Throughput-Delay-Reliability Tradeoff in Ad Hoc Networks

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Abstract—**Delay-reliability (D-R), and throughput-delay-reliability (T-D-R) tradeoffs in an ad hoc network are derived for single hop and multi-hop transmission with automatic repeat request (ARQ) on each hop. The delay constraint is modeled by assuming that each packet is allowed at most D retransmissions end-to-end, and the reliability is defined as the probability that the packet is successfully decoded in at most D retransmissions. The throughput of the ad hoc network is characterized by the transmission capacity, which is defined to be the maximum allowable density of transmitting nodes satisfying a per transmitter receiver rate, and an outage probability constraint, multiplied with the rate of transmission and the success probability. Given an end-to-end retransmission constraint of D, the optimal allocation of the number of retransmissions allowed at each hop is derived that maximizes a lower bound on the transmission capacity. For equidistant hops equally distributing the total retransmission constraint among all the hops is optimal. Optimizing over the number of hops, single hop transmission is shown to be optimal for maximizing a lower bound on the transmission capacity in the sparse network regime.**

I. INTRODUCTION

The transmission capacity of an ad hoc network is the maximum allowable density of transmitting nodes, satisfying a per transmitter receiver rate, and outage probability constraints [1]–[4]. In other words, the transmission capacity characterizes the maximum density of spatial transmissions that can be simultaneously supported in an ad hoc network under a quality of service (QoS) constraint. The transmission capacity is computed under the assumption that the transmitter locations are distributed as a Poisson point process (PPP) using the tools from stochastic geometry [1]–[4]. The transmission capacity framework allows for tractable analysis with different physical layer transmission techniques, such as use of multiple antennas [5]–[8], bandwidth partitioning [9], and successive interference cancelation [2].

Most of the prior work on computing the transmission capacity of ad hoc networks has been limited to single hop communication. Recently, under some assumptions, [10] computed the transmission capacity of ad hoc network with multi-hop transmissions, and automatic repeat request (ARQ) on each hop. To account for retransmissions and multiple hops, [10] normalized the transmission capacity by end-to-end expected delay, and defined the success event as the event that the packet is successfully decoded in at most D retransmissions. Modeling D as delay, the relationship between success probability and D captures the delay-reliability (D-R) tradeoff, while the transmission capacity expression characterizes the throughput-delay-reliability (T-D-R) tradeoff.

The analysis carried out in [10] assumes $D = \infty$, and independent packet success/failure events across time slots. The second assumption can only be justified for very low density of transmitters, and does not hold true otherwise [11]. The result of [10] is useful in determining the optimal number of hops with no retransmissions constraint, however, does not characterize the D-R or the T-D-R tradeoff of the ad hoc network.

To characterize the D-R and T-D-R tradeoffs, in this paper, we derive an exact expression for the transmission capacity with multiple hops and retransmissions. In contrast to [10] to derive the transmission capacity, we

i) use a finite D,

ii) do not assume independence of success/failures of packets across time slots.

iii) assume that each transmitter retransmits using the slotted ALOHA protocol.

Our results are summarized as follows.

- We derive the exact expressions for the success probability, and the transmission capacity for single hop and multi-hop transmissions with finite D.
- We derive tight upper and lower bounds on the success probability using the FKG inequality [12].
- Using the derived bounds, we characterize the D-R, and the T-D-R tradeoff in an ad hoc network. We show that the success probability increases as $1 - x^{D+1}$ ($x < 1$ is a constant) for single hop transmission.
- For equidistant hops we show that equally distributing the total retransmission constraint among all the hops is optimal for maximizing a lower bound on the transmission capacity.
- For multiple equidistant hops, we derive the optimal number of hops that maximize a lower bound on the transmission capacity. In the sparse network regime we show that it is optimal to transmit over a single hop.

