# A Stochastic Geometry Model for the Best Signal Quality in a Wireless Network

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***Abstract*—In a wireless network composed of randomly scattered nodes, the characterization of the distribution of the best signal quality received from a group of nodes is of primary importance for many network design problems. In this paper, using shot noise models for the interference field, we develop a framework for analyzing this distribution. We first identify the joint distribution of the interference and the maximum signal strength. We then represent the best signal quality as a function of these two quantities. Particular practical scenarios are also analyzed in which explicit expressions are obtained.**

***Index Terms*—shot noise, max sinr, joint interference and signal strength.**

## I. Introduction

Wireless communication is constrained by various impairments due to radio propagation characteristics such as propagation loss, fadings, noise and interference. To take these factors into account, the *signal-to-interference-plus-noise ratio* (SINR) is used as a basic metric for the description of the *signal quality*.

In a wireless communication network composed of many spatially scattered nodes, connecting a user to a node or a group of nodes which provides the *best signal quality* is a fundamental requirement.

The signal quality is a function of the signal power strength and the interference. Its analysis requires a model of the interference which is itself already a challenge [1]–[3]. Besides, as the SINRs received from a set of nodes which are sharing a common frequency band suffer from a common interference, they are stochastically dependent. Identifying the distribution of the best signal quality therefore requires some efforts.

In this paper, using a shot noise model for the interference field [4], we develop a framework for the distribution of the best signal quality. We consider that nodes are spatially distributed on the two-dimensional Euclidean plane according to a Poisson point process. Nodes are operating in a wireless network subject to distance-dependent path loss and a generic fading which can capture the effects of multi-path fading, shadowing, or both. We begin with the derivation of the joint distribution of the interference and the maximum signal strength. Using the fact that the best signal quality is a function of the interference and the maximum signal strength, the distribution of the best signal quality is derived. Some particular scenarios are also presented in which explicit expressions of these probabilistic characterizations are obtained.

Section II describes the studied model. Section III gives a representation of the best signal quality as a function of the interference and the maximum signal strength. In Section IV, we develop the joint distribution of these two quantities. The distribution of the best signal quality is analyzed in Section V. Concluding remarks are finally given in Section VI.

## II. System Model

The underlying network is composed of nodes with omnidirectional antennas. The set of all the nodes in the network is denoted by $\Omega$. We construct a model for studying the maximum signal strength, the interference, and the best signal quality after specifying essential parameters of the radio propagation and the spatial distribution of nodes.

In the following, the spatial distribution of nodes will be modeled by a Poisson point process with intensity $\lambda$ in a two dimensional plane $\mathbb{R}^2$.

The signal strength of node $i$ received at a position $\mathbf{y} \in \mathbb{R}^2$ is given by:

$$P_i(\mathbf{y}) = p_i / l(|\mathbf{y} - \mathbf{x}_i|), \tag{1}$$

where $l$ is path-loss function; the typical far-field model is:

$$l(|\mathbf{y} - \mathbf{x}|) = |\mathbf{y} - \mathbf{x}|^\beta \quad \text{for } \mathbf{x},\, \mathbf{y} \in \mathbb{R}^2,$$

with $\beta$ the path loss exponent; here we consider $2 < \beta$. And

$$p_i = AX_i, \tag{2}$$

where $A$ represents the node's transmission power; the variables $\{X_i,\ i = 1, 2, \ldots\}$ refer to fading and are assumed independent and identically distributed. Here, $X_i$ can be used to model slow fading (i.e., shadowing), multi-path fading, or both. Let $F_p$ be the common cumulative distribution function (cdf) of $\{p_i,\ i = 1, 2, \ldots\}$, and let denote $p = p_1$.

It is assumed that all the nodes share a common frequency band. The results can be easily generalized to networks with multiple frequencies. The signal quality is expressed in term of the SINR received at $\mathbf{y} \in \mathbb{R}^2$ from node $i$:

$$\zeta_i(\mathbf{y}) = \frac{P_i(\mathbf{y})}{N_0 + \sum_{j \neq i, j \in \Omega} P_j(\mathbf{y})}, \tag{3}$$

where $N_0$ is the thermal noise average power which is assumed constant. For notational simplicity, consider $A := A/N_0$, then:

$$\zeta_i(\mathbf{y}) = \frac{P_i(\mathbf{y})}{1 + \sum_{j \neq i, j \in \Omega} P_j(\mathbf{y})}. \tag{4}$$

In the following, we will use (4) instead of (3).

## III. Preliminary

Given a set of nodes $S \subset \Omega$, the *best signal quality* received from $S$ at position $\mathbf{y} \in \mathbb{R}^2$, denoted by $Y_S(\mathbf{y})$, is defined as:

$$Y_S(\mathbf{y}) = \max_{i \in S} \zeta_i(\mathbf{y}). \quad (5)$$

***Lemma** 1: In the set $S$ of nodes, the node which provides the maximum signal strength also provides the best signal quality which is*

$$Y_S(\mathbf{y}) = \frac{M_S(\mathbf{y})}{1 + I(\mathbf{y}) - M_S(\mathbf{y})}, \quad \forall \mathbf{y} \in \mathbb{R}^2, \quad (6)$$

*where*

$$M_S(\mathbf{y}) = \max_{i \in S} P_i(\mathbf{y})$$

*is the maximum signal strength received at $\mathbf{y}$ from the set $S$, and*

$$I(\mathbf{y}) = \sum_{i \in \Omega} P_i(\mathbf{y})$$

*is the total interference received at $\mathbf{y}$.*

*Proof:* The proof is straightforward. We can rewrite (4) as $\zeta_i(\mathbf{y}) = P_i(\mathbf{y})/\{1 + I(\mathbf{y}) - P_i(\mathbf{y})\}$. Since $P_i(\mathbf{y}) < I(\mathbf{y})$, (6) follows from the fact that no matter which node $i \in \Omega$ is considered, $I(\mathbf{y})$ is the same; and from the fact that $x/(c-x)$ with $c$ constant is an increasing function of $x < c$. □

From Lemma 1, the distribution of $Y_S$ can be determined by the joint distribution of $M_S$ and $I$. We identify their joint distribution in the following section.

***Remark***. For notational simplicity, the location variable $\mathbf{y}$ appearing in $Y_S(\mathbf{y})$, $M_S(\mathbf{y})$, and $I(\mathbf{y})$ will be omitted in case of no ambiguity.

