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Abstract

We explicitly determine the best uniform polynomial approximation p_{n-1}^* to a class of rational functions of the form $1/(x-c)^2 + K(a,b,c,n)/(x-c)$ on $[a,b]$ represented by their Chebyshev expansion, where a, b, and c are real numbers, $n-1$ denotes the degree of the best approximating polynomial, and K is a constant determined by a, b, c, and n. Our result is based on the explicit determination of a phase angle η in the representation of the approximation error by a trigonometric function. Moreover, we formulate an ansatz which offers a heuristic strategies to determine the best approximating polynomial to a function represented by its Chebyshev expansion. Combined with the phase angle method, this ansatz can be used to find the best uniform approximation to some more functions.

Keywords: Best approximation; Chebyshev polynomial; Uniform norm

1. Introduction

Best approximation by polynomials is an important subject in approximation theory and has a large number of applications. For the numerical computation of a best approximating polynomial, the Remez algorithm [1] can be used. However, to our experiences, the approximations obtained by the Remez algorithm may significantly deviate from the exact best uniformly approximating polynomial due to its bad convergence behavior and the accumulation of floating point errors.

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We focus on situations that allow an explicit determination of the best approximating algebraic polynomial of degree n to a certain class of rational functions with respect to the maximum norm [2]. Despite the fact that Chebyshev's alternation theorem [2] provides a characterization of the best uniformly approximating polynomial, there are only a few classes of functions for which an explicit representation of this polynomial is known.

First, we will introduce the problem of best uniform polynomial approximation [2] and give some definitions.

Let the function f be contained in $C[a,b]$, and let P_n denote the set of polynomials of degree not bigger than n, with real coefficients. For every nonnegative integer n there exists a unique polynomial p_n^* in P_n, such that

$$\max_{a\leqslant x\leqslant b}\left|f(x)-p(x)\right| > \max_{a\leqslant x\leqslant b}\left|f(x)-p_n^*(x)\right| = E_n(f)$$

for all polynomials $p \in P_n$ other than p_n^*. We call p_n^* the best uniform polynomial approximation of degree n to f on $[a,b]$. We can characterize p_n^* via the following theorem.

Chebyshev Alternation Theorem. [2] *Let f be in $C[a,b]$. Let the polynomial p be in P_n, and $\varepsilon(x) = f(x) - p(x)$. Then p is the best uniform approximation p_n^* to f on $[a,b]$ if and only if there exist at least $n+2$ points $x_1,\dots,x_{n+2}$ in $[a,b]$, $x_i < x_{i+1}$, for which $|\varepsilon(x_i)| = \max_{a\leqslant x\leqslant b}|f(x)-p(x)|$, with $\varepsilon(x_{i+1}) = -\varepsilon(x_i)$.*

Now, the Chebyshev polynomials are introduced as usual.

Definition 1. [3] The Chebyshev polynomial on $[-1,1]$ of degree n is denoted by T_n and is defined by $T_n(x) = \cos(n\theta)$, where $x = \cos\theta$.

Definition 2. [3] The Chebyshev polynomial on $[a,b]$ of order n is denoted by T_n^* and defined by $T_n^*(x) = \cos(n\theta)$ where $\cos\theta = \frac{2x-(b+a)}{b-a}$ and $\theta \in [0,\pi]$. According Definition 1, we have $T_n^*(x) = T_n(\frac{2x-(b+a)}{b-a})$.

Definition 3. [2] The Chebyshev polynomial of the second kind on $[-1,1]$ of degree n is denoted by U_n, and is defined by $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$, where $x = \cos\theta$.

