DROP COST AND WAVELENGTH OPTIMAL TWO-PERIOD GROOMING WITH RATIO 4⋆

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DROP COST AND WAVELENGTH OPTIMAL TWO-PERIOD GROOMING WITH RATIO 4*

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Abstract. We study grooming for two-period optical networks, a variation of the traffic grooming problem for wavelength division multiplexed (WDM) ring networks introduced by Colbourn, Quattrocchi, and Syrotiuk. In the two-period grooming problem, during the first period of time there is all-to-all uniform traffic among n nodes, each request using $1/C$ of the bandwidth; and during the second period there is all-to-all uniform traffic only among a subset V of v nodes, each request now being allowed to use $1/C'$ of the bandwidth, where $C' < C$. We determine the minimum drop cost (minimum number of add-drop multiplexers (ADMs)) for any n, v and $C = 4$ and $C' \in \{1, 2, 3\}$. To do this, we use tools of graph decompositions. Indeed the two-period grooming problem corresponds to minimizing the total number of vertices in a partition of the edges of the complete graph K_n into subgraphs, where each subgraph has at most C edges and where furthermore it contains at most C' edges of the complete graph on v specified vertices. Subject to the condition that the two-period grooming has the least drop cost, the minimum number of wavelengths required is also determined in each case.

Key words. traffic grooming, SONET ADM, optical networks, graph decomposition, design theory

AMS subject classifications. 68R10, 05B30, 05C51

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1. Introduction.

Traffic grooming is the generic term for packing low-rate signals into higher speed streams (see the surveys [5, 19, 23, 28, 30]). By using traffic grooming, one can bypass the electronics in the nodes which are not sources or destinations of traffic, and therefore reduce the cost of the network. Here we consider unidirectional SONET/WDM ring networks. In that case, the routing is unique and we have to assign to each request between two nodes a wavelength and some bandwidth on this wavelength. If the traffic is uniform and if a given wavelength has capacity for at least C requests, we can assign to each request at most $\frac{1}{C}$ of the bandwidth. C is known as the *grooming ratio* or the *grooming factor*. Furthermore if the traffic requirement is symmetric, it can be easily shown (by exchanging wavelengths)

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that there always exists an optimal solution in which the same wavelength is given to each pair of symmetric requests. Thus without loss of generality we assign to each pair of symmetric requests, called a *circle*, the same wavelength. Then each circle uses $\frac{1}{C}$ of the bandwidth in the whole ring. If the two end-nodes of a circle are i and j, we need one add-drop multiplexer (ADM) at node i and one at node j. The main point is that if two requests have a common end-node, they can share an ADM if they are assigned the same wavelength. For example, suppose that we have symmetric requests between nodes 1 and 2, and also between 2 and 3. If they are assigned two different wavelengths, then we need 4 ADMs, whereas if they are assigned the same wavelength, we need only 3 ADMs.

The so-called traffic grooming problem consists of minimizing the total number of ADMs to be used in order to reduce the overall cost of the network.

Suppose we have a ring with 4 nodes $\{0, 1, 2, 3\}$ and all-to-all uniform traffic. There are therefore 6 circles (pairs of symmetric requests) $\{i, j\}$ for $0 \leq i < j \leq 3$. If there is no grooming (i.e., $C = 1$), we need 6 wavelengths (one per circle) and a total of 12 ADMs. For grooming factor $C = 2$, we can put two circles on the same wavelength, using 3 or 4 ADMs according to whether they share a node or not. For example, we can put $\{1, 2\}$ and $\{2, 3\}$ together on one wavelength, $\{1, 3\}$ and $\{3, 4\}$ on a second wavelength, and $\{1, 4\}$ and $\{2, 4\}$ on a third one, for a total of 9 ADMs. For grooming factor $C = 3$, we can use only 2 wavelengths. If we put $\{1, 2\}$, $\{2, 3\}$, and $\{3, 4\}$ together on one wavelength and $\{1, 3\}$, $\{2, 4\}$, and $\{1, 4\}$ on the other one, we need 8 ADMs (solution a); but we can do better by putting $\{1, 2\}$, $\{2, 3\}$, and $\{1, 3\}$ on the first wavelength and $\{1, 4\}$, $\{2, 4\}$, and $\{3, 4\}$ on the second one, using 7 ADMs (solution b).

Here we study the problem for a unidirectional SONET ring with n nodes, grooming ratio C, and all-to-all uniform unitary traffic. This problem has been modeled as a graph partition problem in both [4] and [21]. In the all-to-all case the set of requests is modeled by the complete graph K_n. To a wavelength k is associated a subgraph B_k in which each edge corresponds to a pair of symmetric requests (that is, a circle) and each node to an ADM. The grooming constraint, i.e., the fact that a wavelength can carry at most C requests, corresponds to the fact that the number of edges $|E(B_k)|$ of each subgraph B_k is at most C. The cost corresponds to the total number of vertices used in the subgraphs, and the objective is therefore to minimize this number.

Traffic Grooming in the Ring

> Input: Two integers n and C.
>
> Output: Partition $E(K_n)$ into subgraphs B_k, $1 \leq k \leq \Lambda$, s.t. $|E(B_k)| \leq C$ for all k.
>
> Objective: Minimize $\sum_{k=1}^{\Lambda} |V(B_k)|$.

In the example above with $n = 4$ and $C = 3$, solution a consists of a decomposition of K_4 into two paths with four vertices $[1, 2, 3, 4]$ and $[2, 4, 1, 3]$, while solution b corresponds to a decomposition into a triangle $(1, 2, 3)$ and a star with the edges $\{1, 4\}$, $\{2, 4\}$, and $\{3, 4\}$.

With the all-to-all set of requests, optimal constructions for a given grooming ratio C have been obtained using tools of graph and design theory [10], in particular for grooming ratio $C = 3$ [1], $C = 4$ [4, 22], $C = 5$ [3], $C = 6$ [2], $C = 7$ [11], $C = 8$ [12], and $C \geq N(N-1)/6$ [6].

Graph decompositions have been extensively studied for other reasons as well. See [8] for an excellent survey, [17] for relevant material on designs with blocksize three, and [10] for terminology in design theory.

Most of the papers on grooming deal with a single (static) traffic matrix. Some articles consider variable (dynamic) traffic, such as finding a solution which works for the maximum traffic demand [7, 31] or for all request graphs with a given maximum degree [24], but all keep a fixed grooming factor. In [14] an interesting variation of the traffic grooming problem, grooming for two-period optical networks, has been introduced in order to capture some dynamic nature of the traffic. Informally, in the two-period grooming problem each time period supports different traffic requirements. During the first period of time there is all-to-all uniform traffic among n nodes, each request using $1/C$ of the bandwidth; but during the second period there is all-to-all traffic only among a subset V of v nodes, each request now being allowed to use a larger fraction of the bandwidth, namely, $1/C'$, where $C' < C$.

Denote by X the subset of n nodes. Therefore the two-period grooming problem can be expressed as follows.

Two-Period Grooming in the Ring

> Input: Four integers n, v, C, and C'.
>
> Output: A partition $N(n,v;C,C')$ of $E(K_n)$ into subgraphs B_k, $1 \le k \le \Lambda$, such that for all k, $|E(B_k)| \le C$, and $|E(B_k) \cap (V \times V)| \le C'$, with $V \subseteq X$, $|V| = v$.
>
> Objective: Minimize $\sum_{k=1}^{\Lambda} |V(B_k)|$.

Following [14], a grooming is denoted by $N(n,C)$. When the grooming $N(n,C)$ is *optimal*, i.e., minimizes the total ADM cost, then the grooming is denoted by $\mathscr{ON}(n,C)$. Whether general or optimal, the drop cost of a grooming is denoted by *cost* $N(n,C)$ or *cost* $\mathscr{ON}(n,C)$, respectively.

A grooming of a two-period network $N(n,v;C,C')$ with grooming ratios (C,C') coincides with a graph decomposition $(X,\mathcal{B})$ of K_n (using standard design theory terminology, $\mathcal{B}$ is the set of all the *blocks* of the decomposition) such that $(X,\mathcal{B})$ is a grooming $N(n,C)$ in the first time period, and $(X,\mathcal{B})$ faithfully embeds a graph decomposition of K_v such that $(V,\mathcal{D})$ is a grooming $N(v,C')$ in the second time period. Let $V \subseteq X$. The graph decomposition $(X,\mathcal{B})$ *embeds* the graph decomposition $(V,\mathcal{D})$ if there is a mapping $f : \mathcal{D} \to \mathcal{B}$ such that D is a subgraph of $f(D)$ for every $D \in \mathcal{D}$. If f is injective (i.e., one-to-one), then $(X,\mathcal{B})$ *faithfully embeds* $(V,\mathcal{D})$. This concept of faithful embedding has been explored in [13, 26].

We use $\mathscr{ON}(n,v;C,C')$ to denote an optimal grooming $N(n,v;C,C')$.

As it turns out, an $\mathscr{ON}(n,v;C,C')$ does not always coincide with an $\mathscr{ON}(n,C)$. Generally we have *cost* $\mathscr{ON}(n,v;C,C') \ge$ *cost* $\mathscr{ON}(n,C)$ (see Examples 1.2 and 1.3). Of particular interest is the case when *cost* $\mathscr{ON}(n,v;C,C') =$ *cost* $\mathscr{ON}(n,C)$ (see Example 1.1).

Example 1.1. Let $n = 7$, $v = 4$, $C = 4$. Let $V = \{0,1,2,3\}$ and $W = \{a_0,a_1,a_2\}$. An optimal decomposition is given by three triangles $(a_0,0,1)$, $(a_1,1,2)$, and $(a_2,2,3)$, and three 4-cycles $(0,2,a_0,a_1)$, $(0,3,a_0,a_2)$, and $(1,3,a_1,a_2)$, giving a total cost of 21 ADMs.

This solution is valid and optimal for both $C' = 1$ and $C' = 2$, and it is optimal for the classical Traffic Grooming in the Ring problem when $n = 7$ and $C = 4$. Therefore, *cost* $\mathscr{ON}(7,4;4,1) =$ *cost* $\mathscr{ON}(7,4;4,2) =$ *cost* $\mathscr{ON}(7,4) = 21$.

Example 1.2. Let $n = 7$, $v = 5$, $C = 4$, and $C' = 2$. Let $V = \{0,1,2,3,4\}$ and $W = \{a_0,a_1\}$. We see later that an optimal decomposition is given by the five kites $(a_0,1,2;0)$, $(a_0,3,4;1)$, $(a_1,1,3;2)$, $(a_1,2,4;0)$, and $(a_0,a_1,0;1)$, plus the edge $\{0,3\}$, giving a total cost of 22 ADMs. So *cost* $\mathscr{ON}(7,5;4,2) = 22$. Note that this decomposition is not a valid solution for $C' = 1$, since there are subgraphs containing more than one edge with both end-vertices in V.

$$cost\ \mathscr{ON}(v+w,v;4,1)=\begin{cases}\binom{v+w}{2} & \text{if } v\le w+1,\\[2ex] \binom{v+w}{2}+\binom{v}{2}-\left\lfloor\frac{vw}{2}\right\rfloor & \text{if } v\ge w+1;\end{cases}$$

$$cost\ \mathscr{ON}(v+w,v;4,2)=$$

$$\begin{cases}\binom{v+w}{2} & \text{if } v\le 2w,\\[1ex] \binom{v+w}{2}+\left\lceil\frac{1}{2}\binom{v}{2}\right\rceil-\frac{vw}{2}+\delta & \text{if } v>2w \text{ and } v \text{ even},\\[1ex] \qquad\text{where } \delta=\begin{cases}1 & \text{if } w=2, \text{ or}\\ & \text{if } w=4 \text{ and}\\ & \quad v\equiv 0 \pmod 4,\\ 0 & \text{otherwise,}\end{cases} & \\[1ex] \binom{v+w}{2}+\left\lceil\frac{1}{2}\left(\binom{v}{2}-vw-\left\lceil\frac{w}{2}\right\rceil\right)\right\rceil+\delta & \text{if } v>2w \text{ and } v \text{ odd},\\[1ex] \qquad\text{where } \delta=\begin{cases}1 & \text{if } w=3 \text{ and}\\ & \quad v\equiv 3 \pmod 4,\\ 0 & \text{otherwise;}\end{cases} & \end{cases}$$

$$cost\ \mathscr{ON}(v+w,v;4,3)=\binom{v+w}{2}.$$

FIG. 1. *Cost formulas for $n=v+w>4$.*

Example 1.3. Let $n=7$, $v=5$, $C=4$, and $C'=1$. Let again $V=\{0,1,2,3,4\}$ and $W=\{a_0,a_1\}$. We see later that an optimal decomposition is given by the four K_3s $(a_0,1,2)$, $(a_0,3,4)$, $(a_1,0,3)$, and $(a_1,2,4)$, the C_4 $(0,1,a_1,a_0)$, plus the five edges $\{0,4\}$, $\{1,3\}$, $\{0,2\}$, $\{1,4\}$, and $\{2,3\}$, giving a total cost of 26 ADMs. So *cost* $\mathscr{ON}(7,5;4,1)=26$.

