On removable edges in 3-connected cubic graphs

Jean-Luc Fouquet, Henri Thuillier

ON REMOVABLE EDGES IN 3−CONNECTED CUBIC GRAPHS

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Abstract. A removable edge in a 3−connected cubic graph G is an edge $e = uv$ such that the cubic graph obtained from $G \setminus \{u, v\}$ by adding an edge between the two neighbours of u distinct from v and an edge between the two neighbours of v disctinct from u is still 3−connected. Li and Wu [3] showed that a spanning tree in a 3−connected cubic graph avoids at least two removable edges, and Kang, Li and Wu [4] showed that a spanning tree contains at least two removable edges. We show here how to obtain these results easily from the structure of the sets of non removable edges and we give a characterization of the extremal graphs for these two results.

1. Introduction

In 1961 Tutte [5] gave a structural characterization for 3−connected graphs by using the existence of contractible or removable edges. A *cubic graph* is a simple 3-regular graph. From now on, all graphs considered here are cubic graphs. An edge e of o 3−connected cubic graph G is said to be *removable* when the cubic graph obtained from G by the following operations remains to be 3−connected.

- Delete u and v from $V(G)$ and their incident edges from $E(G)$
- Add one edge between the two neighbours of u distinct from v as well as between the two neighbours of v distinct from u

An edge which is not removable is said to be *non removable.* The set of removable edges of G is denoted by $R(G)$ and the set of non removable edges is denoted by $N(G)$.

Conversely, we can get a new 3−connected cubic graph from a 3−connected cubic graph G by *inserting* one edge between two existing edges. More formally, let uv and $u^{'}v^{'}$ be two edges of a 3−connected cubic graph G, we get a new 3−connected cubic graph $G^{'}$ when the three following operations are performed.

- Delete uv and $u^{'}v^{'}$ from $E(G)$
- Add two new adjacent vertices x and y to $V(G)$
- Join x to u and v and y to $u^{'}$ and $v^{'}$.

We shall say that we have proceeded to the *insertion* (of the edge xy). Obviously the new edge xy is removable in the obtained graph.

Li and Wu [3] showed that a spanning tree in a 3−connected cubic graph avoids at least two removable edges:

Theorem 1.1. [3] *Let G be a 3−connected cubic graph with at least six vertices. Then every spanning tree of G avoids at least two removable edges.*

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Kang, Li and Wu [4] showed that a spanning tree contains at least two removable edges:

Theorem 1.2. [4] *Let G be a $3-$connected cubic graph with at least six vertices. Then every spanning tree of G contains at least two removable edges.*

We shall show in Section 3 how to obtain these results easily from the structure of the set of non removable edges (Corollaries 3.4 and 3.7) and we give a characterization of the extremal graphs for these two theorems. More precisely, we shall exhibit two infinite families of $3-$connected cubic graphs, the PR-graphs and the $3T$-graphs (defined below in Subsection 1.2) and we shall prove that a $3-$connected cubic graph having a spanning tree avoiding exactly two removable edges is a PR-graph (Corollary 3.6), and that a $3-$connected cubic graph having a spanning tree containing exactly two removable edges is a $3T$-graph (Corollary 3.8).

1.1. **Edge cut.** Let $\{V_1, V_2\}$ be a partition of the vertex set $V(G)$ of G. The set F of edges joining V_1 to V_2 denoted by (V_1, V_2) is an *edge cut* and the partition $\{V_1, V_2\}$ of $V(G)$ is the *associated partition.* An edge cut F of k edges is a $k-$*edge cut.* An edge cut F is *minimal* if no proper subset of F is an edge cut, it is *trivial* if it is minimal and one component of $G \setminus F$ is a single vertex.