Second, we will introduce those classes of functions for which an explicit representation of the best uniformly approximating polynomial is known. Chebyshev [3] gave the best uniform approximation of $1/(x-a)$ on $[-1,1]$ where $a > 1$. After that, a lot of research was done on classes of functions possessing a certain expansion by Chebyshev polynomials. In 1936, Bernstein [4] showed that if $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$ with $a_k \geqslant 0$, $x \in [-1,1]$, then $p_n^*(x) = \sum_{k=0}^{n} a_k T_k(x)$ is the best uniformly approximating polynomial of degree n for each n, if and only if the ratio $q_i = k_{i+1}/k_i$ of the indices of two successive nonvanishing coefficients $a_{k_i}, a_{k_{i+1}}$ is an odd integer for each i. The approximation is merely a truncation of the series representing the function. In 1962, extending results given by Hornecker [5], Rivlin [6] considered the class of functions

$$f(x) = \sum_{k=0}^{\infty} t^k T_{pk+q}(x) = \frac{T_q(x) - tT_{|q-p|}(x)}{1+t^2-2tT_p(x)},$$

with two integers $p > 0$ and $q \geqslant 0$ and with $-1 < t < 1$. For $t \neq 0$, the best uniform polynomial approximations are shown to be truncations of this expansion with a modification of the last term in the truncated series. This class contains the result of Jokar and Mehri [7]. In 1979, Ollin [2] determined the best uniform approximation to a class of rational functions of the form

$$f(x) = \sum_{j=0}^{\infty} t^j U_{pj+q}(x) = \frac{U_q(x) - tU_{q-p}(x)}{1 + t^2 - 2tT_p(x)},$$

with two integers $p > 0$ and $q \geqslant 0$ and with $-1 < t < 1$.

In 1982, Ollin and Gerst [8] studied a larger class of odd rational functions of the form

$$f(x) = A(x^2)/(xD(x^2)) \quad \text{or} \quad f(x) = xB(x^2)/C(x^2),$$

where A, B, C, and D are arbitrary polynomials.

Third, we will introduce the phase angle method. The method of determining the best uniformly approximating polynomial in the above mentioned cases goes back at least to Bernstein [9]:

Basically, the approximation error $f(x) - p_n^*(x)$ is represented in the form

$$f(x) - p_n^*(x) = b_n \cos(n\theta + \eta), \tag{1}$$

with $b_n \geqslant 0$, where θ is defined via $\cos\theta = X$, and X is the image of x under an affine map from $[a, b]$ onto $[-1, 1]$ preserving the orientation. The phase angle η is a continuous function of x (or X, respectively). If $n\theta + \eta$ covers a sufficiently large interval, the error function possesses a set of alternation points of length $n + 2$ ensuring that p_n^* is the best uniformly approximating polynomial by the Chebyshev alternation theorem [2].

Fourth, we will introduce the new techniques required to obtain our results. In the situation considered here, the representation of the approximation error is different from Eq. (1). Further, the expansion by Chebyshev polynomials which the considered functions possess is also different from the ones presented above. We consider functions of the form $1/(x-c)^2 + K(a, b, c, n)/(x-c)$ on $[a, b]$ with $c < a$ or $b < c$. For such a function f, a polynomial p_{n-1}^* of degree $n - 1$ is constructed by modification of a truncation of the Chebyshev expansion of f. This construction is carried out in such a way that a function η exists, which is independent of n and fulfills

$$f(\cos\theta) - p_{n-1}^*(\cos\theta) = b_n \cos(n\theta + \eta).$$

The Chebyshev expansion of f possesses a portion of the type $\sum_{k=0}^{\infty} kt^k T_k^*(x)$. Hence, these functions are not contained in the class considered by Rivlin [6], and we need to determine a representation of $\sum_{k=n}^{\infty} kt^k T_k^*(x)$ as a rational function in $\cos\theta$ and $\sin\theta$. The above new techniques are used in the proof of Theorem 1 (see Section 2). In Section 4, a heuristic principle is formulated that was applied to find the result of Theorem 1 and that may help to construct the uniformly best approximating polynomial for larger classes of functions.