II. System Model

Consider an ad hoc network where multiple source destinations pairs want to communicate with each other without any centralized control. The location of each source node $\mathcal{S}_m,\ m \in \mathbb{N}$ is modeled as a homogenous Poisson point process (PPP) on a two-dimensional plane with intensity λ_0 [13]. Our model precludes mobility of nodes, and is restricted to averaging with respect to the PPP spatial node distribution. We assume that source $\mathcal{S}_m$ is located at a distance d from its intended receiver $\mathcal{D}_m\ \forall m$, with N relays $R_{nm},\ n = 1, 2, \ldots, N$ (N hops) in between[1]. The distance between relays n and $n+1$ is d_n, such that $\sum_{n=1}^{N} d_n = d$. Thus link m is described by the set of nodes $\{\mathcal{S}_m, \mathcal{R}_{1m}, \ldots, \mathcal{R}_{Nm}, \mathcal{D}_m\}$. We assume that all nodes in the network have single antenna each. The transmission happens hop by hop using the decode and forward strategy, i.e. if the n^{th} relay decodes the packet in k^{th} time slot, it transmits the decoded packet to the $n+1^{th}$ relay in the $k+1^{th}$ time slot. We consider ARQ on each hop, where the receiver informs the transmitter of the success (ack) or failure (nack) of the packet decoding instantly, and without any errors. We assume that at most D end-to-end retransmissions are allowed between $\mathcal{S}_m$ and $\mathcal{D}_m,\ \forall\ m$. This requirement is used to model the delay constraint, which gives rise to the outage event that the packet is not successfully decoded at the destination after D retransmissions. Let D_n be the number of retransmissions used on hop n, then $D = \sum_{n=1}^{N} D_n$. For simplicity, same packet is assumed to be retransmitted (at most D times) with every nack, without any incremental redundancy or rate adaptation.

Following [10], we assume that there is only one active packet on each link[2] i.e. source waits to transmit the next packet until the previous packet has been received by the destination, or the delay constraint has been violated. Therefore, for each link, in any given time slot, the relay closest to the destination that has decoded the packet correctly is allowed to transmit. Let the transmitter on link m in time slot t be T_m^t, $T_m^t \in \{\mathcal{S}_m, \mathcal{R}_{1m}, \ldots, \mathcal{R}_{Nm}\}$. Similarly let the receiver on link m at time t be R_m^t, $R_m^t \in \{\mathcal{R}_{1m}, \ldots, \mathcal{R}_{Nm}, \mathcal{D}_m\}$. Let the set of all transmitters in the network at time slot t be denoted by $\Phi_t = \{T_k^t,\ k \in \mathbb{N}\}$. Then the set of interfering nodes for R_m^t is $\Phi_t^m := \{\Phi_t \backslash T_m^t\}$. Using the Slivnyak's Theorem, the stationarity of the PPP, and the random translation invariance property of the PPP [13], [14], the locations of interferers of Φ_t^m are distributed as a PPP with intensity $\lambda_0,\ \forall\ t, m$ [10].

We consider a slotted ALOHA like random access protocol, where each transmitter (source or any relay) attempts to transmit its packet with an access probability p, independently of all other transmitters. Consequently, the active transmitter process is also a homogenous PPP on a two-dimensional plane with intensity $\lambda := p\lambda_0$. Note that when the active transmitter process is a PPP, the success/failure of packet decoding at different receivers is correlated [11]. Therefore retransmission of packets depending on the nack introduces correlation among the active transmitters process, and it is no longer a PPP. Violating the PPP assumption, however, entails analytical intractability. To satisfy the PPP assumption on the active transmitter locations, we assume that similar to the newly arrived packets in its queue, each transmitter uses a slotted ALOHA protocol with access probability p to retransmit old packets as well.

For the purpose of analysis we consider a typical link $\{\mathcal{S}_0, \mathcal{R}_{10}, \ldots, \mathcal{R}_{N0}, \mathcal{D}_0\}$. It has been shown in [1] that the performance of the typical source destination pair is identical to the network wide performance, using the stationarity of the homogenous Poisson point process, and the Slivnyak's Theorem [13]. For simplicity we refer to link $\{\mathcal{S}_0, \mathcal{R}_{10}, \ldots, \mathcal{R}_{N0}, \mathcal{D}_0\}$ as $\{\mathcal{S}_0, \mathcal{D}_0\}$. Let n^{th} relay ($n = 0$ corresponds to the source $\mathcal{S}_0$) be the active transmitter for the typical link $\{\mathcal{S}_0, \mathcal{D}_0\}$ at time slot t, i.e. $T_0^t = R_{n0}$. Then the received signal at the $n+1^{th}$ relay (defined R_0^t) of link $\{\mathcal{S}_0, \mathcal{D}_0\}$ at time slot t is