Colbourn, Quattrocchi, and Syrotiuk [14, 15] completely solve the cases when $C=2$ and $C=3$ ($C'=1$ or 2). In this article we determine the minimum drop cost of an $N(n,v;4,C')$ for all $n\ge v\ge 0$ and $C'\in\{1,2,3\}$.

We are also interested in determining the minimum number of wavelengths, or *wavecost*, required in an assignment of wavelengths to a decomposition. Among the $\mathscr{ON}(n,4)$'s, one having the minimum wavecost is denoted by $\mathscr{MON}(n,4)$, and the corresponding minimum number of wavelengths by *wavecost* $\mathscr{MON}(n,4)$. We characterize the $\mathscr{ON}(n,v;C,C')$ whose wavecost is minimum among all $\mathscr{ON}(n,v;C,C')$'s and denote one by $\mathscr{MON}(n,v;C,C')$; then *wavecost* $\mathscr{MON}(n,v;C,C')$ denotes its wavecost.

We deal separately with each value of $C'\in\{1,2,3\}$. Figure 1 summarizes the cost formulas for $n=v+w>4$.

2. Notation and preliminaries. We establish some graph-theoretic notation to be used throughout. We denote the edge between u and v by $\{u,v\}$. K_n denotes a complete graph on n vertices and K_X represents the complete graph on the vertex set X. A triangle with edges $\{\{x,y\},\{x,z\},\{y,z\}\}$ is denoted by (x,y,z). A 4-cycle with edges $\{\{x,y\},\{y,z\},\{z,u\},\{u,x\}\}$ is denoted by (x,y,z,u). A *kite* with edges $\{\{x,y\},\{x,z\},\{y,z\},\{z,u\}\}$ is denoted by $(x,y,z;u)$. The groomings to be produced also employ paths; the path on k vertices P_k is denoted by $[x_1,\dots,x_k]$ when it contains edges $\{x_i,x_{i+1}\}$ for $1\le i<k$. Now let $G=(X,E)$ be a graph. If $|X|$ is even, a

set of $|X|/2$ disjoint edges in E is a 1-*factor*; a partition of E into 1-factors is a 1-*factorization*. Similarly, if $|X|$ is odd, a set of $(|X|-1)/2$ disjoint edges in E is a *near* 1-*factor*; a partition of E into near 1-factors is a *near* 1-*factorization*. We also employ well-known results on partial triple systems and group divisible designs with blocksize three; see [17] for background.

The vertices of the set V are the integers modulo v, $\{0, 1, \ldots, v-1\}$. The vertices in $X \setminus V$ form the set W of size $w = n - v$ with $W = \{a_0, \ldots, a_{w-1}\}$, the indices being taken modulo w.

Among graphs with three or fewer edges (i.e., when $C = 3$), the only graph with the minimum ratio (number of vertices over the number of edges) is the triangle. For $C = 4$ three different such graphs have minimum ratio 1: the triangle, the 4-cycle, and the kite. This simplifies the problem substantially. Indeed, in contrast to the lower bounds in [15], in this case the lower bounds arise from easy classification of the edges on V. We recall the complete characterization for optimal groomings with a grooming ratio of four.

THEOREM 2.1 (see [4, 22]). *cost* $\mathscr{O}\mathscr{N}(4,4) = 7$ *and, for* $n \geq 5$, *cost* $\mathscr{O}\mathscr{N}(n,4) = \binom{n}{2}$. *Furthermore an* $\mathscr{M}\mathscr{O}\mathscr{N}(4,4)$ *employs two wavelengths and can be realized by a kite and a* P_3 *(or a* K_3 *and a star), and an* $\mathscr{M}\mathscr{O}\mathscr{N}(n,4)$, $n \geq 5$, *employs* $\lceil \frac{n(n-1)}{8} \rceil$ *wavelengths and can be realized by* t K_3*s and* $\lceil \frac{n(n-1)}{8} - t \rceil$ 4-*cycles or kites, where*

$$t = \begin{cases} 0 & \textit{if } n \equiv 0, 1 \pmod 8, \\ 1 & \textit{if } n \equiv 3, 6 \pmod 8, \\ 2 & \textit{if } n \equiv 4, 5 \pmod 8, \\ 3 & \textit{if } n \equiv 2, 7 \pmod 8. \end{cases}$$

In order to unify the treatment of the lower bounds, in a decomposition $N(v+w, v; 4, C')$ for $C' \in \{1, 2\}$, we call an edge with both ends in V *neutral* if it appears in a triangle, 4-cycle, or kite; we call it *positive* otherwise. An edge with one end in V and one in W is a *cross edge*.

LEMMA 2.2.

1. *In an* $N(v+w, v; 4, C')$ *with* $C' \in \{1, 2\}$, *the number of neutral edges is at most* $\frac{1}{2}C'vw$.
2. *When* v *is odd and* $C' = 2$, *the number of neutral edges is at most* $vw - \frac{w}{2}$.

Proof. Every neutral edge appears in a subgraph having at least two cross edges. Thus the number of subgraphs containing one or more neutral edges is at most $\frac{1}{2}vw$. Each can contain at most C' neutral edges, and hence there are at most $\frac{1}{2}C'vw$ neutral edges. This proves the first statement.

Suppose now that $C' = 2$ and v is odd. Any subgraph containing two neutral edges employs exactly two cross edges incident to the same vertex in W. Thus the number α of such subgraphs is at most $\frac{1}{2}w(v-1)$. Then remaining neutral edges must arise (if present) in triangles, kites, or 4-cycles that again contain two cross edges but only one neutral edge; their number, β, must satisfy $\beta \leq \frac{vw}{2} - \alpha$. Therefore the number of neutral edges, $2\alpha + \beta$, satisfies $2\alpha + \beta \leq \frac{1}{2}w(v-1) + \frac{vw}{2} = vw - \frac{w}{2}$. $\square$

When $C = 3$ there are strong interactions among the decompositions placed on V, on W, and on the cross edges [14, 15]; fortunately here we shall see that the structure on V suffices to determine the lower bounds. Because every $N(v+w, v; 4, C')$ is an $N(v+w, v; 4, C'+1)$ for $1 \leq C' \leq 3$, and $N(v+w, v; 4, 4)$ coincides with $N(v+w, 4)$, *cost* $\mathscr{O}\mathscr{N}(v+w, v; 4, 1) \geq$ *cost* $\mathscr{O}\mathscr{N}(v+w, v; 4, 2) \geq$ *cost* $\mathscr{O}\mathscr{N}(v+w, v; 4, 3) \geq$ *cost* $\mathscr{O}\mathscr{N}(v+w, 4)$. We use these obvious facts to establish lower and upper bounds without further comment.

3. Case $C' = 1$.

3.1. $\mathscr{ON}(n, v; 4, 1)$.

THEOREM 3.1. *Let* $n = v + w \geq 5$.

1. *cost* $\mathscr{ON}(v + w, v; 4, 1) =$ *cost* $\mathscr{ON}(v + w, 4)$ *when* $v \leq w + 1$.
2. *cost* $\mathscr{ON}(v + w, v; 4, 1) = \binom{v+w}{2} + \binom{v}{2} - \lfloor \frac{vw}{2} \rfloor$ *when* $v \geq w + 1$.

Proof. To prove the lower bound, we establish that *cost* $\mathscr{ON}(v + w, v; 4, 1) \geq \binom{v+w}{2} + \binom{v}{2} - \lfloor \frac{vw}{2} \rfloor$. It suffices to prove that the number of subgraphs employed in an $N(v + w, v; 4, 1)$ other than triangles, kites, and 4-cycles is at least $\lceil \binom{v}{2} - \frac{1}{2}vw \rceil = \binom{v}{2} - \lfloor \frac{1}{2}vw \rfloor$. By Lemma 2.2, this is a lower bound on the number of positive edges in any such decomposition; because each positive edge lies in a different subgraph of the decomposition, the lower bound follows.

Now we turn to the upper bounds. For the first statement, because an $\mathscr{ON}(v + w, v; 4, 1)$ is also an $\mathscr{ON}(v+w, v-1; 4, 1)$, it suffices to consider $v \in \{w, w+1\}$. When $v = w$, write $v = 4s+t$ with $t \in \{0, 3, 5, 6\}$. Form on V a complete multipartite graph with s classes of size four and one class of size t. Replace edge $e = \{x, y\}$ of this graph by the 4-cycle (x, y, a_x, a_y). On $\{x_1, \ldots, x_\ell, a_{x_1}, \ldots, a_{x_\ell}\}$ whenever $\{x_1, \ldots, x_\ell\}$ forms a class of the multipartite graph, place a decomposition that is optimal for drop cost and uses 4, 7, 12, and 17 wavelengths when ℓ is 3, 4, 5, or 6, respectively. These are as follows:

$\mathscr{MON}(3+3, 3; 4, 1)$	$\{(0, a_0, 1; a_2), (1, a_1, 2; a_0), (2, a_2, 0; a_1), (a_0, a_1, a_2)\}$;
$\mathscr{MON}(4+4, 4; 4, 1)$	$\{(1, 2, a_3; a_0), (0, 3, a_2; a_1), (a_1, 1, 3; a_0), (a_0, a_2, 1; 0),$ $(a_0, a_1, 2; 0), (a_1, a_3, 0; a_0), (2, 3, a_3, a_2)\}$;
$\mathscr{MON}(5+5, 5; 4, 1)$	$\{(1, 2, a_3; a_0), (0, 3, a_2; a_1), (a_1, 1, 3; a_0), (a_0, a_2, 1; 0),$ $(a_0, a_1, 2; 0), (a_1, a_3, 0; a_0), (2, a_2, 4; a_4), (3, a_3, 4),$ $(a_2, a_3, a_4), (2, 3, a_4), (0, 4, a_0, a_4), (1, 4, a_1, a_4)\}$;
$\mathscr{MON}(6+6, 6; 4, 1)$	$\{(1, 2, a_3; a_0), (0, 3, a_2; a_1), (a_1, 1, 3; a_0), (a_0, a_2, 1; 0),$ $(a_0, a_1, 2; 0), (a_1, a_3, 0; a_0), (4, 5, a_5; a_4), (2, a_2, 4; a_4),$ $(2, 3, a_4; 5), (3, 4, a_3), (a_2, a_3, a_4), (0, 4, a_0, a_4),$ $(1, 4, a_1, a_4), (0, 5, a_0, a_5), (1, 5, a_1, a_5), (2, 5, a_2, a_5),$ $(3, 5, a_3, a_5)\}$.