Obviously, a $3-$connected cubic graph has no $2-$edge cut. Moreover, any non trivial $3-$edge cut F is a matching of three edges and the edges of this edge cut are contained in $N(G)$ (non removable edges). By deleting the edges of F, we get two connected graphs (the subgraphs $G[V_1]$ and $G[V_2]$ of G induced respectively by V_1 and V_2) and we remark that these two subgraphs are $2-$connected. By contracting $G[V_2]$ in a single new vertex u and $G[V_1]$ in a single new vertex v, we get two smaller $3-$connected cubic graphs G_1 and G_2. Conversely, let G_1 and G_2 be two $3-$connected cubic graphs and $u \in V(G_1)$, $v \in V(G_2)$ with $N_u = \{u_1, u_2, u_3\}$ and $N_v = \{v_1, v_2, v_3\}$. We construct a new $3-$connected cubic graph G where $V(G) = (V(G_1 \setminus \{u\}) \cup (V(G_2 \setminus \{v\})$ and $E(G) = (E(G_1) \setminus \{uu_1, uu_2, uu_3\}) \cup (E(G_2) \setminus \{vv_1, vv_2, vv_3\}) \cup \{u_1v_1, u_2v_2, u_3v_3\}$ having $\{u_1v_1, u_2v_2, u_3v_3\}$ as a non trivial $3-$edge cut (note that G may contain other non trivial $3-$edge cuts).

1.2. **Two special families of $3-$connected cubic graphs.**

1.2.1. *The family of PR-graphs.* Let $PR_{0,0}$ be the $3-$connected cubic graph on six vertices formed by two triangles joined by a matching of three edges. Let us remark that these three edges are not removable. Starting from $PR_{0,0}$ we proceed to successive insertions between edges of non trivial 3-edge cuts or insertions of claws (by adding three vertices of degree 2 on the edges of a non trivial 3-edge cut and joining these 3 vertices to a fourth vertex). To proceed to an insertion of an edge, we choose two edges of a $3-$edge cut F and we insert an edge between these two chosen edges. To proceed to an insertion of a claw, we proceed first to the insertion of an edge as previously (let xy be the new edge obtained) and we insert a new edge between xy and the last edge of the considered $3-$edge cut F. Let k_1 and k_2 be two integers such that $k_1 \geq 0, k_2 \geq 0$ and $k_1 + k_2 \geq 1$. A cubic graph obtained from $PR_{0,0}$ by k_1 insertions of edges and k_2 insertions of claws is said to be *a graph of type* PR_{k_1,k_2} (or simply, *a* PR_{k_1,k_2}). More precisely, a graph of type PR_{k_1+1,k_2} is obtained from a PR_{k_1,k_2} by insertion of an edge and a graph of type PR_{k_1,k_2+1} is obtained from a PR_{k_1,k_2} by insertion of a claw. It must be clear that given k_1 and k_2, we may obtain several non isomorphic cubic graphs of type

PR_{k_1,k_2}. Since the operation of insertion of an edge preserves the 3-connectivity, it is easy to see that a PR_{k_1,k_2} is a $3-$connected cubic graph. A $PR-graph$ is a graph G such that there exist integers k_1 and k_2 and G is of type PR_{k_1,k_2}. In Figure 1, we give example of a graph of type $PR_{2,1}$ and a graph of type $PR_{1,2}$.

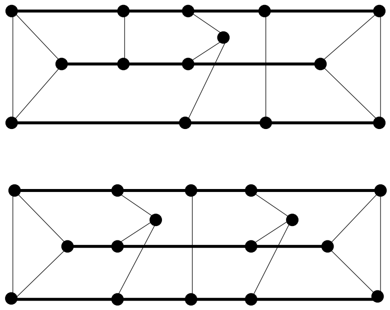

FIGURE 1. Type $PR_{2,1}$ and Type $PR_{1,2}$

It can be easily verified that the only non removable edges of a graph of type PR_{k_1,k_2} are the edges of the three disjoint paths P_1, P_2 and P_3 joining the two triangles (drawn in bold in Figure 1). Then a graph G of type PR_{k_1,k_2} has $n = 2k_1+4k_2+6$ vertices and it verifies $|R(G)| = k_1+3k_2+6$ and $|N(G)| = 2k_1+3k_2+3$.