$$y_0^t = \sqrt{P} d^{-\alpha/2} h_{00}^{nt} x_0^t + \sum_{T_s^t \in \Phi_t^m} \sqrt{P} \mathbf{1}_{T_s^t} d_s^{-\alpha/2} h_{0s}^{nt} x_s^t + z_0^t, \quad (1)$$

where P is the transmit power of each transmitter, $h_{0s}^{nt} \in \mathbb{C}$ is the channel coefficient between T_s^t and R_0^t on hop n, d_s is the distance between T_s^t and R_0^t, α is the path loss exponent $\alpha > 2$, $x_s^t \sim \mathcal{CN}(0,1)$ is the signal transmitted from T_s^t in time slot t, $\mathbf{1}_{T_s^t} = 1$ with probability p, and 0 otherwise, due to ALOHA transmission strategy, and z_0^t is the additive white Gaussian noise. All results in this paper are valid for $\alpha > 2$. We consider the interference limited regime, i.e. noise power is negligible compared to the interference power, and henceforth drop the noise contribution [1]. We also assume $P = 1$, since the signal to interference ratio (SIR) is independent of P. We assume that each h_{0s}^{nt} is independent and identically $\mathcal{CN}(0,1)$ distributed $\forall\ n, t, s$.

Let SIR_t^n denote the SIR on hop n of link $\{\mathcal{S}_0, \mathcal{D}_0\}$ at time slot t. With the received signal model (1), $SIR_t^n := \frac{d_n^{-\alpha}|h_{00}^{nt}|^2}{\sum_{T_s \in \Phi_n^t \backslash \{T_0\}} \mathbf{1}_{T_s} d_{T_s}^{-\alpha} |h_{0s}^{nt}|^2}$. We assume that the rate of transmission for each hop is $R(\beta) = \log(1+\beta)$ bits/sec, therefore, a packet transmitted by T_0^t can be successfully decoded at R_0^t in time slot t on hop n, if $SIR_t^n \geq \beta$.

Let M_n be the random variable denoting the number of transmissions used at hop n, $M_n \leq D_n + 1$. Then the expected delay on the n^{th} hop is $\mathbb{E}\{M_n\}$. Let P_s be the probability that the packet is successfully decoded by the destination $\mathcal{D}_0$ within D retransmissions. Then the transmission capacity of ad hoc network with multi-hop transmission is defined as

$$C := \max_{D_n,\ \sum_{n=1}^{N} D_n \leq D} \frac{P_s \lambda R}{\mathbb{E}\{\sum_{n=1}^{N} M_n\}} \text{ bits/sec/Hz/m}^2, \quad (2)$$

where in contrast to [10] we have not multiplied the transmission distance d. The transmission capacity quantifies the end-to-end rate that can be supported by λ simultaneous transmissions/unit area, with outage probability P_s, and maximum delay $D + N$. Thus, the transmission capacity captures the **T-D-R tradeoff** of ad hoc networks, where throughput $= C$, maximum delay $\leq D + N$, and reliability $= P_s$. Similarly, the definition of P_s captures the **D-R tradeoff** of ad hoc network.

[1]The results of this paper can be generalized for random distances between the source and the destination.

[2]For more discussion on this assumption see Remark 2 [10].

III. Single Hop Transmission

In this section we consider single hop communication $N = 1$. Our goal in this section is to first derive the success probability P_s when at most D retransmissions are allowed for each packet, and then using the derived expression for P_s, derive the transmission capacity C defined by (2).

Towards that end, we can write

$$P_s = \sum_{j=1}^{D+1} P_s^j, \quad (3)$$

where $P_s^j = P(\text{success in the } j^{th} \text{ time slot})$. Note that the event $\{\text{success in the } j^{th} \text{ time slot}\}$ is the union of the events $\{\text{failures in any } k \text{ time slots, success in the } j^{th} \text{ time slot}\}$ for $k = 0, 1, 2, \dots, j-1$, since at each time slot, retransmission happens only with probability p. Note that since SIR_t is identically distributed $\forall\ t$, $P\left(\text{success in } j^{th} \text{ time slot}\right)$ only depends on how many failures have happened before time slot j, and not where those failures happened. Therefore it follows that

$$P_s^j = \sum_{k=0}^{j-1} \binom{j-1}{k} p^k (1-p)^{j-1-k} p \; P\left(SIR_1 < \beta,\ \dots,\ SIR_k < \beta,\ SIR_j \geq \beta\right), \quad (4)$$

by accounting for $k = 0$, or 1, or , $\dots, j-1$ failures before success at the the j^{th} slot. Since SIR's are not independent across times slots, to compute the success probability (3), we first need to compute the joint probability $P\left(SIR_1 < \beta,\ \dots,\ SIR_k < \beta,\ SIR_j \geq \beta\right)$, for each k and $j > k$, which is derived in the following Lemma.