Now let $v = w+1$. Let $V = \{0, \ldots, v-1\}$ and $W = \{a_0, \ldots, a_{v-2}\}$. Form triangles $(i, i+1, a_i)$ for $0 \leq i < v-1$. Then form 4-cycles $(i, j+1, a_i, a_j)$ for $0 \leq i < j \leq v-2$.

When $v \geq w + 2$ and v is even, form a 1-factorization $F_0, \ldots, F_{v-2}$ on V. For $0 \leq i < w$, let $\{e_{ij} : 1 \leq j \leq \frac{v}{2}\}$ be the edges of F_i, and form triangles $T_{ij} = \{a_i\} \cup e_{ij}$. Now for $0 \leq i < w$ and $1 \leq j \leq \lfloor \frac{w}{2} \rfloor$, and furthermore $j \neq \frac{w}{2}$ if $i \geq \frac{w}{2}$ and w is even, adjoin edge $\{a_i, a_{i+j \bmod w}\}$ to T_{ij} to form a kite. All edges of 1-factors $\{F_i : w \leq i < v-1\}$ are taken as K_2's.

When $v \geq w + 2$ and v is odd, form a near 1-factorization $F_0, \ldots, F_{v-1}$ on V, in which F_{v-1} contains the edges $\{\{2h, 2h + 1\} : 0 \leq h < \frac{v-1}{2}\}$, and near 1-factor F_i misses vertex i for $0 \leq i < v$. Then form 4-cycles $(2h, 2h + 1, a_{2h+1}, a_{2h})$ for $0 \leq h < \lfloor \frac{w}{2} \rfloor$. For $0 \leq i < w$, let $\{e_{ij} : 1 \leq j \leq \frac{v-1}{2}\}$ be the edges of F_i, and form triangles $T_{ij} = \{a_i\} \cup e_{ij}$. Without loss of generality we assume that $w - 1 \in e_{01}$; when w is odd, adjoin $\{w - 1, a_{w-1}\}$ to T_{01} to form a kite. Now for $0 \leq i < w$ and $1 \leq j \leq \lfloor \frac{w}{2} \rfloor$, and furthermore $j \neq \frac{w}{2}$ if $i \geq \frac{w}{2}$ and w is even and $j \neq 1$ if $i = 2h$ for $0 \leq h < \lfloor \frac{w}{2} \rfloor$, adjoin edge $\{a_i, a_{i+j \bmod w}\}$ to T_{ij} to form a kite. All edges of near 1-factors $\{F_i : w \leq i < v-1\}$ and the $\frac{v-1}{2} - \lfloor \frac{w}{2} \rfloor$ remaining edges of F_{v-1} are taken as K_2's.

When $v \geq w+1$, each subgraph contains exactly one edge on V and so their number is $\binom{v}{2}$. This fact is later used to prove Theorem 3.3. ∎

3.2. $\mathscr{MON}(n, v; 4, 1)$.

THEOREM 3.2. *Let $v + w \geq 5$. For $C' = 1$ and $v \leq w$,*

$$\textit{wavecost}\, \mathscr{MON}(v+w, v; 4, 1) = \textit{wavecost}\, \mathscr{MON}(v+w, 4).$$

Proof. We need only treat the cases when $v \in \{w, w-1\}$; the case with $v = w$ is handled in the proof of Theorem 3.1. When $v = w-1$, the argument is identical to that proof, except that we choose $v = 4s + t$ with $t \in \{0, 1, 2, 3\}$ and place decompositions on $\{x_1, \ldots, x_\ell, a_{x_1}, \ldots, a_{x_\ell}, a_v\}$ instead, with $1, 3, 6, 9$ wavelengths when $\ell = 1, 2, 3, 4$, respectively. These are as follows:

$\mathscr{MON}(1+2, 1; 4, 1)$	$\{(0, a_0, a_1)\}$;
$\mathscr{MON}(2+3, 2; 4, 1)$	$\{(0, a_0, a_1), (1, a_1, a_2), (0, 1, a_0, a_2)\}$;
$\mathscr{MON}(3+4, 3; 4, 1)$	$\{(0, a_0, a_1), (1, a_1, a_2), (0, 1, a_0, a_2), (2, a_2, a_3), (0, 2, a_0, a_3), (1, 2, a_1, a_3)\}$;
$\mathscr{MON}(4+5, 4; 4, 1)$	$\{(0, 1, a_0; a_3), (0, 2, a_1; a_3), (0, 3, a_2; a_3), (2, 3, a_0; a_4), (1, 3, a_1; a_4), (1, 2, a_3; 3), (0, a_3, a_4; 3), (1, a_2, a_4; 2), (a_0, a_1, a_2; 2)\}$. ∎

THEOREM 3.3. *When $v > w$,*

$$\textit{wavecost}\, \mathscr{MON}(v+w, v; 4, 1) = \binom{v}{2}.$$

Proof. Since every edge on V appears on a different wavelength, $\binom{v}{2}$ is a lower bound. As noted in the proof of Theorem 3.1 the constructions given there meet this bound. ∎

The solutions used from Theorem 3.1 are (essentially) the only ones to minimize the number of graphs in an $\mathscr{ON}(v+w, v; 4, 1)$ with $v > w$. However, perhaps surprisingly they are not the only ones to minimize the number of wavelengths. To see this, consider an $\mathscr{ON}(v+w, v; 4, 1)$ with $v > w > 2$ from Theorem 3.1. Remove edges $\{a_0, a_1\}$, $\{a_0, a_2\}$, and $\{a_1, a_2\}$ from their kites, and form a triangle from them. This does not change the drop cost, so the result is also an $\mathscr{ON}(v+w, v; 4, 1)$. It has one more graph than the original. Despite this, it does not need an additional wavelength, since the triangle (a_0, a_1, a_2) can share a wavelength with an edge on V. In this case, while minimizing the number of connected graphs serves to minimize the number of wavelengths, it is not the only way to do so.

4. Case $C' = 2$.

4.1. $\mathscr{ON}(n, v; 4, 2)$.

THEOREM 4.1. *Let $v + w \geq 5$ and let v be even.*

1. *When $v \leq 2w$, cost $\mathscr{ON}(v+w, v; 4, 2) =$ cost $\mathscr{ON}(v+w, 4)$.*
2. *When $v \geq 2w+2$, cost $\mathscr{ON}(v+w, v; 4, 2) = \binom{v+w}{2} + \lceil \frac{1}{2}\binom{v}{2} \rceil - \frac{vw}{2} + \delta$, where $\delta = 1$ if $w = 4$ or if $w = 2$ and $v \equiv 0 \pmod 4$, and $\delta = 0$ otherwise.*

Proof. By Lemma 2.2, $\binom{v}{2} - vw$ is a lower bound on the number of positive edges in any $N(v+w, v; 4, 2)$; every subgraph of the decomposition containing a positive edge contains at most two positive edges. So the number of subgraphs employed in an $N(v+w, v; 4, 2)$ other than triangles, kites, and 4-cycles is at least $\lceil \frac{1}{2}\left(\binom{v}{2} - vw\right) \rceil$. The lower bound follows for $w \neq 2, 4$.

As in the proof of Lemma 2.2, denote by α (resp., β) the number of subgraphs containing 2 (resp., 1) neutral edges and so at least two cross edges. We have $2\alpha+\beta \leq 2\alpha + 2\beta \leq vw$. Equality in the lower bound, when $v \equiv 0 \pmod 4$, arises only when $\beta = 0$, and therefore to meet the bound an $\mathscr{ON}(w,4)$ must be placed on W, implying that $\delta = 1$ if $w = 2$ or 4. When $v \equiv 2 \pmod 4$, we can have $2\alpha + \beta = vw - 1$ and so $\beta = 1$. We can use an edge on W in a graph with an edge on V. But when $w = 4$, the five edges that would remain on W require drop cost 6, and so $\delta = 1$.

Now we turn to the upper bounds. If $w \geq v-1$, apply Theorem 3.1. Suppose that $w \leq v-2$. Let $V = \{0, \ldots, 2t-1\}$ and $W = \{a_0, \ldots, a_{w-1}\}$. Place an $\mathscr{ON}(w,4)$ on W. Form a 1-factorization on V containing factors $\{F_0, \ldots, F_{w-1}, G_0, \ldots, G_{2t-2-w}\}$ in which the last two 1-factors are $\{\{2h, 2h+1\} : 0 \leq h < t\}$ and $\{\{2h+1, 2h+2 \bmod 2t\} : 0 \leq h < t\}$, whose union is a Hamilton cycle. For $0 \leq i < w$, form triangles T_{ij} by adding a_i to each edge $e_{ij} \in F_i$. For $0 \leq i < \min(w, 2t-1-w)$, observe that $H_i = F_i \cup G_i$ is a 2-factor containing even cycles. Hence there is a bijection σ mapping edges of F_i to edges of G_i so that e and $\sigma(e)$ share a vertex. Adjoin edge $\sigma(e_{ij})$ to the triangle T_{ij} to form a kite. In this way, all edges between V and W appear in triangles or kites, and all edges on V are employed when $v \leq 2w$. When $v \geq 2w+2$, the edges remaining on V are those of the factors $G_w, \ldots, G_{v-2-w}$.

When $v \neq 2w+2$, the union of these edges is connected because the union of the last two is connected, and hence it can be partitioned into P_3's (and one P_2 when $v \equiv 2 \pmod 4$)) [9, 29]. When $w = 2$ and $v \equiv 2 \pmod 4$, the drop cost can be reduced by 1 as follows. Let $\{x, y\}$ be the P_2 in the decomposition, and let $\{x, z\} \in G_0$. Let T be the triangle obtained by removing $\{x, z\}$ from its kite. Add $\{a_0, a_1\}$ to T to form a kite. Add the P_3 $[y, x, z]$. In this way two isolated P_2's are replaced by a P_3, lowering the drop cost by 1.

When $v = 2w+2$, we use a variant of this construction. Let R be a graph with vertex set V that is isomorphic to $\frac{v}{4}$ K_4's when $v \equiv 0 \pmod 4$ and to $\frac{v-6}{4}$ K_4's and one $K_{3,3}$ when $v \equiv 2 \pmod 4$. Let $F_1, \ldots, F_{w-1}, G_1, \ldots, G_{w-1}$ be the 1-factors of a 1-factorization of the complement of R (one always exists [27]). Proceed as above to form kites using a_i for $1 \leq i < w$ and the edges of F_i and G_i. For each K_4 of R with vertices $\{p, q, r, s\}$, form kites $(a_0, q, p; r)$ and $(a_0, r, s; p)$. Then add the P_3 $[r, q, s]$. If R contains a $K_{3,3}$ with bipartition $\{\{p, q, r\}, \{s, t, u\}\}$, add kites $(a_0, s, p; t)$, $(a_0, q, t; r)$, and $(a_0, r, u; p)$. What remains is the P_4 $[r, s; q, u]$, which can be partitioned into a P_2 and a P_3. ◻

In order to treat the odd case, we establish the following easy preliminary result.

LEMMA 4.2. *Let $w > 3$ be a positive integer. The graph on w vertices containing all edges except for $\lfloor \frac{w}{2} \rfloor$ disjoint edges (i.e., $K_w \setminus \lfloor \frac{w}{2} \rfloor K_2$) can be partitioned into*

1. *4-cycles when w is even;*
2. *kites and 4-cycles when $w \equiv 1 \pmod 4$; and*
3. *kites, 4-cycles, and exactly two triangles when $w \equiv 3 \pmod 4$.*

Proof. Let $W = \{a_0, \ldots, a_{w-1}\}$. Form 4-cycles $\{(a_{2i}, a_{2j}, a_{2i+1}, a_{2j+1}) : 0 \leq i < j < \frac{w}{2}\}$ when w is even, leaving uncovered the $\frac{w}{2}$ edges $\{a_{2i}, a_{2i+1}\}$. (This is also a consequence of a much more general result in [20].)