## IV. Joint Distribution of the Interference and the Maximum Signal Strength

Under the model described in Section II, the interference field can be modeled as a shot noise [4, Chap. 2] which is:

- Defined on the two-dimensional Euclidean plane $\mathbb{R}^2$, and taking values in $\mathbb{R}_+$,
- Generated by an independently marked Poisson point process $\widetilde{\Phi}$ of intensity $\lambda$, and a set of marks $\{p_i,\, i = 1, 2...\}$ which are independent and identically distributed on $\mathbb{R}_+$,
- Associated with a response function $L(\mathbf{x}, \mathbf{y}, p) \triangleq p/l(|\mathbf{x} - \mathbf{y}|)$, for $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$, and $p \in \mathbb{R}_+$.

Given a network area $B$, we are interested in a set $S$ of nodes which are selected in $B$ according to some given criterion. An interesting case is when nodes located in $B$ are randomly selected with equal probability $\rho$. In this case, the nodes of set $S$ are distributed in $B$ according to a Poisson point process of intensity $\rho\lambda$, c.f. Proposition 1.3.5 in [4].

***Theorem** 1: Given $B \subset \mathbb{R}^2$ and $\rho \in [0, 1]$, let $S$ be the set of nodes which are randomly selected from $B$ with equal probability $\rho$. For $u \geq 0$ and $s \in \mathbb{C}$ with a non-negative real part, define:*

$$\mathcal{L}_{(I, M_S \leq u)}(s) = \mathbf{E}\{1_{(M_S \leq u)} \exp(-sI)\},$$

*where $\mathbf{E}\{\cdot\}$ is the mathematical expectation. If the interference field is modeled as the shot noise as described above, then:*

$$\begin{aligned} &\mathcal{L}_{(I, M_S \leq u)}(s) = \\ &= \exp\Big(-\lambda \int_{\mathbb{R}^2} \Big(1 - \mathcal{L}_p(\frac{s}{l(|\mathbf{y} - \mathbf{x}|)})\Big) \mathrm{d}\mathbf{x} \\ &\quad - \rho\lambda \int_B \mathbf{E}\Big\{1_{(\frac{p}{l(|\mathbf{y}-\mathbf{x}|)} > u)} \exp(\frac{-sp}{l(|\mathbf{y} - \mathbf{x}|)})\Big\} \mathrm{d}\mathbf{x}\Big). \end{aligned} \quad (7)$$

*where $\mathcal{L}_p(s) \triangleq \mathbf{E}\{\exp(-sp)\}$ is the Laplace transform of the random variable $p = p_1$.*

*Proof:* See Appendix A. □

Theorem 1 may be useful for various studies related to the interference and the maximum signal strength.

Before investigating more properties, let us consider a few special cases. We first look at the special case where $B = \mathbb{R}^2$ and $\rho = 1$. Denote by $M = \max_{i \in \Omega} P_i(\mathbf{y})$ the global maximum signal strength of the network. From (7):

$$\begin{aligned} &\mathcal{L}_{(I, M \leq u)}(s) \\ &= \exp\Big(-2\pi\lambda \int_0^\infty \Big(1 - \mathcal{L}_p(\frac{s}{l(r)}) + \mathbf{E}\{1_{(\frac{p}{l(r)} > u)} e^{\frac{-sp}{l(r)}}\}\Big) r \mathrm{d}r\Big) \\ &= \exp\Big(-2\pi\lambda \int_0^\infty \Big(1 - \mathbf{E}\{1_{(\frac{p}{l(r)} \leq u)} e^{\frac{-sp}{l(r)}}\}\Big) r \mathrm{d}r\Big). \end{aligned} \quad (8)$$

Now using the fact that $\mathbf{P}\{M_S \leq u\} = \mathcal{L}_{(I, M_S \leq u)}(0)$, from (7) we get the cumulative distribution function (cdf) of $M_S$:

$$F_{M_S}(u) = \exp\left(-\rho\lambda \int_B \Big(1 - F_p(l(|\mathbf{y} - \mathbf{x}|)u)\Big) \mathrm{d}\mathbf{x}\right). \quad (9)$$

We find back the well known result on the distribution of the max shot noise (see e.g. Proposition 2.4.2 in [4]).

Notice that, when assuming that $F_p(0) < 1$ ($F_p(0) = 1$ corresponds to the case where the node's transmission power is 0 since the fading $X_i$ satisfies $F_X(0) < 1$) and that $\rho\lambda > 0$, which will be done here and below, we get that

$$\int_B \Big(1 - F_p(l(|y - x|)u)\Big) \mathrm{d}\mathbf{x}|_{(u=0)} = \infty \quad \text{iff}\, B = \mathbb{R}^2.$$

Therefore, from (9) $F_{M_S}(0) = 0$ if and only if (iff) $B = \mathbb{R}^2$. Otherwise, $F_{M_S}(0) > 0$, which means that there is a mass of $M_S$ at the origin, which is

$$F_{M_S}(0) = \exp\left(-\rho\lambda(1 - F_p(0))|B|\right), \quad (10)$$

with $|B|$ the Lebesgue measure of $B$.

Another interesting special case is when $B$ is a disk of radius $R_B$ and the path loss function is $l(r) = r^\beta$. In this case, $F_{M_S}$ is given in closed form as follows:

***Corollary** 1: Under the conditions of Theorem 1, denote by $M_S$ the maximum signal strength received at the center of a disk $B$ of radius $R_B$. If the path loss function is $l(r) = r^\beta$ for $r \in \mathbb{R}_+$, then:*

$$\begin{aligned} F_{M_S}(u) = \exp\Big(&-\pi\rho\lambda R_B^2 \Big(1 - F_p(uR_B^\beta)\Big) \\ &- \pi\rho\lambda \mathbf{E}\{1_{(p \leq uR_B^\beta)} p^\alpha\} u^{-\alpha}\Big), \end{aligned} \quad (11)$$

*where $\alpha = 2/\beta$. In particular, if $B = \mathbb{R}^2$ then:*

$$F_{M_S}(u) = \exp\left(-\pi\rho\lambda\mathbf{E}\{p^\alpha\}u^{-\alpha}\right). \quad (12)$$

Here, note that for some constant $a > 0$:

