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SPECTRAL MEASURE AND APPROXIMATION OF HOMOGENIZED COEFFICIENTS

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Abstract. This article deals with the numerical approximation of effective coefficients in stochastic homogenization of discrete linear elliptic equations. The originality of this work is the use of a well-known *abstract* spectral representation formula to design and analyze effective and *computable* approximations of the homogenized coefficients. In particular, we show that information on the edge of the spectrum of the generator of the environment viewed by the particle projected on the local drift yields bounds on the approximation error, and conversely. Combined with results by Otto and the first author in low dimension, and results by the second author in high dimension, this allows us to prove that for any dimension $d \geq 2$, there exists an explicit numerical strategy to approximate homogenized coefficients which converges at the rate of the central limit theorem.

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1. Introduction

We consider a discrete elliptic operator $-\nabla^* \cdot A\nabla$, where $\nabla^*\cdot$ and ∇ are the discrete backward divergence and forward gradient, respectively. For all $z \in \mathbb{Z}^d$, $A(z)$ is the diagonal matrix whose entries are the conductances $\omega(z, z+\mathbf{e}_i)$ of the edges $(z, z+\mathbf{e}_i)$ starting at z, where $\{\mathbf{e}_i\}_{i\in\{1,\dots,d\}}$ denotes the canonical basis of $\mathbb{Z}^d$. The values of the conductances are random and their realizations are assumed to be independent and identically distributed. In what follows, for any random variable X taking values in $\mathbb{R}^{n\times m}$ $(n, m \geq 1)$ we denote by $\langle X\rangle$ the expectation of X in the underlying probability space componentwise. If X has bounded second moments componentwise, we denote by $\mathrm{var}\,[X]$ its variance, that is $\mathrm{var}\,[X] := \left\langle |X - \langle X\rangle|^2\right\rangle$, where $|\cdot|$ is the Euclidian norm on $\mathbb{R}^{n\times m}$.

Provided that the conductances lie in a compact set of $(0, +\infty)$, standard homogenization results (see for instance [13, Theorem 4] and [12, Corollary 1]) ensure that there exists some *deterministic* matrix A_{hom} such that the solution operator of the deterministic continuous differential operator $-\nabla \cdot A_{\mathrm{hom}}\nabla$ describes the large scale behavior of the solution operator of the random difference operator $-\nabla^* \cdot A\nabla$ almost surely. As a by-product of this homogenization result, one obtains a characterization of the homogenized coefficients A_{hom}: it is shown that for every direction $\xi \in \mathbb{R}^d$, there exists a unique scalar field ϕ such that $\nabla\phi$ is stationary, $\langle\nabla\phi\rangle = 0$ (vanishing expectation), which solves the corrector equation

$$-\nabla^* \cdot A(\xi + \nabla\phi) \;=\; 0 \qquad \text{in } \mathbb{Z}^d, \tag{1.1}$$

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and normalized by $\phi(0) = 0$. With this corrector, the homogenized coefficients A_{hom} can be characterized as

$$\xi \cdot A_{\mathrm{hom}}\xi \;=\; \langle (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi) \rangle \tag{1.2}$$

(since the random field $x \mapsto (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)(x)$ is stationary, the point at which the expectation is taken is irrelevant).

In this article, we are interested in numerical approximations of these effective coefficients A_{hom}. From the practical point of view, (1.2) is not of immediate interest since the corrector equation (1.1) has to be solved

- for every realization of the coefficients ω,
- on the whole $\mathbb{Z}^d$.

Ergodicity allows one to replace the expectation by a spatial average (on increasing domains) almost surely. To approximate ϕ, one usually uses ϕ_R, the unique solution to equation (1.1) on some large but finite domain $Q_R = (-R/2, R/2)^d$, completed by say periodic or homogeneous Dirichlet boundary conditions (see for instance [5]). Yet, the comparison of $\nabla\phi_R$ to $\nabla\phi$ is not obvious since $\nabla\phi_R$ and $\nabla\phi$ are not "jointly stationary". In order to avoid this difficulty, Otto and the first author have used a somewhat different strategy, that proceeds in two steps. First, ϕ is replaced by its standard regularization ϕ_μ, unique stationary solution to the modified corrector equation

$$\mu\phi_\mu - \nabla^* \cdot A(\xi + \nabla\phi_\mu) \;=\; 0 \qquad \text{in } \mathbb{Z}^d$$

for some small $\mu > 0$ (see the original contribution by Papanicolaou and Varadhan in the continuous case [17], and the discrete counterpart by Künnemann [13]). Then, ϕ_μ itself is replaced by $\phi_{\mu,R}$, the unique weak solution to

$$\left\{ \begin{array}{rcll} \mu\phi_{\mu,R} - \nabla^* \cdot A(\xi + \nabla\phi_{\mu,R}) &=& 0 & \text{in } Q_R \cap \mathbb{Z}^d, \\ \phi_{\mu,R} &=& 0 & \text{on } \mathbb{Z}^d \setminus Q_R. \end{array} \right.$$

The advantages are twofold:

- $\nabla\phi$ and $\nabla\phi_\mu$ are jointly stationary, which is of great help for the analysis,
- ϕ_μ is accurately approximated by $\phi_{\mu,R}$ on domains of the form $Q_L = (-L/2, L/2)^d$ provided that $(R-L)\sqrt{\mu} \gg 1$, due to the exponential decay of the Green's function associated with $\mu - \nabla^* \cdot A\nabla$ in $\mathbb{Z}^d$ (see [8]), so that we only focus on ϕ_μ and not $\phi_{\mu,R}$ from now on.

In particular, we may approximate A_{hom} by the following average

$$\xi \cdot A_{\mu,1,L}\xi \;:=\; \int_{Q_L} (\xi + \nabla\phi_\mu) \cdot A(\xi + \nabla\phi_\mu)\chi_L(x)\mathrm{d}x,$$

where χ_L is a smooth mask supported on Q_L and of mass one. Note that $A_{\mu,1,L}$ is a stationary random variable itself. Let us stress the fact that μ will be small (the perturbation is sent to zero) whereas L will be large (to apply the ergodic theorem). In [9], it is observed that the L^2-norm of the error in probability takes the form

$$\left\langle \left(\xi \cdot A_{\mu,1,L}\xi - \xi \cdot A_{\mathrm{hom}}\xi\right)^2 \right\rangle \;=\; \mathrm{var}\left[\xi \cdot A_{\mu,1,L}\xi\right] + \left(\xi \cdot (A_{\mu,1} - A_{\mathrm{hom}})\xi\right)^2, \tag{1.3}$$

where

$$\xi \cdot A_{\mu,1}\xi \;:=\; \langle (\xi + \nabla\phi_\mu) \cdot A(\xi + \nabla\phi_\mu) \rangle \,.$$

The first term of the r. h. s. of (1.3) is stochastic in nature: it measures the fluctuations of $A_{\mu,1,L}$ around its mean value. It is called the *random error*. The second term is a deterministic error related to the fact that we have modified the corrector equation by a zero-order term. It is refered to as the *systematic error*, and will be the main focus of this work.

In [9], it is proved that the random error depends on the dimension: there exists q depending only on the ellipticity constants α, β such that

$$\mathrm{var}\left[A_{\mu,1,L}\right]^{1/2} \lesssim \left| \begin{array}{ll} (L^{-1}+\mu)\ln^q \mu & \text{if} \quad d=2, \\ L^{-d/2}(1+\mu L) & \text{if} \quad d>2, \end{array} \right. \tag{1.4}$$

where $\lesssim$ stands for $\leq$ up to some constant depending only on α, β and d. In particular, in the regime $\mu L \lesssim 1$, the random error has the scaling of the central limit theorem (in other words the energy density of the approximate corrector behaves as if it were independent from site to site). The systematic error has been identified in [10]. It also depends on the dimension for $d<5$, but saturates at $d=5$: there exists q depending only on the ellipticity constants α, β such that

$$|A_{\mu,1} - A_{\mathrm{hom}}| \lesssim \left| \begin{array}{rl} \mu \ln^q \mu^{-1} & \text{if} \quad d=2, \\ \mu^{3/2} & \text{if} \quad d=3, \\ \mu^2 \ln \mu^{-1} & \text{if} \quad d=4, \\ \mu^2 & \text{if} \quad d>4. \end{array} \right. \tag{1.5}$$

These two estimates are optimal (up to some possible logarithmic corrections for $d=2$). In order to use $\phi_{\mu,R}$ as a proxy for ϕ_μ on Q_L, at first order we may take $\mu^{-1} \sim L^2 \sim R^2$. Hence, the random error dominates up to $d=8$, so that the convergence rate of the numerical strategy is optimal (it coincides with the central limit theorem scaling, which is an upper bound):

$$\left\langle \left(\xi \cdot A_{L^{-2},1,L}\xi - \xi \cdot A_{\mathrm{hom}}\xi\right)^2 \right\rangle^{1/2} \lesssim \left| \begin{array}{rl} L^{-1}\ln^q L & \text{if} \quad d=2, \\ L^{-d/2} & \text{if} \quad 2<d\leq 8. \end{array} \right.$$

Yet, for $d>8$, the systematic error dominates and the numerical strategy is not optimal any longer:

$$\left\langle \left(\xi \cdot A_{L^{-2},1,L}\xi - \xi \cdot A_{\mathrm{hom}}\xi\right)^2 \right\rangle^{1/2} \lesssim L^{-4} \quad \text{if} \quad 8<d.$$

The aim of this paper is to introduce new formulas for the approximation of A_{hom} using the modified corrector ϕ_μ (possibly with different μ's) in order to reduce the *systematic error*. In early and seminal papers on stochastic homogenization (for instance [17] and [11]) spectral analysis has been used to prove uniqueness of correctors, and devise a spectral representation formula for A_{hom}. In particular, another point of view on the homogenization of discrete elliptic equations with random i. i. d. coefficients consists in studying a random walk in the random environment described by these i. i. d. conductances on $\mathbb{Z}^d$. Adopting this point of view, Kipnis and Varadhan [11] have proved the convergence of the random walk in the diffusive scaling to a Brownian motion with covariance matrix $2A_{\mathrm{hom}}$ for the annealed law (that is to say, after averaging over the random environment). This result has then been extended in [6] to general integrable conductances. Convergence for almost every environment has been obtained for uniformly elliptic conductances in [18], and has recently been extended to more general conductances in [2, 15, 3, 14, 1].

Denoting by $-\mathcal{L}$ the generator of the environment viewed by the particle, and by $e_{\mathfrak{d}}$ its spectral measure projected on the local drift $\mathfrak{d} = \nabla^* \cdot A\xi$ (see Section 3), Kipnis and

Varadhan [11] have shown that

$$\xi \cdot A_{\rm hom}\xi \;=\; \langle \xi \cdot A\xi\rangle - \int_{\mathbb{R}_+} \frac{1}{\lambda} \mathrm{d}e_{\mathfrak{d}}(\lambda),$$

where $\mathbb{R}_+ = [0,+\infty)$. As noticed by the second author in [16], $A_{\mu,1}$ can also be written in terms of the spectral measure $e_{\mathfrak{d}}$ (see Section 2 for details):

$$\begin{aligned} \xi \cdot A_{\mu,1}\xi &= \langle \xi \cdot A\xi\rangle - \int_{\mathbb{R}_+} \frac{\lambda+2\mu}{(\mu+\lambda)^2} \mathrm{d}e_{\mathfrak{d}}(\lambda) \\ &= \xi \cdot A_{\rm hom}\xi + \mu^2 \int_{\mathbb{R}_+} \frac{1}{\lambda(\mu+\lambda)^2} \mathrm{d}e_{\mathfrak{d}}(\lambda). \end{aligned}$$

The key idea of the present paper is to use this spectral representation in order to design approximations of $A_{\rm hom}$ at an abstract level first, and then go back to physical space and obtain formulas in terms of the modified correctors ϕ_μ. We shall actually introduce, for every integer $k \geq 1$, an approximation $A_{\mu,k}$ of $A_{\rm hom}$ defined in terms of $\phi_\mu, \ldots, \phi_{2^{k-1}\mu}$, and prove that, up to logarithmic corrections, the difference $|A_{\rm hom} - A_{\mu,k}|$ is bounded in our discrete stochastic setting by

$$\left| \begin{array}{rl} \mu^{\min\{2k,d/2\}} & \text{if} \quad d \leq 6, \\ \mu^{\min\{2k,\max(3,d/2-3)\}} & \text{if} \quad d > 6, \end{array} \right. \tag{1.6}$$

(see Theorem 3 for a more precise statement). The systematic error associated with the new approximations can be made of a higher order than (1.5) as soon as $d \geq 4$. The proof of these estimates relies on the observation that the systematic error is controlled by the edge of the spectrum $e_{\mathfrak{d}}((0,\mu))$. In turn, the systematic error also controls the edge of the spectrum (see Theorem 4 for a precise statement), so that estimating the systematic error is equivalent to quantifying $e_{\mathfrak{d}}((0,\mu))$.

As we shall also prove, the variance estimate (1.4) is unchanged if $A_{\mu,1}$ is replaced by $A_{\mu,k}$ for all $k \geq 1$. In particular, if we keep $\mu^{-1} \sim L^2$, we obtain a numerical strategy whose convergence rate is optimal with respect to the central limit theorem scaling in the stochastic case, for any $d \geq 2$. This improves and completes for $d > 8$ the series of papers [9, 10, 8] by Otto and the first author on quantitative estimates in stochastic homogenization of discrete elliptic equations. In turn, we also obtain "optimal" bounds on $e_{\mathfrak{d}}((0,\mu))$ up to $d = 6$ (see Theorem 5), thus improving the corresponding results of the second author in [16].

Note however that the bounds (1.6) are not yet optimal: the systematic error is expected to behave as $\mu^{\min\{2k,d/2\}}$ in any dimension (up to logarithmic corrections), see also Remark 1 for the equivalent statement in terms of the edge of the spectrum. We wish to address this issue in a future work.

The article is organized as follows. Although the main focus of this work is on stochastic homogenization of discrete elliptic equations, we first describe the strategy on the elementary case of periodic homogenization of continuous elliptic equations in Section 2 (this new strategy may indeed be valuable to numerical homogenization methods, see in particular [7] for related issues). We introduce the spectral decomposition formula for the homogenized coefficients. The binomial formula then provides with natural approximations of the homogenized coefficients in terms of the associated spectral measure. We conclude the section by rewriting these formulas in physical space using solutions to the modified corrector equation, which yields new *computable* approximations of the homogenized coefficients. In particular, this generalizes the method introduced in [7] and makes the systematic error decay arbitrarily fast. Some numerical tests displayed in Appendix B illustrate the sharpness of the analysis.

We turn to the core of this article in Section 3: the stochastic homogenization of discrete elliptic equations. We first recall the spectral decomposition of the generator of the environment viewed by the particle. The algebra is the same as in the continuous periodic case, so that the formulas we obtain in Section 2 adapt mutatis mutandis to the discrete stochastic case. Yet, the error analysis is more subtle. We show that the asymptotic behavior of the systematic error is driven by the behavior of the edge of the spectrum of the generator. Using results of [16] in high dimension, and results in the spirit of [10] (see Lemma 5 and Appendix A) in low dimension, we obtain estimates on the edge of this spectrum, which show that the systematic error is effectively reduced in high dimensions (although our bounds are not optimal when $d > 6$). We then note that the variance estimates derived in [9] also hold for these approximations, thus concluding the error analysis of the numerical strategy.

We will make use of the following notation:

- $d \geq 2$ is the dimension;
- In the discrete case, $\int_{\mathbb{Z}^d} \mathrm{d}x$ denotes the sum over $x \in \mathbb{Z}^d$, and $\int_D \mathrm{d}x$ denotes the sum over $x \in \mathbb{Z}^d$ such that $x \in D$, D open subset of $\mathbb{R}^d$;
- $\langle \cdot \rangle$ is the average in the periodic case, and the expectation in the stochastic case;
- $\mathrm{var}\,[\cdot]$ is the variance in the stochastic case;
- $\lesssim$ and $\gtrsim$ stand for $\leq$ and $\geq$ up to a multiplicative constant which only depends on the dimension d and the constants α, β (the ellipticity constants of the matrix A, see Definitions 1 and 5) if not otherwise stated;
- when both $\lesssim$ and $\gtrsim$ hold, we simply write $\sim$;
- we use $\gg$ instead of $\gtrsim$ when the multiplicative constant is (much) larger than 1;
- $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ denotes the canonical basis of $\mathbb{R}^d$.