When w is odd, the proof is by induction on w by adding four new vertices. So we provide two base cases for the induction to cover all odd values of w.

For $w = 5$, partition $K_5 \setminus \{\{a_0, a_1\}, \{a_2, a_3\}\}$ into two kites $(a_2, a_4, a_0; a_3)$ and $(a_3, a_4, a_1; a_2)$.

For $w = 7$, partition $K_7 \setminus \{\{a_0, a_1\}, \{a_2, a_3\}, \{a_4, a_5\}\}$ into kites $(a_3, a_6, a_0; a_5)$, $(a_1, a_6, a_4; a_3)$, and $(a_5, a_6, a_2; a_1)$, and the K_3's (a_0, a_2, a_4) and (a_1, a_3, a_5).

By induction form an optimal decomposition of $K_w - F$, with $F = \{\{a_{2h}, a_{2h+1}\} : 0 \le h < \frac{w-1}{2}\}$. Add four vertices $\{a_w, a_{w+1}, a_{w+2}, a_{w+3}\}$. For $0 \le h < \frac{w-1}{2}$, add the C_4's $(a_{2h}, a_w, a_{2h+1}, a_{w+1})$ and $(a_{2h}, a_{w+2}, a_{2h+1}, a_{w+3})$. Cover the edges of the K_5 on $\{a_{w-1}, a_w, a_{w+1}, a_{w+2}, a_{w+3}\}$ minus the edges $\{a_{w-1}, a_w\}$ and $\{a_{w+1}, a_{w+2}\}$, using two kites as shown for the case when $w = 5$. ∎

THEOREM 4.3. *Let $v + w \ge 5$ and v be odd.*

1. *When $v \le 2w - 1$, cost $\mathscr{ON}(v + w, v; 4, 2) =$ cost $\mathscr{ON}(v + w, 4)$.*
2. *When $v \ge 2w+1$, cost $\mathscr{ON}(v+w, v; 4, 2) = \binom{v+w}{2} + \lceil \frac{1}{2} \left(\binom{v}{2} - vw + \lceil \frac{w}{2} \rceil\right) \rceil + \delta$, where $\delta = 1$ if $w = 3$ and $v \equiv 3 \pmod 4$, 0 otherwise.*

Proof. To prove the lower bound, it suffices to prove that the number of subgraphs employed in an $N(v + w, v; 4, 2)$ other than triangles, kites, and 4-cycles is at least $\lceil \frac{1}{2} \left(\binom{v}{2} - vw + \lceil \frac{w}{2} \rceil\right) \rceil$. As in the proof of Theorem 4.1, this follows from Lemma 2.2. When $w = 3$ and $v \equiv 3 \pmod 4$, at least $\binom{v}{2} - 3v + 2$ edges are positive, an even number. To meet the bound, exactly one cross edge remains and exactly two edges on W remain. These necessitate a further graph that is not a triangle, kite, or 4-cycle.

Now we turn to the upper bounds. By Theorem 4.1, *cost* $\mathscr{ON}((v + 1) + (w - 1), v + 1; 4, 2) =$ *cost* $\mathscr{ON}(v + w, 4)$ when $v \le 2w - 3$. So suppose that $v \ge 2w - 1$. Write $v = 2t + 1$.

When $w = t + 1$, form a near 1-factorization on V consisting of $2t + 1$ near 1-factors, $F_0, \dots, F_t$, $G_0, \dots, G_{t-1}$. Without loss of generality, F_i misses vertex i for $0 \le i \le t$, and F_t contains the edges $\{\{k, t + k + 1\} : 0 \le k < t\}$. The union of any two near 1-factors contains a nonnegative number of even cycles and a path with an even number of edges. For $0 \le i \le t$, form triangles T_{ij} by adding a_i to each edge $e_{ij} \in F_i$. As in the proof of Theorem 4.1, for $0 \le i < t$, use the edges of G_i to convert every triangle T_{ij} into a kite. Then add edge $\{i, a_i\}$ to triangle T_{ti} constructed from edge $\{i, t + 1 + i\}$. What remains is the single edge $\{t, a_t\}$ together with all edges on W.

When $w \notin \{2, 4\}$, place an $\mathscr{ON}(w, 4)$ on W of cost $\binom{w}{2}$ so that a_t appears in a triangle in the decomposition, and use the edge $\{t, a_t\}$ to convert this to a kite. We use a decomposition having $1 \le \delta \le 4$ triangles, therefore getting a solution with at most 3 triangles. Such a decomposition exists by Theorem 2.1 if $w \not\equiv 0, 1 \pmod 8$. If $w \equiv 0, 1 \pmod 8$ we build a solution using 4 triangles as follows. If $w \equiv 1 \pmod 8$, form an $\mathscr{ON}(w - 2, 4)$ on vertices $\{0, \dots, w - 3\}$ with 3 triangles. Add the triangle $(w-3, w-2, w-1)$ and the 4-cycles $\{(2h, w-2, 2h+1, w-1) : 0 \le h < \frac{w-3}{2}\}$. For $w = 8$ a solution with 4 triangles is given as $\mathscr{ON}(8, 4)$: $\mathcal{B} = \{(1, 2, 0; 4)$, $(0, 3, 6; 7)$, $(0, 7, 5; 2)$, $(4, 5, 3; 1)$, $(1, 4, 7)$, $(1, 5, 6)$, $(2, 3, 7)$, $(2, 4, 6)\}$. In general, for $w \equiv 0 \pmod 8$, form an $\mathscr{ON}(w - 8, 4)$ on vertices $\{0, \dots, w - 9\}$ with 4 triangles. Add the 4-cycles $\{(2h, w - 2j, 2h + 1, w - 2j + 1) : 0 \le h < \frac{w-8}{2}\}$ for $1 \le j \le 4$ and an $\mathscr{ON}(8, 4)$ without triangles on the 8 vertices $\{w - 8, \dots, w - 1\}$.

Two values for w remain. When $w = 2$, an $\mathscr{ON}(5, 3; 4, 1)$ is also an $\mathscr{ON}(5, 3; 4, 2)$. The case when $v = 7$ and $w = 4$ is given as $\mathscr{MON}(7 + 4, 7; 4, 2)$: $\mathcal{B} = \{(a_0, 4, 2; 3)$, $(a_0, 3, 6; 0)$, $(a_0, 0, 5; 1)$, $(a_1, 5, 3; 4)$, $(a_1, 4, 6; 1)$, $(a_1, 1, 0; 2)$, $(a_2, 0, 4; 5)$, $(a_2, 6, 5; a_3)$, $(a_2, 1, 2; 5)$, $(0, 3, a_3; 2)$, $(1, a_0, a_2, 3)$, (a_0, a_1, a_2, a_3), $(a_1, 2, 6, a_3)$, $(1, 4, a_3)\}$. The solution given has only 1 triangle.

Henceforth $w \le t$. For $t > 2$, form $\{F_0, \dots, F_{w-1}, G_0, \dots, G_{2t-1-w}\}$, a near 1-factorization of $K_v \backslash C_t$ where C_t is the t-cycle on $(0, 1, \dots, t-1)$; such a factorization exists [25]. Name the factors so that the missing vertex in F_i is $\lfloor i/2 \rfloor$ for $0 \le i < w$ (this can be done, as every vertex i satisfying $0 \le i < t$ is the missing vertex in two of the near 1-factors). Form triangles using $F_0, \dots, F_{w-1}$ and convert to kites using $G_0, \ldots, G_{w-1}$ as before. There remain $2(t-w)$ near 1-factors $G_w, \ldots, G_{2t-1-w}$. For $0 \le h < t-w$, $G_{w+2h} \cup G_{w+2h+1}$ contains even cycles and an even path, and so partitions into P_3's. Then the edges remaining are (1) the edges of the t-cycle; (2) the edges $\{\{\lfloor i/2 \rfloor, a_i\} : 0 \le i < w\}$; and (3) all edges on W. For $0 \le i < \lfloor \frac{w}{2} \rfloor$, form triangle (i, a_{2i}, a_{2i+1}) and add edge $\{i, i+1\}$ to convert it to a kite. Edges $\{\{i, i+1 \bmod t\} : \lfloor \frac{w}{2} \rfloor \le i < t\}$ of the cycle remain from (1); edge $\{\frac{w-1}{2}, a_{w-1}\}$ remains when w is odd, and no edge remains when w is even, from (2); and all edges except for a set of $\lfloor \frac{w}{2} \rfloor$ disjoint edges on W remain.

When $w \neq 3$, we partition the remaining edges in (1) (which form a path of length $t - \lfloor \frac{w}{2} \rfloor$) into P_3's when $t - \lfloor \frac{w}{2} \rfloor$ is even, and into P_3's and the P_2 $\{0, t-1\}$ when $t - \lfloor \frac{w}{2} \rfloor$ is odd. We adjoin edge $\{\frac{w-1}{2}, a_{w-1}\}$ to the P_3 (from the t-cycle) containing the vertex $\frac{w-1}{2}$ to form a P_4. Finally, we apply Lemma 4.2 to exhaust the remaining edges on W.

When $w = 3$, the remaining edges are those of the path $[0, t-1, t-2, \ldots, 2, 1, a_2]$ and edges $\{\{a_2, a_0\}, \{a_2, a_1\}\}$. Include $\{\{1,2\}, \{1, a_2\}, \{a_2, a_0\}, \{a_2, a_1\}\}$ in the decomposition, and partition the remainder into P_3's and, when $v \equiv 3 \pmod 4$, one P_2 $\{0, t-1\}$.

The case when $t = 2$ is shown in Example 1.2 (the construction is that given above, except that we start with a near 1-factorization of $K_5 \setminus \{\{0,1\}, \{0,3\}\}$). $\square$

4.2. $\mathscr{MON}(n, v; 4, 2)$.

Theorem 4.4. *For $C' = 2$ and $v \le 2w$,*

$$wavecost\, \mathscr{MON}(v+w, v; 4, 2) = wavecost\, \mathscr{MON}(v+w, 4).$$

Proof. It suffices to prove the statement for $v \in \{2w-2, 2w-1, 2w\}$. When $v = 2w-1$, apply the construction given in the proof of Theorem 4.3, where we noted that there are at most three triangles. The proof of Theorem 4.3 provides explicit solutions when $w \in \{2, 4\}$.

Now suppose that $v = 2w$. In the proof of Theorem 4.1, $\frac{v}{2} = w$ triangles containing one edge on V and two edges between a vertex of V and a_{w-1} remain. Then convert $w-1$ triangles to kites using edges on W incident to a_{w-1}. That leaves one triangle. When the remaining edges on the $w-1$ vertices of W support an $\mathscr{MON}(w-1, 4)$ that contains at most two triangles, we are done. It remains to treat the cases when $w - 1 \equiv 2, 7 \pmod 8$ or $w - 1 = 4$. For the first case, let x be one vertex of the triangle left containing a_{w-1}, namely, (a_{w-1}, x, y). Consider the pendant edge $\{x, t\} \in G_{w-2}$ used in a kite containing a_{w-2}. Delete $\{x, t\}$ from this kite and adjoin $\{a_{w-3}, a_{w-2}\}$ to the unique triangle so formed, forming another kite. Finally adjoin $\{x, t\}$ to the triangle (a_{w-1}, x, y). Proceed as before, but partition all edges on $\{a_0, \ldots, a_{w-2}\}$ except edge $\{a_{w-3}, a_{w-2}\}$ into 4-cycles and kites. The case when $w - 1 = 4$ is similar, but we leave three of the triangles arising from F_{w-1} and partition $K_5 \setminus P_3$ into two kites.