- If $p$ follows a Rayleigh dist. of parameter $\sigma > 0$:

$$\mathbf{E}\{1_{(p\le a)}p^\alpha\} = 2^{\frac{\alpha}{2}}\sigma^\alpha\Big(\Gamma(1+\frac{\alpha}{2}) - \Gamma(1+\frac{\alpha}{2}, \frac{a^2}{2\sigma^2})\Big),$$

  where $\Gamma(s) \triangleq \int_0^\infty t^{s-1}e^{-t}\mathrm{d}t$ for $\Re(s) > 0$ is the gamma function, and $\Gamma(s,z) \triangleq \int_z^\infty t^{s-1}e^{-t}\mathrm{d}t$ is the upper incomplete gamma function.
- If $p$ follows a Gamma dist. of shape parameter $k > 0$ and scale parameter $\theta > 0$:

$$\mathbf{E}\{1_{(p\le a)}p^\alpha\} = \theta^\alpha\left(\Gamma(k+\alpha) - \Gamma(k+\alpha, a/\theta)\right)/\Gamma(k).$$

- If $p$ follows a Lognormal dist. of parameter $(\mu, \sigma > 0)$:

$$\mathbf{E}\{1_{(p\le a)}p^\alpha\} = \frac{e^{\alpha\mu + \frac{\alpha^2\sigma^2}{2}}}{2}\mathrm{erfc}\left(\frac{\mu + \alpha\sigma^2 - \log(a)}{\sqrt{2}\sigma}\right).$$

*Proof:* Substitute $l(r) = r^\beta$, $\mathrm{d}\mathbf{x} = r\mathrm{d}r\mathrm{d}\theta$ into (9):

$$\int_B \Big(1 - F_p(l(|\mathbf{y}-\mathbf{x}|)u)\Big)\mathrm{d}x = 2\pi\int_0^{R_B}\Big(1 - F_p(ur^\beta)\Big)r\mathrm{d}r.$$

An integration by parts with $1 - F_p(ur^\beta)$ and $2r\mathrm{d}r$ yields

$$2\int_0^{R_B}\left(1 - F_p(ur^\beta)\right)r\mathrm{d}r = R_B^2\left(1 - F_p(uR_B^\beta)\right) + \int_0^{R_B} r^2\mathrm{d}F_p(ur^\beta).$$

where using the change of variable $t = ur^\beta$, we have:

$$\int_0^\infty r^2\mathrm{d}F_p(ur^\beta) = \int_0^{uR_B^\beta}(\frac{t}{u})^{2/\beta}F_p(\mathrm{d}t) = \frac{\mathbf{E}\{1_{(p\le uR_B^\beta)}p^\alpha\}}{u^\alpha}.$$

The proof for the case of $B = \mathbb{R}^2$ is straightforward. □

***Remark** 1:* As shown by (12) where the expression on right hand-side is a Fréchet distribution with shape parameter $\alpha$ and scale parameter $(\pi\rho\lambda\mathbf{E}\{p^\alpha\})^{1/\alpha}$, we find back the result on the regular variation of the max shot noise [5].

Here is another corollary of Theorem 1 for the case of a disk.

***Corollary** 2: For $u \ge 0$ define:*

$$\phi_{(I,M_S\le u)}(w) = \mathbf{E}\{1_{(M_S\le u)}\exp(jwI)\}, \quad \text{for } w \in \mathbb{R}.$$

*Assume the same conditions of Theorem 1, and that $B$ is a disk of radius $R_B$, $l(r) = r^\beta$ for $r \in \mathbb{R}_+$ and $\beta > 2$. Let $M_S$ and $I$ be the max power and the interference received at the center of $B$. Then*

$$\phi_{(I,M_S\le u)}(w) = \exp\Big(-\delta|w|^\alpha\big(1 - j\mathrm{sign}(w)\tan(\frac{\pi\alpha}{2})\big) - \rho\pi\lambda\alpha|w|^\alpha\mathbf{E}\Big\{\int_{\frac{|w|u}{p}}^{+\infty}\frac{1_{(p\le uR_B^\beta)}e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t\Big\} - \rho\pi\lambda\alpha|w|^\alpha\mathbf{E}\Big\{\int_{\frac{|w|}{R_B^\beta}}^{+\infty}\frac{1_{(p>uR_B^\beta)}e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t\Big\}\Big), \quad (13)$$

*where $\delta = \pi\lambda\mathbf{E}\{p^\alpha\}\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})$.*

*Proof:* See Appendix B. □

Denote by $\phi_I$ the characteristic function of $I$. Since $\phi_I(w) = \phi_{(I,M_S\le+\infty)}(w)$, we have:

$$\phi_I(w) = \exp\Big(-\delta|w|^\alpha\big(1 - j\mathrm{sign}(w)\tan(\frac{\pi\alpha}{2})\big)\Big). \quad (14)$$

We find back the same result given in [3].

***Corollary** 3: Under the assumptions of Corollary 2, if $B = \mathbb{R}^2$ then:*

$$\phi_{(I,M_S\le u)}(w) = \exp\Big(-C_1|w|^\alpha\Big(1 - j\mathrm{sign}(w)\tan(\frac{\pi\alpha}{2})\Big) - C_2(w,u) + jC_3(w,u)\Big), \quad (15)$$

*where $C_1 = (1-\rho)\delta$, and*

$$C_2(w,u) = \rho\pi\lambda\mathbf{E}\{p^\alpha\}\frac{{}_1F_2(-\frac{\alpha}{2};\frac{1}{2},1-\frac{\alpha}{2};-\frac{u^2w^2}{4})}{u^\alpha}.$$

$$C_3(w,u) = \rho\pi\lambda\mathbf{E}\{p^\alpha\}\frac{\alpha w}{1-\alpha}\frac{{}_1F_2(\frac{1-\alpha}{2};\frac{3}{2},\frac{3-\alpha}{2};-\frac{u^2w^2}{4})}{u^{\alpha-1}}.$$

*with ${}_1F_2$ denoting the hypergeometric function,*

$${}_1F_2(a_1;b_1,b_2;z) \triangleq \sum_{n=0}^\infty\frac{(a_1)_n}{(b_1)_n(b_2)_n}\frac{z^n}{n!}, \quad (16)$$

*where $(a)_n$ represents the Pochhammer symbol:*

$$(a)_n = a(a+1)(a+2)\dots(a-n+1), \quad \text{and } (a)_0 = 1. \quad (17)$$

*Proof:* See Appendix C. □

***Remark** 2:* In many practical scenarios, the case where $R_B$ is large is of interest. For example, macro cellular networks are often deployed for scattered traffic areas and have a low density of base stations. Considering $B_N$ as a disk with radius $R_N = \sqrt{N/(\pi\lambda)}$, which has in average $N$ base stations; its radius increases rapidly as $N$ increases. And so, one can consider that $R_N \approx \infty$ for some moderate value of $N$. In such a situation, one can use (15) as an approximation of $\phi_{(I,M_S\le u)}(w)$.

