, ..., K$. Then the corresponding channel gain is denoted as $g_{kl} = |\underline{h}_{kl}|^2$. Fig. 1 shows a K transmitter-receiver pairs scenario where σ_k^2 is the noise power at receiver k. Then

$$\mathbf{G} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1K} \\ g_{21} & g_{22} & \cdots & g_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ g_{K1} & g_{K2} & \cdots & g_{KK} \end{pmatrix}$$

is the gain matrix where the matrix's non-diagonal elements correspond to the interfering links. Let $\mathbf{p} = (p_1, \cdots, p_K)^T$ be the vector of the transmitted powers. The main concern here is how to allocate powers to the individual transmitters $\mathbf{p}$ under the total power constraint

$$\sum_{k=1}^{K} p_k = p_{\text{tot}}, \; p_k \geq 0 \tag{1}$$

which directly reflects the resulting total interference. The sum rate is used as a measure of performance which is calculated based on a simplified assumption that the interference is treated as white Gaussian noise:

$$C = \sum_{k=1}^{K} \text{ld}\left(1 + \frac{g_{kk} p_k}{\sigma_k^2 + \sum\limits_{l \neq k} g_{kl} p_l}\right). \tag{2}$$

III. Problem Statement

Based on the assumptions stated on Section II, the optimum power allocation vector $\mathbf{p}_{\text{opt}}$ is found by solving the following maximization problem:

$$\mathbf{p}_{\text{opt}} = \underset{\mathbf{p}}{\text{argmax}} \left\{ \sum_{k=1}^{K} \text{ld}\left(1 + \frac{g_{kk} p_k}{\sigma_k^2 + \sum\limits_{l \neq k} g_{kl} p_l}\right) \right\} \tag{3}$$

subject to

$$\sum_{k=1}^{K} p_k = p_{\text{tot}}, \; p_k \geq 0. \tag{4}$$

This problem is non-convex and a closed form solution is not known. Using a quotient property of the logarithms which states that $\log(A/B) = \log(A) - \log(B)$ for $A, B > 0$, the sum rate function can be written as a difference of two concave functions. Therefore, the maximization problem of (3)-(4) can be reformulated as a D.C. problem:

$$\mathbf{p}_{\text{opt}} = \underset{\mathbf{p}}{\text{argmax}} \{\text{f}(\mathbf{p}) - \text{g}(\mathbf{p})\} \tag{5}$$

subject to

$$\sum_{k=1}^{K} p_k = p_{\text{tot}}, \; p_k \geq 0 \tag{6}$$

where

$$\text{f}(\mathbf{p}) = \sum_{k=1}^{K} \text{ld}\left(\sigma_k^2 + \sum_{l=1}^{K} g_{kl} p_l\right), \tag{7}$$

$$\text{g}(\mathbf{p}) = \sum_{k=1}^{K} \text{ld}\left(\sigma_k^2 + \sum_{l \neq k} g_{kl} p_l\right). \tag{8}$$

Both functions of (7) and (8) are concave functions.

IV. Branch and Bound Algorithm

In this section, an algorithm based on the branch and bound technique is described. The algorithm basically finds a global optimum power allocation by searching over a full binary tree.

A. Constructing the Tree

Definition 1. *A K-simplex T has the $K+1$ vertices $v^{(0)}, \cdots, v^{(K)}$ with $v^{(0)}$ being the vertex at the origin and the representation $x = \sum_{k=0}^{K} \lambda^{(k)} v^{(k)}$ is unique for all $x \in T$ where $\sum_k \lambda^{(k)} = 1$ and $0 \leq \lambda^{(k)} \leq 1$ for $k = 0, \cdots, K$.*

The root of the tree corresponds to an initial $(K-1)$-dimensional face $F^{(1)}$, i.e., $F^{(1)}$ is obtained by setting $\lambda^{(0)} = 0$ in a K-simplex, which covers exactly the whole feasible region of the power allocations and the vertices of $F^{(1)}$ are the corners of the feasible region, i.e.,

$$\begin{aligned} F^{(1)} &= [(p_{\text{tot}}, 0, \cdots, 0), (0, p_{\text{tot}}, 0, \cdots, 0), \\ &\quad \cdots, (0, \cdots, 0, p_{\text{tot}})] \\ &= \left[\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \cdots, \mathbf{p}^{(K)}\right]. \end{aligned} \tag{9}$$