Now suppose that $v = 2w - 2$. We do a construction similar to that above. In the proof of Theorem 4.1, there remain $3\frac{v}{2} = 3(w-1)$ triangles joining a_{w-3} (resp., a_{w-2}, a_{w-1}) to F_{w-3} (resp., F_{w-2}, F_{w-1}). Then convert the $w-1$ triangles containing a_{w-1} to kites using edges on W incident to a_{w-1}, $w-2$ triangles containing a_{w-2} to kites using the remaining edges on W incident to a_{w-2}, and $w-3$ triangles containing a_{w-3} to kites using edges on W incident to a_{w-3}. That leaves three triangles. So, if $w - 3 \equiv 0, 1 \pmod 8$, we are done. Otherwise, as above, choose in each of the three remaining triangles vertices x_1, x_2, x_3; consider the edges $\{x_1, t_1\}$ (resp., $\{x_2, t_2\}$) appearing in the kites containing a_{w-4} and x_1 (resp., a_{w-4} and x_2), and the edge

$\{x_3, t_3\}$ in the kite containing a_{w-5} and x_3. Delete these edges and adjoin them to the three remaining triangles. Finally adjoin the edges $\{a_{w-4}, a_{w-5}\}$ and $\{a_{w-4}, a_{w-6}\}$ to the two triangles obtained from the two kites containing a_{w-4}, and adjoin the edge $\{a_{w-5}, a_{w-6}\}$ to the triangle obtained from the kite containing a_{w-5}. Proceed as before, but partition all edges on $\{a_0, \dots, a_{w-4}\}$ except the triangle $(a_{w-6}, a_{w-5}, a_{w-4})$ into 4-cycles and kites. $\square$

THEOREM 4.5.

1. *When $v > 2w$ is even,*

$$\textit{wavecost}\, \mathscr{MON}(v+w, v; 4, 2) = \left\lceil \left(2\binom{v}{2} + \binom{w}{2}\right) \Big/ 4 \right\rceil.$$

2. *When $v > 2w$ is odd,*

$$\textit{wavecost}\, \mathscr{MON}(v+w, v; 4, 2) = \left\lceil \left(2\binom{v}{2} + \frac{(w-1)(w+1)}{2}\right) \Big/ 4 \right\rceil.$$

Proof. First we treat the case when v is even. Then (by Theorem 4.1) an $\mathscr{ON}(v+w, v; 4, 2)$ must employ vw or $vw-1$ neutral edges, using all vw edges between V and W. Each such graph uses two edges on V and none on W, except that a single graph may use one on V and one on W. Now the edges of V must appear on $\lceil \frac{1}{2}\binom{v}{2} \rceil$ different wavelengths, and these wavelengths use at most one edge on W (when $v \equiv 2 \pmod 4$). Thus at least $\lceil \binom{w}{2}/4 \rceil$ additional wavelengths are needed when $v \equiv 0 \pmod 4$, for a total of $\lceil \binom{v}{2}/2 + \binom{w}{2}/4 \rceil$. When $v \equiv 2 \pmod 4$, at least $\lceil (\binom{w}{2} - 1)/4 \rceil$ additional wavelengths are needed; again the total is $\lceil \binom{v}{2}/2 + \binom{w}{2}/4 \rceil$. Theorem 4.1 realizes this bound.

When v is odd, first suppose that w is even. In order to realize the bound of Theorem 4.3 for drop cost, by Lemma 2.2, $\frac{w}{2}$ neutral edges appear in subgraphs with one neutral edge and all other neutral edges appear in subgraphs with two. In both cases, two edges between V and W are consumed by such a subgraph. When two neutral edges are used, no edge on W can be used; when one neutral edge is used, one edge on W can also be used. It follows that the number of wavelengths is at least $\frac{1}{2}(\binom{v}{2} - \frac{w}{2}) + \frac{w}{2} + \frac{1}{4}(\binom{w}{2} - \frac{w}{2})$. This establishes the lower bound. The case when w is odd is similar. The proof of Theorem 4.3 gives constructions with at most 3 triangles and so establishes the upper bound except when $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, $w \neq 3$, where the construction employs one more graph than the number of wavelengths permitted. However, one graph included is the P_2 $\{0, t-1\}$, and in the decomposition on W, there is a triangle. These can be placed on the same wavelength to realize the bound. $\square$

When $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, $w \neq 3$, we place a disconnected graph, $P_2 \cup K_3$, on one wavelength in order to meet the bound. The construction of Theorem 4.3 could be modified to avoid this by instead using a decomposition of $K_w \setminus (K_3 \cup \frac{w-3}{2} K_2)$ into 4-cycles and kites, and using the strategy used in the case for $w = 3$. In this way, one could prove the slightly stronger result that the number of (connected) subgraphs in the decomposition matches the lower bound on number of wavelengths needed.

In Theorem 3.3, the number of wavelengths and the drop cost are minimized simultaneously by the constructions given; each constructed $\mathscr{ON}(v+w, v; 4, 1)$ has not only the minimum drop cost but also the minimum number of wavelengths over all $N(v+w, v; 4, 1)$'s. This is not the case in Theorem 4.5. For example, when $v > (1+\sqrt{2})w$, it is easy to construct an $N(v+w, v; 4, 2)$ that employs only $\lceil \binom{v}{2}/2 \rceil$

wavelengths, which is often much less than are used in Theorem 4.5. We emphasize therefore that an $\mathscr{MON}(v+w,v;4,2)$ minimizes the number of wavelengths over all $\mathscr{ON}(v+w,v;4,2)$'s, *not necessarily* over all $N(v+w,v;4,2)$'s.

5. Case $C' = 3$.

5.1. $\mathscr{ON}(n,v;4,3)$.

THEOREM 5.1. *Let $v+w \geq 5$.*

1. *When $w \geq 1$, cost $\mathscr{ON}(v+w,v;4,3) =$ cost $\mathscr{ON}(v+w,4)$.*
2. *cost $\mathscr{ON}(v+0,v;4,3) =$ cost $\mathscr{ON}(v,3)$.*

Proof. The second statement is trivial. Moreover *cost* $\mathscr{ON}(n,4) =$ *cost* $\mathscr{ON}(n,3)$ when $n \equiv 1,3 \pmod 6$, and hence the first statement holds when $v+w \equiv 1,3 \pmod 6$. To complete the proof it suffices to treat the upper bound when $w=1$.

When $v+1 \equiv 5 \pmod 6$, there is a maximal partial triple system $(X,\mathcal{B})$ with $|X| = v+1$ covering all edges except those in the 4-cycle (r,x,y,z). Set $W = \{r\}$, $V = X \setminus W$, and add the 4-cycle to the decomposition to obtain an $\mathscr{ON}(v+1,v;4,3)$.

When $v \equiv 1,5 \pmod 6$, set $\ell = v-1$, and when $v \equiv 3 \pmod 6$, set $\ell = v-3$. Then ℓ is even. Form a maximal partial triple system $(V,\mathcal{B})$, $|V| = v$, covering all edges except those in an ℓ-cycle $(0,1,\dots,\ell-1)$ [16]. Add a vertex a_0 and form kites $(a_0, 2i, 2i+1; (2i+2) \bmod \ell)$ for $0 \leq i < \frac{\ell}{2}$. For $i \in \{\ell,\dots,v-1\}$, choose a triple $B_i \in \mathcal{B}$ so that $i \in B_i$ and $B_i = B_j$ only if $i = j$. Add $\{a_0, i\}$ to B_i to form a kite. This yields an $\mathscr{ON}(v+1,v;4,3)$. $\square$

5.2. $\mathscr{MON}(n,v;4,3)$.

We focus first on lower bounds in section 5.2.1, and then we provide constructions attaining these lower bounds in section 5.2.2.

5.2.1. Lower bounds.

When $C' = 3$, Theorem 5.1 makes no attempt to minimize the number of wavelengths. We focus on this case here. Except when $n \in \{2,4\}$ or $v = n$, *cost* $\mathscr{ON}(n,v;4,3) = \binom{n}{2}$, and every graph in an $\mathscr{ON}(n,v;4,3)$ is a triangle, kite, or 4-cycle. Let δ, κ, and γ denote the numbers of triangles, kites, and 4-cycles in the grooming, respectively. Then $3\delta + 4\kappa + 4\gamma = \binom{n}{2}$, and the number of wavelengths is $\delta + \kappa + \gamma$. Thus in order to minimize the number of wavelengths, we must minimize the number δ of triangles. We focus on this equivalent problem henceforth.

In an $\mathscr{ON}(n,v;4,3)$, for $0 \leq i \leq 3$ and $0 \leq j \leq 4$, let δ_{ij}, κ_{ij}, and γ_{ij} denote the number of triangles, kites, and 4-cycles, respectively, each having i edges on V and j edges between V and W. The only counts that can be nonzero are δ_{00}, δ_{02}, δ_{12}, δ_{30}; κ_{00}, κ_{01}, κ_{02}, κ_{03}, κ_{12}, κ_{13}, κ_{22}, κ_{31}; γ_{00}, γ_{02}, γ_{04}, γ_{12}, γ_{22}. We write $\sigma_{ij} = \kappa_{ij} + \gamma_{ij}$ when we do not need to distinguish kites and 4-cycles. Our objective is to minimize $\delta_{00} + \delta_{02} + \delta_{12} + \delta_{30}$ subject to certain constraints; we adopt the strategy of [15] and treat this as a linear program.

Let $\varepsilon = 0$ when $v \equiv 1,3 \pmod 6$, $\varepsilon = 2$ when $v \equiv 5 \pmod 6$, and $\varepsilon = \frac{v}{2}$ when $v \equiv 0 \pmod 2$. We specify the linear program in Table 1. The first row lists the primal variables. The second lists coefficients of the objective function to be minimized. The third lists the coefficients of linear inequalities, with the final column providing the *lower bound* on the linear combination specified. The first inequality states that the number of edges on V used is at least the total number on V, while the second specifies that the number of edges used between V and W is at most the total number between V and W. For the third, when $v \equiv 5 \pmod 6$ at least four edges on V are not in triangles, and so at least two graphs containing edges of V do not have a triangle on V; when $v \equiv 0 \pmod 2$ every graph can induce at most two odd degree vertices on V, yet all are odd in the decomposition.

TABLE 1
The linear program for $\mathscr{ON}(n,v;4,3)$.

δ_{30}	δ_{12}	δ_{02}	δ_{00}	κ_{31}	σ_{22}	κ_{13}	σ_{12}	γ_{04}	κ_{03}	σ_{02}	κ_{01}	σ_{00}	
1	1	1	1	0	0	0	0	0	0	0	0	0	
3	1	0	0	3	2	1	1	0	0	0	0	0	$\binom{v}{2}$
0	-2	-2	0	-1	-2	-3	-2	-4	-3	-2	-1	0	$-vw$
0	1	0	0	0	1	1	1	0	0	0	0	0	ε

We do not solve this linear program. Rather we derive lower bounds by considering its dual. Let y_1, y_2, and y_3 be the dual variables. A dual feasible solution has $y_1 = \frac{1}{3}$, $y_2 = 1$, and $y_3 = \frac{4}{3}$, yielding a dual objective function value of $\frac{1}{6}v(v-1) - vw + \frac{4}{3}\varepsilon$. Recall that every dual feasible solution gives a lower bound on all primal feasible solutions.