We have identified $F_{M_S}$ and $\phi_I$ as well as the joint distribution of $M_S$ and $I$. In the following, we will further investigate some more properties of this joint distribution when assuming that it admits a density.

***Theorem** 2: Assume that the conditions of Theorem 1 hold and that $l(r) = r^\beta$ for $r \in \mathbb{R}_+$, $\beta > 2$, and $0 < \mathbf{E}\{p_i^\alpha\} < \infty$. Then:*

(i) *$|\phi_{(I,M_S\le u)}(w)|^q \in L$ with respect to (w.r.t.) $w$ for all $q = 1, 2, \dots$, $\forall u > 0$.*

(ii) *If $F_p$ admits a continuous density $f_p$, then $\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w)$ exists, and $|\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w)|^q \in L$ w.r.t. $w$ for all $q = 1, 2, \dots$, $\forall u > 0$.*

*where $L$ denotes the space of absolutely integrable functions.*

*Proof:* See Appendix D □

***Corollary** 4: Assume that the conditions of Theorem 2 hold and that $F_p$ admits a continuous density $f_p$. Then $(I, M_S)$ admits a joint density $f_{(I,M_S)}(v,u)$ on $(\mathbb{R}_+^*)^2$ and the function*

$v \to f_{(I,M_S)}(v,u)$ *is bounded, continuous, square-integrable. In addition, for all* $u > 0$,

$$f_{(I,M_S)}(v,u) = \int_{-\infty}^{+\infty} \frac{e^{-jwv}}{2\pi} \frac{\partial}{\partial u} \phi_{(I,M_S \le u)}(w) \mathrm{d}w. \quad (18)$$

*Proof:* By definition,

$$\frac{\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w)}{\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(0)}$$

is the characteristic function of $I$ conditional on $M_S = u$. From (ii) of Theorem 2, for all $u > 0$, $w \to |\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w)| \in L$. So by Theorem 3 in [6, p.509], for all $u > 0$, the law of $I$ conditional on $M_S = u$ admits a density and $(I, M_S)$ hence admits a joint density $f_{(I,M_S)}(v,u)$ at $v$ which is bounded and continuous w.r.t. $v$, and which is given by (18). Secondly, since $|\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w)|^2 \in L$ as shown by (ii) of Theorem 2, $f_{(I,M_S)}(v,u)$ is square-integrable w.r.t. $v$ [6, p.510]. □

## V. Distribution of the Best Signal Quality

The next theorem gives the tail distribution function $\bar{F}_{Y_S}$ of the best signal quality.

***Theorem** 3: Assume that the conditions of Theorem 2 hold and that $F_p$ admits a continuous density $f_p$. Then for all $\gamma > 0$*

$$\bar{F}_{Y_S}(\gamma) = \frac{1}{2\pi}\int_{u=\gamma}^{+\infty}\int_{-\infty}^{+\infty} \phi_{(I,M_S\le u)}(w) g(w,u) \mathrm{d}w\mathrm{d}u, \quad (19)$$

*where*

$$g(w,u) = e^{-jwu} - (1+\gamma^{-1}) e^{jw(1-\frac{1+\gamma}{\gamma}u)} \quad (20)$$

*and*

$$\bar{F}_{Y_S}(0) = 1 - \exp\left(-\rho\lambda(1 - F_p(0))|B|\right).$$

*Proof:* See Appendix E. □

***Corollary** 5: Under the assumptions of Theorem 2, if $B = \mathbb{R}^2$, then $\bar{F}_{Y_S}(0) = 1$, and*

$$\begin{aligned}\bar{F}_{Y_S}(\gamma) = &\int_{u=\gamma}^{+\infty}\int_{w=0}^{+\infty} \frac{1}{\pi}\exp\Big\{-(C_1 w^\alpha + C_2(w,u))\Big\} \\ &\times\Big(-\frac{1+\gamma}{\gamma}\cos\Big\{C_1 w^\alpha \tan(\frac{\pi\alpha}{2}) + C_3(w,u) + C_4(w,u)\Big\} \\ &+\cos\Big\{C_1 w^\alpha \tan(\frac{\pi\alpha}{2}) + C_3(w,u) - wu\Big\}\Big)\mathrm{d}w\mathrm{d}u, \quad (21)\end{aligned}$$

*for $\gamma > 0$ where $C_4(w,u) = w(1 - \frac{1+\gamma}{\gamma}u)$.*

*Proof:* See Appendix F. □

## VI. Conclusion

Using a shot noise model for the interference field, we have analyzed the joint distribution of the interference and the maximum signal strength, and thereby identified the distribution of the best signal quality. In addition, this framework also allows one to obtain marginal distributions of the interference, and the maximum signal strength. These results are expected to enable various studies related to the stochastic modeling of communication networks.

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## Appendix A
## Proof of Theorem 1

Under the provided assumptions, the nodes of set $S$ are distributed in $B$ according to a Poisson point process of intensity $\rho\lambda$. And the nodes located in $B$ but not retained in $S$ are distributed $B$ according to an independent Poisson point process of intensity $(1-\rho)\lambda$. Thus, we can decompose the marked Poisson point process defined on $\mathbb{R}^2$ into three independent marked Poisson point processes such that: the first $\tilde{\Phi}_1$ is defined on $B$ and has intensity $\rho\lambda$, the second $\tilde{\Phi}_2$ is defined on $B$ and has intensity $(1-\rho)\lambda$, and the third $\tilde{\Phi}_3$ is defined on $\bar{B} = \mathbb{R}^2 \setminus B$ and has intensity $\lambda$. Note that for a node $i : i \notin S$, then $(\mathbf{x}_i, p_i) \in \tilde{\Phi}_2 \cup \tilde{\Phi}_3$.