Finally, the paper is organized as follows: After some definitions and a brief history of the explicit determination of the best approximating algebraic polynomial to a rational function with respect to the maximum norm have been given in this section, we state our main theorem and provide a proof for it in Section 2. In Section 3, we present two numerical examples. In Section 4, an ansatz is formulated providing a heuristic strategy to find the best approximating polynomial to a function given by a series of Chebyshev polynomials. Finally, we present plans for future work.

2. Best approximation of $1/(x-c)^2 + K/(x-c)$ on $[a, b]$

For the proof of Theorem 1, some auxiliary lemmas are required. Lemma 1 is the result of Jokar and Mehri's [7]. Although it is contained in Rivlin's result [6], we list it here because our research is based on it. In Lemma 2, a representation of the derivative of a Chebyshev polynomial on $[-1,1]$ is presented. This result is transferred to an arbitrary interval $[a,b]$ in Lemma 3. Lemma 4 provides an expansion of $1/(x-c)^2$ in terms of polynomials $T_j^*(x)$. Finally, a representation of $\sum_{j=n}^{\infty} jt^j T_j^*(x)$ by trigonometric functions is given in Lemma 5.

Lemma 1. [7] *The best approximation out of P_n to $1/(x-c)$, on $[a,b]$, where $b<c$ or $c<a$, and P_n is a function space of polynomials of degree n, is*

$$p_n^*(x) = \frac{-4t}{(t^2-1)(b-a)} + \frac{8t}{(t^2-1)(b-a)} \sum_{j=0}^{n-1} t^j T_j^*(x) - \frac{8t^{n+1}}{(t^2-1)^2(b-a)} T_n^*(x),$$

with

$$E_n(f) = \frac{8|t|^{n+2}}{(t^2-1)^2(b-a)},$$

where $E_n(f)$ is the maximum norm of the error $f-p_n^*$, and

$$t = \begin{cases} \frac{2c-(a+b)-2\sqrt{(c-a)(c-b)}}{b-a} & \textit{if } (c>b), \\ \frac{2c-(a+b)+2\sqrt{(c-a)(c-b)}}{b-a} & \textit{if } (c<a). \end{cases}$$

Remark 1. The method of Jokar and Mehri's proof [7] is based on the construction of a phase angle η in the error representation

$$f(\cos\theta) - p_n^*(\cos\theta) = b_n \cos(n\theta + \eta).$$

Although there are some mistakes in connection with the use of the equation

$$\cos(n\theta+\eta) = \cos\eta\cos n\theta - \sin\eta\sin n\theta,$$

this lemma is correct.

Lemma 2. *Let $T_n'(x)$ be the derivative of $T_n(x)$. Then we have*

$$T_n'(x) = \begin{cases} 2n(T_{n-1}(x) + T_{n-3}(x) + \cdots + T_0(x)) - n, & n = 2k+1, \\ 2n(T_{n-1}(x) + T_{n-3}(x) + \cdots + T_1(x)), & n = 2k, \end{cases}$$

where $x \in [-1,1]$ and $k = 0,1,2,\ldots$.

Proof. From Definition 1, we have $\cos\theta = x$, and thus we obtain $(-\sin\theta)\theta' = 1$ and $\theta' = \frac{-1}{\sin\theta}$, where θ' denotes the derivative of θ with respect to x.

Therefore, we obtain $T_n'(x) = (\cos n\theta)' = -n\theta' \sin n\theta = \frac{n \sin n\theta}{\sin\theta}$ (see [1]).