1.2.2. *The family of 3T-graphs.* A *fundamental* $3T_{k+2}$ (with $k \geq 0$) is a cubic graph obtained from three isomorphic trees T_1, T_2 and T_3 of maximum degree 3 and no vertex of degree 2 with $k+2$ vertices of degree one and k vertices of degree three each. Each triple of pendent vertices (one in each tree) mapped by the isomorphism are joined by a triangle. It must be clear that a fundamental $3T_{k+2}$ is 3-connected.

A $p-extended$ $3T_{k+2}$ is a cubic graph obtained from a fundamental $3T_{k+2}$ by insertion of p edges. The family of $p-$extended $3T_{k+2}$ shall be denoted by $3T_{k+2,p}$. As above, to proceed to the insertion of an edge, we choose a non trivial $3-$edge cut F and two distinct edges of F in the graph in construction. Note that given $k \geq 0$ and $p \geq 2$, the family $3T_{k+2,p}$ may contain several non isomorphic cubic graphs. Since the operation of insertion of an edge preserves the 3-connectivity, a $p-$extended $3T_{k+2}$ is a $3-$connected cubic graph. In Figure 2 we give a fundamental $3T_4$ and in Figure 3 a $3-$extended $3T_4$.

A fundamental $3T_{k+2}$ can be seen as a $0-$extended $3T_{k+2}$. A $3T-graph$ is a graph that belongs to the union $\bigcup_{k\geq 0, p\geq 0} 3T_{k+2,p}$.

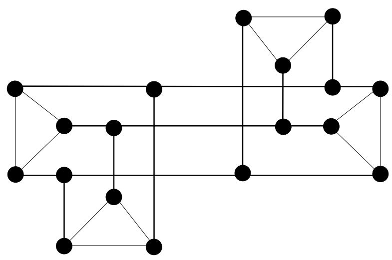

FIGURE 2. A fundamental $3T_4$

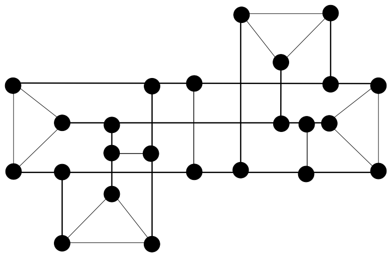

FIGURE 3. A $3-$extended $3T_4$

Remark 1.3. A $p-$extended $3T_{k+2}$, G, is obtained from a fundamental $3T_{k+2}$, H, by insertion of p edges. The three isomorphic trees of H (T_1, T_2 and T_3 in our description) are transformed into induced trees of G (still denoted T_1, T_2 and T_3). The edges of these three trees are the only non removable edges of G. Then, G has $n = 6k + 2p + 6$ vertices and it verifies $|R(G)| = 3k + p + 6 = \frac{n+6}{2}$ and $|N(G)| = 6k + 2p + 3$.

2. Some technical lemmas

Throughout this section G is a $3-$connected cubic graph and $F = \{e, f, g\}$ and $F^{'} = \{e^{'}, f^{'}, g^{'}\}$ are two distinct non trivial $3-$edge cuts of G. The two associated partitions are $\{V_1, V_2\}$ and $\{V_1^{'}, V_2^{'}\}$. Moreover these two edge cuts partition the vertex set of G in four sets $V_1 \cap V_1^{'}$, $V_1 \cap V_2^{'}$, $V_2 \cap V_1^{'}$ and $V_2 \cap V_2^{'}$.