Each node i has two children. They are constructed by splitting $F^{(i)}$ over its longest edge. This process is called branching. Now, consider a node i with $F^{(i)} = \left[\mathbf{p}^{(1)}, \cdots, \mathbf{p}^{(K)}\right]$ which has K vertices and the longest edge is in between the vertices $\mathbf{p}^{(x)}$ and $\mathbf{p}^{(y)}$. Then the new vertex which is shared by $F^{(2i)}$ and $F^{(2i+1)}$ is $\mathbf{p}^{(k)} = \frac{1}{2}\left(\mathbf{p}^{(x)} + \mathbf{p}^{(y)}\right)$. The two new $(K-1)$-dimensional faces are

$$F^{(2i)} = \left[\mathbf{p}^{(1)}, \cdots, \mathbf{p}^{(x)}, \mathbf{p}^{(k)}, \cdots, \mathbf{p}^{(K)}\right], \tag{10}$$

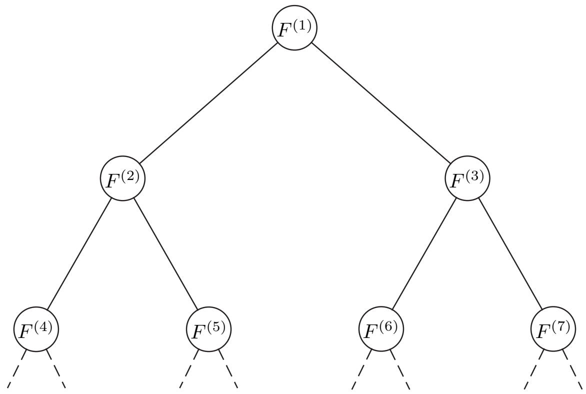
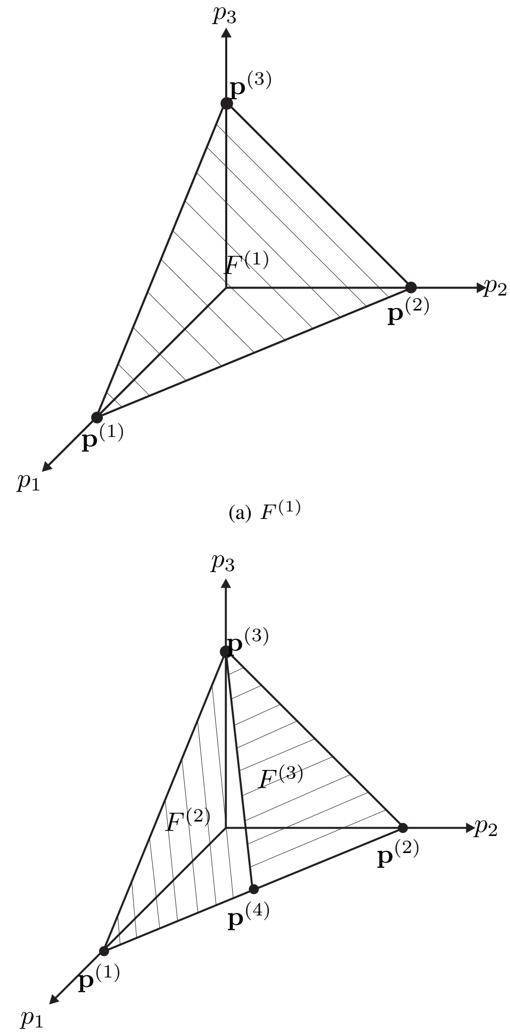


Fig. 2: The structure of the tree where every node corresponds to a face covering the whole or part of the feasible region.



(a) $F^{(1)}$

(b) $F^{(2)}$ and $F^{(3)}$

Fig. 3: An example of splitting a 2-dimensional face.

and

$$F^{(2i+1)} = \left[\mathbf{p}^{(1)}, \cdots, \mathbf{p}^{(y)}, \mathbf{p}^{(k)}, \cdots, \mathbf{p}^{(K)}\right]. \tag{11}$$

The structure of the resulting tree is shown in Fig. 2. For example, in a three-user scenario, the initial 2-dimensional face is

$$\begin{aligned} F^{(1)} &= \left[(p_{\mathrm{tot}}, 0, 0), (0, p_{\mathrm{tot}}, 0), (0, 0, p_{\mathrm{tot}})\right] \\ &= \left[\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \mathbf{p}^{(3)}\right]. \end{aligned} \tag{12}$$

It is split into two faces

$$F^{(2)} = \left[\mathbf{p}^{(1)}, \mathbf{p}^{(3)}, \mathbf{p}^{(4)}\right], \tag{13}$$

and

$$F^{(3)} = \left[\mathbf{p}^{(2)}, \mathbf{p}^{(3)}, \mathbf{p}^{(4)}\right] \tag{14}$$

where

$$\mathbf{p}^{(4)} = \frac{1}{2}\left(\mathbf{p}^{(1)} + \mathbf{p}^{(2)}\right). \tag{15}$$

This process is illustrated in Fig. 3.