When $n = 2k$, we have

$$\begin{aligned} &2n\big(T_{n-1}(x) + T_{n-3}(x) + \cdots + T_1(x)\big) \\ &\quad = \frac{n}{\sin\theta}\big(2\sin\theta\cos(n-1)\theta + 2\sin\theta\cos(n-3)\theta + \cdots + 2\sin\theta\cos 3\theta + 2\sin\theta\cos\theta\big) \\ &\quad = \frac{n}{\sin\theta}\big(\big(\sin n\theta - \sin(n-2)\theta\big) + \big(\sin(n-2)\theta - \sin(n-4)\theta\big) + \cdots \\ &\qquad + (\sin 4\theta - \sin 2\theta) + (\sin 2\theta)\big) \\ &\quad = \frac{n\sin n\theta}{\sin\theta}. \end{aligned}$$

When $n = 2k + 1$, we have

$$\begin{aligned}
&2n\big(T_{n-1}(x) + T_{n-3}(x) + \cdots + T_0(x)\big) - n \\
&\quad = \frac{n}{\sin\theta}\big(\big(2\sin\theta\cos(n-1)\theta + 2\sin\theta\cos(n-3)\theta + \cdots \\
&\qquad + 2\sin\theta\cos 2\theta + 2\sin\theta\big) - \sin\theta\big) \\
&\quad = \frac{n}{\sin\theta}\big(\big(\big(\sin n\theta - \sin(n-2)\theta\big) + \big(\sin(n-2)\theta - \sin(n-4)\theta\big) + \cdots \\
&\qquad + (\sin 3\theta - \sin\theta) + 2\sin\theta\big) - \sin\theta\big) \\
&\quad = \frac{n\sin n\theta}{\sin\theta}.
\end{aligned}$$

Hence the lemma is proved. □

Lemma 3.

$$\big(T_n^*(x)\big)' = \begin{cases} \frac{2}{b-a}(2n(T_{n-1}^*(x) + T_{n-3}^*(x) + \cdots + T_0^*(x)) - n), & n = 2k+1, \\ \frac{2}{b-a}(2n(T_{n-1}^*(x) + T_{n-3}^*(x) + \cdots + T_1^*(x))), & n = 2k, \end{cases}$$

where $(T_n^*(x))'$ is the derivative of $T_n^*(x)$, $x \in [a,b]$, $a < b$, $a, b \in R$, and $k = 0, 1, 2, \ldots$.

Proof. From Definition 2, we have

$$T_n^*(x) = T_n\left(\frac{2x-(b+a)}{b-a}\right), \quad \text{and}$$

$$\big(T_n^*(x)\big)' = \left(T_n\left(\frac{2x-(b+a)}{b-a}\right)\right)' = \frac{2}{b-a}T_n'\left(\frac{2x-(b+a)}{b-a}\right).$$

Hence, from Lemma 2, this lemma is proved. □

Lemma 4.

$$\begin{aligned}
\frac{1}{(x-c)^2} = {}& \frac{32t^2}{(b-a)^2(1-t^2)^2}\sum_{j=0}^{\infty} jt^jT_j^*(x) + \frac{32t^4+32t^2}{(b-a)^2(1-t^2)^3}\sum_{j=0}^{\infty} t^jT_j^*(x) \\
& - \frac{16t^4+16t^2}{(b-a)^2(1-t^2)^3},
\end{aligned}$$

where $x \in [a,b]$, $a < b$, $a, b \in R$, $b < c$ or $c < a$, and

$$t = \begin{cases} \frac{2c-(a+b)-2\sqrt{(c-a)(c-b)}}{b-a} & \text{if } (c > b), \\ \frac{2c-(a+b)+2\sqrt{(c-a)(c-b)}}{b-a} & \text{if } (c < a). \end{cases}$$

Proof. Due to

$$\frac{1}{x-c} = -\frac{4t}{(t^2-1)(b-a)} + \frac{8t}{(t^2-1)(b-a)}\sum_{j=0}^{\infty} t^jT_j^*(x) \quad \big(\text{see [7]}\big), \tag{2}$$

we have

$$\left(\frac{1}{x-c}\right)' = \left(\frac{8t}{(b-a)(t^2-1)}\sum_{j=0}^{\infty} t^j T_j^*(x) + \frac{-4t}{(b-a)(t^2-1)}\right)'$$
$$= \frac{8t}{(b-a)(t^2-1)}\sum_{j=0}^{\infty} t^j \big(T_j^*(x)\big)'.$$