Lemma 2.1. *If a cycle C intersects F then C contains exactly two edges of F*

Proof An edge cut is a so called co-cycle and it is well known that the intersection of a cycle and a co-cycle is an even set. The result follows. □

Lemma 2.2. *One of the two subgraphs $G[V_1]$ or $G[V_2]$ either contains the three edges of F' or contains exactly two edges of F' and the remaining edge of F' is an edge of F.*

Proof Without loss of generality, suppose that the subgraph $G[V_2]$ contains an edge of $F^{'}$. Since this subgraph is $2-$connected it has no isthmus, and hence $|F^{'} \cap E(G[V_2])| \neq 1$. Then, either $G[V_2]$ contains the three edges of $F^{'}$ or it contains exactly two edges of $F^{'}$. Since as above $|F^{'} \cap E(G[V_1])| \neq 1$, the remaining edge of $F^{'}$ must be an edge of F. □

Lemma 2.3. *There is a partition of $V(G)$ in three sets A, B and C such that*

- *A and B are connected by the three edges of F and B and C are connected by the three edges of $F^{'}$*
- *or A and B are connected by two edges of F, B and C are connected by two edges of $F^{'}$ while the edge of $F \cap F^{'}$ connects A and C.*

Proof There exists an edge of the 3-edge cut $F^{'}$ that is contained in $G[V_1]$ or in $G[V_2]$ (say in $G[V_2]$). By Lemma 2.2, either $G[V_2]$ contains the three edges of F' ($F \cap F^{'} = \emptyset$) or it contains exactly two edges of F' and the remaining edge of F' is also an edge of F ($|F \cap F^{'}| = 1$). It is easy to see that, one of the sets $V_1^{'}$ or $V_2^{'}$ (say $V_1^{'}$) contains the whole set V_1, so $V_1 \cap V_2^{'} = \emptyset$. We let $A = V_1 \cap V_1^{'}$ (hence $A = V_1$), $B = V_2 \cap V_1^{'}$ and $C = V_2 \cap V_2^{'}$ (hence $C = V_2^{'}$) and we can check that the first item is verified when $F \cap F^{'} = \emptyset$ while the second item is verified when $|F \cap F^{'}| = 1$. $\square$

An *end* 3-*edge cut* is a $3-$edge cut such that every edge of the subgraph induced on one of the two sets of the associated partition is removable. This subgraph without any non removable edge will be called an *extremity* (it may happens that the two sets of the associated partition are extremities). Let us remark that an extremity of a $3-$connected cubic graph G is a $2-$connected induced subgraph of G.

Lemma 2.4. *Each set of the associated partition of any* $3-$*edge cut* F *contains an extremity.*

Proof If every edge of $G[V_2]$ is a removable edge then V_2 is an extremity. If $G[V_2]$ contains a non removable edge e then let F' be a $3-$edge cut containing e. Clearly, F' is disctinct from F. By Lemma 2.3, we have the partition $A = V_1 \cap V_1^{'} = V_1$, $B = V_2 \cap V_1^{'}$ and $C = V_2 \cap V_2^{'} = V_2^{'}$. We have thus obtained a refining of the partition $\{V_1, V_2\}$. If every edge of $G[V_2^{'}]$ is removable then $V_2^{'}$ is an extremity, otherwise we can proceed to a new refinement of $V_2^{'}$. Since the number of $3-$edge cuts is finite, we shall be left with an extremity in V_2. The same holds for V_1 and the Lemma follows. $\square$

Lemma 2.5. *Let* $P = u_1u_2 \dots u_k$ $(k \geq 3)$ *be a path contained in* $N(G)$ *and let* F *be a* $3-$*edge cut of* G. *Then* F *has at most one edge in* P.

Proof Assume to the contrary that there exists a 3-edge cut F containing two edges of P, u_iu_{i+1} and u_ju_{j+1} ($i \neq j$, $1 \leq i \leq k-2$, $i+1 \leq j \leq k-1$). Since F is a matching, the edge $u_{i+1}u_{i+2}$ is distinct from u_ju_{j+1}. Assume moreover that the subpath $P^{'} = u_{i+1}u_{i+2} \dots u_j$ of P does not contained the third edge of F. We can suppose that F has been chosen in such a way that the distance on P between u_iu_{i+1} and u_ju_{j+1} is as short as possible.