We have:

$$\begin{aligned}&\mathbf{E}\{1_{(M_S\le u)}\exp(-sI)\} = \\ &= \mathbf{E}\Big\{1_{(M_S\le u)}\exp(-s\sum_{i\in\Omega}P_i(\mathbf{y}))\Big\} \\ &= \mathbf{E}\Big\{\exp\Big(\sum_{i\in S}\big(\ln 1_{(P_i(\mathbf{y})\le u)} - sP_i(\mathbf{y})\big) - s\sum_{i\notin S}P_i(\mathbf{y})\Big)\Big\} \\ &= \mathbf{E}\Big\{\exp\Big(\sum_{(\mathbf{x}_i,p_i)\in\tilde{\Phi}_1}\big(\ln 1_{(P_i(\mathbf{y})\le u)} - sP_i(\mathbf{y})\big) \\ &\qquad - s\sum_{(\mathbf{x}_i,p_i)\in\tilde{\Phi}_2\cup\tilde{\Phi}_3}P_i(\mathbf{y})\Big)\Big\}.\end{aligned}$$

Using Proposition 2.2.4 in [4], we have:

$$\begin{aligned}&\mathbf{E}\{1_{(M_S\le u)}\exp(-sI)\} = \\ &= \exp\Big(-\rho\lambda\int_B\Big(1 - \mathbf{E}\{1_{(\frac{p}{l(|\mathbf{y}-\mathbf{x}|)}\le u)}e^{\frac{-sp}{l(|\mathbf{y}-\mathbf{x}|)}}\}\Big)\mathrm{d}\mathbf{x} \\ &\qquad -(1-\rho)\lambda\int_B\Big(1 - \mathbf{E}\{e^{\frac{-sp}{l(|\mathbf{y}-\mathbf{x}|)}}\}\Big)\mathrm{d}\mathbf{x} \\ &\qquad -\lambda\int_{\bar{B}}\Big(1 - \mathbf{E}\{e^{\frac{-sp}{l(|\mathbf{y}-\mathbf{x}|)}}\}\Big)\mathrm{d}\mathbf{x}\Big). \quad (22)\end{aligned}$$

Note that $\mathbf{E}\{e^{\frac{-sp}{l(|\mathbf{y}-\mathbf{x}|)}}\} = \mathcal{L}_p(\frac{s}{l(|\mathbf{y}-\mathbf{x}|)})$, this gives us:

$$\begin{aligned}&\mathbf{E}\{1_{(M_S\le u)}\exp(-sI)\} = \\ &= \exp\Big(-\lambda\int_{\mathbb{R}^2}\Big(1-\mathcal{L}_p(\frac{s}{l(|\mathbf{y}-\mathbf{x}|)})\Big)\mathrm{d}\mathbf{x} \\ &\quad -\rho\lambda\int_B \mathbf{E}\Big\{1_{(\frac{p}{l(|\mathbf{y}-\mathbf{x}|)}>u)}e^{\frac{-sp}{l(|\mathbf{y}-\mathbf{x}|)}}\Big\}\mathrm{d}\mathbf{x}\Big).\end{aligned}$$

□

## APPENDIX B
## PROOF OF COROLLARY 2

By definition we have:

$$\phi_{(I,M_S\le u)}(w) = \mathcal{L}_{(I,M_S\le u)}(-jw).$$

Substitute $l(r) = r^\beta$ and $\mathrm{d}\mathbf{x} = r\mathrm{d}r\mathrm{d}\theta$ into (7) of Theorem 1; this gives us

$$\begin{aligned}\phi_{(I,M_S\le u)}(w) &= \exp\Big(-2\pi\lambda\int_0^{+\infty}(1-\mathbf{E}\{\exp(\frac{jwp}{r^\beta})\})r\mathrm{d}r \\ &\quad -2\pi\rho\lambda\int_0^{R_B}\mathbf{E}\{1_{(p>ur^\beta)}\exp(\frac{jwp}{r^\beta})\}r\mathrm{d}r\Big). \quad (23)\end{aligned}$$

Using the change of variable $t = |w|r^{-\beta}$ and $\alpha = 2/\beta$:

$$\begin{aligned}&\int_0^{+\infty}(1-\mathbf{E}\{\exp(\frac{jwp}{r^\beta})\})r\mathrm{d}r \\ &\quad = \frac{\alpha|w|^\alpha}{2}\int_0^{+\infty}\frac{1-\mathbf{E}\{e^{j\mathrm{sign}(w)tp}\}}{t^{\alpha+1}}\mathrm{d}t. \quad (24)\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^{R_B}\mathbf{E}\{1_{(p>ur^\beta)}\exp(\frac{jwp}{r^\beta})\}r\mathrm{d}r \\ &\quad = \frac{\alpha|w|^\alpha}{2}\int_{\frac{|w|}{R_B^\beta}}^{+\infty}\frac{\mathbf{E}\{1_{(t>\frac{u|w|}{p})}e^{j\mathrm{sign}(w)tp}\}}{t^{\alpha+1}}\mathrm{d}t. \quad (25)\end{aligned}$$

For $0<\alpha<1$ we have:

$$\int_0^{+\infty}\frac{1-e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t = -\Gamma(-\alpha)p^\alpha e^{-j\mathrm{sign}(w)\frac{\pi\alpha}{2}}. \quad (26)$$

Taking expectations on both sides of (26), we get

$$\int_0^{+\infty}\frac{1-\mathbf{E}\{e^{j\mathrm{sign}(w)tp}\}}{t^{\alpha+1}}\mathrm{d}t = -\mathbf{E}\{p^\alpha\}\Gamma(-\alpha)e^{-j\mathrm{sign}(w)\frac{\pi\alpha}{2}}.$$

From (24) we have

$$\begin{aligned}&2\pi\lambda\int_0^{+\infty}(1-\mathbf{E}\{\exp(\frac{jwp}{r^\beta})\})r\mathrm{d}r \\ &\quad = \pi\lambda\mathbf{E}\{p^\alpha\}\Gamma(1-\alpha)(-jw)^\alpha. \quad (27)\end{aligned}$$

And for (25), we simply have

$$\begin{aligned}&\int_{\frac{|w|}{R_B^\beta}}^{+\infty}\frac{\mathbf{E}\{1_{(t>\frac{u|w|}{p})}e^{j\mathrm{sign}(w)tp}\}}{t^{\alpha+1}}\mathrm{d}t \\ &\quad = \mathbf{E}\Big\{\int_{\frac{|w|}{R_B^\beta}}^{+\infty}\frac{1_{(p>uR_B^\beta)}e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t\Big\} \\ &\quad + \mathbf{E}\Big\{\int_{\frac{|w|u}{p}}^{+\infty}\frac{1_{(p\le uR_B^\beta)}e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t\Big\}. \quad (28)\end{aligned}$$