Lemma 1:

$$P\left(SIR_1 < \beta,\ \dots,\ SIR_k < \beta,\ SIR_j \geq \beta\right) = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \times \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \left(\frac{p}{1 + d^\alpha \beta x^{-\alpha}} + 1 - p\right)^{\ell+1} \mathbf{d}x\right).$$

For lack of space we delete all the proofs in this paper and refer the reader to an extended version of this paper [15].

Proposition 1: The success probability P_s is given by

$$P_s = \sum_{j=1}^{D+1} \sum_{k=0}^{j-1} \binom{j-1}{k} p^k (1-p)^{j-1-k} p \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \times \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \left(\frac{p}{1 + d^\alpha \beta x^{-\alpha}} + 1 - p\right)^{\ell+1} \mathbf{d}x\right).$$

Proof follows by substituting Lemma 1 in (3).

Remark 1: The exact expression for P_s involves an integral which can computed for any given value of ℓ. The integral, however, cannot be written down easily in terms of variable ℓ.

Recall that M is the random variable denoting the number of retransmissions required. Note that M takes values in $[0 : D+1]$ with probability $P(M = j) = P_s^j,\ j = 0, 1, 2, \dots, D$, and $P(M = D+1) = P_s^{D+1} + \sum_{j=0}^{D} P_s^j = P_s^{D+1} + (1 - P_s)$. The second term in $P(M = D+1)$ is to account for delay incurred by packets that are not decoded even after D retransmissions. Using Lemma 1, the expected number of retransmissions $\mathbb{E}\{M\} = \sum_{j=1}^{D+1} j P_s^j + (D+1)(1 - P_s)$ is computed as follows.

Proposition 2: The expected delay $\mathbb{E}\{M\}$ in a single hop ad hoc network with at most D retransmissions is

$$\mathbb{E}\{M\} = \sum_{j=1}^{D+1} \sum_{k=0}^{j-1} \sum_{\ell=0}^{k} j \binom{j-1}{k} p^{k+1} (1-p)^{j-1-k} (-1)^\ell \binom{k}{\ell} \times \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \left(\frac{p}{1 + d^\alpha \beta x^{-\alpha}} + 1 - p\right)^{\ell+1} \mathbf{d}x\right) + (D+1)(1 - P_s).$$

Theorem 1: The transmission capacity of a single hop ad hoc network with at most D retransmissions is

$$C = \lambda R \frac{P_s}{\mathbb{E}\{M\}},$$

where P_s is given by Proposition 1, and $\mathbb{E}\{M\}$ is given by Proposition 2.

Proposition 1 and Theorem 1 give an exact expression for the success probability, and the transmission capacity, respectively, of an ad hoc network with single hop transmission, and retransmissions constraint of D. Because of the correlation of $SIR's$ across different time slots with PPP distributed transmitter locations, the derived expressions are complicated, and do not allow for a simple closed form expression for P_s, and C, as a function of D. To get more insights on the dependence of P_s, and C, on D (to obtain simple D-R and T-D-R tradeoffs), we next derive tight lower and upper bounds on P_s^j, and consequently on P_s, and the transmission capacity C. To derive the bounds we make use of the FKG inequality [12] as stated below. Before stating the FKG inequality, we need the following definition.

Definition 1: Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the probability space. Let A be an event in $\mathcal{F}$, and $\mathbf{1}_A$ be the indicator function of A. Then the event $A \in \mathcal{F}$ is called increasing if $\mathbf{1}_A(\omega) \leq \mathbf{1}_A(\omega')$, whenever $\omega \leq \omega'$. The event A is called decreasing if its complement A^c is increasing.