Let $F^{'}$ be a $3-$edge cut containing $u_{i+1}u_{i+2}$. The choice of F forces $F^{'}$ to have no other edge between u_iu_{i+1} and u_ju_{j+1}. We consider that u_{i+1} and u_j are in V_1 (hence, $P^{'}$ is a path in $G[V_1]$ and u_i and u_{j+1} are in V_2). Let Q be a path in $G[V_2]$ joining u_i to u_{j+1} and consider the cycle obtained by concatenation of u_iu_{i+1}, $P^{'}$, u_ju_{j+1} and Q. By Lemma 2.1, this cycle contains an edge e of $F^{'}$ distinct from $u_{i+1}u_{i+2}$. By the choice of F, this edge e must be on Q. We do not know the exact position of the third edge of F', but we are certain that at least one of the two 2-connected subgraphs $G[V_1]$ or $G[V_2]$ contains exactly one edge of F'. Hence $G[V_1]$ or $G[V_2]$ has an isthmus, a contradiction. $\square$

Lemma 2.6. *Let* $P = u_1u_2 \dots u_k$ $(k \geq 3)$ *be a path contained in* $N(G)$. *Then* P *is an induced path of* G.

Proof Assume to the contrary that u_iu_j is an edge of G ($i \neq j$, $1 \leq i \leq k-1$ $i+2 \leq j \leq k$). Then the concatenation of the subpath $P^{'}$ of P with ends u_i and u_j together with the edge u_iu_j gives a cycle of G. This cycle intersects a $3-$cut edge containing the edge $u_i u_{+1}$. By Lemma 2.1, a second edge of this $3-$edge cut must be contained in $P^{'}$, a contradiction with Lemma 2.5. □

3. On the set of non removable edges

Theorem 3.1. [1] *The subgraph of a $3-$connected cubic graph G induced by the set $N(G)$ of non removable edges is an induced forest with at least three trees. Each $3-$edge cut intersects three distinct trees of that forest.*

Proof Assume that the edge-induced subgraph on $N(G)$ (denoted also $N(G)$) contains a cycle C and let $e \in C$. By Lemma 2.1, any 3-edge cut containing e must intersect C at least twice. Then two edges of this 3-edge cut are contained in a path P of $N(G)$, a contradiction with Lemma 2.5. Hence, $N(G)$ is a forest as claimed and, by Lemma 2.6, it is clear that this forest is an induced forest.

Let F be a $3-$edge cut. If two edges of F are contained in the same tree of $N(G)$ then we can find a path contained in $N(G)$ joining these two edges, again a contradiction with Lemma 2.5. The theorem follows. □

Remark 3.2. Since $N(G)$ has at most $n-3$ edges (with $n = |V(G)|$), the graph G contains at least $\frac{n+6}{2}$ removable edges. By Remark 1.3, we see that the $3T$-graphs are extremal for these numbers. More precisely, we have proved in [2] that the family of $3T$-graphs is exactly the family of $3-$connected cubic graphs having the minimum number of removable edges.

Corollary 3.3. *Let G be a $3-$connected cubic graph and let C be $C = u_0u_1 \dots u_ku_0$ be a cycle of G. Then C contains at least two removable edges.*

Proof Since by Theorem 3.1 $N(G)$ is a forest, C contains at least one removable edge. Assume that C contains only one removable edge. Let P be the path obtained from C by deleting this edge and let e be an edge of P. Since P is contained in $N(G)$ there is a $3-$edge cut F containing e. By Lemma 2.1, F contains exactly one other edge of F, a contradiction with Lemma 2.5. □

Corollary 3.4. [3] *Let G be a $3-$connected cubic graph with at least six vertices. Then every spanning tree of G avoids at least two removable edges.*