From Lemma 3, we obtain

$$\left(\frac{1}{x-c}\right)' = \frac{16t}{(b-a)^2(t^2-1)}\left(\sum_{j=0}^{\infty}\left(2T_j^*(x)\sum_{k=0}^{\infty}(j+1+2k)t^{j+1+2k}\right) - \sum_{i=0}^{\infty} t^{2i+1}(2i+1)\right).$$

With

$$s = \sum_{k=0}^{\infty}(j+1+2k)t^{j+1+2k} = (j+1)t^{j+1} + (j+3)t^{j+3} + \cdots,$$

one obtains

$$t^2 s = (j+1)t^{j+3} + (j+3)t^{j+5} + \cdots.$$

From the definition of t, we obtain $|t| < 1$. Hence, we have

$$\big(1-t^2\big)s = (j+1)t^{j+1} + \frac{2t^{j+3}}{1-t^2}.$$

Therefore, we obtain

$$\frac{-1}{(x-c)^2} = \frac{16t}{(b-a)^2(t^2-1)}\left(\sum_{j=0}^{\infty}\left(2T_j^*(x)\left(\frac{(j+1)t^{j+1}}{1-t^2} + \frac{2t^{j+3}}{(1-t^2)^2}\right)\right) - \left(\frac{t}{1-t^2} + \frac{2t^3}{(1-t^2)^2}\right)\right).$$

Hence this lemma is proved. □

Lemma 5.

$$\sum_{j=n}^{\infty} jt^j T_j^*(x) = \frac{nt^n(\cos n\theta - t\cos(n-1)\theta)}{1+t^2-2t\cos\theta} + \frac{t^{n+1}(\cos(n+1)\theta - 2t\cos n\theta + t^2\cos(n-1)\theta)}{(1+t^2-2t\cos\theta)^2},$$

where $x \in [a,b]$, $a < b$, $a, b \in R$, $\cos\theta = \frac{2x-(b+a)}{b-a}$, $\theta \in [0,\pi]$, *and*

$$t = \begin{cases} \frac{2c-(a+b)-2\sqrt{(c-a)(c-b)}}{b-a} & \textit{if } (c > b), \\ \frac{2c-(a+b)+2\sqrt{(c-a)(c-b)}}{b-a} & \textit{if } (c < a). \end{cases}$$

Proof. We expand the left-hand side of the equation to be proved as follows:

$$\sum_{j=n}^{\infty} jt^j T_j^*(x) = \sum_{j=n}^{\infty} nt^j T_j^*(x) + \sum_{j=n+1}^{\infty} t^j T_j^*(x) + \sum_{j=n+2}^{\infty} t^j T_j^*(x) + \cdots.$$

Since

$$\sum_{j=n}^{\infty} t^j T_j^*(x) = t^n \frac{\cos n\theta(1 - t\cos\theta) - t\sin n\theta \sin\theta}{1 + t^2 - 2t\cos\theta} \quad \bigl(\text{see [7]}\bigr), \tag{3}$$

we have

$$\sum_{j=n}^{\infty} t^j T_j^*(x) = t^n \frac{\cos n\theta - t\cos(n-1)\theta}{1 + t^2 - 2t\cos\theta},$$