On the other hand, $3\delta \equiv \binom{n}{2} \pmod 4$ and so $\delta \equiv 9\delta \equiv 3\binom{n}{2} \pmod 4$. The value of $3\binom{n}{2} \pmod 4$ is in fact the value of t given in Theorem 2.1. Therefore if x is a lower bound on δ in an $\mathscr{ON}(n,v;4,3)$, so is $\langle x\rangle_n$, where $\langle x\rangle_n$ denotes the smallest nonnegative integer $\overline{x}$ such that $\overline{x} \geq x$ and $\overline{x} \equiv 3\binom{n}{2} \pmod 4$.

The discussion above proves the general lower bound on the number of triangles.

THEOREM 5.2. *Let $v+w \geq 5$, and let*

$$L(v,w) = \begin{cases} \frac{1}{6}v(v-1) - vw & \textit{if} \quad v \equiv 1,3 \pmod 6, \\ \frac{1}{6}v(v-1) - vw + \frac{8}{3} & \textit{if} \quad v \equiv 5 \pmod 6, \\ \frac{1}{6}v(v+3) - vw & \textit{if} \quad v \equiv 0 \pmod 2. \end{cases}$$

Then the number of triangles in an $\mathscr{ON}(v+w,v;4,3)$ is at least

$$\delta_{\min}(v,w) = \langle L(v,w)\rangle_{v+w}.$$

Remark 5.3. In particular, if v is odd and $w \geq \lceil\frac{v-1}{6}\rceil$ or if v is even and $w \geq \lceil\frac{v-4}{6}\rceil$, then $L(v,w) \leq 0$ and the minimum number of triangles is $\delta_{\min}(v,w) = \langle 0\rangle_{v+w} \leq 3$.

5.2.2. Upper bounds. We first state two simple lemmas to be used intensively in the proof of Theorem 5.7. The following result shows that in fact we do not need to check *exactly* that the number of triangles of an optimal construction meets the bound of Theorem 5.2.

LEMMA 5.4. *Any $\mathscr{ON}(v+w,v;4,3)$ is an $\mathscr{MON}(v+w,v;4,3)$ if the number of triangles that it contains is at most* $\max(3, \lceil L(v,w)\rceil + 3)$.

Proof. In the closed interval $[\lceil L(v,w)\rceil, \lceil L(v,w)\rceil + 3]$ there is exactly one integer congruent to $3\binom{n}{2} \pmod 4$, and so exactly one integer equal to $\delta_{\min}(v,w)$. ∎

Combining Remark 5.3 and Lemma 5.4 we deduce that when v is odd and $w \geq \lceil\frac{v-1}{6}\rceil$ or if v is even and $w \geq \lceil\frac{v-4}{6}\rceil$, to prove the optimality of a construction it is enough to check that there are at most three triangles.

As a prelude to the constructions, let $(V,\mathcal{B})$ be a partial triple system, $V = \{0,\ldots,v-1\}$, and $\mathcal{B} = \{B_1,\ldots,B_b\}$. Let r_i be the number of blocks of $\mathcal{B}$ that contain $i \in V$. A *headset* is a multiset $S = \{s_1,\ldots,s_b\}$ so that $s_k \in B_k$ for $1 \leq k \leq b$, and for $0 \leq i \leq v-1$ the number of occurrences of i in S is $\lfloor\frac{r_i}{3}\rfloor$ or $\lceil\frac{r_i}{3}\rceil$.

LEMMA 5.5. *Every partial triple system has a headset.*

Proof. Form a bipartite graph Γ with vertex set $V \cup \mathcal{B}$, and an edge $\{v,B\}$ for $v \in V$ and $B \in \mathcal{B}$ if and only if $v \in B$. The graph Γ admits an equitable 3-edge-coloring [18]; that is, the edges can be colored green, white, and red so that every vertex of degree d is incident with either $\lfloor d/3 \rfloor$ or $\lceil d/3 \rceil$ edges of each color. Then for $1 \le k \le b$, B_k is incident to exactly three edges, and hence to exactly one edge $\{i_k, B_k\}$ that is green; set $s_k = i_k$. Then $(s_1, \ldots, s_b)$ forms the headset. $\square$

LEMMA 5.6. *There exist an* $\mathscr{MON}(13+3, 13; 4, 3)$, $\mathscr{MON}(15+4, 15; 4, 3)$, $\mathscr{MON}(17+3, 17; 4, 3)$, *and* $\mathscr{MON}(17+4, 17; 4, 3)$.

Proof. $\mathscr{MON}(13+3, 13; 4, 3)$:

$$\begin{aligned}\mathcal{B} = \{&(5+i, 4+i, 1+i; a_1) \mid i = 0, 1, \ldots, 9\}\\ &\cup\{(1+i, 5+i, 4+i; a_0) \mid i = 10, 11, 12\}\\ &\cup\{(3+i, 1+i, 9+i; a_2) \mid i = 6, 7, \ldots, 12\}\\ &\cup\{(9+i, 1+i, 3+i; a_0) \mid i = 1, 2, \ldots, 5\}\\ &\cup\{(9, 3, 1; a_2), (0, a_1, a_2; 12), (12, a_1, a_0; 0),\\ &\qquad (a_0, 9, a_2, 10), (a_0, a_2, 11; a_1), (a_0, 9, a_2, 10)\},\end{aligned}$$

where the sums are computed modulo 13.

$\mathscr{MON}(15+4, 15; 4, 3)$:

$\mathcal{B} = \{(1,2,3)$	$(a_0,4,a_1,5)$	$(a_0,10,a_1,11)$	$(5,4,1;a_3)$	$(7,1,6;a_1)$
$(6,4,2;a_3)$	$(7,5,2;a_2)$	$(4,7,3;a_2)$	$(6,5,3;a_3)$	$(9,1,8;a_1)$
$(10,1,14;a_0)$	$(11,1,0;a_2)$	$(13,1,12;a_2)$	$(10,2,8;a_2)$	$(11,2,9;a_0)$
$(12,2,14;a_2)$	$(0,2,13;a_3)$	$(8,3,11;a_3)$	$(10,3,12;a_0)$	$(13,3,9;a_2)$
$(14,3,0;a_3)$	$(12,8,4;a_2)$	$(11,4,13;a_0)$	$(0,10,4;a_3)$	$(9,4,14;a_1)$
$(8,5,13;a_2)$	$(0,5,12;a_3)$	$(14,11,5;a_3)$	$(10,9,5;a_2)$	$(8,6,0;a_0)$
$(14,13,6;a_3)$	$(9,6,12;a_1)$	$(10,6,11;a_2)$	$(14,8,7;a_3)$	$(9,7,0;a_1$
$(10,7,13;a_1)$	$(12,11,7;a_0)$	$(6,a_2,a_0;2)$	$(7,a_2,a_1;3)$	$(10,a_3,a_2;1)$
$(1,a_0,a_1;2)$	$(9,a_1,a_3;14)$	$(8,a_3,a_0;3)\}$.		

$\mathscr{MON}(17+3, 17; 4, 3)$:

$\mathcal{B} = \{(7,16,0)$	$(a_0,a_2,0)$	$(a_0,1,2;3)$	$(a_0,3,4;1)$	$(4,5,2;a_1)$
$(1,3,5;a_0)$	$(16,a_0,a_1;a_2)$	$(6,10,1;a_1)$	$(9,14,1;a_2)$	$(15,1,7;a_2)$
$(1,8,12;a_2)$	$(1,0,13;a_2)$	$(1,16,11;a_1)$	$(2,11,6;a_1)$	$(2,16,8;a_2)$
$(10,15,2;a_2)$	$(9,2,13;a_1)$	$(0,2,12;a_1)$	$(2,7,14;a_2)$	$(6,13,3;a_1)$
$(11,3,7;a_1)$	$(12,3,16;a_2)$	$(9,0,3;a_2)$	$(3,10,14;a_1)$	$(8,3,15;a_1)$
$(14,6,4;a_2)$	$(4,11,15;a_2)$	$(7,12,4;a_1)$	$(13,4,8;a_1)$	$(4,16,9;a_2)$
$(0,4,10;a_1)$	$(5,12,6;a_2)$	$(7,13,5;a_2)$	$(8,14,5;a_1)$	$(15,5,9;a_1)$
$(5,16,10;a_2)$	$(5,0,11;a_2)$	$(9,7,6;a_0)$	$(10,8,7;a_0)$	$(11,9,8;a_0)$
$(12,10,9;a_0)$	$(13,11,10;a_0)$	$(14,12,11;a_0)$	$(15,13,12;a_0)$	$(16,14,13;a_0)$
$(0,15,14;a_0)$	$(6,16,15;a_0)$	$(8,6,0;a_1)\}$.		

$\mathscr{MON}(17+4, 17; 4, 3)$:

$\mathcal{B} = \{(2, 9, 11)$	$(9, 12, 16)$	$(a_0, 13, 14; 15)$	$(a_0, 15, 16; 13)$	$(16, 0, 14; a_1)$
$(13, 15, 0; a_0)$	$(13, 2, 1; a_3)$	$(13, 12, 3; a_3)$	$(13, 11, 4; a_3)$	$(5, 10, 13; a_1)$
$(6, 9, 13; a_2)$	$(7, 8, 13; a_3)$	$(14, 4, 2; a_3)$	$(14, 12, 5; a_3)$	$(11, 14, 6; a_3)$
$(14, 10, 7; a_3)$	$(1, 3, 14; a_2)$	$(9, 8, 14; a_3)$	$(1, 4, 15; a_1)$	$(3, 5, 15; a_2)$
$(2, 6, 15; a_3)$	$(15, 7, 12; a_3)$	$(15, 11, 8; a_3)$	$(1, 16, 5; a_1)$	$(6, 4, 16; a_2)$
$(3, 7, 16; a_3)$	$(2, 8, 16; a_1)$	$(10, 16, 11; a_3)$	$(1, 6, 0; a_1)$	$(4, 8, 0; a_2)$
$(10, 15, 9, ; a_3)$	$(2, 10, 0; a_3)$	$(5, 0, 7; a_1)$	$(3, 0, 9; a_1)$	$(12, 0, 11; a_1)$
$(1, a_0, 7; 6)$	$(8, 6, a_0; a_3)$	$(9, a_0, 5; 11)$	$(10, a_0, 4; 9)$	$(11, a_0, 3; 10)$
$(2, a_0, 12; 8)$	$(8, a_1, 1; 11)$	$(10, a_1, 6; 3)$	$(12, 4, a_1; a_3)$	$(3, a_1, 2; 7)$
$(1, a_2, 9; 7)$	$(10, a_2, 8; 3)$	$(11, a_2, 7; 4)$	$(12, a_2, 6; 5)$	$(2, a_2, 5; 8)$
$(3, a_2, 4; 5)$	$(a_1, a_0, a_2; a_3)$	$(12, 1, 10; a_3)\}$.	$\square$	

THEOREM 5.7. *Let $v + w \geq 5$. When $w \geq 1$,*

$$wavecost\,\mathscr{MON}(v+w, v; 4, 3) = \left\lceil \left(\binom{v+w}{2} + \delta_{\min}(v, w) \right) \Big/ 4 \right\rceil.$$

Proof. The lower bound follows from Theorem 5.2, so we focus on the upper bound.

When $w \geq 1$, an $\mathscr{ON}(v+w, v; 4, 3)$ of cost $\binom{v+w}{2}$ is an $\mathscr{ON}(v+w, v-1; 4, 3)$. Let us show that it suffices to prove the statement for $w \leq \frac{v+9}{6}$ when v is odd, and for $w \leq \frac{v+4}{6}$ when v is even. Equivalently, we show that if it is true for these values of w, then it follows for any w. Note that $\delta_{\min}(v, w) \leq 3$ if $\delta_{\min}(v+1, w-1) \leq 3$.