2. The continuous periodic case

Definition 1. Let $A : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ be a $Q = (0,1)^d$-periodic symmetric diffusion matrix which is uniformly continuous and coercive with constants $\beta \geq \alpha > 0$: for almost all $x \in Q$ and all $\xi \in \mathbb{R}^d$, $|A\xi| \leq \beta|\xi|$ and $\xi \cdot A\xi \geq \alpha|\xi|^2$. The associated homogenized matrix A_{hom} is characterized for all $\xi \in \mathbb{R}^d$ by

$$\begin{aligned}\xi \cdot A_{\mathrm{hom}}\xi &= \langle (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi) \rangle \\ &= \int_Q (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi) \mathrm{d}x,\end{aligned}$$

where $\langle \cdot \rangle$ denotes the average on the periodic cell Q, and ϕ is the unique Q-periodic weak solution to

$$-\nabla \cdot A(\xi + \nabla\phi) = 0 \tag{2.1}$$

with zero average $\langle \phi \rangle = 0$.

Let us define $\mathcal{E} : H^1_{\mathrm{per}}(Q) \times H^1_{\mathrm{per}}(Q) \to \mathbb{R}, (\psi, \chi) \mapsto \int_Q \nabla\psi \cdot A\nabla\chi \mathrm{d}x$ the Dirichlet form associated with A. We (abusively) refer to the quadratic form $\psi \to \mathcal{E}(\psi,\psi)$ as the *Dirichlet form* as well. One may write the homogenized matrix as

$$\xi \cdot A_{\mathrm{hom}}\xi \,=\, \langle \xi \cdot A\xi \rangle - \mathcal{E}(\phi,\phi). \tag{2.2}$$

Indeed, the weak formulation of (2.1) implies

$$\int_Q \nabla\phi \cdot A(\xi + \nabla\phi)\mathrm{d}x \,=\, 0, \tag{2.3}$$

and therefore

$$\int_Q \xi \cdot A\nabla\phi \mathrm{d}x \,=\, \int_Q \nabla\phi \cdot A\xi \mathrm{d}x \overset{(2.3)}{=} -\int_Q \nabla\phi \cdot A\nabla\phi \mathrm{d}x \,=\, -\mathcal{E}(\phi,\phi).$$

The objective of this section is to use a spectral decomposition to design approximations for $\mathcal{E}(\phi,\phi)$.

2.1. **Spectral decomposition.**

Definition 2. Let A be as in Definition 1. We let $\mathcal{L}^{-1}$ denote the inverse of the elliptic operator $\mathcal{L} = -\nabla \cdot A\nabla$ with periodic boundary conditions on $L^2_0(Q) = \{v \in L^2(Q) \,|\, \int_Q v(x)dx = 0\}$. It is a well-defined compact operator by generalized Poincaré's inequality, Riesz's, and Rellich's theorems.

By Hilbert-Schmidt's theorem, there exist an orthonormal basis $\{\psi_i\}_{i>0}$ of $L^2_0(Q)$ and positive eigenvalues $\{\lambda_i\}_{i>0}$ (in increasing order) such that for all $i > 0$, $\mathcal{L}^{-1}\psi_i = \frac{1}{\lambda_i}\psi_i$. By definition, $\psi_i \in H^1_{\mathrm{per}}(Q)$. Setting $\psi_0 \equiv 1$ and $\lambda_0 = 0$, one may then characterize $H^1_{\mathrm{per}}(Q)$ as

$$H^1_{\mathrm{per}}(Q) \,=\, \left\{ u = \sum_{i\in\mathbb{N}} \alpha_i\psi_i \,\middle|\, \sum_{i\in\mathbb{N}} (1+\lambda_i)\alpha_i^2 < \infty \right\}.$$

By Riesz's representation theorem, this also implies for the dual $H^{-1}_{\mathrm{per}}(Q)$ of $H^1_{\mathrm{per}}(Q)$:

$$H^{-1}_{\mathrm{per}}(Q) \,=\, \left\{ f = \sum_{i\in\mathbb{N}} \beta_i\psi_i \,\middle|\, \sum_{i\in\mathbb{N}} \frac{\beta_i^2}{1+\lambda_i} < \infty \right\}. \tag{2.4}$$

Hence, for all $f \in H^{-1}_{\mathrm{per}}(Q)$ such that $\langle f, 1\rangle_{H^{-1}_{\mathrm{per}}, H^1_{\mathrm{per}}} = 0$, the unique weak solution $u \in H^1_{\mathrm{per}}(Q)$ to

$$-\nabla \cdot A\nabla u \,=\, f$$

is given by

$$u \,=\, \sum_{i\in\mathbb{N}\setminus\{0\}} \frac{\langle f, \psi_i\rangle_{H^{-1}_{\mathrm{per}}, H^1_{\mathrm{per}}}}{\lambda_i}\psi_i.$$

For all $f \in H^{-1}_{\mathrm{per}}(Q)$ such that $\langle f, 1\rangle_{H^{-1}_{\mathrm{per}}, H^1_{\mathrm{per}}} = 0$, we define the spectral measure e_f of $\mathcal{L}$ projected on f by

$$e_f \,=\, \sum_{i\in\mathbb{N}} \langle f, \psi_i\rangle_{H^{-1}_{\mathrm{per}}, H^1_{\mathrm{per}}} \delta_{\lambda_i}, \tag{2.5}$$

where δ_{λ_i} is the Dirac mass on λ_i. The above characterizations of H^1 and H^{-1} then allow us to give a mathematical meaning to the formal functional calculus

$$\langle f, \Psi(\mathcal{L})(f)\rangle_{H^{-1}_{\rm per},H^1_{\rm per}} = \int_{\mathbb{R}_+} \Psi(\lambda)\mathrm{d}e_f(\lambda),$$

for every continuous function $\Psi : [0,+\infty) \to \mathbb{R}$ such that $\lambda\Psi(\lambda) \lesssim 1$ as $\lambda \to \infty$.

We are now in position to express the Dirichlet form of the corrector ϕ in terms of the spectral measure projected on the "local drift"

$$\mathfrak{d} := \nabla \cdot A\xi \in H^{-1}_{\rm per}(Q). \tag{2.6}$$

In particular,

$$\begin{aligned}
\mathcal{E}(\phi,\phi) &= \langle \mathcal{L}\phi, \phi\rangle_{H^{-1}_{\rm per},H^1_{\rm per}} \\
&= \left\langle \mathcal{L}\mathcal{L}^{-1}\mathfrak{d}, \mathcal{L}^{-1}\mathfrak{d}\right\rangle_{H^{-1}_{\rm per},H^1_{\rm per}} \\
&= \int_{\mathbb{R}_+} \frac{1}{\lambda}\mathrm{d}e_{\mathfrak{d}}(\lambda).
\end{aligned} \tag{2.7}$$

Let us then turn to the approximation of $A_{\rm hom}$ used in [7], that is

$$\xi \cdot A_\mu \xi := \langle (\xi + \nabla\phi_\mu) \cdot A(\xi + \nabla\phi_\mu)\rangle, \tag{2.8}$$

where $\phi_\mu \in H^1_{\rm per}(Q)$ is the unique weak solution to the modified corrector equation

$$\mu\phi_\mu - \nabla \cdot A(\xi + \nabla\phi_\mu) = 0, \tag{2.9}$$

that we more compactly write as $(\mu + \mathcal{L})\phi_\mu = \mathfrak{d}$. In this case, the weak formulation of the equation implies

$$\int_Q \nabla\phi_\mu \cdot A(\xi + \nabla\phi_\mu)\mathrm{d}x = -\mu \int_Q \phi_\mu^2 \mathrm{d}x, \tag{2.10}$$

so that the defining formula for A_μ turns into

$$\xi \cdot A_\mu \xi = \langle \xi \cdot A\xi\rangle - \mathcal{E}(\phi_\mu, \phi_\mu) - 2\mu \left\langle \phi_\mu^2\right\rangle. \tag{2.11}$$

Proceeding as above, we rewrite the last two terms of the r. h. s. of (2.11) as

$$\begin{aligned}
\mathcal{E}(\phi_\mu,\phi_\mu) &= \left\langle \mathcal{L}(\mu+\mathcal{L})^{-1}\mathfrak{d}, (\mu+\mathcal{L})^{-1}\mathfrak{d}\right\rangle_{H^{-1}_{\rm per},H^1_{\rm per}} \\
&= \left\langle \mathfrak{d}, \mathcal{L}(\mu+\mathcal{L})^{-2}\mathfrak{d}\right\rangle_{H^{-1}_{\rm per},H^1_{\rm per}} \\
&= \int_{\mathbb{R}_+} \frac{\lambda}{(\mu+\lambda)^2}\mathrm{d}e_{\mathfrak{d}}(\lambda),
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
\left\langle \phi_\mu^2\right\rangle &= \left\langle (\mu+\mathcal{L})^{-1}\mathfrak{d}, (\mu+\mathcal{L})^{-1}\mathfrak{d}\right\rangle_{L^2,L^2} \\
&= \left\langle \mathfrak{d}, (\mu+\mathcal{L})^{-2}\mathfrak{d}\right\rangle_{H^{-1}_{\rm per},H^1_{\rm per}} \\
&= \int_{\mathbb{R}_+} \frac{1}{(\mu+\lambda)^2}\mathrm{d}e_{\mathfrak{d}}(\lambda).
\end{aligned} \tag{2.13}$$

The combination of (2.2), (2.7), (2.11), (2.12) & (2.13) allows us to express the difference between A_μ and $A_{\rm hom}$ in terms of the spectral measure of $\mathcal{L}$ projected on the local drift $\mathfrak{d}$, as observed in [16, Addendum]:

$$\begin{aligned}\xi\cdot(A_\mu - A_{\rm hom})\xi &= \mathcal{E}(\phi,\phi) - \mathcal{E}(\phi_\mu,\phi_\mu) - 2\mu\left\langle \phi_\mu^2\right\rangle \\ &= \int_{\mathbb{R}_+}\left(\frac{1}{\lambda} - \frac{\lambda}{(\mu+\lambda)^2} - \frac{2\mu}{(\mu+\lambda)^2}\right) \mathrm{d}e_{\mathfrak{d}}(\lambda) \\ &= \int_{\mathbb{R}_+}\frac{\mu^2}{\lambda(\mu+\lambda)^2}\mathrm{d}e_{\mathfrak{d}}(\lambda).\end{aligned}$$

Not only does this identity suggest that $|A_\mu - A_{\rm hom}| \sim \mu^2$ (as proved by a different approach in [7]), but it also gives a strategy to construct approximations of $A_{\rm hom}$ at any order. In particular, for all $k\in\mathbb{N}$, we write

$$\begin{aligned}\xi\cdot A_{\rm hom}\xi &= \langle \xi\cdot A\xi\rangle - \int_{\mathbb{R}_+}\frac{(\mu+\lambda)^{2k}}{\lambda(\mu+\lambda)^{2k}}\mathrm{d}e_{\mathfrak{d}}(\lambda) \\ &= \langle \xi\cdot A\xi\rangle - \int_{\mathbb{R}_+}\frac{(\mu+\lambda)^{2k}-\mu^{2k}}{\lambda(\mu+\lambda)^{2k}}\mathrm{d}e_{\mathfrak{d}}(\lambda) - \int_{\mathbb{R}_+}\frac{\mu^{2k}}{\lambda(\mu+\lambda)^{2k}}\mathrm{d}e_{\mathfrak{d}}(\lambda)\end{aligned}$$

and set

$$\begin{aligned}\xi\cdot \tilde A_{\mu,k}\xi &:= \langle \xi\cdot A\xi\rangle - \int_{\mathbb{R}_+}\frac{(\mu+\lambda)^{2k}-\mu^{2k}}{\lambda(\mu+\lambda)^{2k}}\mathrm{d}e_{\mathfrak{d}}(\lambda) \\ &= \langle \xi\cdot A\xi\rangle - \sum_{j=1}^{2k-1}\binom{2k}{j}\int_{\mathbb{R}_+}\frac{\mu^j\lambda^{2k-1-j}}{(\mu+\lambda)^{2k}}\mathrm{d}e_{\mathfrak{d}}(\lambda). \qquad (2.14)\end{aligned}$$

Note that the only operator which has to be inverted to compute $\tilde A_{\mu,k}$ is indeed $\mu+\mathcal{L}$, and not $\mathcal{L}$, as desired. In addition, this definition implies

$$\xi\cdot(\tilde A_{\mu,k} - A_{\rm hom})\xi = \int_{\mathbb{R}_+}\frac{\mu^{2k}}{\lambda(\mu+\lambda)^{2k}}\mathrm{d}e_{\mathfrak{d}}(\lambda),$$

which suggests that the error is now of order μ^{2k}.

In view of formula (2.14), the effective computation of $(\mu+\mathcal{L})^{-k}\mathfrak{d}$ is needed in practice to obtain $\tilde A_{\mu,k}$. This is a big handicap for the numerical method since the numerical inversion of $\mu+\mathcal{L}$ has to be iterated k times, which dramatically magnifies the numerical error. Fortunately, one may use a sligthly different approximation of $A_{\rm hom}$ which avoids this drawback, as shown in the following subsection.

2.2. **Abstract approximations.** Let us first introduce functions $\mathfrak{d}_{\mu,k}$, which are defined as linear combinations of $\phi_\mu,\dots,\phi_{2^{k-1}\mu}$ (and therefore easily computable) and will serve as substitutes for $(\mu+\mathcal{L})^{-k}\mathfrak{d}$.

Definition 3. Let A and Q, $\mathcal{L}$, and $\mathfrak{d}$ be as in Definition 1, Definition 2, and (2.6), respectively. For all $\mu>0$, the sequence of functions $\mathfrak{d}_{\mu,k}\in H^1_{\rm per}(Q)$ is defined by its first term

$$\mathfrak{d}_{\mu,1} = \phi_\mu = (\mu+\mathcal{L})^{-1}\mathfrak{d}, \quad c_1 = 1, \qquad (2.15)$$

and by the induction rule

$$\mathfrak{d}_{\mu,k+1} = c_k \mu^{-1}(\mathfrak{d}_{\mu,k} - \mathfrak{d}_{2\mu,k}), \quad c_{k+1} = \left(\frac{2}{c_k} + 1\right)^{-1}. \tag{2.16}$$

Defined this way, the functions $\mathfrak{d}_{\mu,k}$ satisfy the following fundamental properties:

Proposition 1. *Let $\mathfrak{d}_{\mu,k}$ be as in Definition 3, then for all $\mu > 0$ and $k \geq 1$, we have*

$$\mathfrak{d}_{\mu,k+1} = (\mu + \mathcal{L})^{-1}\mathfrak{d}_{2\mu,k}, \tag{2.17}$$

$$\mathcal{L}\mathfrak{d}_{\mu,k+1} = (1 + c_k)\mathfrak{d}_{2\mu,k} - c_k\mathfrak{d}_{\mu,k}. \tag{2.18}$$

Proof. Identity (2.18) is a direct consequence of (2.16) & (2.17), and we only need to prove the latter. We proceed by induction. Let us first check that it is indeed true for $k = 1$. By definition of $\mathfrak{d}_{\mu,1}$, we have

$$(\mu + \mathcal{L})\mathfrak{d}_{\mu,1} = \mathfrak{d}, \tag{2.19}$$

and as a consequence,

$$(\mu + \mathcal{L})\mathfrak{d}_{2\mu,1} = \mathfrak{d} - \mu\, \mathfrak{d}_{2\mu,1}. \tag{2.20}$$

Combining (2.19) and (2.20), one obtains :

$$(\mu + \mathcal{L})(\mathfrak{d}_{\mu,1} - \mathfrak{d}_{2\mu,1}) = \mu\, \mathfrak{d}_{2\mu,1},$$

from which it follows that $\mathfrak{d}_{\mu,2} = (\mu+\mathcal{L})^{-1}\mathfrak{d}_{2\mu,1}$. Let us now assume that (2.17) is satisfied at level $k \geq 1$. Similarly, we have

$$(\mu + \mathcal{L})\mathfrak{d}_{\mu,k+1} = \mathfrak{d}_{2\mu,k},$$

$$(\mu + \mathcal{L})\mathfrak{d}_{2\mu,k+1} = \mathfrak{d}_{4\mu,k} - \mu\, \mathfrak{d}_{2\mu,k+1}.$$

Using these equalities, together with the definition (2.16) of $\mathfrak{d}_{\mu,k+1}$, we are led to

$$(\mu + \mathcal{L})(\mathfrak{d}_{\mu,k+1} - \mathfrak{d}_{2\mu,k+1}) = \mathfrak{d}_{2\mu,k} - \mathfrak{d}_{4\mu,k} + \mu\, \mathfrak{d}_{2\mu,k+1} = \mu\left(\frac{2}{c_k} + 1\right)\mathfrak{d}_{2\mu,k+1},$$

and thus, $\mathfrak{d}_{\mu,k+2} = (\mu + \mathcal{L})^{-1}\mathfrak{d}_{2\mu,k+1}$. □

In order to be consistent with (2.17), we set $\mathfrak{d}_{\mu,0} = \mathfrak{d}$ for all $\mu > 0$.