Example 1: Success event $\{SIR > \beta\}$ is a decreasing event. For the PPP under consideration, let $\omega = (a_1, a_2, \dots,)$ where for $n \in \mathbb{N}$,

$$a_n = \begin{cases} 1 & \text{if } Tx_n \text{ is active,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\omega' \geq \omega$, if $a_n' \geq a_n,\ \forall\ n$, i.e. configuration ω' contains at least those interferers which are present in configuration ω. Recall our definition of $SIR_t = \frac{d^{-\alpha}|h_{00}^t|^2}{\sum_{T_n^t \in \Phi_t \setminus \{T_0\}} d_{Tn}^{-\alpha}|h_{0n}^t|^2}$. Since SIR decreases with more interferers, it follows that if $\omega' \in \{SIR > \beta\}$, then $\omega \in \{SIR > \beta\}$ if $\omega' \geq \omega$.

Lemma 2: (FKG Inequality [12]) If both $A, B \in \mathcal{F}$ are increasing or decreasing events then $P(AB) \geq P(A)P(B)$.

Upper bound on the success probability P_s.

Proposition 3: The success probability P_s with single hop transmission in an ad hoc network with at most D retransmissions is upper bounded by

$$P_s \quad \leq \quad 1-(pq+1-p)^{D+1}, \tag{5}$$

where $q := P\left(SIR_1 < \beta\right) = 1 - \exp\left(-\frac{\lambda 2\pi^2 d^2 \beta^{\frac{2}{\alpha}} Csc\left(\frac{2\pi}{\alpha}\right)}{\alpha}\right)$ [4].

Lower bound on the success probability P_s.

Proposition 4: The success probability P_s with single hop transmission in an ad hoc network with at most D retransmissions is lower bounded by

$$\begin{aligned} P_s \quad \geq \quad & P\left(SIR_{D+1} \geq \beta \mid SIR_1 < \beta, \ \ldots, \ SIR_D < \beta\right) \times \\ & \frac{1-(pq+1-p)^{D+1}}{1-q}. \end{aligned} \tag{6}$$

For small values of D we can analytically show that

$$P\left(SIR_{D+1} \geq \beta \mid SIR_1 < \beta, \ \ldots, \ SIR_D < \beta\right) \approx 1-q,$$

and hence our derived bounds on P_s are tight. For higher values of D also, the bounds can be shown to be tight using simulations in the sparse network regime i.e. small λ or $\lambda \rightarrow 0$. Thus, from here on in this paper we assume that $P_s = c\left(1-(pq+1-p)^{D+1}\right)$ where $c < 1$ is a constant.

D-R tradeoff: From the upper and lower bounds,

$$P_s \quad = \quad c(1-(1-p+pq)^{D+1}). \tag{7}$$

Thus the success probability increases as $1 - x^{D+1}$, where $x < 1$ is a constant. Using the derived upper and lower bound the expected delay is

$$\mathbb{E}\{M\} \quad = \quad c\left[\frac{1-(pq+1-p)^{D+1}}{(1-q)}\right] + (D+1)(1-c) \tag{8}$$

T-D-R tradeoff: Using the derived expression for P_s (7), and $\mathbb{E}\{M\}$ (8), we get

$$C = \frac{c(1-(pq+1-p)^{D+1})\lambda R}{c\left[\frac{1-(pq+1-p)^{D+1}}{(1-q)}\right] + (D+1)(1-c)} \text{ bits/sec/Hz/}m^2.$$

Discussion: In this section we first derived the exact D-R, and the T-D-R tradeoffs in an ad hoc network when the destination can be directly reached by its source in one hop. The exact expressions are fairly complicated, and do not yield a simple relationship between C, P_s, and D, for any arbitrary D. To obtain a simple relationship between P_s, D, and C, we then derived tight upper and lower bounds on the success probability, and showed that the bounds are tight for small D, or in the sparse network regime.

IV. Multi-Hop Transmissions

In this section we consider multi-hop communication (arbitrary N). We analyze the case of $N = 2$. The analysis for $N > 2$ follows similarly. With at most D_1 retransmissions on the first hop, and D_2 retransmissions on the second hop ($D_1 + D_2 = D$), the success probability P_s for transmission between T_0 and R_0 is $P_s = \sum_{j=1}^{D_1+1}\sum_{k=1}^{D_2+1} P_s^{jk}$ where $P_s^{jk} = P$(success in j^{th} time slot on hop 1, success in k^{th} time slot on hop 2).