Indeed, let v be even. If $w = \lfloor \frac{v+4}{6} \rfloor + 1$, the result follows from the case for $v+1$ (odd) and $w - 1 = \lfloor \frac{v+4}{6} \rfloor \leq \frac{v+1+9}{6}$, in which case $\delta_{\min}(v+1, w-1) = \langle 0 \rangle_{v+w}$. If $w = \lfloor \frac{v+4}{6} \rfloor + 2$, it follows from the case for $v+1$ (odd) and $w-1 = \lfloor \frac{v+4}{6} \rfloor + 1 \leq \frac{v+1+9}{6}$, and $\delta_{\min}(v+1, w-1) = \langle 0 \rangle_{v+w}$. If $w \geq \lfloor \frac{v+4}{6} \rfloor + 3$, it follows from the case for $v+2$ (even) and $w-2$.

Let v be odd. If $w = \lfloor \frac{v+9}{6} \rfloor + 1$, it follows from the case for $v+1$ (even) and $w-1$, which has been already proved (in this case also $\delta_{\min}(v+1, w-1) = \langle 0 \rangle_{v+w}$). If $w \geq \lfloor \frac{v+9}{6} \rfloor + 2$, it follows from the case for $v+2$ (odd) and $w-2$.

In each case, we use the same general prescription. Given a partial triple system $(V, \mathcal{B})$, a headset $S = \{s_1, \ldots, s_b\}$ is formed using Lemma 5.5. Add vertices $W = \{a_0, \ldots, a_{w-1}\}$, a set disjoint from V of size $w \geq 1$. For each i let D_i be a subset of $\{0, \ldots, w-1\}$, which is specified for each subcase, and that satisfies the following property: $|D_i|$ is at most the number of occurrences of i in the headset S. Among the blocks B_k such that $s_k = i$, we choose $|D_i|$ of them, namely, the subset $\{B_k^j : j \in D_i\}$, and form $|D_i|$ kites by adding for each $j \in D_i$ the edge $\{a_j, i\}$ to the block B_k^j.

The idea behind the construction is that if we can choose $|D_i| = w$, we use all the edges between V and W leaving a minimum number of triangles in the partition of V (see Case O1a). Unfortunately it is not always possible to choose $|D_i| = w$, in particular when w is greater than the number of occurrences of i in the headset. So we distinguish different cases.

Case O1a. $v = 6t+1$ or $6t+3$ and $w \leq \frac{v-1}{6}$. Let $(V, \mathcal{B})$ be a Steiner triple system. For $0 \leq i < v$, let $D_i = \{0, \ldots, w-1\}$. Apply the general prescription.

If $v = 6t+1$, i appears t times in S and $w \le \frac{v-1}{6} = t$. If $v = 6t+3$, i appears t or $t+1$ times in S and $w \le t$. In both cases $|D_i|$ is at most the number of occurrences of i in S, so the construction applies and all the edges between V and W are used in the kites. All the edges on V are used and $\frac{v(v-1)}{6} - vw$ triangles remain. Finally, it remains to partition the edges of W. When $w \notin \{2,4\}$, form an $\mathscr{MON}(w,4)$ on W, and doing so we have at most $\delta_{\min}$ triangles. If $w = 2$ or $w = 4$, remove edges $\{a_0, 0\}$ and $\{a_1, 0\}$ from their kites and partition K_W together with these edges into a triangle ($w = 2$) or two kites ($w = 4$).

Case O1b. $v = 6t+5$ and $w \le \frac{v-1}{6}$. Form a partial triple system $(V, \mathcal{B})$ covering all edges except those in the C_4 $(0,1,2,3)$. For $0 \le i \le 3$, let $D_i = \{0, \ldots, w-2\}$, and for $4 \le i < v$, $D_i = \{0, \ldots, w-1\}$. Apply the general prescription. Add the kites $(a_{w-1}, 1, 2; 3)$ and $(a_{w-1}, 3, 0; 1)$. Here again i appears at least t times in S and $w \le t$. So D_i is at most the number of occurrences of i in S. Again we have used all the edges on V and all the edges between V and W. It remains to partition the edges of W, and this can be done as in the Case O1a.

Case O2. $v = 6t+3$ and $w = t+1$, $v > 3$. Form a partial triple system covering all edges except those on the v-cycle $\{\{i, (i+1) \bmod v\} : 0 \le i < v\}$ [16]. Set $D_i = \{1, \ldots, w-1\}$ for all i. Apply the general prescription. Adjoin edges from a_0 to a partition of the cycle, minus edge $\{0, v-1\}$, into P_3's. The only edge between V and W that remains is $\{a_0, v-1\}$. When an $\mathscr{ON}(w,4)$ exists having 1, 2, 3, or 4 triangles, this edge is used to convert a triangle to a kite. This handles all cases except when $w \in \{2,4\}$. In these cases, remove the pendant edge $\{a_1, v-1\}$ from its kite. When $w = 2$, $\{a_0, a_1, v-1\}$ forms a triangle. When $w = 4$, partition the edges on W together with $\{a_0, v-1\}$ and $\{a_1, v-1\}$ into two kites.

Case O3. $v = 6t+1$ and $w = t+1$. When $t = 1$, an $\mathscr{MON}(7+2, 7; 4, 3)$ has $\mathcal{B} = \{(0, a_1, a_0; 6),\ (2, 0, 6; a_1),\ (3, 0, 4; a_1),\ (1, 0, 5; a_1),\ (3, 6, 5; a_0),\ (4, 6, 1; a_1),$ $(3, 2, 1; a_0),\ (5, 2, 4; a_0),\ (a_0, 2, a_1, 3)\}$.

A solution with $t = 2$ is given in Lemma 5.6.

When $t \ge 3$, form a 3-GDD of type 6^t with groups $\{\{6p+q : 0 \le q < 6\} : 0 \le p < t\}$. Let $D_{6p+q} = \{0, \ldots, w-2\} \setminus \{p\}$ for $0 \le p < t$ and $0 \le q < 6$. Apply the general prescription. For $0 \le p < t$, on $\{6p+q : 0 \le q < 6\} \cup \{v-1\} \cup \{a_{w-1}, a_p\}$ place an $\mathscr{MON}(7+2, 7; 4, 3)$ obtained from the solution $\mathcal{B}$ for $t = 1$, by replacing q by $6p+q : 0 \le q < 6$, 6 by $v-1$ a_0 by a_{w-1} and a_1 by a_p; then omit the kite $(a_p, 6p, a_{w-1}; v-1)$. All edges on W remain; the edges $\{a_{w-1}, 6p\}$ and $\{a_p, 6p\}$ remain for $0 \le p < t$, and the edge $\{a_{w-1}, v-1\}$ remains.

Add the kites $(a_{w-2}, 6(w-2), a_{w-1}; v-1)$, and for $0 \le j < w-2 = t-1$, $(6j, a_{w-1}, a_j; a_{w-2})$. If $w-2 \notin \{2,4\}$, that is, $t \notin \{3,5\}$, place an $\mathscr{MON}(w-2, 4)$ on $W - a_{w-2} - a_{w-1}$. Note that, as $3\binom{w-2}{2} \equiv 3\binom{v+w}{2} \pmod 4$, we have the right number of triangles (at most 3). If $w-2 \in \{2,4\}$, remove edges $\{a_0, w-2\}$ and $\{a_1, w-2\}$ from their kites, and partition K_w together with these edges.

Case O4. $v = 6t+5$ and $w = t+1$. For $t = 0$, an $\mathscr{MON}(5+1, 5; 4, 3)$ has kites $(3, a_0, 0; 1)$, $(1, a_0, 2; 3)$, $(1, 3, 4; a_0)$ and triangle $(0, 2, 4)$.

For $t = 1$, let $V = \{0, \ldots, 10\}$ and $W = \{a_0, a_1\}$. An $\mathscr{MON}(11+2, 11; 4, 3)$ is formed by using an $\mathscr{MON}(5+1, 5; 4, 3)$ on $\{0,1,2,3,4\} \cup \{a_0\}$, and a partition of the remaining edges, denoted by $\mathcal{Q}$, into 15 kites and a triangle. So we have two triangles, attaining $\delta_{min}(11, 2)$ as $13 \equiv 5 \pmod 8$. The partition of $\mathcal{Q}$ is as follows: the triangle $(a_0, a_1, 10)$ and the kites $(0, 6, 5; a_0)$, $(1, 8, 6; a_0)$, $(2, 9, 7; a_0)$, $(3, 10, 8; a_0)$, $(4, 6, 9; a_0)$,

$(8, 9, 0; a_1)$, $(5, 7, 1; a_1)$, $(5, 8, 2; a_1)$, $(6, 7, 3; a_1)$, $(5, 10, 4; a_1)$, $(3, 9, 5; a_1)$, $(2, 10, 6; a_1)$, $(0, 10, 7; a_1)$, $(4, 7, 8; a_1)$, and $(1, 10, 9; a_1)$.

For $t = 2$, an $\mathscr{MON}(17 + 3, 17; 4, 3)$ is given in Lemma 5.6.

For $t \geq 3$, form a 3-GDD of type 6^t with groups $\{\{6p+q : 0 \leq q < 6\} : 0 \leq p < t\}$. Let $D_{6p+q} = \{0, \ldots, w-2\} \setminus \{p\}$ for $0 \leq p < t$ and $0 \leq q < 6$. Apply the general prescription. There remain uncovered for each p the edges of the set $\mathcal{Q}_p$ obtained from the complete graph on the set of vertices $\{6p+q : 0 \leq q < 6\} \cup \{v-5, v-4, v-3, v-2, v-1\} \cup \{a_{w-1}, a_p\}$ minus the complete graph on $\{v-5, v-4, v-3, v-2, v-1\} \cup \{a_{w-1}\}$.

To deal with the edges of $\mathcal{Q}_p$, we start from a partition of $\mathcal{Q}$, where we replace pendant edges in kites as follows: Replace $\{a_1, 4\}$ by $\{a_1, 10\}$, $\{a_0, 8\}$ by $\{a_0, 10\}$, and $\{a_1, 2\}$ by $\{a_0, 8\}$. We delete the triangle $(a_0, a_1, 10)$, resulting in a new partition of $\mathcal{Q}$ into 15 kites and the 3 edges $\{a_0, a_1\}$, $\{a_1, 2\}$, and $\{a_1, 4\}$. Then we obtain a partition of $\mathcal{Q}_p$ by replacing $\{0, 1, 2, 3, 4\}$ by $\{v-5, v-4, v-3, v-2, v-1\}$, $q+5$ by $6p+q$ for $0 \leq q < 6$, a_0 by a_{w-1}, and a_1 by a_p. At the end we get a partition of $\mathcal{Q}_p$ into 15 kites plus the 3 edges $\{a_{w-1}, a_p\}$, $\{a_p, v-3\}$, and $\{a_p, v-1\}$.

Now the $3t$ edges $\{\{a_{w-1}, a_p\}, \{a_p, v-3\}, \{a_p, v-1\} : 0 \leq p < t\}$ plus the uncovered edges of K_W form a K_{t+3} missing a triangle on $\{a_{w-1}, v-3, v-1\}$. If $t + 3 \equiv 2, 3, 4, 5, 6, 7 \pmod 8$, use Theorem 2.1 to form an $\mathscr{ON}(t+3, 4)$ having a triangle $(v-3, v-1, a_{w-1})$ and 0, 1, or 2 other triangles; remove the triangle $(v-3, v-1, a_{w-1})$ to complete the solution with 1, 2, or 3 triangles (the triangle $(v-5, v-3, v-1)$ is still present). A variant is needed when $t+3 \equiv 0, 1 \pmod 8$. In these cases, form an $\mathscr{ON}(t+3, 4)$ (having no triangles) in which $(v-3, a_{w-1}, v-1; a_1)$ is a kite. Remove all edges of this kite, and use edge $\{a_1, v-1\}$ to convert triangle $(v-5, v-3, v-1)$ to a kite.