We are now in position to define a suitable approximation of A_{hom}. The idea is to use the identity

$$\frac{1}{\lambda} = \frac{(\mu+\lambda)^2(2\mu+\lambda)^2 \dots (2^{k-1}\mu+\lambda)^2}{\lambda(\mu+\lambda)^2(2\mu+\lambda)^2 \dots (2^{k-1}\mu+\lambda)^2}$$

in (2.7), expand, and take advantage of Proposition 1 to efficiently compute terms of the form $\mathfrak{d}_{\mu,k} = (\mu + \mathcal{L})^{-1}(2\mu + \mathcal{L})^{-1} \dots (2^{k-1}\mu + \mathcal{L})^{-1}\mathfrak{d}$. This gives rise to the following (abstract) approximations of A_{hom}, and systematic errors:

Theorem 1. *Let A be as in Definition 1, and A_{hom} be the associated homogenized diffusion matrix. For any fixed $\xi \in \mathbb{R}^d$ such that $|\xi| = 1$, we denote by $e_{\mathfrak{d}}$ the spectral measure (2.5) of $\mathcal{L} = -\nabla\cdot A\nabla$ projected on the local drift $\mathfrak{d} = \nabla\cdot A\xi$. For all $k \in \mathbb{N}$, we let $P_k : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be the polynomial given by*

$$P_k(\mu,\lambda) = \lambda^{-1}\left((\mu+\lambda)^2(2\mu+\lambda)^2 \cdots (2^{k-1}\mu+\lambda)^2 - 2^{k(k-1)}\mu^{2k}\right), \tag{2.21}$$

and for all $\mu > 0$, we define the approximation $A_{\mu,k}$ of A_{hom} by

$$\xi \cdot A_{\mu,k}\xi = \langle \xi \cdot A\xi \rangle - \int_{\mathbb{R}_+} \frac{P_k(\mu,\lambda)}{(\mu+\lambda)^2(2\mu+\lambda)^2 \cdots (2^{k-1}\mu+\lambda)^2} de_{\mathfrak{d}}(\lambda). \tag{2.22}$$

Then the systematic error satisfies

$$0 \le \xi \cdot (A_{\mu,k} - A_{\mathrm{hom}})\xi \le 2^{k(k-1)} \langle |A|^2 \rangle \left(\frac{1}{4\alpha\pi^2} \right)^{2k} \mu^{2k}. \tag{2.23}$$

Proof. Starting point is the identity

$$\begin{aligned}
\xi \cdot (A_{\mu,k} - A_{\mathrm{hom}})\xi &= \int_{\mathbb{R}_+} \left(\frac{1}{\lambda} - \frac{P_k(\mu,\lambda)}{(\lambda+\mu)^2 \cdots (\lambda + 2^{k-1}\mu)^2} \right) de_{\mathfrak{d}}(\lambda) \\
&= \int_{\mathbb{R}_+} \frac{2^{k(k-1)}\mu^{2k}}{\lambda(\lambda+\mu)^2 \cdots (\lambda+2^{k-1}\mu)^2} \, de_{\mathfrak{d}}(\lambda),
\end{aligned}$$

which is a direct consequence of (2.2), (2.7), and (2.22). From this identity, and using Definition 2, we infer that the systematic error is smaller than and asymptotically equivalent to $C\mu^{2k}$ (as μ tends to 0), where $C > 0$ is given by

$$C := 2^{k(k-1)} \int_{\mathbb{R}_+} \frac{1}{\lambda^{2k+1}} \, de_{\mathfrak{d}}(\lambda) = 2^{k(k-1)} \sum_{i \in \mathbb{N}} \frac{\langle \mathfrak{d}, \psi_i \rangle^2_{H^{-1}_{\mathrm{per}}, H^1_{\mathrm{per}}}}{\lambda_i^{2k+1}}. \tag{2.24}$$

In order to estimate C via (2.24), we compare the spectral gap λ_1 of $\mathcal{L}$ to the spectral gap λ_1^0 of $-\triangle$ on $H^1_{\mathrm{per}}(Q)$. By comparison of the two Dirichlet forms, we have

$$\lambda_1 \ge \alpha \lambda_1^0.$$

The spectrum of the Laplace operator on $H^1_{\mathrm{per}}(Q)$ is explicitly known, and the spectral gap given by $\lambda_1^0 = 4\pi^2$. Hence, recalling that $\langle \mathfrak{d}, 1 \rangle_{H^{-1}_{\mathrm{per}}, H^1_{\mathrm{per}}} = 0$ and using the characterization (2.4) of $H^{-1}_{\mathrm{per}}(Q)$, one may bound the r. h. s. of (2.24) by

$$\begin{aligned}
C &\le 2^{k(k-1)} \frac{1}{(\alpha\lambda_1^0)^{2k}} \sum_{i \in \mathbb{N}} \frac{\langle \mathfrak{d}, \psi_i \rangle^2_{H^{-1}_{\mathrm{per}}, H^1_{\mathrm{per}}}}{\lambda_i} \\
&= 2^{k(k-1)} \|\mathfrak{d}\|^2_{H^{-1}_{\mathrm{per}}} \left(\frac{1}{4\alpha\pi^2} \right)^{2k} \\
&\le 2^{k(k-1)} \langle |A|^2 \rangle \left(\frac{1}{4\alpha\pi^2} \right)^{2k},
\end{aligned}$$

as desired. □

2.3. **New formulas for the approximation of homogenized coefficients.** In this subsection, we show how to rewrite the approximations $A_{\mu,k}$ of A_{hom} introduced in Theorem 1 in terms of the modified correctors $\phi_\mu, \phi_{2\mu}, \dots, \phi_{2^{k-1}\mu}$. We proceed by induction.

Proposition 2. *Let c_k be as in Definition 3. We define the sequence $\{a_{k,i}\}_{k \ge 1, i \in \{0,\dots,k-1\}}$ by $a_{1,0} = 1$ and the induction rules*

$$\begin{aligned}
a_{k+1,0} &= c_k a_{k,0}, \\
a_{k+1,i} &= c_k a_{k,i} - 2^{1-k} c_k a_{k,i-1} \qquad \textit{for } i \in \{1, k-1\}, \\
a_{k+1,k} &= -2^{1-k} c_k a_{k,k-1}.
\end{aligned}$$

Within the assumptions and notation of Theorem 1, the approximations $A_{\mu,k}$ *of* A_{hom} *satisfy the formula: for all* $\xi \in \mathbb{R}^d$,

$$\begin{aligned}\xi \cdot A_{\mu,k}\xi &= \langle (\xi + \nabla\phi_\mu) \cdot A(\xi + \nabla\phi_\mu)\rangle \\ &\qquad + \mu \sum_{i=0}^{k-1} \eta_{k,i} \left\langle \phi_{2^i\mu}^2 \right\rangle + \mu \sum_{i=0}^{k-1}\sum_{j>i}^{k-1} \nu_{k,i,j} \left\langle \phi_{2^i\mu}\phi_{2^j\mu}\right\rangle, \qquad (2.25)\end{aligned}$$

where the $\{\phi_{2^i\mu}\}_{i\in\mathbb{N}}$ *are the modified correctors associated with* ξ *through* (2.9)*, and the coefficients* $\{\eta_{k,i}\}_{k\geq 1, 0\leq i<k}$*, and* $\{\nu_{k,i,j}\}_{k\geq 2, 0\leq j<k, 0\leq i<j}$ *are defined by the initial value* $\eta_{1,0} = 0$*, and the induction rules*

$$\begin{aligned}\eta_{k+1,i} &= \eta_{k,i} + (2^{k(k-1)+i} - 2^{k^2+1})a_{k+1,i}^2 \qquad \textit{for } i \in \{0, k-1\}, \\ \eta_{k+1,k} &= -2^{k^2} a_{k+1,k}^2, \\ \nu_{k+1,i,k} &= \big(2^{k(k-1)+i} - 3 \times 2^{k^2}\big) a_{k+1,i}a_{k+1,k}, \\ \nu_{k+1,i,j} &= \nu_{k,i,j} + \big(2^{k(k-1)}(2^i + 2^j) - 2^{k^2+2}\big) a_{k+1,i}a_{k+1,j} \qquad \textit{for } j \in \{0, k-1\}.\end{aligned}$$

Note that $\{\nu_{k,i,j}\}_{k\geq 1, 0\leq j<k, 0\leq i<j}$ *does not require further initialization.*

Proof. We proceed in four steps.

Step 1. Proof that for all $k \geq 1$,

$$\xi \cdot A_{\mu,k+1}\xi = \xi \cdot A_{\mu,k}\xi - 2^{k(k-1)}\mu^{2k}\mathcal{E}(\mathfrak{d}_{\mu,k+1}, \mathfrak{d}_{\mu,k+1}) - 2^{k^2+1}\mu^{2k+1}\left\langle \mathfrak{d}_{\mu,k+1}^2\right\rangle. \qquad (2.26)$$

In order to prove (2.26), we first note that the polynomials P_k defined in (2.21) satisfy the identity

$$P_{k+1}(\mu,\lambda) = (2^k\mu + \lambda)^2 P_k(\mu,\lambda) + 2^{k(k-1)}\lambda\mu^{2k} + 2^{k^2+1}\mu^{2k+1}. \qquad (2.27)$$

Hence, formula (2.22) implies

$$\begin{aligned}\xi \cdot A_{\mu,k+1}\xi &= \langle \xi \cdot A\xi\rangle - \int_{\mathbb{R}_+} \frac{P_{k+1}(\mu,\lambda)}{(\mu+\lambda)^2(2\mu+\lambda)^2\cdots(2^k\mu+\lambda)^2} de_{\mathfrak{d}}(\lambda) \\ &\overset{(2.27)}{=} \langle \xi \cdot A\xi\rangle - \int_{\mathbb{R}_+} \frac{(2^k\mu+\lambda)^2 P_k(\mu,\lambda)}{(\mu+\lambda)^2(2\mu+\lambda)^2\cdots(2^k\mu+\lambda)^2} de_{\mathfrak{d}}(\lambda) \\ &\qquad - \int_{\mathbb{R}_+} \frac{2^{k(k-1)}\lambda\mu^{2k} + 2^{k^2+1}\mu^{2k+1}}{(\mu+\lambda)^2(2\mu+\lambda)^2\cdots(2^k\mu+\lambda)^2} de_{\mathfrak{d}}(\lambda) \\ &\overset{(2.22)}{=} \xi \cdot A_{\mu,k}\xi - \int_{\mathbb{R}_+} \frac{2^{k(k-1)}\lambda\mu^{2k} + 2^{k^2+1}\mu^{2k+1}}{(\mu+\lambda)^2(2\mu+\lambda)^2\cdots(2^k\mu+\lambda)^2} de_{\mathfrak{d}}(\lambda).\end{aligned}$$

From (2.17) in Proposition 1, we infer that $(\mu+\mathcal{L})^{-1}\cdots(2^k\mu+\mathcal{L})^{-1}\mathfrak{d} = \mathfrak{d}_{\mu,k+1}$, so that the above identity turns into

$$\begin{aligned}\xi \cdot A_{\mu,k+1}\xi &= \xi \cdot A_{\mu,k}\xi - 2^{k(k-1)}\mu^{2k}\left\langle \mathcal{L}\mathfrak{d}_{\mu,k+1}, \mathfrak{d}_{\mu,k+1}\right\rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} \\ &\qquad - 2^{k^2+1}\mu^{2k+1}\left\langle \mathfrak{d}_{\mu,k+1}, \mathfrak{d}_{\mu,k+1}\right\rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} \\ &= \xi \cdot A_{\mu,k}\xi - 2^{k(k-1)}\mu^{2k}\mathcal{E}(\mathfrak{d}_{\mu,k+1}, \mathfrak{d}_{\mu,k+1}) - 2^{k^2+1}\mu^{2k+1}\left\langle \mathfrak{d}_{\mu,k+1}^2\right\rangle,\end{aligned}$$

as desired.

Step 2. Proof that for all $k \geq 1$,

$$\mu^{k-1}\mathfrak{d}_{\mu,k} = \sum_{i=0}^{k-1} a_{k,i}\phi_{2^i\mu}. \tag{2.28}$$

We proceed by induction, and assume that (2.28) holds at step k. The induction rule (2.16) then yields at step $k+1$

$$\begin{aligned}
\mu^k\mathfrak{d}_{\mu,k+1} &= \mu^{k-1}c_k(\mathfrak{d}_{\mu,k} - \mathfrak{d}_{2\mu,k}) \\
&= c_k\mu^{k-1}\mathfrak{d}_{\mu,k} - c_k 2^{1-k}(2\mu)^{k-1}\mathfrak{d}_{2\mu,k} \\
&= c_k\sum_{i=0}^{k-1} a_{k,i}\phi_{2^i\mu} - c_k 2^{1-k}\sum_{i=0}^{k-1} a_{k,i}\phi_{2^{i+1}\mu} \\
&= c_k a_{k,0}\phi_\mu + \left(\sum_{i=1}^{k-1} c_k(a_{k,i} - 2^{1-k}a_{k,i-1})\phi_{2^i\mu}\right) - c_k 2^{1-k}a_{k,k-1}\phi_{2^k\mu},
\end{aligned}$$

so that $\mu^k\mathfrak{d}_{\mu,k+1} = \sum_{i=0}^k a_{k,i}\phi_{2^i\mu}$, as desired. It remains to recall that $\mathfrak{d}_{\mu,1} = \phi_\mu$ to conclude the proof of (2.28).

Note that $a_{2,0} = 1$ and $a_{2,1} = -1$. In particular, $a_{2,0} + a_{2,1} = 0$ and the property

$$\sum_{i=0}^{k-1} a_{k,i} = 0 \tag{2.29}$$

follows by induction, for all $k \geq 2$.