Proof Let n be the number of vertices of G. Since G has $3\frac{n}{2}$ edges, a spanning tree avoids $\frac{n+2}{2} \geq 4$ edges. If every edge of G is removable, the result is immediate

Now, assume that $N(G) \neq \emptyset$. By Lemma 2.4, G contains at least two extremities. Let F be an end 3-edge cut with associated partition $\{V_1, V_2\}$ such that $G[V_1]$ is an extremity. The subgraph $G[V_1]$ contains $2p+1$ vertices (with $p \geq 1$) and $3p$ edges. The trace T_1 on $G[V_1]$ of a spanning tree T of G is a spanning forest of this extremity having k trees (with $1 \leq k \leq 3$). Hence, T_1 has $2p-k+1$ edges and avoids $p+k-1 \geq p$ edges of $G[V_1]$. Thus, T must avoid at least one edge in each extremity and the theorem follows. □

Lemma 3.5. *Let G be a $3-$connected cubic graph with at least six vertices having a spanning tree avoiding exactly two removable edges. Then G has exactly two extremities and these extremities are isomorphic to a triangle.*

Proof Let n be the number of vertices of G. Since we know that there are $\frac{n+2}{2} \geq 4$ edges outside any spanning tree, if a spanning tree T avoids exactly two removable edges then $N(G)$ is not empty. By Lemma 2.4, G has $k \geq 2$ extremities $H_1, H_2, \cdots, H_k$. We have seen in the proof of Corollary 3.4 that if H_i $(i = 1, \cdots, k)$ is an extremity having $2p_i + 1$ vertices then a spanning tree T of G avoids at least $p_i \geq 1$ edges of H_i. Hence, T must avoid at least $p_1 + p_2 + \cdots + p_k$ removable edges. Since T avoids exactly two removable edges, $p_1 + p_2 + \cdots + p_k = 2$. Hence $k = 2$ and $p_1 = p_2 = 1$, that is the graph G has exactly two extremities and each extremity has three vertices. $\square$

Corollary 3.6. *Let G be a $3-$connected cubic graph. Then G has a spanning tree T avoiding exactly two removable edges if and only if G is a PR-graph.*

Proof Assume that G is isomorphic to some $3-$connected cubic graph of type PR_{k_1,k_2} $(k_1 + k_2 \geq 0)$. Let M be the set of edges involved in the insertions operated from $PR_{0,0}$ in order to obtain G. Assume that the two triangles are a, b, c and $a^{'}, b^{'}, c^{'}$. Let $M^{'} = M \cup \{ab, bc, a^{'}b^{'}, b^{'}c^{'}\}$. We can easily find a spanning tree T containing the edges of $M^{'}$ (perform the greedy Kruskal's algorithm to find a minimum spanning tree of G when the edges of $M^{'}$ are placed at the beginning of the ordering of $E(G)$). Since the removable edges of G are the edges of M and the six edges contained in the two triangles, exactly two removable edges are outside this spanning tree.

We prove now by induction on the number of vertices $n \geq 6$, that whenever G is a $3-$connected cubic graph having a spanning tree avoiding exactly two removable edges then G is isomorphic to some graph of type PR_{k_1,k_2} $(k_1 + k_2 \geq 0)$.

When $n = 6$, $PR_{0,0}$ is the only graph with that property. Assume that the result holds for any $3-$connected cubic graph with $6 \leq n^{'} < n$ vertices having a spanning tree avoiding exactly two removable edges.

Let G be a $3-$connected cubic graph with n vertices having a spanning tree avoiding exactly two removable edges. By lemma 3.5, G has exactly two extremities isomorphic to a triangle. Assume that these triangles are Δ_1 and Δ_2. If T is a spanning tree of G avoiding exactly two removable edges, one of this edge (say e_1) must be in Δ_1 and the other (say e_2) is in Δ_2.