Finally, substituting this and (27) into (23), we obtain (13). □

## APPENDIX C
## PROOF OF COROLLARY 3

Condition $B = \mathbb{R}^2$ induces $R_B = \infty$. Take this into account for (13), we have:

$$\mathbf{E}\Big\{\int_{\frac{|w|}{R_B^\beta}}^{+\infty}\frac{1_{(p>uR_B^\beta)}e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t\Big\} = 0, \quad (29)$$

And

$$\mathbf{E}\Big\{\int_{\frac{|w|u}{p}}^{+\infty}\frac{1_{(p\le uR_B^\beta)}e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t\Big\} = \mathbf{E}\Big\{\int_{\frac{|w|u}{p}}^{+\infty}\frac{e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t\Big\}.$$

For $0<\alpha<1$, the integral on the right hand-side of this can be evaluated with help of a symbolic calculation software like Mathematica:

$$\begin{aligned}&\int_{\frac{|w|u}{p}}^{+\infty}\frac{e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t = p^\alpha\Gamma(-\alpha)e^{-j\mathrm{sign}(w)\frac{\pi\alpha}{2}} \\ &\quad -j\frac{wu}{1-\alpha}\Big(\frac{p}{|w|u}\Big)^\alpha {}_1F_2(\frac{1-\alpha}{2};\frac{3}{2},\frac{3-\alpha}{2};-\frac{u^2w^2}{4}) \\ &\quad +\frac{1}{\alpha}\Big(\frac{p}{|w|u}\Big)^\alpha {}_1F_2(-\frac{\alpha}{2};\frac{1}{2},1-\frac{\alpha}{2};-\frac{u^2w^2}{4}).\end{aligned}$$

Taking expectations on both sides, we obtain

$$\begin{aligned}&\mathbf{E}\Big\{\int_{\frac{|w|u}{p}}^{+\infty}\frac{e^{j\mathrm{sign}(w)tp}}{t^{\alpha+1}}\mathrm{d}t\Big\} = \mathbf{E}\{p^\alpha\}\Gamma(-\alpha)e^{-j\mathrm{sign}(w)\frac{\pi\alpha}{2}} \\ &\quad -j\mathbf{E}\{p^\alpha\}\frac{w}{1-\alpha}\frac{{}_1F_2(\frac{1-\alpha}{2};\frac{3}{2},\frac{3-\alpha}{2};-\frac{u^2w^2}{4})}{|w|^\alpha u^{\alpha-1}} \\ &\quad +\mathbf{E}\{p^\alpha\}\frac{{}_1F_2(-\frac{\alpha}{2};\frac{1}{2},1-\frac{\alpha}{2};-\frac{u^2w^2}{4})}{\alpha|w|^\alpha u^\alpha}. \quad (30)\end{aligned}$$

Substituting (30) and (29) back to (13), we get (15). □

## APPENDIX D
## PROOF OF THEOREM 2

We begin with the proof of (i). Use (7) of Theorem 1:

$$\begin{aligned}&\phi_{(I,M_S\le u)}(w) = \mathcal{L}_{(I,M_S\le u)}(-jw) \\ &\quad = \exp\Big(-2\pi\lambda\int_{\mathbb{R}_+}\Big(1-\mathbf{E}\{\exp(\frac{jwp}{l(r)})\}\Big)r\mathrm{d}r \\ &\quad -\rho\lambda\int_B\mathbf{E}\{1_{(p>ul(|\mathbf{y}-\mathbf{x}|))}\exp(\frac{jwp}{l(|\mathbf{y}-\mathbf{x}|)})\}\mathrm{d}\mathbf{x}\Big). \quad (31)\end{aligned}$$

It follows that

$$|\phi_{(I,M_S\le u)}(w)|^q = \exp\Big(-2q\pi\lambda\int_{\mathbb{R}_+}\Big(1-\mathbf{E}\{\cos(\frac{wp}{l(r)})\}\Big)r\mathrm{d}r \\ - q\rho\lambda\int_B \mathbf{E}\{1_{(p>ul(|\mathbf{y}-\mathbf{x}|))}\cos(\frac{wp}{l(|\mathbf{y}-\mathbf{x}|)})\}\mathrm{d}\mathbf{x}\Big). \quad (32)$$

For the first term in the exponential, substituting $l(r)=r^\beta$ and noting that

$$\int_{\mathbb{R}_+}\Big(1-\mathbf{E}\{\cos(\frac{wp}{r^\beta})\}\Big)r\mathrm{d}r \\ = \Re\Big\{\int_{\mathbb{R}_+}\Big(1-\mathbf{E}\{\exp(\frac{jwp}{r^\beta})\}\Big)r\mathrm{d}r\Big\},$$

we get from (27) that:

$$-2q\pi\lambda\int_{\mathbb{R}_+}\Big(1-\mathbf{E}\{\cos(\frac{wp}{r^\beta})\}\Big)r\mathrm{d}r \\ = -q\pi\lambda\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})\mathbf{E}\{p^\alpha\}|w|^\alpha. \quad (33)$$

For the second integral of (32), we have:

$$\begin{aligned} -&\int_B \mathbf{E}\{1_{(p>ul(|\mathbf{y}-\mathbf{x}|))}\cos(\frac{wp}{l(|\mathbf{y}-\mathbf{x}|)})\}\mathrm{d}\mathbf{x} \\ =&\ -\int_B\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty \cos(\frac{wt}{l(|\mathbf{y}-\mathbf{x}|)})F_p(\mathrm{d}t)\mathrm{d}\mathbf{x} \\ =&\ -\int_B\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty \Big(1-1+\cos(\frac{wt}{l(|\mathbf{y}-\mathbf{x}|)})\Big)F_p(\mathrm{d}t)\mathrm{d}\mathbf{x} \\ =&\ \int_B\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty F_p(\mathrm{d}t)\mathrm{d}\mathbf{x} \\ &\ -\int_B\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty \Big(1+\cos(\frac{wt}{l(|\mathbf{y}-\mathbf{x}|)})\Big)F_p(\mathrm{d}t)\mathrm{d}\mathbf{x}, \end{aligned}$$

where, since $1+\cos(\cdot)\ge 0$, we have:

$$\begin{aligned} -&\int_B \mathbf{E}\{1_{(p>ul(|\mathbf{y}-\mathbf{x}|))}\cos(\frac{wp}{l(|\mathbf{y}-\mathbf{x}|)})\}\mathrm{d}\mathbf{x} \\ &\le \int_B\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty F_p(\mathrm{d}t)\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^2}\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty F_p(\mathrm{d}t)\mathrm{d}\mathbf{x} \\ &= 2\pi\int_{\mathbb{R}_+}\big(1-F_p(ur^\beta)\big)r\mathrm{d}r. \end{aligned} \quad (34)$$