Step 3. Proof that for all $i, j \geq 1$,

$$\begin{aligned}
\mathcal{E}(\phi_{2^i\mu}, \phi_{2^j\mu}) &= \langle \xi \cdot A\xi \rangle - \frac{1}{2}\Big(\big\langle (\xi + \nabla\phi_{2^i\mu}) \cdot A(\xi + \nabla\phi_{2^i\mu}) \big\rangle \\
&\qquad + \big\langle (\xi + \nabla\phi_{2^j\mu}) \cdot A(\xi + \nabla\phi_{2^j\mu}) \big\rangle \Big) \\
&\quad -\mu 2^{i-1}\left\langle \phi_{2^i\mu}(\phi_{2^i\mu} + \phi_{2^j\mu}) \right\rangle - \mu 2^{j-1}\left\langle \phi_{2^j\mu}(\phi_{2^i\mu} + \phi_{2^j\mu}) \right\rangle.
\end{aligned} \tag{2.30}$$

We can easily see that (2.30) holds when $i = j$. From (2.8) and (2.11), we have indeed that

$$\mathcal{E}(\phi_{2^i\mu}, \phi_{2^i\mu}) = \langle \xi \cdot A\xi \rangle - \big\langle (\xi + \nabla\phi_{2^i\mu}) \cdot A(\xi + \nabla\phi_{2^i\mu}) \big\rangle - 2^{i+1}\mu\left\langle \phi_{2^i\mu}^2 \right\rangle, \tag{2.31}$$

as desired. For general $i, j \in \mathbb{N}$, we have, using (2.10) first for $\phi_{2^j\mu}$ and then for $\phi_{2^i\mu}$,

$$\begin{aligned}
\mathcal{E}(\phi_{2^i\mu}, \phi_{2^j\mu}) &= \left\langle \nabla\phi_{2^i\mu} \cdot A\nabla\phi_{2^j\mu} \right\rangle \\
&= \left\langle \nabla\phi_{2^i\mu} \cdot A(\xi + \nabla\phi_{2^j\mu}) \right\rangle - \left\langle \nabla\phi_{2^i\mu} \cdot A\xi \right\rangle \\
&\overset{(2.10)}{=} -2^j\mu\left\langle \phi_{2^i\mu}\phi_{2^j\mu} \right\rangle - \left\langle \nabla\phi_{2^i\mu} \cdot A(\xi + \nabla\phi_{2^i\mu}) \right\rangle + \left\langle \nabla\phi_{2^i\mu} \cdot A\nabla\phi_{2^i\mu} \right\rangle \\
&\overset{(2.10)}{=} -2^j\mu\left\langle \phi_{2^i\mu}\phi_{2^j\mu} \right\rangle + 2^i\mu\left\langle \phi_{2^i\mu}^2 \right\rangle + \mathcal{E}(\phi_{2^i\mu}, \phi_{2^i\mu}) \\
&\overset{(2.31)}{=} -2^j\mu\left\langle \phi_{2^i\mu}\phi_{2^j\mu} \right\rangle - 2^i\mu\left\langle \phi_{2^i\mu}^2 \right\rangle \\
&\qquad + \langle \xi \cdot A\xi \rangle - \big\langle (\xi + \nabla\phi_{2^i\mu}) \cdot A(\xi + \nabla\phi_{2^i\mu}) \big\rangle.
\end{aligned}$$

We conclude the proof of (2.30) by changing the roles of i and j.

Step 4. Proof of (2.25).
In view of (2.26), we have to estimate two terms. We begin with the Dirichlet form: inserting (2.28) in the integral yields

$$\mu^{2k}\mathcal{E}(\mathfrak{d}_{\mu,k+1},\mathfrak{d}_{\mu,k+1}) \;=\; \sum_{i=0}^{k}\sum_{j=0}^{k} a_{k+1,i}a_{k+1,j}\mathcal{E}(\phi_{2^i\mu},\phi_{2^j\mu}).$$

We then appeal to (2.30) to turn this identity into

$$\begin{aligned}
&\mu^{2k}\mathcal{E}(\mathfrak{d}_{\mu,k+1},\mathfrak{d}_{\mu,k+1})\\
&\quad= \Big(\sum_{i=0}^{k} a_{k+1,i}\Big)^2\langle \xi\cdot A\xi\rangle - \sum_{i=0}^{k}\sum_{j>i}^{k}\mu(2^i+2^j)a_{k+1,i}a_{k+1,j}\left\langle \phi_{2^i\mu}\phi_{2^j\mu}\right\rangle\\
&\qquad\qquad -\sum_{i=0}^{k}\mu 2^i a_{k+1,i}^2\left\langle\phi_{2^i\mu}^2\right\rangle\\
&\qquad -\sum_{i=0}^{k} a_{k+1,i}\Big(\sum_{j=0}^{k}a_{k+1,j}\Big)\Big(2^i\mu\left\langle\phi_{2^i\mu}^2\right\rangle+\left\langle(\xi+\nabla\phi_{2^i\mu})\cdot A(\xi+\nabla\phi_{2^i\mu})\right\rangle\Big).
\end{aligned}$$

Taking into account (2.29), we finally have

$$\begin{aligned}
&\mu^{2k}\mathcal{E}(\mathfrak{d}_{\mu,k+1},\mathfrak{d}_{\mu,k+1})\\
&\quad= -\sum_{i=0}^{k}\sum_{j>i}^{k}\mu(2^i+2^j)a_{k+1,i}a_{k+1,j}\left\langle\phi_{2^i\mu}\phi_{2^j\mu}\right\rangle-\sum_{i=0}^{k}\mu 2^i a_{k+1,i}^2\left\langle\phi_{2^i\mu}^2\right\rangle. \qquad (2.32)
\end{aligned}$$

We now turn to the last term of the r. h. s. of (2.26) and appeal to (2.28):

$$\mu^{2k+1}\left\langle\mathfrak{d}_{\mu,k+1}^2\right\rangle \;=\; \sum_{i=0}^{k}\mu a_{k+1,i}^2\left\langle\phi_{2^i\mu}^2\right\rangle+2\sum_{i=0}^{k}\sum_{j>i}^{k}\mu a_{k+1,i}a_{k+1,j}\left\langle\phi_{2^i\mu}\phi_{2^j\mu}\right\rangle. \qquad (2.33)$$

We then prove (2.25) by induction, recalling that

$$A_{\mu,1} \;=\; \langle(\xi+\nabla\phi_\mu)\cdot A(\xi+\nabla\phi_\mu)\rangle\,.$$

Let us assume that (2.25) holds at step $k\geq 1$. Then, using (2.25) at step $k\geq 1$, and (2.32) & (2.33), the identity (2.26) turns into

$$\begin{aligned}
\xi\cdot A_{\mu,k+1}\xi \;&\overset{(2.26)}{=}\; \xi\cdot A_{\mu,k}\xi-2^{k(k-1)}\mu^{2k}\mathcal{E}(\mathfrak{d}_{\mu,k+1},\mathfrak{d}_{\mu,k+1})-2^{k^2+1}\mu^{2k+1}\left\langle\mathfrak{d}_{\mu,k+1}^2\right\rangle\\
&=\; \langle(\xi+\nabla\phi_\mu)\cdot A(\xi+\nabla\phi_\mu)\rangle\\
&\qquad\qquad +\mu\sum_{i=0}^{k-1}\eta_{k,i}\left\langle\phi_{2^i\mu}^2\right\rangle+\mu\sum_{i=0}^{k-1}\sum_{j>i}^{k-1}\nu_{k,i,j}\left\langle\phi_{2^i\mu}\phi_{2^j\mu}\right\rangle\\
&\quad +2^{k(k-1)}\mu\sum_{i=0}^{k}\sum_{j>i}^{k}(2^i+2^j)a_{k+1,i}a_{k+1,j}\left\langle\phi_{2^i\mu}\phi_{2^j\mu}\right\rangle+2^{k(k-1)}\mu\sum_{i=0}^{k}2^i a_{k+1,i}^2\left\langle\phi_{2^i\mu}^2\right\rangle\\
&\quad -2^{k^2+1}\mu\sum_{i=0}^{k}a_{k+1,i}^2\left\langle\phi_{2^i\mu}^2\right\rangle-2^{k^2+2}\mu\sum_{i=0}^{k}\sum_{j>i}^{k}a_{k+1,i}a_{k+1,j}\left\langle\phi_{2^i\mu}\phi_{2^j\mu}\right\rangle,
\end{aligned}$$

from which we deduce that (2.25) holds at step $k+1$. □

Proposition 2 yields the following formulas for the first four approximations of A_{hom}:

$$\begin{aligned}
\xi\cdot A_{\mu,1}\xi &= \langle(\xi+\nabla\phi_\mu)\cdot A(\xi+\nabla\phi_\mu)\rangle\,,\\
\xi\cdot A_{\mu,2}\xi &= \langle(\xi+\nabla\phi_\mu)\cdot A(\xi+\nabla\phi_\mu)\rangle-3\mu\left\langle\phi_\mu^2\right\rangle-2\mu\left\langle\phi_{2\mu}^2\right\rangle+5\mu\left\langle\phi_\mu\phi_{2\mu}\right\rangle,\\
\xi\cdot A_{\mu,3}\xi &= \langle(\xi+\nabla\phi_\mu)\cdot A(\xi+\nabla\phi_\mu)\rangle-\frac{55}{9}\mu\left\langle\phi_\mu^2\right\rangle-8\mu\left\langle\phi_{2\mu}^2\right\rangle-\frac{4}{9}\mu\left\langle\phi_{4\mu}^2\right\rangle\\
&\qquad+\frac{41}{3}\mu\left\langle\phi_\mu\phi_{2\mu}\right\rangle-\frac{22}{9}\mu\left\langle\phi_\mu\phi_{4\mu}\right\rangle+\frac{10}{3}\mu\left\langle\phi_{2\mu}\phi_{4\mu}\right\rangle,\\
\xi\cdot A_{\mu,4}\xi &= \langle(\xi+\nabla\phi_\mu)\cdot A(\xi+\nabla\phi_\mu)\rangle-\frac{3655}{441}\mu\left\langle\phi_\mu^2\right\rangle-\frac{128}{9}\mu\left\langle\phi_{2\mu}^2\right\rangle-\frac{16}{9}\mu\left\langle\phi_{4\mu}^2\right\rangle\\
&\qquad-\frac{8}{441}\mu\left\langle\phi_{8\mu}^2\right\rangle+\frac{1325}{63}\mu\left\langle\phi_\mu\phi_{2\mu}\right\rangle-\frac{370}{63}\mu\left\langle\phi_\mu\phi_{4\mu}\right\rangle+\frac{184}{441}\mu\left\langle\phi_\mu\phi_{8\mu}\right\rangle\\
&\qquad+\frac{82}{9}\mu\left\langle\phi_{2\mu}\phi_{4\mu}\right\rangle-\frac{44}{63}\mu\left\langle\phi_{2\mu}\phi_{8\mu}\right\rangle+\frac{20}{63}\mu\left\langle\phi_{4\mu}\phi_{8\mu}\right\rangle.
\end{aligned}$$

2.4. **Complete error estimate.** In this subsection, we combine the approximation formulas $A_{\mu,k}$ with the filtering method used in [7]. The filters are defined as follows.

Definition 4. A function $\chi:[-1,1]\to\mathbb{R}_+$ is said to be a filter of order $p\geq 0$ if
- (i) $\chi\in C^p([-1,1])\cap W^{p+1,\infty}((-1,1))$,
- (ii) $\int_{-1}^1\chi(x)\mathrm{d}x=1$,
- (iii) $\chi^{(k)}(-1)=\chi^{(k)}(1)=0$ for all $k\in\{0,\dots,p-1\}$.

The associated mask $\chi_L:[-L,L]^d\to\mathbb{R}_+$ in dimension $d\geq 1$ is then defined for all $L>0$ by

$$\chi_L(x)\;:=\;L^{-d}\prod_{i=1}^d\chi(L^{-1}x_i),$$

where $x=(x_1,\dots,x_d)\in\mathbb{R}^d$.

Let now A and A_{hom} be as in Definition 1. For all $k\geq 1$, $\mu>0$, $p\geq 0$, and $R\geq L>0$, we define the approximation $A_{\mu,k,R,L}$ of A_{hom} as

$$\begin{aligned}
\xi\cdot A_{\mu,k,R,L}\xi\;:=\;&\langle\!\langle(\xi+\nabla\phi_{\mu,R})\cdot A(\xi+\nabla\phi_{\mu,R})\rangle\!\rangle_L\\
&+\mu\sum_{i=0}^{k-1}\eta_{k,i}\langle\!\langle\phi_{2^i\mu,R}^2\rangle\!\rangle_L+\mu\sum_{i=0}^{k-1}\sum_{j>i}^{k-1}\nu_{k,i,j}\langle\!\langle\phi_{2^i\mu,R}\phi_{2^j\mu,R}\rangle\!\rangle_L,
\end{aligned}\tag{2.34}$$

where the coefficients $\eta_{k,i}$ and $\nu_{k,i,j}$ are as in Proposition 2, the modified correctors $\phi_{2^i\mu,R}$ are the unique weak solutions in $H^1_0(Q_R)$ to

$$2^i\mu\phi_{2^i\mu,R}-\nabla\cdot A(\xi+\nabla\phi_{2^i\mu,R})\;=\;0,$$

and $\langle\!\langle\cdot\rangle\!\rangle_L$ denotes the average with mask χ_L:

$$\langle\!\langle h\rangle\!\rangle_L\;:=\;\int_{\mathbb{R}^d}h(x)\chi_L(x)\mathrm{d}x.$$

The combination of [7, Theorem 1] with Theorem 1 and Proposition 2 then yields

Theorem 2. *Let $d \geq 2$, A and A_{hom} be as in Definition 1, $k \geq 1$, χ be a filter of order $p \geq 0$, and $A_{\mu,k,R,L}$ be the approximation* (2.34) *of the homogenization matrix, where $R^2 \gtrsim \mu^{-1} \gtrsim R$, $R \geq L \sim R \sim R-L$. Then, there exists $c > 0$ depending only on α, β and d such that we have*

$$|A_{\mu,k,R,L} - A_{\text{hom}}| \lesssim L^{-(p+1)} + \mu^{2k} + \mu^{-1/4} \exp\left(-c\sqrt{\mu}(R-L)\right), \qquad (2.35)$$

where the multiplicative constant depends on k, next to α, β and d.

In order to illustrate Theorem 2, we provide the results of numerical tests in a periodic discrete case in Appendix B. They confirm the sharpness of the analysis.

3. The discrete stochastic case

We start this section by defining the discrete stochastic model we wish to consider.

3.1. **Notation and preliminaries.** We say that x, y in $\mathbb{Z}^d$ are neighbors, and write $x \sim y$, whenever $|y - x| = 1$. This relation turns $\mathbb{Z}^d$ into a graph, whose set of (non-oriented) edges we will denote by $\mathbb{B}$. We now turn to the definition of the associated diffusion coefficients, and their statistics.

Definition 5 (environment)**.** Let $\Omega = [\alpha, \beta]^{\mathbb{B}}$. An element $\omega = (\omega_e)_{e \in \mathbb{B}}$ of Ω is called an *environment*. With any edge $e = (x, y) \in \mathbb{B}$, we associate the *conductance* $\omega_{x,y} := \omega_e$ (by construction $\omega_{x,y} = \omega_{y,x}$). Let ν be a probability measure on $[\alpha, \beta]$. We endow Ω with the product probability measure $\mathbb{P} = \nu^{\otimes \mathbb{B}}$. In other words, if ω is distributed according to the measure $\mathbb{P}$, then $(\omega_e)_{e \in \mathbb{B}}$ are independent random variables of law ν. We denote by $L^2(\Omega)$ the set of real square integrable functions on Ω for the measure $\mathbb{P}$, and write $\langle \cdot \rangle$ for the expectation associated with $\mathbb{P}$.

In the framework of Definition 5, we can introduce a notion of stationarity.

Definition 6 (stationarity)**.** For all $z \in \mathbb{Z}^d$, we let $\theta_z : \Omega \to \Omega$ be such that for all $\omega \in \Omega$ and $(x, y) \in \mathbb{B}$, $(\theta_z\, \omega)_{x,y} = \omega_{x+z,y+z}$. This defines an additive action group $\{\theta_z\}_{z \in \mathbb{Z}^d}$ on Ω which preserves the measure $\mathbb{P}$, and is ergodic for $\mathbb{P}$.
We say that a function $f : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ is stationary if and only if for all $x, z \in \mathbb{Z}^d$ and $\mathbb{P}$-almost every $\omega \in \Omega$,

$$f(x+z, \omega) \;=\; f(x, \theta_z\, \omega).$$

In particular, with all $f \in L^2(\Omega)$, one may associate the stationary function (still denoted by f) $\mathbb{Z}^d \times \Omega \to \mathbb{R}, (x, \omega) \mapsto f(\theta_x\, \omega)$. In what follows we will not distinguish between $f \in L^2(\Omega)$ and its stationary extension on $\mathbb{Z}^d \times \Omega$.