Expanding P_s^{jk},

$$\begin{aligned} P_s \quad = \quad & \sum_{j=1}^{D_1+1}\sum_{k=2}^{D_2+1}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}p^{\ell}(1-p)^{j-1-\ell} \\ & \sum_{m=0}^{k-1}\binom{k-1}{m}p^m(1-p)^{k-1-m}p^2 \times \\ & P\left(SIR_1^1 < \beta, \ \ldots SIR_\ell^1 < \beta, \ SIR_j^1 \geq \beta,\right. \\ & \left. SIR_1^2 < \beta, \ \ldots SIR_m^2 < \beta, \ SIR_k^2 \geq \beta\right). \end{aligned} \tag{9}$$

P_s can be computed by deriving an exact expression for the joint probability similar to Lemma 1.

The expected delay $\mathbb{E}\{M\}$ for $N = 2$ can be computed easily by using the linearity of expectation, since

$$\mathbb{E}\{M\} = \mathbb{E}\left\{M_1 + M_2\right\} = \mathbb{E}\left\{M_1\right\} + \mathbb{E}\left\{M_2\right\},$$

where $\mathbb{E}\left\{M_n\right\}$ is given by Proposition 2.

Therefore using the transmission capacity definition (2) for multi-hop transmissions we get the following Theorem.

Theorem 2: The transmission capacity of an ad hoc network with $N = 2$-hop transmission and end-to-end retransmission constraint of D is

$$C = \max_{D_n, \ \sum_{n=1}^{N} D_n \leq D} \frac{\lambda R P_s}{\sum_{n=1}^{2}\mathbb{E}\left\{M_n\right\}} \quad \text{bits/sec/Hz/}m^2,$$

where P_s is given by (9), and $\mathbb{E}\left\{M_n\right\}$ is given by Proposition 2.

Here again similar to the single hop case (Section III) we see that finding a closed form expression for P_s in terms of D_1 and D_2 is not possible due to the complicated expression for the joint probability of success on the two hops. To gain more insight into the dependence of D_1, and D_2 on P_s, and C, we derive a lower bound on P_s as follows.

A. Lower Bound

By definition

$$\begin{aligned} P_s \quad = \quad & P(\text{success in less than } D_1 \text{ retransmissions on the } \\ & 1^{st} \text{ hop, success in less than } D_2 \text{ retransmissions} \\ & \text{on the } 2^{nd} \text{ hop}). \end{aligned}$$

The event $S_{D_n} :=$ {success in less than D_n retransmissions on n^{th} hop} is a decreasing event since for $\omega' \geq \omega$ (ω as defined in Example 1), if $\mathbf{1}_{S_{D_n}}(\omega') = 1$ then automatically $\mathbf{1}_{S_{D_n}}(\omega) = 1$. Therefore, from the FKG inequality (Lemma 2)

$$\begin{aligned} P_s \quad \geq \quad & P(\text{success in less than } D_1 \text{ retransmissions on the } \\ & 1^{st} \text{ hop}) \times P(\text{ success in less than } D_2 \\ & \text{retransmissions on the } 2^{nd} \text{ hop}), \\ \stackrel{(a)}{=} \quad & c^2\prod_{n=1}^{2}(1-(pq_{d_n}+1-p)^{D_n+1}), \end{aligned}$$

where (a) is obtained by using the lower and upper bound derived in the last section for a single hop transmission, and $q_{d_n} = q(d_n)$ i.e. value of q with $d = d_n$.

Remark 2: The lower bound on the success probability corresponds to the case when the success event on the first and second hop are independent. Since the spatial correlation of interference in a PPP is zero with path-loss model of $d^{-\alpha}$ [11], the derived lower bound is expected to be tight (also shown using simulations).

Lemma 3: ($D-R$ tradeoff of N hop ad hoc network) The success probability in an ad hoc network with N hop transmission is lower bounded by

$$P_s \ge c^n \prod_{n=1}^{N} \left(1 - (pq_{d_n} + 1 - p)^{D_n+1}\right).$$

Proof: Since S_{D_n} is a decreasing event for each $n = 1, \dots, N$, results follows from the FKG inequality. ∎

The end-to-end transmissions/delay is $M := \sum_{n=1}^{N} M_n$, and by linearity of expectation $\mathbb{E}\{M\} = \sum_{n=1}^{N} \mathbb{E}\{M_n\}$. From (8),

$$\mathbb{E}\{M_n\} = \left[\frac{c\left(1 - (pq_{d_n} + 1 - p)^{D_n+1}\right)}{(1 - q_{d_n})} + (D_n + 1)(1 - c)\right]. \tag{10}$$

Remark 3: Since $M_n \le D_n + 1$, a simple upper bound on the expected end-to-end delay $\mathbb{E}\{M\}$ is $\sum_{n=1}^{N} D_n + 1 = D + N$. We will use this upper bound in next two sections to find the optimal D_n's ($\sum_{n=1}^{N} D_n = D$), and N that maximize a lower bound on the transmission capacity.