B. Searching Over the Tree

Apply a breath-first search through the tree. For every node i in the tree, compute an upper bound $u^{(i)}$ of $\max\{\mathrm{f}(\mathbf{p}) - \mathrm{g}(\mathbf{p})\}$ and a lower bound $l^{(i)}$ of $\max\{\mathrm{f}(\mathbf{p}) - \mathrm{g}(\mathbf{p})\}$ with the corresponding power allocation $\mathbf{p}_{\mathrm{LB},i}$. The computation of an upper bound and a lower bound is described in details in Sections IV-C and IV-D, respectively. This process is usually called bounding.

Then update the global lower bound as $\beta^{(i)} = \max\left\{\beta^{(i-1)}, l^{(i)}\right\}$ where $\beta^{(1)} = l^{(1)}$ and its corresponding power allocation is $\mathbf{p}_{\mathrm{opt},i}$. Accordingly, there are three cases:

1) If $u^{(i)} - \beta^{(i)} < 0$ the power allocation which maximizes $\mathrm{f}(\mathbf{p}) - \mathrm{g}(\mathbf{p})$ is not in $F^{(i)}$. So, there is no need to inspect the children.
2) If $u^{(i)} - \beta^{(i)} > \epsilon$ for some tolerance value $\epsilon > 0$, the face $F^{(i)}$ may contain a power allocation corresponding to the global maximum but a lower bound which is close to the upper bound $u^{(i)}$ with some arbitrary small difference ϵ still has to be found. So the two children nodes $2i$ and $2i+1$ need to be inspected.
3) If $0 \le u^{(i)} - \beta^{(i)} \le \epsilon$ the lower bound $\beta^{(i)}$ is the local maximum of $F^{(i)}$ with some acceptable precision if $\mathbf{p}_{\mathrm{opt},i} \in F^{(i)}$ and it can be a global maximum if no other nodes with higher lower bounds are found.

Finally, the algorithm terminates when no more nodes have to be inspected with a global optimum power allocation $\mathbf{p}_{\mathrm{opt},I}$ corresponding to $\beta^{(I)}$ where I represents the index of the last checked node.

C. Computing an Upper Bound

Let $M^{(i)} = M^{(i-1)} \bigcup \{\mathbf{p}_{\max,i-1}\}$ with $M^{(1)} = V\left(F^{(1)}\right)$ representing a set of feasible power allocations where $\mathbf{p}_{\max,i-1}$ is a feasible power allocation which is found when calculating the upper bound $u^{(i-1)}$ and $V\left(F^{(i)}\right)$ is the set of vertices of the face $F^{(i)}$.

$$\mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p}) = \min_{\mathbf{p}_j \in M^{(i)}} \left\{(\mathbf{p} - \mathbf{p}_j)^T \partial \mathrm{f}(\mathbf{p}_j) + \mathrm{f}(\mathbf{p}_j)\right\} \tag{16}$$

as an approximation of the concave function $\mathrm{f}(\mathbf{p})$ with the following properties:

- $\mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p})$ is piecewise linear and concave.
- $\mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p}) \geq \mathrm{f}(\mathbf{p})$ is an outer approximation.
- $\mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p}) = \mathrm{f}(\mathbf{p}), \forall \mathbf{p} \in M^{(i)}$.
- From $M^{(i-1)} \subset M^{(i)}$ follows $\mathrm{f}_{\mathrm{M}}^{(i-1)}(\mathbf{p}) > \mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p})$.
- From $\mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p}) \geq \mathrm{f}(\mathbf{p})$ follows:

$$\max\{\mathrm{f}(\mathbf{p}) - \mathrm{g}(\mathbf{p})\} \leq \max\left\{\mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p}) - \mathrm{g}(\mathbf{p})\right\}. \tag{17}$$