It remains to define the conductivity matrix on $\mathbb{Z}^d$.

Definition 7 (conductivity matrix)**.** Let Ω, $\mathbb{P}$, and $\{\theta_z\}_{z \in \mathbb{Z}^d}$ be as in Definitions 5 and 6. The stationary diffusion matrix $A : \mathbb{Z}^d \times \Omega \to \mathcal{M}_d(\mathbb{R})$ is defined by

$$A(x, \omega) = \operatorname{diag}(\omega_{x,x+\mathbf{e}_i}, \ldots, \omega_{x,x+\mathbf{e}_d}).$$

For each $\omega \in \Omega$, we may consider the discrete elliptic equation whose operator is

$$-\nabla^* \cdot A(\cdot, \omega)\nabla,$$

where ∇ and ∇^* are defined for all $u : \mathbb{Z}^d \to \mathbb{R}$ by

$$\nabla u(x) := \begin{bmatrix} u(x+\mathbf{e}_1) - u(x) \\ \vdots \\ u(x+\mathbf{e}_d) - u(x) \end{bmatrix}, \quad \nabla^* u(x) := \begin{bmatrix} u(x) - u(x-\mathbf{e}_1) \\ \vdots \\ u(x) - u(x-\mathbf{e}_d) \end{bmatrix}, \tag{3.1}$$

and the backward divergence is denoted by $\nabla^*\cdot$, as usual. The standard stochastic homogenization theory for such discrete elliptic operators (see for instance [13], [12]) ensures that there exist homogeneous and deterministic coefficients A_{hom} such that the solution operator of the continuum differential operator $-\nabla \cdot A_{\mathrm{hom}}\nabla$ describes $\mathbb{P}$-almost surely the large scale behavior of the solution operator of the discrete differential operator $-\nabla^* \cdot A(\cdot,\omega)\nabla$. As for the periodic case, the definition of A_{hom} involves the so-called correctors $\phi : \mathbb{Z}^d \times \Omega \to \mathbb{R}$, which are solutions (in a sense made precise below) to the equations

$$-\nabla^* \cdot A(x,\omega)(\xi + \nabla\phi(x,\omega)) \;=\; 0, \qquad x \in \mathbb{Z}^d, \tag{3.2}$$

for $\xi \in \mathbb{R}^d$. The following lemma gives the existence and uniqueness of the corrector ϕ.

Lemma 1 (corrector)**.** *Let Ω, $\mathbb{P}$, $\{\theta_z\}_{z\in\mathbb{Z}^d}$, and A be as in Definitions 5, 6, and 7. Then, for all $\xi \in \mathbb{R}^d$, there exists a unique measurable function $\phi : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ such that $\phi(0,\cdot) \equiv 0$, $\nabla\phi$ is stationary, $\langle \nabla\phi\rangle = 0$, and ϕ solves* (3.2) *$\mathbb{P}$-almost surely. Moreover, the symmetric homogenized matrix A_{hom} is characterized by*

$$\xi \cdot A_{\mathrm{hom}}\xi \;=\; \langle (\xi + \nabla\phi) \cdot A(\xi+\nabla\phi)\rangle\,. \tag{3.3}$$

As mentioned in the introduction, the standard proof of Lemma 1 makes use of the regularization of (3.2) by a zero-order term $\mu > 0$:

$$\mu\phi_\mu(x,\omega) - \nabla^* \cdot A(x,\omega)(\xi + \nabla\phi_\mu(x,\omega)) \;=\; 0, \qquad x \in \mathbb{Z}^d. \tag{3.4}$$

Lemma 2 (modified corrector)**.** *Let Ω, $\mathbb{P}$, $\{\theta_z\}_{z\in\mathbb{Z}^d}$, and A be as in Definitions 5, 6, and 7. Then, for all $\mu > 0$ and $\xi \in \mathbb{R}^d$, there exists a unique stationary function $\phi_\mu \in L^2(\Omega)$ which solves* (3.4) *$\mathbb{P}$-almost surely.*

In order to proceed as in the periodic case and use a spectral approach, one needs to suitably define an elliptic operator on $L^2(\Omega)$ (which is the stochastic counterpart to the space $H^1_{\mathrm{per}}(Q)$ of Section 2). Stationarity is crucial here. Following [17], we introduce difference operators on $L^2(\Omega)$: for all $u \in L^2(\Omega)$, we set

$$\mathrm{D}\,u(\omega) := \begin{bmatrix} u(\theta_{\mathbf{e}_1}\omega) - u(\omega) \\ \vdots \\ u(\theta_{\mathbf{e}_d}\omega) - u(\omega) \end{bmatrix}, \quad \mathrm{D}^*\,u(\omega) := \begin{bmatrix} u(\omega) - u(\theta_{-\mathbf{e}_1}\omega) \\ \vdots \\ u(\omega) - u(\theta_{-\mathbf{e}_d}\omega) \end{bmatrix}. \tag{3.5}$$

We are in position to define the stochastic counterpart to the operator of Definition 2.

Definition 8. Let Ω, $\mathbb{P}$, $\{\theta_z\}_{z\in\mathbb{Z}^d}$, and A be as in Definitions 5, 6, and 7. We define $\mathcal{L} : L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{aligned} \mathcal{L}u(\omega) &= -\,\mathrm{D}^* \cdot A(\omega)\,\mathrm{D}\,u(\omega) \\ &= \sum_{z\sim 0} \omega_{0,z}(u(\omega) - u(\theta_z\,\omega)) \end{aligned}$$

where D and D^* are as in (3.5).

In probabilistic terms, the operator $-\mathcal{L}$ is the generator of the Markov process called the "environment viewed by the particle". This process is defined to be $(\theta_{X_t}\ \omega)$, where (X_t) is a random walk whose jump rate from x to a neighbor y is given by $\omega_{x,y}$.

Using Definition 8 and the stationarity of ϕ_μ, Lemma 2 implies that ϕ_μ is the unique solution in $L^2(\Omega)$ to the equation

$$(\mu + \mathcal{L})\phi_\mu \;=\; \mathfrak{d}, \tag{3.6}$$

where

$$\mathfrak{d}(\omega) \;:=\; \mathrm{D}^* \cdot A(\omega)\xi. \tag{3.7}$$

At the level of the corrector ϕ itself (which is not stationary), the weak form of (3.6) survives for $\mu = 0$: for every $\psi \in L^2(\Omega)$, we have

$$\langle \mathrm{D}\,\psi \cdot A\,\mathrm{D}\,\phi\rangle \;=\; \langle \mathrm{D}\,\psi \cdot A\xi\rangle\,. \tag{3.8}$$

For all $f \in L^2(\Omega)$, we let $\mathcal{E}(f,f)$ be the Dirichlet form associated with $\mathcal{L}$, defined by

$$\mathcal{E}(f,f) = \langle \mathcal{L}f \cdot f\rangle = \langle \mathrm{D}\,f \cdot A\,\mathrm{D}\,f\rangle = \frac{1}{2}\sum_{z\sim 0}\left\langle \omega_{0,z}(f(\theta_z\ \omega) - f(\omega))^2\right\rangle. \tag{3.9}$$

As in the periodic case, the homogenized diffusion matrix satisfies the identity

$$\xi \cdot A_{\mathrm{hom}}\xi = \langle \xi \cdot A\xi\rangle - \mathcal{E}(\phi,\phi). \tag{3.10}$$

The proof is formally the same as for (2.2), provided we use the weak form (3.8) of the corrector equation, which holds for ϕ in place of ψ (although ϕ is not stationary).

We refer the reader to [13] for the proofs of the statements above.

3.2. **Spectral representation and approximations of the homogenized coefficients.** The operator $\mathcal{L}$ is bounded, positive, and self-adjoint on $L^2(\Omega)$. By the spectral theorem, for any function $f \in L^2(\Omega)$, we can define the spectral measure e_f of $\mathcal{L}$ projected on f, that is such that for any bounded continuous function $\Psi : \mathbb{R}_+ \to \mathbb{R}$, one has

$$\langle f \cdot \Psi(\mathcal{L})f\rangle = \int_{\mathbb{R}_+} \Psi(\lambda)\ \mathrm{d}e_f(\lambda).$$

As in the periodic case, we can express the homogenized diffusion matrix in terms of the spectral measure projected on $\mathfrak{d}$.

Lemma 3. *Let Ω, $\mathbb{P}$, $\{\theta_z\}_{z\in\mathbb{Z}^d}$, A, and $\mathcal{L}$ be as in Definitions 5, 6, 7, and 8. We let A_{hom} denote the associated homogenized diffusion matrix* (3.3)*, and $\mathfrak{d}$ be the local drift* (3.7)*. Then, the following identity holds*

$$\xi \cdot A_{\mathrm{hom}}\xi = \langle \xi \cdot A\xi\rangle - \int_{\mathbb{R}_+} \frac{1}{\lambda}\ \mathrm{d}e_{\mathfrak{d}}(\lambda),$$

where $e_{\mathfrak{d}}$ is the spectral measure of $\mathcal{L}$ projected on $\mathfrak{d}$.

Proof. In view of formula (3.10), we need to show that

$$\mathcal{E}(\phi,\phi) = \int_{\mathbb{R}_+} \frac{1}{\lambda}\ \mathrm{d}e_{\mathfrak{d}}(\lambda).$$

This is either a consequence of Kipnis and Varadhan's arguments (see in particular [16, Theorem 8.1]), or a consequence of [10, Corollary 1 & Remark 2]. We detail the second argument. [10, Corollary 1 & Remark 2] imply that $\lim_{\mu\to 0} \nabla\phi_\mu = \nabla\phi$ strongly in $L^2(\Omega)$, hence

$$\lim_{\mu\to 0} \mathcal{E}(\phi_\mu,\phi_\mu) \;=\; \mathcal{E}(\phi,\phi).$$

Besides, for all $\mu>0$, we have by definition of the spectral decomposition

$$\mathcal{E}(\phi_\mu,\phi_\mu) = \int_{\mathbb{R}_+} \frac{\lambda}{(\lambda+\mu)^2}\, \mathrm{d}e_f(\lambda),$$

and the result follows by the monotone convergence theorem. □

From Lemma 3, we deduce that the approximations $A_{\mu,k}$ introduced in Theorem 1 and further characterized in Proposition 2 may also be used in this discrete stochastic case, provided the notation $\langle\cdot\rangle$ is understood as the expectation (instead of periodic average).

3.3. **Suboptimal estimate of the systematic error.** We let $\mathfrak{d}_{\mu,k}$, P_k, and $A_{\mu,k}$ be as in Section 2. In order to quantify the systematic error, we introduce, for any $D,q,k\ge 0$, the function $\mathrm{Err}_{D,q,k}:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\mathrm{Err}_{D,q,k}(\mu) = \left|\begin{array}{ll} \mu^{2k} & \text{if } k<D/4,\\ \mu^{2k}\ln_+^{1+q}(\mu^{-1}) & \text{if } k=D/4,\\ \mu^{D/2}\ln_+^{q}(\mu^{-1}) & \text{if } k>D/4,\end{array}\right.$$

where we write $\ln_+(x)=\max\{\ln x,1\}$. The purpose of this section is to show the following theorem.

Theorem 3. *Let Ω, $\mathbb{P}$, $\{\theta_z\}_{z\in\mathbb{Z}^d}$, A, and $\mathcal{L}$ be as in Definitions 5, 6, 7, and 8, and $e_{\mathfrak{d}}$ be as in Lemma 3. We let A_{hom} denote the associated homogenized diffusion matrix* (3.3)*, and $A_{\mu,k}$ be the approximation* (2.22) *of A_{hom} for $\mu>0$, and $k\ge 1$. Then, there exists $q\ge 0$ (depending on α and β) such that for all $\xi\in\mathbb{R}^d$ with $|\xi|=1$, and $\mu>0$,*

$$0\le \xi\cdot(A_{\mu,k}-A_{\mathrm{hom}})\xi \lesssim \left|\begin{array}{ll} \mathrm{Err}_{2,q,k}(\mu) & \text{if } d=2,\\ \mathrm{Err}_{d,0,k}(\mu) & \text{if } 5\ge d>2,\\ \mathrm{Err}_{6,1,k}(\mu) & \text{if } d=6,\\ \mathrm{Err}_{6,0,k}(\mu) & \text{if } 12\ge d>6,\\ \mathrm{Err}_{d-6,0,k}(\mu) & \text{if } d>12,\end{array}\right.$$

where the multiplicative constants depends on k, next to α,β, and d.

In order to prove Theorem 3, we need to introduce some vocabulary. For all $\gamma>1$ and $q\ge 0$, we say that the spectral exponents of a function $f\in L^2(\Omega)$ are at least $(\gamma,-q)$ if we have

$$\int_0^\mu \mathrm{d}e_f(\lambda) \lesssim \mu^\gamma \ln_+^q(\mu^{-1}).$$

Note that, if $(\gamma',-q')\le(\gamma,-q)$ for the lexicographical order, and if the spectral exponents of f are at least $(\gamma,-q)$, then they are at least $(\gamma',-q')$. Hence, the phrasing is consistent.

In order to prove Theorem 3, we first express the systematic error in terms of the spectral exponents of $\mathfrak{d}$. This is the object of Theorem 4. We then prove estimates on these exponents in Theorem 5, which concludes the proof of Theorem 3.

Theorem 4. *Within the notation and assumptions of Theorem 3, the following two statements hold: for all $\xi\in\mathbb{R}^d$ with $|\xi|=1$, and $\mu>0$,*

(1) *If the spectral exponents of* $\mathfrak{d}$ *are at least* $(\gamma,-q)$*, then*

$$0 \le \xi\cdot(A_{\mu,k}-A_{\hom})\xi \lesssim \left| \begin{array}{ll} \mu^{2k} & \text{if } \gamma > 2k+1, \\ \mu^{2k}\ \ln_+^{1+q}(\mu^{-1}) & \text{if } \gamma = 2k+1, \\ \mu^{\gamma-1}\ \ln_+^{q}(\mu^{-1}) & \text{if } \gamma < 2k+1, \end{array}\right.$$

where the multiplicative constant depends on γ *and* k *next to* α,β*, and* d*.*

(2) *Conversely,*

$$\xi\cdot(A_{\mu,k}-A_{\hom})\xi \gtrsim \mu^{2k}+\mu^{-1}\int_0^{\mu} \mathrm{d}e_{\mathfrak{d}}(\lambda),$$

where the multiplicative constant depends on k *next to* α,β*, and* d*.*

This theorem extends [16, Proposition 9.1]. We begin by proving the following result.