When there is no $3-$edge cut distinct from the $3-$edge cut incident to Δ_1 or to Δ_2, it is not difficult to see that G is isomorphic to $PR_{0,0}$ or to $PR_{1,0}$ or to $PR_{0,1}$ (see $PR_{0,1}$ in Figure 4).

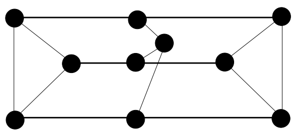

FIGURE 4. $PR_{0,1}$

Let $F = \{x_1x_2, y_1y_2, z_1z_2\}$ be a $3-$edge cut of G distinct from the end $3-$edge cuts F_1 and F_2 respectively incident to Δ_1 and to Δ_2. Let $\{V_1, V_2\}$ be the associated partition of F. We can construct a new $3-$connected cubic graph G_1 by replacing in G the subgraph $G[V_2]$ by the triangle $\Delta_1^{'} = \{x_2, y_2, z_2\}$. In the same way, we construct G_2 by replacing $G[V_1]$ by the triangle $\Delta_2^{'} = \{x_1, y_1, z_1\}$. Clearly, for $i = 1, 2$ $R(G_i) = (R(G) \cap E(G[V_i])) \cup E(\Delta_i^{'})$.

Let U_i be the trace of the spanning tree T on $G[V_i]$ $(i = 1, 2)$. Note that U_i is a spanning forest of $G[V_i]$ having at most three trees and that U_i avoids exactly one removable edge in $E(G[V_i])$ (the edge e_i in Δ_i). By using the trace U_i we will construct a spanning tree T_i of G_i avoiding exactly two removable edges in G_i.

Following the number of edges of F in $E(T)$ there are three cases :
Case 1 : $|E(T) \cap F| = 1$. Assume that $E(T) \cap F = \{x_1x_2\}$. We see that U_1 and U_2 are trees. Hence, $T_1 = U_1 + \{x_1x_2, x_2y_2, x_2z_2\}$ is a spanning tree of G_1 and $T_2 = U_2 + \{x_1x_2, x_1y_1, x_1z_1\}$ is a spanning tree of G_2.
Case 2 : $|E(T) \cap F| = 2$. Assume that $E(T) \cap F = \{x_1x_2, y_1y_2\}$. Consider the unique path P in T connecting x_1x_2 to y_1y_2. Then either P is a subpath of $G[V_1]$ having x_1 and y_1 as end vertices or P is a subpath of $G[V_2]$ having x_2 and y_2 as end vertices. If P is a subpath of $G[V_1]$ then U_1 is a tree and there is no path in U_2 connecting x_2 to y_2. Then U_2 is a forest of two trees, one of them containing x_2 and the other containing y_2. We see that $T_1 = U_1 + \{x_1x_2, x_2y_2, x_2z_2\}$ and $T_2 = U_2 + \{x_1x_2, y_1y_2, x_1y_1, x_1z_1\})$ are respectively spanning trees of G_1 and G_2. Ananlogously, if P is a subpath of $G[V_2]$ then U_2 is a tree and U_1 is a forest of two trees, one of them containing x_1 and the other containing y_1. Hence, $T_1 = U_1 + \{x_1x_2, y_1y_2, x_2y_2, x_2z_2\}$ and $T_2 = U_2 + \{x_1x_2, x_1y_1, x_1z_1\})$ are spanning trees of G_1 and G_2.
Case 3 : $F \subset E(T)$. Up to symmetries, there are two subcases:
Subcase 3.1 : U_1 is a tree and U_2 is a forest of three trees (the first containing x_2, the second containing y_2 and the third containing z_2). We consider $T_1 = U_1 + \{x_1x_2, x_2y_2, x_2z_2\}$ and $T_2 = U_2 + \{x_1x_2, y_1y_2, z_1z_2, x_1y_1, x_1z_1\}$.
Subcase 3.2 : U_1 is a forest of two trees (one of them containing x_1 and the other containing y_1 and z_1) and U_2 is a forest of two trees (one of them containing x_2 and z_2 and the other containing y_2). We consider $T_1 = U_1 + \{x_1x_2, y_1y_2, x_2y_2, x_2z_2\}$ and $T_2 = U_2 + \{x_1x_2, y_1y_2, x_1y_1, x_1z_1\})$.