As a result, $\max\left\{\mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p}) - \mathrm{g}(\mathbf{p})\right\}$ is an upper bound of $\max\{\mathrm{f}(\mathbf{p}) - \mathrm{g}(\mathbf{p})\}$. The maximization problem $\max\left\{\mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p}) - \mathrm{g}(\mathbf{p})\right\}$ with the total power constraint can be reformulated as a convex maximization problem:

$$(t_{\mathrm{u}}, \mathbf{p}_{\mathrm{u}}) = \underset{t,\mathbf{p}}{\operatorname{argmax}}\{t - \mathrm{g}(\mathbf{p})\} \tag{18}$$

subject to

$$\sum_{k=1}^{K} p_k = p_{\mathrm{tot}},\ p_k \geq 0, \tag{19}$$

and

$$t - \mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p}) \leq 0. \tag{20}$$

The two constraints in (19) and (20) represent a new feasible region which is described as a polytope

$$E^{(i)} = \left\{(\mathbf{p}, t) : \mathbf{a}^{(i)}t - \mathbf{A}^{(i)}\mathbf{p} \leq \mathbf{b}^{(i)}, \sum_{k=1}^{K} \mathbf{p}_k = p_{\mathrm{tot}}\right\}, \tag{21}$$

i.e., the region is closed and has linear boundaries, where

$$\mathbf{A}^{(i)} = \begin{pmatrix} I_{K\times K} \\ \partial\mathrm{f}(\mathbf{p}_1)^T \\ \vdots \\ \partial\mathrm{f}\left(\mathbf{p}_{|M^{(i)}|}\right)^T \end{pmatrix}, \tag{22}$$

$$\mathbf{a}^{(i)} = \begin{pmatrix} 0_{K\times 1} \\ 1_{|M^{(i)}|\times 1} \end{pmatrix}, \tag{23}$$

and

$$\mathbf{b}^{(i)} = \begin{pmatrix} 0_{K\times 1} \\ \mathrm{f}(\mathbf{p}_1) - (\mathbf{p}_1)^T \partial\mathrm{f}(\mathbf{p}_1) \\ \vdots \\ \mathrm{f}\left(\mathbf{p}_{|M^{(i)}|}\right) - \left(\mathbf{p}_{|M^{(i)}|}\right)^T \partial\mathrm{f}\left(\mathbf{p}_{|M^{(i)}|}\right) \end{pmatrix}. \tag{24}$$

Fig. 4 shows the feasible region $E^{(i)}$ of a two user scenario. The optimum point $(\mathbf{p}, t)$ in the polytope $E^{(i)}$ which maximizes $t - \mathrm{g}(\mathbf{p})$ has to be at the convex envelope of $E^{(i)}$ where $t = \mathrm{f}_{\mathrm{M}}^{(i)}(\mathbf{p})$. Now consider the points $(\mathbf{p}, t)$ which lead to a constant value $t - \mathrm{g}(\mathbf{p}) = c^{(i)}$ where

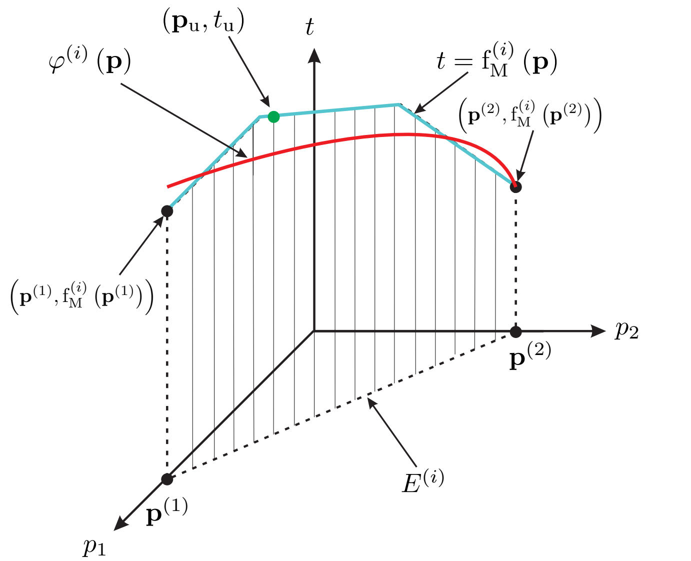


Fig. 4: The feasible region $E^{(i)}$ of the convex maximization problem of a two-user scenario.