Using an integration by parts with $1-F_p(ur^\beta)$ and $2r\mathrm{d}r$:

$$2\int_{\mathbb{R}_+}\big(1-F_p(ur^\beta)\big)r\mathrm{d}r = \int_0^\infty r^2\mathrm{d}F_p(ur^\beta) \\ = \int_0^\infty (\frac{t}{u})^\alpha F_p(\mathrm{d}t) = \frac{\mathbf{E}\{p^\alpha\}}{u^\alpha}.$$

Taking this into account in (34), and then substituting (34) and (33) into (32), we get:

$$|\phi_{(I,M_S\le u)}(w)|^q \le \exp\Big(q\pi\lambda\rho u^{-\alpha}\mathbf{E}\{p^\alpha\}\Big) \\ \times\exp\Big(-p\pi\lambda\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})\mathbf{E}\{p^\alpha\}|w|^\alpha\Big). \quad (35)$$

Under the condition that $\beta>2$, we have $0<\alpha<1$ and so $\cos(\frac{\pi\alpha}{2})>0$ and $\Gamma(1-\alpha)>0$. Hence, provided that $0<\mathbf{E}\{p^\alpha\}<\infty$, the right hand-side of (35) is integrable w.r.t. $w$, $\forall q=1,2,\ldots$, and $u>0$. This proves claim (i).

We now prove (ii). We first show that $\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w)$ exists for all $u>0$. From (31), we can have that the differentiability of $\phi_{(I,M_S\le u)}(w)$ w.r.t. $u$ is implied by that of the second term in the exponential of (31) and that when the latter holds,

$$\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w) = -\rho\lambda\phi_{(I,M_S\le u)}(w) \\ \times\frac{\partial}{\partial u}\int_B\mathbf{E}\{1_{(p>ul(|\mathbf{y}-\mathbf{x}|))}\exp(\frac{jwp}{l(|\mathbf{y}-\mathbf{x}|)})\}\mathrm{d}\mathbf{x}, \quad (36)$$

where

$$\int_B\mathbf{E}\{1_{(p>ul(|\mathbf{y}-\mathbf{x}|))}\exp(\frac{jwp}{l(|\mathbf{y}-\mathbf{x}|)})\}\mathrm{d}\mathbf{x} \\ = \int_B\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty e^{\frac{jwt}{l(|\mathbf{y}-\mathbf{x}|)}}F_p(\mathrm{d}t)\mathrm{d}\mathbf{x}. \quad (37)$$

Under the condition that $F_p$ admits a continuous density $f_p$:

$$\frac{\partial}{\partial u}\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty e^{\frac{jwt}{l(|\mathbf{y}-\mathbf{x}|)}}F_p(\mathrm{d}t) = \frac{\partial}{\partial u}\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty e^{\frac{jwt}{l(|\mathbf{y}-\mathbf{x}|)}}f_p(t)\mathrm{d}t \\ = -e^{jwu}l(|\mathbf{y}-\mathbf{x}|)f_p(ul(|\mathbf{y}-\mathbf{x}|)),$$

and the last expression is continuous w.r.t. $u$. Taking this into account in (37), we have:

$$\begin{aligned} &\frac{\partial}{\partial u}\int_B\mathbf{E}\{1_{(p>ul(|\mathbf{y}-\mathbf{x}|))}\exp(\frac{jwp}{l(|\mathbf{y}-\mathbf{x}|)})\}\mathrm{d}\mathbf{x} \\ &= \int_B\frac{\partial}{\partial u}\int_{ul(|\mathbf{y}-\mathbf{x}|)}^\infty e^{\frac{jwt}{l(|\mathbf{y}-\mathbf{x}|)}}f_p(t)\mathrm{d}t\mathrm{d}\mathbf{x} \\ &= -\int_B e^{jwu}l(|\mathbf{y}-\mathbf{x}|)f_p(ul(|\mathbf{y}-\mathbf{x}|))\mathrm{d}\mathbf{x}. \end{aligned}$$

Thus under the above assumptions, $\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w)$ exists and

$$|\frac{\partial}{\partial u}\phi_{(I,M_S\le u)}(w)|^q = |\rho\lambda\phi_{(I,M_S\le u)}(w)|^q \\ \times\Big(\int_B l(|\mathbf{y}-\mathbf{x}|)f_p(ul(|\mathbf{y}-\mathbf{x}|))\mathrm{d}\mathbf{x}\Big)^q. \quad (38)$$

Moreover, we have:

$$\begin{aligned} &\int_B l(|\mathbf{y}-\mathbf{x}|)f_p(ul(|\mathbf{y}-\mathbf{x}|))\mathrm{d}\mathbf{x} \\ &\le \int_{\mathbb{R}^2} l(|\mathbf{y}-\mathbf{x}|)f_p(ul(|\mathbf{y}-\mathbf{x}|))\mathrm{d}\mathbf{x} \\ &= 2\pi\int_{R_+} r^\beta f_p(ur^\beta)r\mathrm{d}r \\ &= \pi\alpha u^{-\alpha-1}\int_{\mathbb{R}_+} t^\alpha f_p(t)\mathrm{d}t \\ &= \pi\alpha u^{-\alpha-1}\mathbf{E}\{p^\alpha\}<\infty, \text{ provided } \mathbf{E}\{p^\alpha\}<\infty. \end{aligned}$$

We complete the proof of (ii) when taking this into account in (38) and using (i). □

## APPENDIX E
## PROOF OF THEOREM 3

The mass in 0 was evaluated in (10). Now let us consider $\gamma > 0$. We have:

$$\begin{aligned}\bar{F}_{Y_S}(\gamma) &= \mathbf{P}\{\frac{M_S}{1+I-M_S} > \gamma\} = \mathbf{P}\{M_S > \frac{\gamma}{1+\gamma}(1+I)\}\\ &= \int_{v=\gamma}^{\infty}\int_{u=\frac{\gamma(1+v)}{1+\gamma}}^{v} f_{(I,M_S)}(v,u)\mathrm{d}u\mathrm{d}v,\\ &= \int_{v=\gamma}^{\infty}\int_{u=0}^{v} f_{(I,M_S)}(v,u)\mathrm{d}u\mathrm{d}v,\\ &\quad - \int_{v=\gamma}^{\infty}\int_{u=0}^{\frac{\gamma(1+v)}{1+\gamma}} f_{(I,M_S)}(v,u)\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{39}$$

where $f_{(I,M_S)}$ is the joint density function of $I$ and $M_S$, which exists at all points $(i,m)$ with $m > 0$ according to Corollary 4.