and thus we obtain

$$\begin{aligned}
\sum_{j=n}^{\infty} jt^j T_j(x) &= nt^n \frac{\cos n\theta - t\cos(n-1)\theta}{1 + t^2 - 2t\cos\theta} + \sum_{j=n+1}^{\infty} t^j \frac{\cos j\theta - t\cos(j-1)\theta}{1 + t^2 - 2t\cos\theta} \\
&= nt^n \frac{\cos n\theta - t\cos(n-1)\theta}{1 + t^2 - 2t\cos\theta} + \frac{\sum_{j=n+1}^{\infty} t^j \cos j\theta}{1 + t^2 - 2t\cos\theta} - \frac{t^2 \sum_{j=n}^{\infty} t^j \cos j\theta}{1 + t^2 - 2t\cos\theta} \\
&= \frac{nt^n(\cos n\theta - t\cos(n-1)\theta)}{1 + t^2 - 2t\cos\theta} \\
&\quad + \frac{t^{n+1}(\cos(n+1)\theta - 2t\cos n\theta + t^2\cos(n-1)\theta)}{(1 + t^2 - 2t\cos\theta)^2}.
\end{aligned}$$

Hence this lemma is proved. □

Theorem 1. *The best approximation out of P_{n-1} to $f(x)$, on $[a, b]$, where $b < c$ or $c < a$, and*

$$f(x) = \frac{1}{(x-c)^2} - \frac{4t(nt^2 - 3t^2 - n - 1)}{(b-a)(1-t^2)^2(x-c)}$$

is

$$\begin{aligned}
p_{n-1}^*(x) = \frac{4}{(b-a)^2}\Biggl(&\frac{8t^2}{(1-t^2)^2}\sum_{j=0}^{n-1} jt^j T_j^*(x) + \frac{8t^2(nt^2 - 2t^2 - n)}{(1-t^2)^3}\sum_{j=0}^{n-1} t^j T_j^*(x) \\
&- \frac{4t^4 + 4t^2}{(1-t^2)^3} - \frac{4t^2(nt^2 - 3t^2 - n - 1)}{(1-t^2)^3} - \frac{8t^{n+5}}{(1-t^2)^4} T_{n-1}^*(x)\Biggr),
\end{aligned} \tag{4}$$

with

$$E_{n-1}(f) = \frac{32|t|^{n+3}}{(b-a)^2(1-t^2)^4},$$

where $E_{n-1}(f)$ is the maximum norm of the error $f - p_{n-1}^*$, and

$$t = \begin{cases} \frac{2c-(a+b)-2\sqrt{(c-a)(c-b)}}{b-a} & \text{if } (c > b), \\ \frac{2c-(a+b)+2\sqrt{(c-a)(c-b)}}{b-a} & \text{if } (c < a). \end{cases}$$

Proof. Since the proof of the case $c < a$ is similar to that of the case $c > b$, it suffices to prove the theorem in the case $c > b$. We only need to prove that

$$e_{n-1}(x) = f(x) - p_{n-1}^*(x)$$

has at least $n+1$ alternating points in $[a, b]$. From Lemma 4 and Eq. (2), we have

$$\begin{aligned} e_{n-1}(x) &= \frac{32t^2}{(b-a)^2(1-t^2)^2}\sum_{j=n}^{\infty} jt^j T_j^*(x) + \frac{32t^4+32t^2}{(b-a)^2(1-t^2)^3}\sum_{j=n}^{\infty} t^j T_j^*(x) \\ &\quad - \frac{8t(nt^2-3t^2-n-1)}{(b-a)^2(1-t^2)^2}\left(\frac{4t}{t^2-1}\sum_{j=n}^{\infty} t^j T_j^*(x)\right) + \frac{32t^{n+5}}{(b-a)^2(1-t^2)^4} T_{n-1}^*(x). \end{aligned}$$

From Definition 2, Lemma 5, and Eq. (3), we obtain

$$\begin{aligned} e_{n-1}(x) &= \frac{32t^2}{(b-a)^2(1-t^2)^2}\left(\frac{nt^n(\cos n\theta - t\cos(n-1)\theta)}{1+t^2-2t\cos\theta}\right. \\ &\quad \left. + \frac{t^{n+1}(\cos(n+1)\theta - 2t\cos n\theta + t^2\cos(n-1)\theta)}{(1+t^2-2t\cos\theta)^2}\right) \\ &\quad + \frac{32t^4+32t^2}{(b-a)^2(1-t^2)^3}\frac{t^n(\cos n\theta - t\cos(n-1)\theta)}{1+t^2-2t\cos\theta} \\ &\quad - \frac{8t(nt^2-3t^2-n-1)}{(b-a)^2(1-t^2)^2}\left(\frac{4t}{t^2-1}\frac{t^n(\cos n\theta - t\cos(n-1)\theta)}{1+t^2-2t\cos\theta}\right) \\ &\quad + \frac{32t^{n+5}}{(b-a)^2(1-t^2)^4}\cos(n-1)\theta. \end{aligned}$$