Finally, place an $\mathscr{MON}(5+1, 5; 4, 3)$ on $\{v-5, v-4, v-3, v-2, v-1\} \cup \{a_0\}$. Altogether we have a partition of all the edges using at most 3 triangles.

Case O5. $\boldsymbol{v = 6t + 5}$ **and** $\boldsymbol{w = t + 2}$**.** When $t = 0$, partition all edges on $\{0, 1, 2, 3, 4\} \cup \{a_0, a_1\}$ except $\{a_0, a_1\}$ into kites $(3, 1, a_0; 0)$, $(3, 2, a_1; 0)$, $(a_1, 1, 4; 2)$, $(0, 1, 2; a_0)$, and $(3, 0, 4; a_0)$. Then an $\mathscr{MON}(5 + 2, 5; 4, 3)$ is obtained by removing pendant edges $\{a_0, 0\}$ and $\{a_1, 0\}$ and adding triangle $(a_0, a_1, 0)$.

When $t = 1$, an $\mathscr{MON}(11 + 3, 11; 4, 3)$ on $\{0, \ldots, 10\} \cup \{a_0, a_1, a_2\}$ is obtained by taking the above partition on $\{0, 1, 2, 3, 4\} \cup \{a_0, a_1\}$, the triangle (a_0, a_1, a_2), and a partition of the remaining edges (which form a graph called $\mathcal{Q}$) into 11 kites and 6 4-cycles as follows: kites $(2, 9, 7; a_0)$, $(4, 5, 10; a_0)$, $(2, 10, 6; a_1)$, $(4, 6, 9; a_2)$, $(7, 10, 0; a_2)$, $(6, 8, 1; a_2)$, $(5, 8, 2; a_2)$, $(5, 9, 3; a_2)$, $(7, 8, 4; a_2)$, $(6, 7, 5; a_2)$, and $(9, 10, 8; a_1)$; and 4-cycles $(0, 6, a_0, 5)$, $(0, 8, a_0, 9)$, $(1, 5, a_1, 7)$, $(1, 9, a_1, 10)$, $(3, 6, a_2, 7)$, and $(3, 8, a_2, 10)$.

A solution with $t = 2$ is given in Lemma 5.6.

When $t \geq 3$, form a 3-GDD of type 6^t with groups $\{\{6p + q : 0 \leq q < 6\} : 0 \leq p < t\}$. Let $D_{6p+q} = \{0, \ldots, w-3\} \setminus \{p\}$ for $0 \leq p < t$ and $0 \leq q < 6$. Apply the general prescription. Add a partition of the complete graph on $\{v-5, v-4, v-3, v-2, v-1\} \cup \{a_{w-2}, a_{w-1}\}$ as in the case when $t = 0$. It remains to partition, for each p, $0 \leq p < t$. The graph $\mathcal{Q}_p$ is obtained from the complete graph on $\{6p+q : 0 \leq q < 6\} \cup \{v-5, v-4, v-3, v-2, v-1\} \cup \{a_{w-2}, a_{w-1}, a_p\}$ minus the complete graph on $\{v-5, v-4, v-3, v-2, v-1\} \cup \{a_{w-2}, a_{w-1}\}$. This partition is obtained from that of $\mathcal{Q}$ by replacing $\{0, 1, 2, 3, 4\}$ by $\{v-5, v-4, v-3, v-2, v-1\}$, a_0 by a_{w-2}, a_1 by a_{w-1}, and a_2 by a_p. What remains is precisely the edges on W, so place an $\mathscr{MON}(w, 4)$ on W to complete the construction.

Case O6. $\boldsymbol{v = 6t + 3}$ **and** $\boldsymbol{w = t + 2}$**.** When $t = 0$, an $\mathscr{MON}(3 + 2, 3; 4, 3)$ has triangles $(a_0, 0, 1)$ and $\{a_1, 1, 2\}$ and 4-cycle $(0, 2, a_0, a_1)$.

When $t = 1$, on $\{0,\dots,8\} \cup \{a_0,a_1,a_2\}$, place kites $(2,6,4;a_0)$, $(0,8,4;a_1)$, $(0,5,7;a_1)$, $(3,6,0;a_2)$, $(1,7,4;a_2)$, $(5,8,2;a_2)$, $(1,6,5;a_2)$, $(2,7,3;a_2)$, $(3,8,1;a_2)$, $(3,5,a_0;a_2)$, $(7,a_0,6;a_2)$, $(6,8,a_1;a_2)$, $(7,a_2,8;a_0)$, and 4-cycle $(3,4,5,a_1)$. Adding the blocks of an $\mathscr{MON}(3+2,3;4,3)$ forms an $\mathscr{MON}(9+3,9;4,3)$.

A solution with $t = 2$ is given in Lemma 5.6.

When $t \geq 3$, form a 3-GDD of type 6^t with groups $\{\{6p+j : 0 \leq q < 6\} : 0 \leq p < t\}$. Let $D_{6p+q} = \{0,\dots,w-3\} \setminus \{p\}$ for $0 \leq p < t$ and $0 \leq q < 6$. Apply the general prescription. For $0 \leq p < t$, on $\{6p+q : 0 \leq q < 6\} \cup \{v-3,v-2,v-1\} \cup \{a_{w-2},a_{w-1},a_p\}$ place an $\mathscr{MON}(9+3,9;4,3)$, omitting an $\mathscr{MON}(3+2,2;4,3)$ on $\{a_{w-2},a_{w-1},v-3,v-2,v-1\}$. Place an $\mathscr{MON}(3+2,2;4,3)$ on $\{a_{w-2},a_{w-1},v-3, v-2,v-1\}$. Remove edges $\{a_0,a_{w-2}\}$ and $\{a_1,a_{w-1}\}$ from their kites, and convert the two triangles in the $\mathscr{MON}(3+2,2;4,3)$ to kites using these. What remains is all edges on $\{a_0,\dots,a_{w-3}\}$ and everything is in kites or 4-cycles excepting one triangle involving a_0 and one involving a_1. If $w-2 \equiv 0,1,3,6 \pmod 8$, place a $\mathscr{MON}(w-2,4)$ on $\{a_0,\dots,a_{w-3}\}$. Otherwise partition all edges on $\{a_0,\dots,a_{w-3}\}$ except $\{a_0,a_2\}$ and $\{a_1,a_2\}$ into kites, 4-cycles, and at most one triangle, and use the last two edges to form kites with the excess triangles involving a_0 and a_1. The partition needed is easily produced for $w-2 \in \{4,5,7,9\}$ and hence by induction for all the required orders.

Case E1. $\boldsymbol{v \equiv 0 \pmod 2}$ **and** $\boldsymbol{w \leq \frac{v+2}{6}}$**.** Write $v = 6t+s$ for $s \in \{0,2,4\}$. Let $L = (V,E)$ be a graph with edges

$$\{\{3i,3i+1\},\{3i,3i+2\},\{3i+1,3i+2\} : 0 \leq i < t\} \cup \{\{i,3t+i\} : 0 \leq i < 3t\},$$

together with $\{6t,6t+1\}$ when $s=2$ and with $\{\{6t,6t+1\},\{6t,6t+2\},\{6t,6t+3\}\}$ when $s = 4$. Let $(V,\mathcal{B})$ be a partial triple system covering all edges except those in L (this is easily produced). Let $D_i = \{0,\dots,w-2\}$ for $0 \leq i < v$. Apply the general prescription. For $0 \leq i < t$ and $j \in \{0,1,2\}$, form the 4-cycle $(a_{w-1},3i+((j+1) \bmod 3),3i+j,3t+3i+j)$. When $s=4$, form 4-cycle $(a_{w-1},6t+2,6t,6t+3)$. When $s \in \{2,4\}$, form a triangle $(a_{w-1},6t,6t+1)$. All edges on V are used and all edges on W remain. All edges between V and W are used. Except when $w \in \{2,4\}$, or $w \equiv 2,7 \pmod 8$ and $v \equiv 2,4 \pmod 6$, form an $\mathscr{MON}(w,4)$ on W to complete the proof. When $w \equiv 2,7 \pmod 8$ and $v \equiv 2,4 \pmod 6$, convert $\{a_{w-1},6t,6t+1\}$ to a kite using an edge of the K_w, and partition the $K_w \setminus K_2$ into kites and 4-cycles. When $w \in \{2,4\}$, remove edges $\{a_0,0\}$ and $\{a_1,0\}$ from their kites, and partition K_w together with these edges.

Case E2. $\boldsymbol{v \equiv 2 \pmod 6}$ **and** $\boldsymbol{w = \frac{v+4}{6}}$**.** Choose m as large as possible so that $m \leq \frac{v}{2}$, $m \leq \binom{w}{2}$, and $\binom{w}{2} - m \equiv 0 \pmod 4$. Partition the $\binom{w}{2}$ edges on W into sets E_c and E_o with $|E_c| = m$, so that the edges on E_o can be partitioned into kites and 4-cycles; this is easily done. Place these kites and 4-cycles on W. Then let $\{e_i : 0 \leq i < m\}$ be the edges in E_c; let $a_{f_i} \in e_i$ when $0 \leq i < m$; $f_i = 0$ when $m \leq i < \frac{v-2}{2}$; and $f_{(v-2)/2} = 1$ if $m < \frac{v}{2}$. Next form a 3-GDD of type $2^{v/2}$ on V so that $\{\{2i,2i+1\} : 0 \leq i < \frac{v}{2}\}$ forms the groups, and $\mathcal{B}$ forms the blocks. For $0 \leq i < \frac{v}{2}$, let $D_{2i} = D_{2i+1} = \{0,\dots,w-1\} \setminus \{f_i\}$. Apply the general prescription. Now for $0 \leq i < \frac{v}{2}$, form the triangle $(a_{f_i},2i,2i+1)$ and for $0 \leq i < m$ add edge e_i to form a kite. At most three triangles remain *except when* $v \in \{14,20\}$, where four triangles remain. To treat these cases, we reduce the number of triangles; without loss of generality, the 3-GDD contains a triple $\{v-8,v-6,v-4\}$ in a kite with edge $\{a_1,v-8\}$. Remove this kite, and form kites $(a_0,v-7,v-8;v-6)$, $(a_0,v-5,v-6;v-4)$, $(a_0,v-3,v-4;v-8)$, and $(v-2,v-1,a_1;v-8)$. $\square$

COROLLARY 5.8. *Let $v \geq 4$ and let $\mu_3(v)$ be defined by $\mu_3(4) = \mu_3(6) = \mu_3(9) = 1$, $\mu_3(10) = 2$, and otherwise*

v	$6t, t \geq 2$	$1+6t$	$2+6t$	$3+6t, t \geq 2$	$4+6t, t \geq 2$	$5+6t$
$\mu_3(v)$	$1+t$	t	$1+t$	$1+t$	$2+t$	$1+t$

Then wavecost $\mathscr{MON}(v+w, v; 4, 3) = \left\lceil \frac{(v+w)(v+w-1)}{8} \right\rceil$ *if and only if* $w \geq \mu_3(v)$.

6. Conclusions. The determination of *cost* $\mathscr{ON}(n, v; C, C')$ appears to be easier when $C' = 4$ than the case for $C' = 3$ settled in [14, 15]. Nevertheless the very flexibility in choosing kites, 4-cycles, or triangles also results in a wide range of numbers of wavelengths among decompositions with optimal drop cost. This leads naturally to the question of minimizing the drop cost and the number of wavelengths simultaneously. In many cases, the minima for both can be realized by a single decomposition. However, it may happen that the two minimization criteria compete. Therefore we have determined the minimum number of wavelengths among all decompositions of lowest drop cost for the specified values of n, v, and C'.

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