Using Lemma 3 and (10), we obtain the following Theorem

Theorem 3: The transmission capacity of an ad hoc network with multi-hop transmission, and an end-to-end retransmission constraint of D is lower bounded by

$$C \ge \frac{\lambda R c^n \prod_{n=1}^{N} \left(1 - (pq_{d_n} + 1 - p)^{D_n+1}\right)}{\sum_{n=1}^{N} \frac{c(1-(pq_{d_n}+1-p)^{D_n+1})}{(1-q_{d_n})} + (D_n + 1)(1 - c)},$$

bits/sec/Hz/m^2.

Remark 4: Note that an upper bound on transmission capacity has been computed in [10] for $D \to \infty$, and under the assumption that SIR_t^n are independent $\forall\, t, n$, in which case $P_s = 1$, and $\mathbb{E}\{M\} = \frac{N}{P_s^1}$. Thus our result subsumes the result of [10], since with SIR_t^n independent $\forall\, t, n$, $c = 1$, and we have an equality in Lemma 3, and Theorem 3.

Discussion: In this section we first derived the D-R, and the T-D-R tradeoffs in an ad hoc network with multi-hop transmission from the source to its intended destination. The exact tradeoff expressions are quite complicated and to get more insights we derived a lower bound on the success probability P_s, and the transmission capacity C. Next, we derive an analytically tractable lower bound on the transmission capacity using Remark 3, and find the optimal D_n's that maximize the lower bound.

V. Optimal Per Hop Retransmissions

In this section we derive a lower bound on the transmission capacity[3], and then find the optimal D_n's that maximize the lower bound. From Remark 3, $\sum_{n=1}^{N} \mathbb{E}\{M_n\} \le \sum_{n=1}^{N} D_n + 1 = D + N$, thus using the lower bound on P_s (Lemma 3), and the definition of transmission capacity (2)

$$C \ge \max_{D_n,\ \sum_{n=1}^{N} D_n \le D} \frac{\lambda R c^n \prod_{n=1}^{N} \left(1 - (pq_{d_n} + 1 - p)^{D_n+1}\right)}{D + N}. \tag{11}$$

Notice that for maximization with respect to D_n's, the above optimization problem is equal to

$$\max_{D_n,\ \sum_{n=1}^{N} D_n \le D} c^n \prod_{n=1}^{N} \left(1 - (pq_{d_n} + 1 - p)^{D_n+1}\right).$$

The next proposition gives a sufficient condition for solving this optimization problem.

Proposition 5: The optimal $D_n^\star$'s that maximize the lower bound on the transmission capacity satisfy

$$D_n^\star + 1 = \frac{ln\left(\frac{\gamma}{ln(\hat{q}_{d_n}) + \gamma}\right)}{ln(\hat{q}_{d_n})},$$

where γ is such that $\sum_{n=1}^{N} D_n = D$. For equidistant hops $d_n = d/N,\ \forall\, n,\ D_n^\star = D/N$.

Discussion: In this section we first derived an analytically tractable lower bound on the transmission capacity, and then found sufficient conditions for finding the optimal D_n's that maximize the derived lower bound. The optimization function is concave in D_n's, and hence using the KKT conditions we derived the sufficient conditions for optimality. For the special case of equidistant hops, $d_n = \frac{d}{N}$, we derived that equally distributing D (the end-to-end delay constraint) among the N hops, maximizes the success probability.

VI. Optimal Number of Hops N

In this section we want to find the optimal number of equidistant hops N that maximizes the lower bound on the transmission capacity for a fixed D, with $D_n = \lfloor D/N \rfloor,\ \forall\, n$.