$c^{(i)} = \max\left\{\mathrm{f}_{\mathrm{M}}^{(i)}\left(\mathbf{p}^{(k)}\right) - \mathrm{g}\left(\mathbf{p}^{(k)}\right)\right\}$. These points form a K-dimensional concave function $\varphi^{(i)}(\mathbf{p}) = t = c^{(i)} + \mathrm{g}(\mathbf{p})$ at the range of t with $\mathbf{p} = \left\{\mathbf{p} : \sum_{k=1}^{K} p_k = p_{\mathrm{tot}}\right\}$ as shown in Fig. 4. Then the maximum of $t - \mathrm{g}(\mathbf{p})$ corresponds to the point $(\mathbf{p}_{\mathrm{u}}, t_{\mathrm{u}})$ on the envelope of $E^{(i)}$ which has the greatest distance to $\varphi^{(i)}(\mathbf{p})$ with respect to t-axis.

A K-dimensional subspace $\Upsilon^{(i)}$ is uniquely defined by the points $\left(\mathbf{p}^{(k)}, t^{(k)}\right)$ and the origin as

$$t - \mathbf{h}^T\mathbf{p} = 0, \tag{25}$$

where $\mathbf{h}$ is a K-dimensional vector and the points $\left(\mathbf{p}^{(k)}, t^{(k)}\right)$ are found as

$$t^{(k)} = c^{(i)} + \mathrm{g}\left(\mathbf{p}^{(k)}\right), \tag{26}$$

for $k = 1, \cdots, K$. For the sake of simplicity, consider the subspace $\Upsilon^{(i)}$ instead of the concave function $\varphi^{(i)}(\mathbf{p})$. Since $\varphi^{(i)}(\mathbf{p})$ is concave and $\Upsilon^{(i)}$ intersects $\varphi^{(i)}(\mathbf{p})$ at the points $\left(\mathbf{p}^{(k)}, t^{(k)}\right)$, every point $(\mathbf{p}, t)$ in $\Upsilon^{(i)}$ with $\mathbf{p} = \left\{\mathbf{p} : \sum_{k=1}^{K} p_k = p_{\mathrm{tot}}\right\}$ leads to $t - \mathrm{g}(\mathbf{p}) \leq c^{(i)}$. Fig. 5 shows the intersection between the feasible region $E^{(i)}$ and the subspace $\Upsilon^{(i)}$ which is spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$. It shows that this intersection is upper bounded by $\varphi^{(i)}(\mathbf{p})$. On the other hand, the greatest distance $\Delta^{(i)}$ in the direction of t-axis between a point $(\mathbf{p}, t)$ on the envelope of $E^{(i)}$ and a point $(\mathbf{p}, t)$ in $\Upsilon^{(i)}$ with $\mathbf{p} = \left\{\mathbf{p} : \sum_{k=1}^{K} p_k = p_{\mathrm{tot}}\right\}$ is larger than the distance between $\varphi^{(i)}(\mathbf{p})$ and any point on the envelope of $E^{(i)}$ in t-axis direction. Therefore, $\Delta^{(i)} + c^{(i)}$ is a suitable upper bound of $t - \mathrm{g}(\mathbf{p})$.

Proposition 1. *The greatest distance $\Delta_{\max}^{(i)}$ between a point $(\mathbf{p}, t)$ on the envelope of $E^{(i)}$ and a point $(\mathbf{p}, t)$ in $\Upsilon^{(i)}$ with*

$\mathbf{p} = \left\{ \mathbf{p} : \sum_{k=1}^{K} p_k = p_{\text{tot}} \right\}$ *is found using the linear program*

$$\left(\Lambda_{\max}^{(i)}, t_{\max} \right) = \underset{\Lambda^{(i)}, t}{\operatorname{argmax}} \left\{ t - \sum_{k=1}^{K} \lambda^{(k)} t^{(k)} \right\} \tag{27}$$

subject to

$$\mathbf{a}^{(i)} t - \mathbf{A}^{(i)} \mathbf{P}^{(i)} \Lambda^{(i)} \leq \mathbf{b}^{(i)}, \tag{28}$$

and

$$\left(1_{1 \times K} \mathbf{P}^{(i)} \Lambda^{(i)} \right) = p_{\text{tot}}, \tag{29}$$

where $\Lambda^{(i)} = \left(\lambda^{(1)}, \cdots, \lambda^{(K)} \right)^T$ *is a vector of weighting factors* $\lambda^{(k)}$ *of the vertices* $\mathbf{p}^{(k)}$ *and* $\mathbf{P}^{(i)}$ *being a matrix with columns* $\mathbf{p}^{(k)}, \forall k$.