Lemma 4. *If the spectral exponents of* $\mathfrak{d}$ *are at least* $(\gamma,-q)$*, then for all* $\mu>0$*,*

$$0 \le \xi\cdot(A_{\mu,k}-A_{\hom})\xi \lesssim \mu^{2k}+\mu^{\gamma-1}\int_0^{\mu^{-1}} \frac{u^{\gamma-2}}{(1+u)^{2k}}\ln^q((\mu u)^{-1})\ \mathrm{d}u \tag{3.11}$$

where the multiplicative constant depends on k*, next to* α,β*, and* d*.*

Proof of Lemma 4. First, recall that

$$\xi\cdot(A_{\mu,k}-A_{\hom})\xi = 2^{k(k-1)}\mu^{2k}\int_{\mathbb{R}_+}\frac{1}{\lambda(\mu+\lambda)^2\cdots(2^{k-1}\mu+\lambda)^2}\ \mathrm{d}e_{\mathfrak{d}}(\lambda).$$

The integral of the r. h. s. is non-negative and bounded by

$$\int_{\mathbb{R}_+}\frac{1}{\lambda(\mu+\lambda)^{2k}}\ \mathrm{d}e_{\mathfrak{d}}(\lambda). \tag{3.12}$$

We perform a sort of integration by parts on this integral. To this aim, we let $f'(\lambda)$ be given by

$$f'(\lambda) = -\frac{\partial}{\partial\lambda}\frac{1}{\lambda(\mu+\lambda)^{2k}} = \frac{(\mu+\lambda)^{2k-1}(\mu+(2k+1)\lambda)}{\lambda^2(\mu+\lambda)^{4k}}.$$

We then rewrite the integral (3.12) in terms of f', and use Fubini's theorem:

$$\begin{aligned}\int_{\mathbb{R}_+}\frac{1}{\lambda(\mu+\lambda)^{2k}}\ \mathrm{d}e_{\mathfrak{d}}(\lambda) &= \int_{\lambda=0}^{+\infty}\int_{\delta=\lambda}^{+\infty} f'(\delta)\ \mathrm{d}\delta\ \mathrm{d}e_{\mathfrak{d}}(\lambda) \\ &= \int_{\delta=0}^{+\infty} f'(\delta)\int_{\lambda=0}^{\delta}\mathrm{d}e_{\mathfrak{d}}(\lambda)\ \mathrm{d}\delta.\end{aligned}$$

We split this double integral in two parts, and treat the cases $\delta\in(1,+\infty)$ and $\delta\in(0,1]$ separately. We begin with the case when δ ranges in $(1,+\infty)$. We bound the inner integral

$$\int_{\lambda=0}^{\delta}\mathrm{d}e_{\mathfrak{d}}(\lambda) \le \int_{\lambda=0}^{\infty}\mathrm{d}e_{\mathfrak{d}}(\lambda) = \langle\mathfrak{d}^2\rangle \le 4\beta^2 \lesssim 1,$$

by definition of the projection of the spectral measure on $\mathfrak{d}$. This yields for the first part of the double integral

$$\int_{\delta=1}^{+\infty} f'(\delta)\int_{\lambda=0}^{\delta}\mathrm{d}e_{\mathfrak{d}}(\lambda)\ \mathrm{d}\delta \lesssim \frac{1}{(\mu+1)^{2k}} \lesssim 1.$$

We now turn to the case when δ ranges in $(0,1]$. The assumption on the spectral exponents of $\mathfrak{d}$ implies

$$\int_{\delta=0}^{1} f'(\delta)\int_{\lambda=0}^{\delta} \mathrm{d}e_{\mathfrak{d}}(\lambda)\ \mathrm{d}\delta \ \le\ \int_0^1 f'(\delta)\delta^{\gamma}\ln^q(\delta^{-1})\ \mathrm{d}\delta. \tag{3.13}$$

Noting that

$$f'(\delta)\le(2k+1)\frac{1}{\delta^2(\mu+\delta)^{2k}},$$

we bound the r. h. s. of (3.13) by $(2k+1)$ times

$$\int_0^1 \frac{\delta^{\gamma-2}}{(\mu+\delta)^{2k}}\ln^q(\delta^{-1})\ \mathrm{d}\delta.$$

A change of variables yields the announced result. □

Proof of part (1) of Theorem 4. We first assume that $\gamma>2k+1$. In that case, we let γ' be such that $2k+1<\gamma'<\gamma$. Since the spectral exponents of $\mathfrak{d}$ are at least $(\gamma',0)$, Lemma 4 ensures that

$$0\le\xi\cdot(A_{\mu,k}-A_{\mathrm{hom}})\xi\ \lesssim\ \mu^{2k}+\mu^{\gamma'-1}\int_0^{\mu^{-1}}\frac{u^{\gamma'-2}}{(1+u)^{2k}}\ \mathrm{d}u\ \lesssim\ \mu^{2k}.$$

We now turn to the case when $\gamma\le 2k+1$. We need to estimate the integral of the r. h. s. of (3.11). To this aim, we note that

$$\ln^q((\mu u)^{-1})=\big(\ln(\mu^{-1})-\ln(u)\big)^q\le 2^q\big(\ln^q(\mu^{-1})+|\ln(u)|^q\big),$$

so that the integral in (3.11) may be estimated by

$$\int_0^{\mu^{-1}}\frac{u^{\gamma-2}}{(1+u)^{2k}}\big(\ln^q(\mu^{-1})+|\ln(u)|^q\big)\ \mathrm{d}u\ \lesssim\ \left|\begin{array}{ll}\ln^{q+1}(\mu^{-1}) & \text{if } \gamma=2k+1,\\ \ln^q(\mu^{-1}) & \text{if } \gamma<2k+1,\end{array}\right.$$

as desired. □

Proof of part (2) of Theorem 4. Let $\delta>0$ be such that

$$\int_0^{\delta}\mathrm{d}e_{\mathfrak{d}}(\lambda)>0.$$

By the non-negativity of the spectrum and of the integrand,

$$\begin{aligned}\xi\cdot(A_{\mu,k}-A_{\mathrm{hom}})\xi\ &=\ \int_{\mathbb{R}_+}\frac{2^{k(k-1)}\mu^{2k}}{\lambda(\mu+\lambda)^2\cdots(2^{k-1}\mu+\lambda)^2}\ \mathrm{d}e_{\mathfrak{d}}(\lambda)\\ &\ge\ \frac{2^{k(k-1)}\mu^{2k}}{\delta(\mu+\delta)^2\cdots(2^{k-1}\mu+\delta)^2}\int_0^{\delta}\mathrm{d}e_{\mathfrak{d}}(\lambda).\end{aligned}$$

Hence,

$$\xi\cdot(A_{\mu,k}-A_{\mathrm{hom}})\xi\ \gtrsim\ \mu^{2k}.$$

In addition, there exists $C>0$ such that for all $\lambda\in(0,\mu]$, one has

$$\frac{\mu^{2k}}{\lambda(\mu+\lambda)^2\cdots(2^{k-1}\mu+\lambda)^2}\ge\frac{C}{\mu}.$$

Therefore,

$$\xi\cdot(A_{\mu,k}-A_{\mathrm{hom}})\xi\gtrsim\mu^{-1}\int_0^{\mu}\mathrm{d}e_{\mathfrak{d}}(\lambda),$$

which concludes the proof of the theorem. □

It remains to estimate the spectral exponents of $\mathfrak{d}$.

Theorem 5. *Within the notation and assumptions of Theorem 3, there exists $q \geq 0$ depending only on the ellipticity constants α and β such that the spectral exponents of $\mathfrak{d}$ are at least*

$$\left| \begin{array}{ll} (2,-q) & \text{if } d=2, \\ (d/2+1,0) & \text{if } 5 \geq d > 2, \\ (4,-1) & \text{if } d=6, \\ (4,0) & \text{if } 12 \geq d > 6, \\ (d/2-2,0) & \text{if } d > 12. \end{array} \right.$$

Remark 1. We conjecture that the spectral exponents of $\mathfrak{d}$ are in fact $(d/2+1,0)$ for $d>2$. If true, this would imply that the systematic error is in fact bounded by $\mathrm{Err}_{d,0,k}(\mu)$ for any $d>2$ and k.

In order to prove Theorem 5, we will make use of the following result.

Lemma 5. *Within the notation and assumptions of Theorem 3, there exists $q \geq 0$ depending only on the ellipticity constants α and β such that for all $\mu > 0$,*

$$\langle (\mathfrak{d}_{\mu,2})^2 \rangle = \int_{\mathbb{R}_+} \frac{1}{(\mu+\lambda)^2(2\mu+\lambda)^2}\, \mathrm{d}e_{\mathfrak{d}}(\lambda) \lesssim \left| \begin{array}{ll} \mu^{-2}\ln_+^q(\mu^{-1}) & \text{if } d=2, \\ \mu^{d/2-3} & \text{if } 5 \geq d > 2, \\ \ln_+(\mu) & \text{if } d=6, \\ 1 & \text{if } d>6, \end{array} \right.$$

where $\mathfrak{d}_{\mu,2}$ is as in Definition 3.

Lemma 5 is a consequence of the results of [10]. Its proof, which is slightly technical, is deferred to Appendix A.

Proof of Theorem 5. For all $\lambda \leq \mu$, one has

$$\frac{\mu^4}{(\mu+\lambda)^2(2\mu+\lambda)^2} \geq \frac{1}{36}.$$

Hence,

$$\int_0^\mu \mathrm{d}e_{\mathfrak{d}}(\lambda) \leq 36\mu^4 \int_{\mathbb{R}_+} \frac{1}{(\mu+\lambda)^2(2\mu+\lambda)^2}\, \mathrm{d}e_{\mathfrak{d}}(\lambda).$$

The announced bounds then follow from Lemma 5 for $d \leq 12$.

For $d \geq 13$, we use instead [16, Theorems 2.3 and 2.4], which ensure that there exist $C>0$ such that for all $\mu > 0$,

$$\int_0^\mu \lambda^{-1} \mathrm{d}e_{\mathfrak{d}}(\lambda) \leq C\mu^{d/2-3}.$$

This shows that the spectral exponents of $\mathfrak{d}$ are at least $(d/2-2,0)$, since

$$\int_0^\mu \mathrm{d}e_{\mathfrak{d}}(\lambda) \leq \mu \int_0^\mu \lambda^{-1}\mathrm{d}e_{\mathfrak{d}}(\lambda).$$

□

3.4. **Complete error analysis.** As for the periodic case, ϕ_μ can be accurately replaced by $\phi_{\mu,R}$, the solution of the modified corrector equation on a finite box Q_R with homogeneous Dirichlet boundary conditions. We refer the reader to [8] for details.

In order to perform a complete error estimate, one still needs to estimate the variance term in the r. h. s. of the identity corresponding to (1.3). This is the object of the following theorem.

Theorem 6. *Let Ω, $\mathbb{P}$, $\{\theta_z\}_{z\in\mathbb{Z}^d}$, and A be as in Definitions 5, 6, and 7. We let A_{hom} denote the associated homogenized diffusion matrix* (3.3)*, and for all $k \geq 1$, $\mu > 0$, and $L > 0$, we define the approximation $A_{\mu,k,L}$ of A_{hom} as*

$$\begin{aligned}\xi\cdot A_{\mu,k,L}\xi \;\;:=&\;\; \langle\!\langle(\xi+\nabla\phi_\mu)\cdot A(\xi+\nabla\phi_\mu)\rangle\!\rangle_L \\ &+\mu\sum_{i=0}^{k-1}\eta_{k,i}\langle\!\langle\phi_{2^i\mu}^2\rangle\!\rangle_L+\mu\sum_{i=0}^{k-1}\sum_{j>i}^{k-1}\nu_{k,i,j}\langle\!\langle\phi_{2^i\mu}\phi_{2^j\mu}\rangle\!\rangle_L,\end{aligned}$$

where the coefficients $\eta_{k,i}$ and $\nu_{k,i,j}$ are as in Proposition 2, the modified correctors $\phi_{2^i\mu}$ are as in Lemma 2, and $\langle\!\langle\cdot\rangle\!\rangle_L$ denotes the spatial average

$$h\mapsto\langle\!\langle h\rangle\!\rangle_L \;:=\; \int_{\mathbb{Z}^d}h(x)\chi_L(x)\mathrm{d}x,$$

where $x\mapsto\chi_L(x)$ is an averaging function on $(-L,L)^d$ such that $\int_{\mathbb{Z}^d}\chi_L(x)\mathrm{d}x=1$ and $\|\nabla\chi_L\|_{L^\infty}\lesssim L^{-d-1}$. Then, there exists an exponent $q>0$ depending only on α,β such that for all $L\gg1$ and all $\mu>0$,

$$\mathrm{var}\,[A_{\mu,k,L}] \;\lesssim\; \left|\begin{array}{rl}(L^{-2}+\mu^2)\ln_+^q\mu^{-1} & \textit{if}\quad d=2,\\ L^{-d}+\mu^2L^{-d+2} & \textit{if}\quad d>2.\end{array}\right.$$

Theorem 6 is a direct consequence of [9, Theorem 2.1 & Remark 2.1] applied to each term of $A_{\mu,k}$ in the form (2.25) of Proposition 2.

3.5. **Polynomial decay of the variance along the semi-group.** We end this section with a short remark concerning some results of [16]. Let $(S_t)_{t\geq0}$ be the semi-group associated with the infinitesimal generator $-\mathcal{L}$ introduced in Definition 8. In [16], the asymptotic decay to 0 of the variance of S_tf is investigated. A slight modification of [16, Theorem 2.4] reads as follows.

Theorem 7. *Let $f\in L^2(\Omega)$ be such that $\langle f\rangle=0$, and let $\gamma>1$, $q\geq0$. The following two statements are equivalent :*

(1) *The spectral exponents of f are at least $(\gamma,-q)$;*
(2)
$$\left\langle(S_tf)^2\right\rangle\lesssim t^{-\gamma}\;\;\ln_+^q(t).$$

From Theorem 5, we thus obtain the following result, which strengthens [16, Theorem 2.3 and Corollary 9.3] when $4\leq d<12$.

Corollary 1. *Within the notation and assumptions of Theorem 3, there exists $q \geq 0$ depending only on the ellipticity constants α and β such that*

$$\left\langle (S_t\mathfrak{d})^2 \right\rangle \lesssim \left| \begin{array}{ll} t^{-2}\ \ln_+^q(t) & \textit{if } d=2, \\ t^{-(d/2+1)} & \textit{if } 5 \geq d > 2, \\ t^{-4}\ \ln_+(t) & \textit{if } d=6, \\ t^{-4} & \textit{if } 12 \geq d > 6, \\ t^{-(d/2-2)} & \textit{if } d > 12. \end{array} \right.$$

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Appendix A. Proof of Lemma 5

As already mentioned, Lemma 5 is a refinement of results by Otto and the first author in [9, 10]. The present proof heavily relies on auxiliary results and estimates obtained in these papers. In particular, this proof is unfortunately not self-contained and cannot be read without the companion articles [9, 10].

To ease the reading, we adopt the notation of [10]. In particular, we set $T = \mu^{-1}$, and for every edge $e = (x, x+\mathbf{e}_i)$ we rename the conductivities as $a(e) := \omega(x, x+\mathbf{e}_i)$. We denote by G_T the Green's function associated with the elliptic operator $T^{-1} - \nabla^* \cdot A\nabla$, that is: for all $x \in \mathbb{Z}^d$, $G_T(x, \cdot)$ is the unique weak solution in $L^2(\mathbb{Z}^d)$ to

$$T^{-1}G_T(x,y) - \nabla_y^* \cdot A(y)\nabla_y G_T(x,y) \;=\; \delta(x-y).$$

We also let ϕ_T denote the associated modified corrector, unique stationary solution to

$$T^{-1}\phi_T(x) - \nabla^* \cdot A(x)(\xi + \nabla\phi_T(x)) \;=\; 0,$$

and we set $\psi_T := -\mathfrak{d}_{\mu,2}$, that is

$$\psi_T \;=\; T(\phi_{2T} - \phi_T).$$

Note that ψ_T solves

$$T^{-1}\psi_T(x) - \nabla^* \cdot A(x)\nabla\psi_T(x) \;=\; \frac{1}{2}\phi_{2T}(x).$$

In particular, G_T, ϕ_T and ψ_T depend on the diffusion coefficients A. The claim of the lemma is equivalent to

$$\left\langle \psi_T^2 \right\rangle \lesssim \left| \begin{array}{rll} T^2\ln^q T & \text{if} & d=2, \\ T^{3-d/2} & \text{if} & 5 \geq d > 2, \\ \ln T & \text{if} & d=6, \\ 1 & \text{if} & d>6. \end{array} \right. \tag{A.1}$$

For $d = 2$, (A.1) is a direct consequence of [9, Proposition 2.1], so that we now only focus on the case $d \geq 3$.

Since $\langle\psi_T\rangle = 0$, it holds that $\langle\psi_T^2\rangle = \mathrm{var}\,[\psi_T]$. From the identity $\psi_T = T(\phi_{2T} - \phi_T)$ we learn that ψ_T depends continuously on the diffusion coefficients by [9, Lemma 2.6] so that one may apply the variance estimate of [9, Lemma 2.3]. In particular,

$$\langle\psi_T^2\rangle \;=\; \mathrm{var}\,[\psi_T] \lesssim \sum_e \left\langle \sup_{a(e)} \left|\frac{\partial\psi_T(0)}{\partial a(e)}\right|^2 \right\rangle, \tag{A.2}$$

where the sum runs over the edges of $\mathbb{Z}^d$, and where for any measurable function X of the conductivities a_i, $\sup_{a_i}\left|\frac{\partial X}{\partial a_i}\right|$ denotes the supremum of the modulus of the i-th partial derivative

$$\frac{\partial X}{\partial a_i}(a_1,\cdots,a_{i-1},a_i,a_{i+1},\cdots)$$

of X with respect to the variable $a_i \in [\alpha,\beta]$.