For $u > 0$ let

$$h(v,u) := \int_0^u f_{(I,M_S)}(v,t)\mathrm{d}t + g_I(v)\mathbf{P}\{M_S = 0\}, \tag{40}$$

where $g_I(v)$ is the density of $I$ at $v$ given that $M_S = 0$. Intuitively, $h$ is the density of $(1_{(M_S\le u)}I)$. The characteristic function of $v \to h(v,u)$ is $\phi_{(I,M_S\le u)}(w)$. From Theorem 2, we have $\phi_{(I,M_S\le u)}(w) \in L$ w.r.t. $w$ for $u > 0$. So by Theorem 3 in [6, p.509], we have:

$$h(v,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jwv}\phi_{(I,M_S\le u)}(w)\mathrm{d}w.$$

Therefore

$$h(v,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jwv}\phi_{(I,M_S\le v)}(w)\mathrm{d}w.$$

And so:

$$\int_\gamma^\infty h(v,v)\mathrm{d}v = \frac{1}{2\pi}\int_\gamma^\infty\int_{-\infty}^{\infty} e^{-jwv}\phi_{(I,M_S\le v)}(w)\mathrm{d}w\mathrm{d}v. \tag{41}$$

Similarly, we have:

$$h(v,\frac{\gamma(1+v)}{1+\gamma}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jwv}\phi_{(I,M_S\le\frac{\gamma(1+v)}{1+\gamma})}(w)\mathrm{d}w.$$

Hence

$$\begin{aligned}\int_\gamma^\infty & h(v,\frac{\gamma(1+v)}{1+\gamma})\mathrm{d}v\\ &= \frac{1}{2\pi}\int_\gamma^\infty\int_{-\infty}^{\infty} e^{-jwv}\phi_{(I,M_S\le\frac{\gamma(1+v)}{1+\gamma})}(w)\mathrm{d}w\mathrm{d}v.\end{aligned}$$

Conducting the change of variable $x = \frac{\gamma(1+v)}{1+\gamma}$, we get:

$$\begin{aligned}\int_\gamma^\infty & h(v,\frac{\gamma(1+v)}{1+\gamma})\mathrm{d}v\\ &= \frac{1}{2\pi}\frac{1+\gamma}{\gamma}\int_\gamma^\infty\int_{-\infty}^{\infty} e^{-jw(\frac{1+\gamma}{\gamma}x-1)}\phi_{(I,M_S\le x)}(w)\mathrm{d}w\mathrm{d}x.\end{aligned} \tag{42}$$

Moreover, by (40) we have:

$$\begin{aligned}\int_\gamma^\infty\int_0^v f_{(I,M_S)}(v,u)\mathrm{d}u\mathrm{d}v - \int_\gamma^\infty\int_0^{\frac{\gamma(1+v)}{1+\gamma}} f_{(I,M_S)}(v,u)\mathrm{d}u\mathrm{d}v\\ = \int_\gamma^\infty h(v,v)\mathrm{d}v - \int_\gamma^\infty h(v,\frac{\gamma(1+v)}{1+\gamma})\mathrm{d}v.\end{aligned}$$

Substituting (41) and (42) into this, then by (39) we get:

$$\bar{F}_{Y_S}(\gamma) = \frac{1}{2\pi}\int_\gamma^{+\infty}\int_{-\infty}^{+\infty}\phi_{(I,M_S\le x)}(w)g(w,x)\mathrm{d}w\mathrm{d}x.$$

where

$$g(w,x) = e^{-jwx} - \frac{1+\gamma}{\gamma}e^{-jw(\frac{1+\gamma}{\gamma}x-1)}.$$

□

## APPENDIX F
## PROOF OF COROLLARY 5

From Theorem 3, $\bar{F}_{Y_S}$ is given by (19). For this, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} & \phi_{(I,M_S\le u)}(w)g(w,u)\mathrm{d}w =\\ &= \int_0^{+\infty}\Big(\phi_{(I,M_S\le u)}(w)g(w,u)\\ &\qquad + \phi_{(I,M_S\le u)}(-w,u)g(-w,u)\Big)\mathrm{d}w.\end{aligned} \tag{43}$$

where $\phi_{(I,M_S\le u)}$ is given by Corollary 3 and $g$ is given by (20). Thus, we have for $w \in [0,+\infty)$:

$$\begin{aligned}\phi_{(I,M_S\le u)}&(w,u)g(w,u)\\ &= e^{-(C_1w^\alpha + C_2(w,u))}\Big(e^{j(C_1w^\alpha\tan(\frac{\pi\alpha}{2})+C_3(w,u)-wu)}\\ &\qquad - \frac{1+\gamma}{\gamma}e^{j(C_1w^\alpha\tan(\frac{\pi\alpha}{2})+C_3(w,u)+C_4(w,u))}\Big).\end{aligned}$$

where $C_4(w,u) = w(1-\frac{1+\gamma}{\gamma}u)$. Similarly, for $w \in [0,+\infty)$:

$$\begin{aligned}\phi_{(I,M_S\le u)}&(-w,u)g(-w,u)\\ &= e^{-(C_1w^\alpha + C_2(w,u))}\Big(e^{-j(C_1w^\alpha\tan(\frac{\pi\alpha}{2})+C_3(w,u)-wu)}\\ &\qquad - \frac{1+\gamma}{\gamma}e^{-j(C_1w^\alpha\tan(\frac{\pi\alpha}{2})+C_3(w,u)+C_4(w,u))}\Big).\end{aligned}$$

These result in

$$\begin{aligned}\phi_{(I,M_S\le u)}&(w,u)g(w,u) + \phi_{(I,M_S\le u)}(-w,u)g(-w,u)\\ &= 2e^{-(C_1w^\alpha + C_2(w,u))}\\ &\times\Big(-\frac{1+\gamma}{\gamma}\cos\big(C_1w^\alpha\tan(\frac{\pi\alpha}{2}) + C_3(w,u) + C_4(w,u)\big)\\ &\quad + \cos\big(C_1w^\alpha\tan(\frac{\pi\alpha}{2}) + C_3(w,u) - wu\big)\Big).\end{aligned}$$

Substituting this into (43) and then by (19) we obtain (21). □