From (11), for equidistant hops, and optimal $D_n = \lfloor D/N \rfloor,\ \forall\, n$,

$$C \ge \frac{\lambda R c^N \prod_{n=1}^{N} 1 - (1 - p + pq_d)^{\lfloor D/N \rfloor + 1}}{D + N}.$$

Finding the optimal N is a hard problem for arbitrary λ and D. Next we show that in the sparse network regime $\lambda \to 0$, we can find an exact solution for the optimal N.

Proposition 6: For a sparse network $\lambda \to 0$, $N = 1$ maximizes the transmission capacity for $p \approx 1$.

Proof idea: Using the Taylor series approximation for $\lambda \to 0$ show that the transmission capacity lower bound is a decreasing function in N.

Discussion: In this section we showed that in a sparse network regime, it is optimal to transmit over a single hop.

[3]The exact transmission capacity expression is far too complicated for analysis.

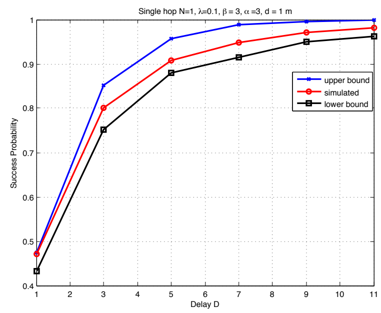

Fig. 1. Success probability as a function of D for N=1

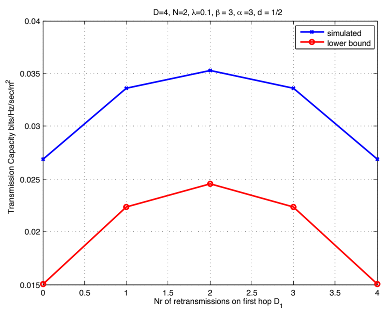

Fig. 2. Transmission capacity as a function of D_1 with $D = 4$ for equidistant hops

The physical interpretation of this result is that in a sparse network with few interferers, the decrease in transmission capacity due to the end-to-end delay (linear in N) outweighs the increase in transmission capacity due to the reduced per hop distance $\left(\frac{d}{N}\right)$. Our result is in agreement with [10], where the transmission capacity (eq. 12) is a decreasing function of N for small values of λ.

VII. Simulations

In all the simulation results we use path-loss exponent $\alpha = 3$, SIR threshold $\beta = 3$ corresponding to $R = 2$ bits/sec/Hz, access probability $p = 1/2$, and transmission density $\lambda = 0.1$. In Fig. 1, for $N = 1$, we plot the success probability P_s together with the upper and lower bound as a function of D. We can see that the upper and lower bound are tight. In Figs. 2, and 3, we plot the transmission capacity, and the derived

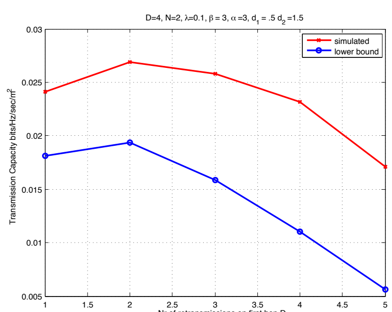

Fig. 3. Transmission capacity as a function of D_1 with $D = 4$ for non equidistant hops

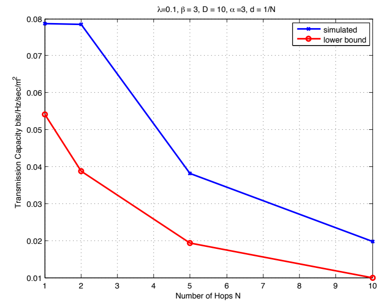

Fig. 4. Transmission capacity as a function of number of hops N for $\lambda = 0.1$

lower bound for two-hop communication $N = 2$, with respect to D_1, with $D = 4$, for equidistant hops $d_1 = d_2 = 1m$, and non equidistant hops $d_1 = 0.5m,\ d_2 = 1.5m$, respectively. The transmission capacity (simulated and the lower bound) is maximized at $D_1 = D_2 = 2$ for $d_1 = d_2 = 1m$, and $D_1 = 1,\ D_2 = 3$ for $d_1 = 0.5m,\ d_2 = 1.5m$ which is in accordance with Proposition 5. In Fig. 4, we plot the transmission capacity as the function of the number of hops N with $D = 10$ for $\lambda = 0.1$. For $\lambda = 0.1$, as derived in Proposition 6, optimal $N = 1$.

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