Our strategy is to use (A.2) to prove (A.1). From [10, (3.10) & (3.21)] and [9, (2.14) & (2.16)], we infer that

$$\begin{aligned}\sup_{a(e)}\left|\frac{\partial\psi_T(0)}{\partial a(e)}\right| \;\lesssim\;& \big(|\nabla\psi_T(z)| + \nu_d(T)(1+|\nabla\phi_{2T}(z)|)\big)G_T(0,e)\\ &+(1+|\nabla\phi_{2T}(z)|)\int_{\mathbb{Z}^d} G_T(0,w)G_T(e,w)\mathrm{d}w, \end{aligned}\tag{A.3}$$

where $e = (z, z+\mathbf{e}_i)$, $G_T(0,e) := G_T(0,z+\mathbf{e}_i) - G_T(0,z)$, $G_T(e,w) = G_T(z+\mathbf{e}_i,w) - G_T(z,w)$, and

$$\nu_d(T) \;:=\; 1 + \left|\begin{array}{rl} \sqrt{T} & \text{if} \quad d=3,\\ \ln T & \text{if} \quad d=4,\\ 1 & \text{if} \quad d>4.\end{array}\right.$$

Hence, using in addition Young's inequality and replacing the summation on the edges e by d times the summation on the sites $z\in\mathbb{Z}^d$, (A.2) turns into

$$\begin{aligned}\mathrm{var}\,[\psi_T] \lesssim& \int_{\mathbb{Z}^d}\big\langle\big(|\nabla\psi_T(z)|^2 + \nu_d(T)^2(1+|\nabla\phi_{2T}(z)|^2)\big)G_T(0,e)^2\big\rangle\,\mathrm{d}z\\ &+\int_{\mathbb{Z}^d}\left\langle(1+|\nabla\phi_{2T}(z)|^2)\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} G_T(0,w)G_T(0,w')|G_T(e,w)||G_T(e,w')|\mathrm{d}w\mathrm{d}w'\right\rangle\mathrm{d}z.\end{aligned}\tag{A.4}$$

Combined with the following two estimates

$$\begin{aligned}&\int_{\mathbb{Z}^d}\big\langle\big(|\nabla\psi_T(z)|^2 + \nu_d(T)^2(1+|\nabla\phi_{2T}(z)|^2)\big)G_T(0,e)^2\big\rangle\,\mathrm{d}z\\ &\qquad\lesssim 1 + \left|\begin{array}{rl} T^{3/2} & \text{if} \quad d=3,\\ \ln^2 T & \text{if} \quad d=4,\\ 1 & \text{if} \quad d>4,\end{array}\right.\end{aligned}\tag{A.5}$$

and

$$\int_{\mathbb{Z}^d}\left\langle(1+|\nabla\phi_{2T}(z)|^2)\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}G_T(0,w)G_T(0,w')|G_T(e,w)||G_T(e,w')|\mathrm{d}w\mathrm{d}w'\right\rangle\mathrm{d}z$$
$$\lesssim 1+\left|\begin{array}{rll} T^{3-d/2} & \text{if} & 5\geq d>2,\\ \ln T & \text{if} & d=6,\\ 1 & \text{if} & d>6,\end{array}\right. \tag{A.6}$$

(A.2) & (A.4) imply the claim of Lemma 5. To conclude the proof of Lemma 5, it remains to prove (A.5) and (A.6), which we do in the following two steps.

Step 1. Proof of (A.5).
To prove (A.5), we first replace the gradient of the Green's function by the Green's function itself and appeal to the deterministic optimal pointwise estimate of [10, Lemma 4]:

$$|G_T(0,e)|\;\leq\;G_T(0,z)+G_T(0,z+\mathbf{e}_i)\;\lesssim\;(1+|z|)^{2-d}\min\{1,\sqrt{T}|z|^{-1}\}^4,$$

for $d>2$. By stationarity, $\langle|\nabla\psi_T(z)|^2\rangle=\langle|\nabla\psi_T(0)|^2\rangle$, and $\langle|\nabla\phi_{2T}(z)|^2\rangle\lesssim 1$ by the a priori estimate of [9, Lemma 2.1]. Hence, bounding $\langle|\nabla\psi_T(0)|^2\rangle$ by [10, (3.27) & (3.29)], that is

$$\langle|\nabla\psi_T|^2\rangle\;\lesssim\;\left|\begin{array}{rll}\sqrt{T} & \text{if} & d=3,\\ \ln T & \text{if} & d=4,\\ 1 & \text{if} & d>4,\end{array}\right.$$

and computing the integral yield

$$\begin{aligned}&\int_{\mathbb{Z}^d}\big\langle\big(|\nabla\psi_T(z)|^2+\nu_d(T)^2(1+|\nabla\phi_{2T}(z)|^2)\big)G_T(0,e)^2\big\rangle\,\mathrm{d}z\\ \lesssim\;&(\langle|\nabla\psi_T(0)|^2\rangle+\nu_d(T)^2)\int_{\mathbb{Z}^d}(1+|z|)^{2(2-d)}\min\{1,\sqrt{T}|z|^{-1}\}^8\mathrm{d}z\\ \lesssim\;&1+\left|\begin{array}{rll}(\sqrt{T}+\sqrt{T}^2)\sqrt{T} & \text{if} & d=3,\\ \ln^2 T & \text{if} & d=4,\\ 1 & \text{if} & d>4,\end{array}\right.\end{aligned}$$

which yields (A.5).

Step 2. Proof of (A.6).
We first replace the Green's function by the deterministic pointwise estimate of [10, Lemma 4] for $d>2$:

$$\begin{aligned}&\int_{\mathbb{Z}^d}\left\langle(1+|\nabla\phi_{2T}(z)|^2)\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}G_T(0,w)G_T(0,w')|G_T(e,w)||G_T(e,w')|\mathrm{d}w\mathrm{d}w'\right\rangle\mathrm{d}z\\ \lesssim\;&\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}(1+|w|)^{2-d}(1+|w'|)^{2-d}\min\{1,\sqrt{T}|w|^{-1}\}^4\min\{1,\sqrt{T}|w'|^{-1}\}^4\\ &\times\int_{\mathbb{Z}^d}\big\langle(1+|\nabla\phi_{2T}(z)|^2)|G_T(e,w)||G_T(e,w')|\big\rangle\,\mathrm{d}z\mathrm{d}w\mathrm{d}w'.\end{aligned}\tag{A.7}$$

We then deal with the inner integral, and appeal to the Meyers' estimate of [9, Lemma 2.9] and the bounds $\langle|\phi_T|^q\rangle\lesssim 1$ (for all $q>0$) of [9, Proposition 2.1] on the moments of the modified correctors for $d>2$. We let $p>2$ be the Meyers' exponent. By Hölder's inequality in probability with exponents $((p-2)/p, 2/p)$, Cauchy-Schwarz' inequality, and stationarity of ∇G_T, we have

$$\begin{aligned}
&\int_{\mathbb{Z}^d} \left\langle (1+|\nabla\phi_{2T}(z)|^2)|G_T(e,w)||G_T(e,w')|\right\rangle \mathrm{d}z \\
\lesssim\ & \int_{\mathbb{Z}^d} \left\langle 1+|\nabla\phi_{2T}(z)|^{2p/(p-2)}\right\rangle \langle |G_T(e,w)|^p\rangle^{1/p} \left\langle |G_T(e,w')|^p\right\rangle^{1/p} \mathrm{d}z \\
\le\ & \int_{\mathbb{Z}^d} \left\langle 1+|\nabla\phi_{2T}(z)|^{2p/(p-2)}\right\rangle \langle |\nabla_z G_T(z-w,0)|^p\rangle^{1/p} \left\langle |\nabla_z G_T(z-w',0)|^p\right\rangle^{1/p} \mathrm{d}z \\
\lesssim\ & \int_{\mathbb{Z}^d} \langle |\nabla_z G_T(z-w,0)|^p\rangle^{1/p} \left\langle |\nabla_z G_T(z-w',0)|^p\right\rangle^{1/p} \mathrm{d}z. \qquad (A.8)
\end{aligned}$$

The combination of (A.7) & (A.8) with [9, Lemma 2.9] thus yields

$$\begin{aligned}
&\int_{\mathbb{Z}^d} \left\langle (1+|\nabla\phi_{2T}(z)|^2) \int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} G_T(0,w)G_T(0,w')|G_T(e,w)||G_T(e,w')|\mathrm{d}w\mathrm{d}w' \right\rangle \mathrm{d}z \\
\lesssim\ & \int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} g_T(|w|)g_T(|w'|)h_T(z-w)h_T(z-w')\mathrm{d}z\mathrm{d}w\mathrm{d}w',
\end{aligned}$$

where $g_T(t) = (1+t)^{2-d}\min\{1,\sqrt{T}t^{-1}\}^4$, and $h_T(w) = \langle |\nabla_z G_T(w,0)|^p\rangle^{1/p}$ is such that: for $R \sim 1$,

$$\int_{|x|\le R} h_T(x)^2 \lesssim 1,$$

and for all $R \gg 1$ and all $j \in \mathbb{N}$,

$$\int_{2^jR\le|x|<2^{j+1}R} h_T(x)^2\mathrm{d}x \lesssim (2^jR)^{d-2(d-1)}.$$

We are now in position to apply Lemma 6 below, which proves (A.6). This concludes the proof of Lemma 5.

Lemma 6 (double convolution estimate)**.** *Let $d>2$, $T \gg 1$, and let $g_T : \mathbb{R}^+ \to \mathbb{R}^+, t \mapsto (1+t)^{2-d}\min\{1,\sqrt{T}t^{-1}\}^4$, and $h_T : \mathbb{Z}^d \to \mathbb{R}^+$ be such that*

$$\int_{|x|\le R} h_T(x)^2 \lesssim 1,$$

and for all $R \gg 1$ and all $j \in \mathbb{N}$,

$$\int_{2^jR\le|x|<2^{j+1}R} h_T(x)^2\mathrm{d}x \lesssim (2^jR)^{d-2(d-1)}.$$

Then,

$$\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} g_T(|w|)g_T(|w'|)h_T(z-w)h_T(z-w')\mathrm{d}z\mathrm{d}w\mathrm{d}w'$$

$$\lesssim 1 + \left|\begin{array}{rl} T^{3-d/2} & \text{if } \ 5\ge d>2, \\ \ln T & \text{if } \ d=6, \\ 1 & \text{if } \ d>6. \end{array}\right. \qquad (A.9)$$

Proof of Lemma 6. The proof of (A.9) is made technical because the bound

$$h_T(z) \text{ “} \lesssim \text{” } (1+|z|)^{1-d}$$

does hold integrated on dyadic annuli, but *not pointwise.* In line with the bounds on h_T, we prove the claim by using a doubly dyadic decomposition of $\mathbb{Z}^d \times \mathbb{Z}^d$ combined with the results of [9, Proof of Lemma 2.10, Steps 1, 2 & 4], that we recall for the reader's convenience for $d > 2$: there exists $R \sim 1$ such that for all $i \in \mathbb{N}$,

$$\int_{2^iR<|x|\le 2^{i+1}R} \int_{\mathbb{Z}^d} h_T(z)h_T(z-x)\mathrm{d}z\mathrm{d}x \quad \lesssim \quad (2^iR)^2 \tag{A.10}$$

$$\int_{|x|\le 4R} \int_{\mathbb{Z}^d} h_T(z)h_T(z-x)\mathrm{d}z\mathrm{d}x \lesssim 1. \tag{A.11}$$

We first use the symmetry with respect to w and w' to restrict the set of integration to $|w'| \ge |w|$, and we make a change of variables

$$\begin{aligned}
&\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} g_T(|w|)g_T(|w'|)h_T(z-w)h_T(z-w')\mathrm{d}z\mathrm{d}w'\mathrm{d}w \\
&\quad\le\quad 2\int_{w\in\mathbb{Z}^d}\int_{w'\in\mathbb{Z}^d,|w'|\ge|w|}\int_{\mathbb{Z}^d} g_T(|w|)g_T(|w'|)h_T(z-w)h_T(z-w')\mathrm{d}z\mathrm{d}w'\mathrm{d}w \\
&\quad=\quad 2\int_{w\in\mathbb{Z}^d}\int_{w-w'\in\mathbb{Z}^d,|w'|\ge|w|}\int_{\mathbb{Z}^d} g_T(|w|)g_T(|w'|)h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w,
\end{aligned}$$

followed by the associated doubly dyadic decomposition of space

$$\begin{aligned}
&\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} g_T(|w|)g_T(|w'|)h_T(z-w)h_T(z-w')\mathrm{d}z\mathrm{d}w'\mathrm{d}w \\
&\quad\lesssim\quad \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\int_{2^iR<|w|\le 2^{i+1}R}\int_{\substack{2^jR\,<\,|w-w'|\,\le\,2^{j+1}R\\ |w'|\,\ge\,|w|}} g_T(|w|)g_T(|w'|) \\
&\qquad\qquad\qquad\qquad \times\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w \qquad (\mathrm{A.12}) \\
&\qquad + \sum_{j\in\mathbb{N}}\int_{|w|\le R}\int_{\substack{2^jR\,<\,|w-w'|\,\le\,2^{j+1}R\\ |w'|\,\ge\,|w|}} g_T(|w|)g_T(|w'|) \\
&\qquad\qquad\qquad\qquad \times\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w \qquad (\mathrm{A.13}) \\
&\qquad + \sum_{i\in\mathbb{N}}\int_{2^iR<|w|\le 2^{i+1}R}\int_{\substack{|w-w'|\,\le\,R\\ |w'|\,\ge\,|w|}} g_T(|w|)g_T(|w'|) \\
&\qquad\qquad\qquad\qquad \times\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w \qquad (\mathrm{A.14}) \\
&\qquad + \int_{|w|\le R}\int_{\substack{|w-w'|\,\le\,R\\ |w'|\,\ge\,|w|}} g_T(|w|)g_T(|w'|) \\
&\qquad\qquad\qquad\qquad \times\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w, \qquad (\mathrm{A.15})
\end{aligned}$$

where $R \sim 1$ is as above. We begin with the last term (A.15) of the sum, and appeal to (A.11), and the fact that $\sup_{\mathbb{R}^+} g_T \lesssim 1$:

$$\begin{aligned}
&\int_{|w|\le R}\int_{\substack{|w-w'|\le R\\ |w'|\ge|w|}} g_T(|w|)g_T(|w'|)\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w\\
&\lesssim \int_{|w-w'|\le R}\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\\
&\lesssim 1. \qquad \text{(A.16)}
\end{aligned}$$

We continue with (A.14). By monotonicity of g_T, from the inequality $|w'|\ge|w|$ we infer that $g_T(|w'|)\le g_T(|w|)$. Hence, using (A.11) and the definition of g_T, we have

$$\begin{aligned}
&\sum_{i\in\mathbb{N}}\int_{2^iR<|w|\le 2^{i+1}R}\int_{\substack{|w-w'|\le R\\ |w'|\ge|w|}} g_T(|w|)g_T(|w'|)\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w\\
&\le \sum_{i\in\mathbb{N}}\int_{2^iR<|w|\le 2^{i+1}R}\int_{|w-w'|\le R} g_T(|w|)^2\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w\\
&= \left(\sum_{i\in\mathbb{N}}\int_{2^iR<|w|\le 2^{i+1}R} g_T(|w|)^2\mathrm{d}w\right)\left(\int_{|\tilde w'|\le R}\int_{\mathbb{Z}^d} h_T(z)h_T(z-\tilde w')\mathrm{d}z\mathrm{d}\tilde w'\right)\\
&\overset{\text{(A.11)}}{\lesssim} \int_{\mathbb{Z}^d}\frac{1}{1+|w|^{2(d-2)}}\min\{1,\sqrt{T}|w|^{-1}\}^8\mathrm{d}w\\
&\lesssim 1+\left|\begin{array}{rl}\sqrt{T} & \text{if} \quad d=3,\\ \ln T & \text{if} \quad d=4,\\ 1 & \text{if} \quad d>4.\end{array}\right. \qquad \text{(A.17)}
\end{aligned}$$