Proof: The greatest distance $\Delta_{\max}^{(i)}$ is described as

$$(\mathbf{p}_{\max,i}, t_{\max}) = \underset{(\mathbf{p}, t)}{\operatorname{argmax}} \left\{ t - \mathbf{h}^T \mathbf{p} \right\} \tag{30}$$

subject to

$$(\mathbf{p}, t) \in E^{(i)}. \tag{31}$$

Based on definition 1, every power allocation $\mathbf{p} \in F^{(i)}$ is uniquely representable as

$$\mathbf{p} = \sum_{k=1}^{K} \lambda^{(k)} \mathbf{p}^{(k)} \tag{32}$$

with $\sum_{k=1}^{K} \lambda^{(k)} = 1$, $\lambda^{(k)} \geq 0$, $k = 1, \cdots, K$. Then substituting (32) to $t - \mathbf{h}^T \mathbf{p}$ gives

$$t - \mathbf{h}^T \sum_{k=1}^{K} \lambda^{(k)} \mathbf{p}^{(k)} = t - \sum_{k=1}^{K} \lambda^{(k)} \mathbf{h}^T \mathbf{p}^{(k)}. \tag{33}$$

Since $\Upsilon^{(i)} \bigcap \varphi^{(i)}(\mathbf{p})$ at the points $\left(\mathbf{p}^{(k)}, t^{(k)} \right)$, these points satisfy

$$\mathbf{h}^T \mathbf{p}^{(k)} = t^{(k)} - c^{(i)}. \tag{34}$$

Substituting (34) in (33) gives

$$t - \sum_{k=1}^{K} \lambda^{(k)} \left(t^{(k)} - c^{(i)} \right) = t - \sum_{k=1}^{K} \lambda^{(k)} t^{(k)} + c^{(i)} = \Delta^{(i)} + c^{(i)} \tag{35}$$

where $\sum_{k} \lambda^{(k)} = 1$. Using the result of (35), the optimization problem of (30)-(31) is equivalent to the linear program of (27)-(29). ■

The linear program of (27)-(29) can be solved using the active-set method with an initial point $\left(\mathbf{p}^{(k)}, t^{(k)} \right)$ corresponding to $c^{(i)}$ [6]. Then the upper bound is

$$u^{(i)} = c^{(i)} + \Delta_{\max}^{(i)}, \tag{36}$$

where

$$\Delta_{\max}^{(i)} = t_{\max} - \sum_{k=1}^{K} \lambda_{\max}^{(k)} t^{(k)}, \tag{37}$$

and

$$(\mathbf{p}_{\max,i}, t_{\max}) = \left(\mathbf{P}^{(i)} \Lambda_{\max}^{(i)}, \mathrm{f}\left(\mathbf{P}^{(i)} \Lambda_{\max}^{(i)} \right) \right) \tag{38}$$

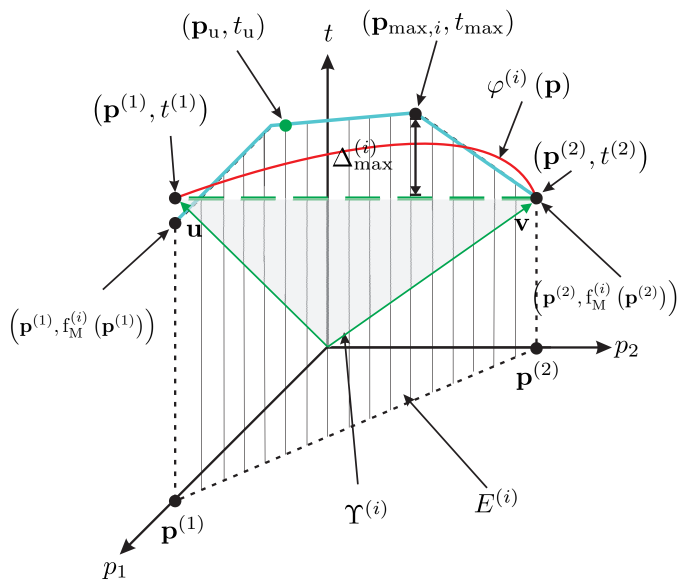


Fig. 5: The subspace $\Upsilon^{(i)}$ with basis vectors $\mathbf{u}$ and $\mathbf{v}$ intersects the feasible region $E^{(i)}$ in between the points $\left(\mathbf{p}^{(k)}, t^{(k)} \right)$ where this intersection is lower than the concave function $\varphi^{(i)}(\mathbf{p})$ in t-axis direction.

is the corner point on the envelope of $E^{(i)}$ with the greatest distance to $\Upsilon^{(i)}$ with $\mathbf{p} = \left\{ \mathbf{p} : \sum_{k=1}^{K} p_k = p_{\text{tot}} \right\}$ in the direction of t-axis and $\mathbf{p}_{\max,i} = \mathbf{P}^{(i)} \Lambda_{\max}^{(i)}$ is a power allocation which can be applied as a lower bound candidate.