In every case, we have constructed a spanning tree T_1 of G_1 (respectively T_2 of G_2) avoiding exactly two removable edges in G_1 (respectively G_2), the edges e_1 and y_2z_2 (resp. e_2 and y_1z_1).

By the induction hypothesis, G_1 is isomorphic to a graph of type PR_{p_1,q_1} and G_2 is isomorphic to a graph of type PR_{p_2,q_2}. At last, G itself is isomorphic to a graph of type $PR_{p_1+p_2,q_1+q_2}$. □

Corollary 3.7. [4] *Let G be a $3-$connected cubic graph with at least six vertices. Then every spanning tree contains at least two removable edges.*

Proof A spanning tree T of G containing at most one removable edge e contains only edges in $N(G) \cup \{e\}$. Since $N(G)$ has at most $n-3$ edges, this is impossible. □

Corollary 3.8. *Let G be a $3-$connected cubic graph with at least six vertices. Then there is a spanning tree containing exactly two removable edges if and only if G is a $3T$-graph.*

Proof Assume that G is isomorphic to some $p-$extended $3T_{k+2}$ $(k \geq 0, p \geq 0)$. Following the notation of Remark 1.3, let T_1, T_2 and T_3 be the three trees of $N(G)$. By adding to $N(G)$ two edges of any given triangle of G we get a spanning tree containing exactly two removable edges.

We prove now by induction on $n \geq 6$ that, if G a $3-$connected cubic graph on n vertices spanned by a tree T containing exactly two removable edges, then it is isomorphic to some $p-$extended $3T_{k+2}$ $(k \geq 0, p \geq 0)$.

When $n = 6$, G is isomorphic to $3T_{2,0}$ (that is, $PR_{0,0}$) and the result is obvious. Assume that the result holds for any $3-$connected cubic graph with $6 \leq n' < n$ vertices having a spanning tree containing exactly two removable edges.

Since $|T| = n - 1$ and $|N(G)| \leq n - 3$, we need to have $|N(G)| = n - 3$ (that is $N(G)$ is a spanning forest and is formed of exactly three trees, T_1, T_2 and T_3) and every edge of $N(G)$ must be contained in T. If no $3-$edge cut distinct from an end $3-$edge cut exists then G is isomorphic either to $PR_{0,0}$ (that is, $3T_{2,0}$) or to $PR_{1,0}$ (that is, $3T_{2,1}$) or to the graph $3T_{3,0}$ depicted in Figure 5.

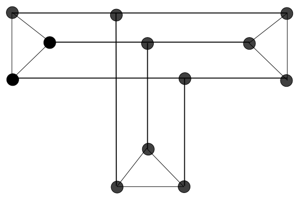

Figure 5. $3T_{3,0}$

Let F be a $3-$edge cut distinct from an end $3-$edge cut. Let $\{V_1, V_2\}$ be the associated partition of F. We can construct a new $3-$connected cubic graph G_1 by replacing in G the subgraph $G[V_2]$ by a triangle Δ_1'. In the same way, we construct G_2 by replacing $G[V_1]$ by a triangle Δ_2'. The trace of the forest $N(G)$ in G_1 gives a spanning forest of three trees of non removable edges. If we add two edges of Δ_1' to these trees, we get a spanning tree of G_1 containing exactly two removable edges. By the induction hypothesis, G_1 is isomorphic to a p_1-extended $3T_{k_1+2}$ and, in the same way G_2 is isomorphic to a p_2-extended $3T_{k_2+2}$. The reconstruction of G gives a $(p_1 + p_2)-$extended $3T_{k_1+k_2+2}$, and the result follows.

□

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