For (A.13) we note that $|w|\le R$ and $2^jR<|w-w'|\le 2^{j+1}R$ imply that $|w'|\ge 2^{j-1}R$ and therefore

$$g_T(|w'|)\le g_T(2^{j-1}R) = (2^{j-1}R)^{2-d}\min\{1,\sqrt{T}(2^{j-1}R)^{-1}\}^4.$$

Bounding $g_T(|w|)$ by $\sup_{\mathbb{R}^+} g_T\lesssim 1$ and appealing to (A.10), we thus have

$$\begin{aligned}
&\sum_{j\in\mathbb{N}}\int_{|w|\le R}\int_{\substack{2^jR<|w-w'|\le 2^{j+1}R\\ |w'|\ge|w|}} g_T(|w|)g_T(|w'|)\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w\\
&\lesssim \int_{|w|\le R}\sum_{j\in\mathbb{N}} 2^{j(2-d)}R^{2-d}\min\{1,\sqrt{T}(2^jR)^{-1}\}^4\\
&\qquad\times\int_{2^jR<|w-w'|\le 2^{j+1}R}\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w\\
&\lesssim R^2\sum_{j\in\mathbb{N}} 2^{j(2-d)}\min\{1,\sqrt{T}(2^jR)^{-1}\}^4\int_{2^jR<|\tilde w'|\le 2^{j+1}R}\int_{\mathbb{Z}^d} h_T(z)h_T(z-\tilde w')\mathrm{d}z\mathrm{d}\tilde w'\\
&\overset{\text{(A.10)}}{\lesssim} R^2\sum_{j\in\mathbb{N}} 2^{j(2-d)}\min\{1,\sqrt{T}(2^jR)^{-1}\}^4\times(2^jR)^2\\
&\overset{R\sim 1}{\lesssim} 1+\left|\begin{array}{rl}\sqrt{T} & \text{if} \quad d=3,\\ \ln T & \text{if} \quad d=4,\\ 1 & \text{if} \quad d>4.\end{array}\right. \qquad \text{(A.18)}
\end{aligned}$$

The dominant term is (A.12). We split the double sum into two parts according to the range of i and j: $j \leq i$, and $j > i$.

For $j \leq i$, we use the monotonicity of g_T in the form of the inequality $g_T(|w'|) \leq g_T(|w|)$ for $|w| \leq |w'|$, and of

$$g_T(|w|) \;\leq\; (2^i R)^{2-d} \min\{1, \sqrt{T}(2^i R)^{-1}\}^4$$

for all $2^i R < |w| \leq 2^{i+1} R$. In particular, using (A.10), this yields

$$\begin{aligned}
&\sum_{i\in\mathbb{N}} \sum_{j\leq i} \int_{2^i R<|w|\leq 2^{i+1}R} \int_{\substack{2^j R \,<\, |w\,-\,w'| \,\leq\, 2^{j+1}R \\ |w'| \,\geq\, |w|}} g_T(|w|) g_T(|w'|) \\
&\qquad\qquad \times \int_{\mathbb{Z}^d} h_T(z) h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w \\
\leq\;& \sum_{i\in\mathbb{N}} \int_{2^i R<|w|\leq 2^{i+1}R} g_T(|w|)^2 \sum_{j\leq i} \int_{2^j R<|w-w'|\leq 2^{j+1}R} \\
&\qquad\qquad \times \int_{\mathbb{Z}^d} h_T(z) h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w \\
\lesssim\;& \sum_{i\in\mathbb{N}} (2^i R)^d (2^i R)^{2(2-d)} \min\{1, \sqrt{T}(2^i R)^{-1}\}^8 \\
&\qquad\qquad \sum_{j\leq i} \int_{2^j R<|\tilde w'|\leq 2^{j+1}R} \int_{\mathbb{Z}^d} h_T(z) h_T(z-\tilde w')\mathrm{d}z\mathrm{d}\tilde w' \\
\overset{(A.10)}{\lesssim}\;& \sum_{i\in\mathbb{N}} (2^i R)^{d+2(2-d)} \min\{1, \sqrt{T}(2^i R)^{-1}\}^8 \sum_{j\leq i} (2^j R)^2 \\
\lesssim\;& \sum_{i\in\mathbb{N}} (2^i R)^{d+2(2-d)} \min\{1, \sqrt{T}(2^i R)^{-1}\}^8 (2^i R)^2 \\
=\;& R^{6-d} \sum_{i\in\mathbb{N}} (2^i)^{6-d} \min\{1, \sqrt{T}(2^i R)^{-1}\}^8 \\
\overset{R\sim 1}{\lesssim}\;& 1 + \left| \begin{array}{rl} \sqrt{T}^3 & \text{if} \quad d=3, \\ T & \text{if} \quad d=4, \\ \sqrt{T} & \text{if} \quad d=5, \\ \ln T & \text{if} \quad d=6, \\ 1 & \text{if} \quad d>6. \end{array} \right.
\end{aligned} \tag{A.19}$$

We now treat the last term of the sum, that is those integers i, j such that $j > i$. From the triangle inequality in the form of $|w-w'| - |w| \leq |w'|$, we deduce that $|w'| \geq 2^{j-1}R$ for all $2^i R < |w| \leq 2^{i+1}R$ and $2^j R < |w-w'| \leq 2^{j+1}R$ with $j > i$, so that by monotonicity of g_T,

$$\begin{aligned}
g_T(|w|) \;&\leq\; (2^i R)^{2-d}, \\
g_T(|w'|) \;&\leq\; (2^{j-1} R)^{2-d} \min\{1, \sqrt{T}(2^{j-1} R)^{-1}\}^4.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{i\in\mathbb{N}}\sum_{j>i}\int_{2^iR<|w|\le 2^{i+1}R}\int_{\substack{2^jR\,<\,|w-w'|\,\le\,2^{j+1}R\\ |w'|\,\ge\,|w|}} g_T(|w|)g_T(|w'|)\\
&\qquad\qquad\qquad\times\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w\\
\lesssim\ &\sum_{i\in\mathbb{N}}(2^iR)^d(2^iR)^{2-d}\sum_{j>i}(2^{j-1}R)^{2-d}\min\{1,\sqrt{T}(2^{j-1}R)^{-1}\}^4\\
&\qquad\qquad\qquad\times\int_{2^jR<|\tilde w'|\le 2^{j+1}R}\int_{\mathbb{Z}^d} h_T(z)h_T(z-\tilde w')\mathrm{d}z\mathrm{d}\tilde w'\\
=\ &\sum_{j\in\mathbb{N}}\Bigg((2^jR)^{2-d}\min\{1,\sqrt{T}(2^jR)^{-1}\}^4\\
&\qquad\times\int_{2^{j+1}R<|\tilde w'|\le 2^{j+2}R}\int_{\mathbb{Z}^d} h_T(z)h_T(z-\tilde w')\mathrm{d}z\mathrm{d}\tilde w'\Bigg)\sum_{i\le j}(2^iR)^2
\end{aligned}$$

by switching the order of the sums and making the change of variables $j\rightsquigarrow j+1$. Computing the sum in i and using (A.10), this yields

$$\begin{aligned}
&\sum_{i\in\mathbb{N}}\sum_{j>i}\int_{2^iR<|w|\le 2^{i+1}R}\int_{\substack{2^jR\,<\,|w-w'|\,\le\,2^{j+1}R\\ |w'|\,\ge\,|w|}} g_T(|w|)g_T(|w'|)\\
&\qquad\qquad\qquad\times\int_{\mathbb{Z}^d} h_T(z)h_T(z-(w-w'))\mathrm{d}z\mathrm{d}w'\mathrm{d}w\\
\overset{(A.10)}{\lesssim}\ &\sum_{j\in\mathbb{N}}(2^jR)^{2-d}\min\{1,\sqrt{T}(2^jR)^{-1}\}^4\times(2^jR)^2\times(2^jR)^2\\
=\ &R^{6-d}\sum_{j\in\mathbb{N}}2^{j(6-d)}\min\{1,\sqrt{T}(2^jR)^{-1}\}^4\\
\overset{R\sim 1}{\lesssim}\ &1+\left|\begin{array}{rl}\sqrt{T}^3 & \text{if}\quad d=3,\\ T & \text{if}\quad d=4,\\ \sqrt{T} & \text{if}\quad d=5,\\ \ln T & \text{if}\quad d=6,\\ 1 & \text{if}\quad d>6.\end{array}\right.
\end{aligned}\tag{A.20}$$

Estimate (A.9) then follows from the combination of (A.12)–(A.15) with (A.16)–(A.20). □

Appendix B. Numerical tests in the discrete periodic case

Numerical tests of [7] have confirmed the sharpness of Theorem 2 for the approximation $A_{\mu,1,R,L}$ on a discrete periodic example. In the present work, we consider the same discrete example, and numerically check the asymptotic convergence of $A_{\mu,2,R,L}$ to $A_{\rm hom}$. As expected, the systematic error is reduced, and the limiting factor rapidly becomes the machine precision. The discrete corrector equation we consider is

$$-\nabla^*\cdot A(\xi+\nabla\phi)=0\qquad\text{in }\mathbb{Z}^2,$$

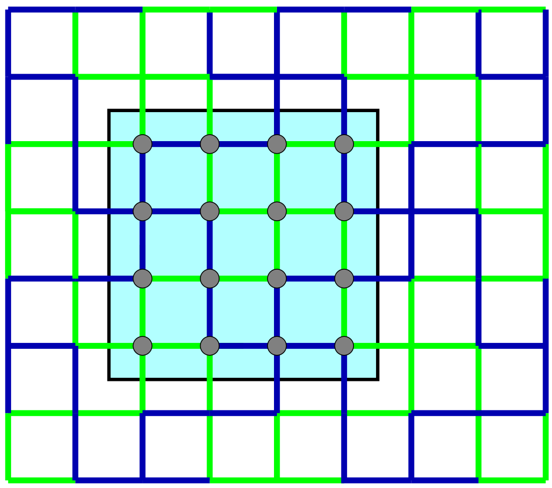

FIGURE 1. Periodic cell in the discrete case

where ∇ and ∇^* are as in (3.1), and

$$A(x) := \mathrm{diag}\left[a(x, x + \mathbf{e}_1), a(x, x + \mathbf{e}_2)\right].$$

The matrix A is $[0, 4)^2$-periodic, and sketched on a periodic cell on Figure 1. In the example considered, $a(x, x + \mathbf{e}_1)$ and $a(x, x + \mathbf{e}_2)$ represent the conductivities 1 or 100 of the horizontal edge $[x, x+\mathbf{e}_1]$ and the vertical edge $[x, x+\mathbf{e}_2]$ respectively, according to the colors on Figure 1. The homogenization theory for such discrete elliptic operators is similar to the continuous case (see for instance [19] in the two-dimensional case dealt with here). By symmetry arguments, the homogenized matrix associated with A is a multiple of the identity. It can be evaluated numerically (note that we do not make any other error than the machine precision). Its numerical value is $A_{\mathrm{hom}} = 26.240099009901\ldots$. To illustrate Theorem 2 in its discrete version (which is similar, see [7] for related arguments), we have conducted a series of tests for $A_{\mu,2}$. In particular, we have taken $\mu \sim R^{-3/2}$, $L = R/3$, and a filter of infinite order. In this case, the convergence rate is expected to be of order 3 for $A_{\mu,1}$, and of order 6 for $A_{\mu,2}$. This is indeed the case, as can be seen on Figure 2, where R denotes the number of periodic cells and ranges from 6 to 400 (that is $\log(R)$ up to 2.6).

REFERENCES

[1] M.T. Barlow, and J.-D. Deuschel. Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.*, 38(1), 234-276, 2010.

[2] N. Berger, and M. Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields*, 137(1-2), 83–120, 2007.

[3] M. Biskup, and T.M. Prescott. Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.*, 12(49), 1323–1348, 2007.

[4] A. Bourgeat and A. Piatnitski. Approximations of effective coefficients in stochastic homogenization. *Ann. I. H. Poincaré*, 40:153–165, 2005.

[5] P. Caputo and D. Ioffe. Finite volume approximation of the effective diffusion matrix: the case of independent bond disorder. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(3):505–525, 2003.

[6] A. De Masi, P.A. Ferrari, S. Goldstein, and W.D. Wick. An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Statist. Phys.*, 55(3-4), 787–855, 1989.

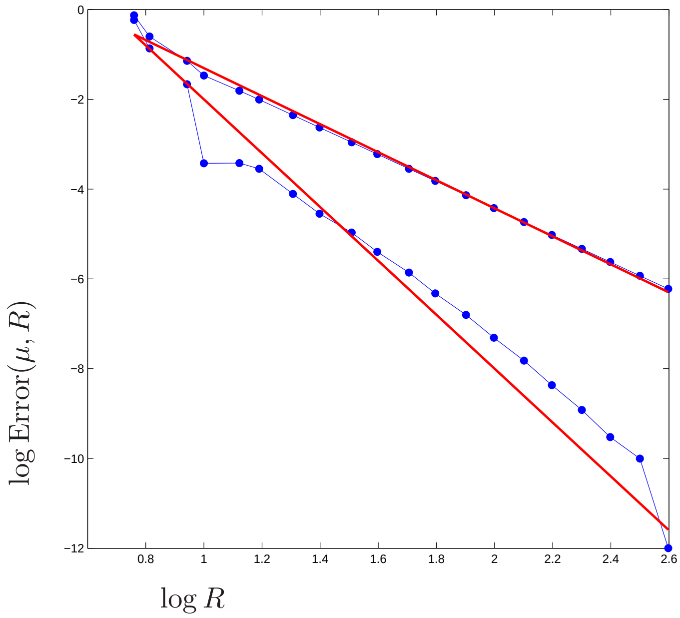

FIGURE 2. Absolute error in log scale for $\mu = 250\, R^{-3/2}$, $A_{\mu,1,R,L}$ (slope -3.1) and $A_{\mu,2,R,L}$ (slope -6), filter of infinite order.

[7] A. Gloria. Reduction of the resonance error - Part 1: Approximation of homogenized coefficients. *Math. Models Methods Appl. Sci.*, to appear. Preprint available at http://hal.archives-ouvertes.fr/inria-00457159/en/.

[8] A. Gloria. Numerical approximation of effective coefficients in stochastic homogenization of discrete elliptic equations. Submitted.

[9] A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.*, 39(3), 779–856, 2011.

[10] A. Gloria and F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.*, to appear. Preprint available at http://hal.archives-ouvertes.fr/inria-00457020/en/.

[11] C. Kipnis and S.R.S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Commun. Math. Phys.*, 104:1–19, 1986.

[12] S.M. Kozlov. Averaging of difference schemes. *Math. USSR Sbornik*, 57(2):351–369, 1987.

[13] R. Künnemann. The diffusion limit for reversible jump processes on $\mathbb{Z}^d$ with ergodic random bond conductivities. *Commun. Math. Phys.*, 90:27–68, 1983.

[14] P. Mathieu. Quenched invariance principles for random walks with random conductances. *J. Statist. Phys.*, 130(5), 1025–1046, 2008.

[15] P. Mathieu and A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. A*, 463(2085), 2287–2307, 2007.

[16] J.-C. Mourrat. Variance decay for functionals of the environment viewed by the particle. *Ann. Inst. H. Poincaré Probab. Statist.*, 47(11), 294–327, 2011.

[17] G.C. Papanicolaou and S.R.S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam, 1981.

[18] V. Sidoravicius and A.-S. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields*, 129(2), 219–244, 2004.

[19] M. Vogelius. A homogenization result for planar, polygonal networks. *RAIRO Modél. Math. Anal. Numér.*, 25(4):483–514, 1991.

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