D. Computing a Lower Bound

For each node i, a lower bound is computed as

$$l^{(i)} = \max \left\{ \mathrm{f}(\mathbf{p}) - \mathrm{g}(\mathbf{p}) \right\}, \tag{39}$$

and the corresponding power allocation is

$$\mathbf{p}_{\mathrm{LB},i} = \underset{\mathbf{p}}{\operatorname{argmax}} \left\{ \mathrm{f}(\mathbf{p}) - \mathrm{g}(\mathbf{p}) \right\}, \tag{40}$$

with

$$\mathbf{p} \in \left\{ \mathrm{V}\left(F^{(i)} \right), \mathbf{p}_{\max,i} \right\}. \tag{41}$$

V. Numerical Results

In this section, the performance of the proposed algorithm is demonstrated as a function of the pseudo signal to noise ratio γ_{pSNR} which is defined as the ratio of the total transmit power to the noise power at the receivers in decibel $\gamma_{\mathrm{pSNR}} = 10\log(p_{\text{tot}}/\sigma_k^2)$, $\forall k$.

For the following, well known sub-optimal power allocation schemes are used as benchmarks. The first scheme is the greedy power allocation which serves only the user with the highest channel gain:

$$p_k = \begin{cases} p_{\text{tot}} & k = k_{\max} \\ 0 & \text{otherwise} \end{cases}, \forall k \tag{42}$$

with

$$k_{\max} = \underset{k}{\operatorname{argmax}} \{g_{kk}\}, \forall k. \tag{43}$$

Also the equal power allocation which serves all users with equal powers is considered:

$$p_k = \frac{p_{\text{tot}}}{K}, \forall k. \tag{44}$$

The third scheme is the signal to interference ratio balancing scheme (SIR balancing) which equalizes the signal to interference ratios at all receivers:

$$\gamma^{(k)} = \frac{g_{kk} p_k}{\sum_{l \neq k}^{K} g_{kl} p_l} = \gamma \tag{45}$$

where γ is the resulting signal to interference ratio. As introduced in [1], the waterfilling scheme neglects the interference part in (2) and solves the resulting convex problem of (3)-(4) using the Lagrangian multiplier method. The allocated power at k-th user is

$$p_k = \max\left\{0, p_{\text{w}} - \frac{\sigma_k^2}{g_{kk}}\right\} \tag{46}$$

where p_{w} is the water level. So it assigns powers to users based on the noise power to the channel gain ratio (NCR). Finally, an iterative algorithm which calculates the interference power based on the power allocation of the previous iteration is considered as proposed in [2]. At the j-th iteration, the interference part in (2) is constant, i.e., calculated from the power allocation in iteration $j-1$. The optimization problem of (3)-(4) is solvable using the Lagrangian multiplier method. The assigned power to the k-th user at j-th iteration is

$$p_k^{(j)} = \max\left\{0, p_{\text{w}}^{(j)} - \frac{\sigma_k^2 + \sum_{l \neq k} g_{kl} p_l^{(j-1)}}{g_{kk}}\right\} \tag{47}$$

where $p_{\text{w}}^{(j)}$ is the water level in iteration j. It is shown in [2] that excluding some users, i.e., allocates no power to some users, and applying this algorithm to the other users increases the sum rate significantly. Therefore, we implement this algorithm in such a way that it allocates powers to a subset of users aiming at achieving the highest sum rate. Apart from the conventional distributed iterative waterfilling proposed in [7], this algorithm is also called iterative waterfilling in the sense that it refills powers to users on the top of the noise plus interference to the channel gain ratios (NICR), i.e., similar to waterfilling.

Assuming equal noise powers for all users, a scenario of three transmitter-receiver pairs is considered with

$$\mathbf{G} = \begin{pmatrix} 10.01 & 10 & 0.01 \\ 0.11 & 0.5 & 0.06 \\ 10^{-5} & 10^{-6} & 0.41 \end{pmatrix}. \tag{48}$$

Fig. 6 shows the sum rate achieved at different γ_{pSNR} using different power allocation schemes. Because the SIR balancing scheme serves all users and the high interference to the first user caused by serving the second user, the achieved sum rate is low as compared to the other schemes especially at high γ_{pSNR}. Moreover, applying equal power allocation, the achieved sum rate is low but increases monotonically with γ_{pSNR} till it saturates at high γ_{pSNR}.

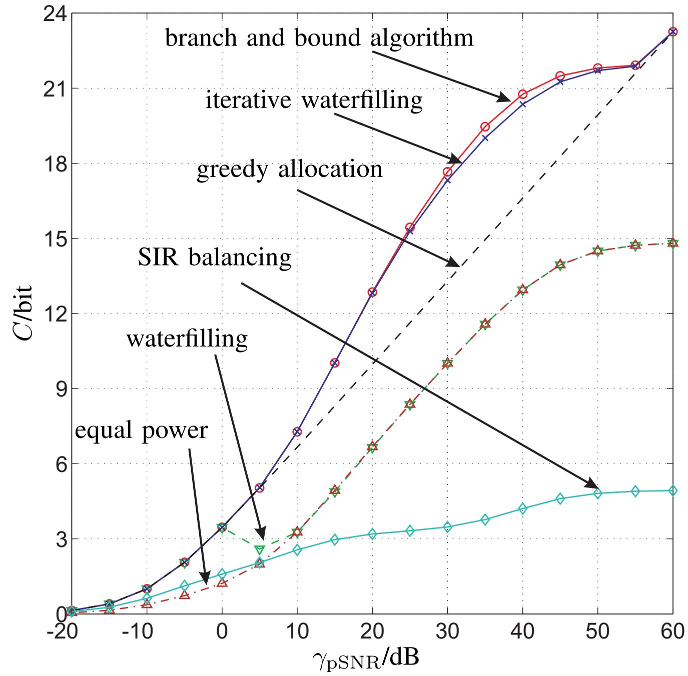


Fig. 6: Sum rate versus the pseudo signal to noise ratio γ_{pSNR} for different power allocation schemes.

Waterfilling serves the first user or the first two users at low γ_{pSNR}, i.e., the third user has a higher NCR as compared to the water level. But at high γ_{pSNR} it serves all users with almost equal powers because the NCR's for all users are very small as compared to the assigned powers. Therefore, waterfilling and equal power allocation are aligned at high γ_{pSNR}.

Greedy power allocation achieves high sum rates by serving only the first user at low γ_{pSNR}. But at moderate interference, both iterative waterfilling and the branch and bound algorithm achieve higher sum rates by serving both the first user and the third user.

Finally, because the iterative waterfilling is a sub-optimal power allocation scheme, it achieves lower sum rates as compared to the branch and bound algorithm which outperforms the other schemes and reaches the global optimum power allocation.

VI. Conclusion

In this paper, a D.C. (difference of two concave functions) formulation of the non-convex optimization problem of power allocation aiming at maximizing the sum rate with a total power constraint is presented. A branch and bound algorithm with a good estimation of the bounds is proposed. For an upper bound, both the convex maximization formulation of the problem and the piecewise linear approximation of the constraints relax the problem into a linear program problem. The results show that the proposed algorithm reaches the global optimum as well as it outperforms other suboptimum schemes in all SNR values.

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REFERENCES

[1] T. Cover and J. Thomas, *Elements of Information Theory*, 2nd ed. John Wiley & Sons, July 2006.

[2] H. Al-Shatri, N. Palleit, and T. Weber, "Transmitter power allocation for optimizing sum capacity interference channels," in *14th International OFDM-Workshop*, September 2009, pp. 73–77.

[3] R. Horst, P. Pardalos, and N. Thoai, *Introduction to Global Optimization*, 2nd ed., ser. Nonconvex Optimization and its Applications. Kluwer Acadimic Publishers, January 2000, vol. 48.

[4] R. Horst and V. Thoai, "DC programming: Overview,," in *Journal of Optimization Theory and Applications*, vol. 103, no. 1, October 1999, pp. 1–43.

[5] Y. Xu, T. Le-Ngoc, and S. Panigrahi, "Global concave minimization for optimal spectrum balancing in multi-user DSL networks," in *IEEE Transactions on Signal Processing*, vol. 56, no. 7, July 2008, pp. 2875–2885.

[6] P. Gill, W. Murray, M. Saunders, and M. Wright, "Procedures for optimization problems with a mixture of bounds and general linear constraints," in *ACM Transactions on Mathematical Software*, vol. 10, no. 3, September 1984, pp. 282–298.

[7] W. Yu, G. Ginis, and J. Cioffi, "Distributed multiuser power control for digital subscriber lines," in *IEEE Journal on Selected Areas in Communications*, vol. 20, no. 5, June 2002, pp. 1105–1115.