With

$$X = \frac{2x-(b+a)}{b-a},$$

we obtain $|X| \leqslant 1$, $\cos\theta = X$, and $\sin\theta = \sqrt{1-X^2}$. Thus, we have

$$\begin{aligned} &e_{n-1}(x) \\ &\quad = \frac{32t^{n+2}}{(b-a)^2(1-t^2)^3}\left(\frac{n(1-t^2)(\cos n\theta - tX\cos n\theta - t\sqrt{1-X^2}\sin n\theta)}{1+t^2-2tX}\right. \\ &\qquad + \frac{t(1-t^2)(X\cos n\theta - \sqrt{1-X^2}\sin n\theta - 2t\cos n\theta + t^2X\cos n\theta + t^2\sqrt{1-X^2}\sin n\theta)}{(1+t^2-2tX)^2} \\ &\qquad + \frac{(nt^2-2t^2-n)(\cos n\theta - tX\cos n\theta - t\sqrt{1-X^2}\sin n\theta)}{1+t^2-2tX} \\ &\qquad \left. + \frac{t^3}{1-t^2}\big(X\cos n\theta + \sqrt{1-X^2}\sin n\theta\big)\right) \\ &\quad = \frac{32t^{n+3}}{(b-a)^2(1-t^2)^4}\left(\frac{-3t^4X+6t^2X-4t+X+4t^3-8t^3X^2+4t^4X^3}{(1+t^2-2tX)^2}\cos n\theta\right. \\ &\qquad \left. - \frac{(1-6t^2+8t^3X-4t^4X^2+t^4)\sqrt{1-X^2}}{(1+t^2-2tX)^2}\sin n\theta\right). \end{aligned}$$

Let

$$g_1(X)=\frac{-3t^4X+6t^2X-4t+X+4t^3-8t^3X^2+4t^4X^3}{(1+t^2-2tX)^2}\quad\text{and}$$

$$g_2(X)=\frac{(1-6t^2+8t^3X-4t^4X^2+t^4)\sqrt{1-X^2}}{(1+t^2-2tX)^2}.$$

Then we have $(g_1(X))^2+(g_2(X))^2=1$, $g_1(-1)=-1$, $g_2(-1)=0$, $g_1(1)=1$, and $g_2(1)=0$. So we can find an angle η in the interval $[0,2\pi)$ which satisfies $\cos\eta=g_1(X)$ and $\sin\eta=g_2(X)$. Therefore, we have

$$\begin{aligned}e_{n-1}(x)&=\frac{32t^{n+3}}{(b-a)^2(1-t^2)^4}(\cos\eta\cos n\theta-\sin\eta\sin n\theta)\\&=\frac{32t^{n+3}}{(b-a)^2(1-t^2)^4}\cos(n\theta+\eta).\end{aligned}$$

Writing $h(X)=n\theta+\eta$, we obtain a continuous function h on $[-1,1]$ with $h(-1)=(n+1)\pi$ and $h(1)=0$.

Now, if x varies from a to b continuously, then X varies from -1 to 1 continuously, θ varies from π to 0 continuously, and η varies from π to 0 continuously. Hence, when x varies from a to b, h varies from $(n+1)\pi$ to 0, and consequently, $\cos(n\theta+\eta)$ possesses at least $n+2$ extremal points, where it assumes alternately the values ± 1.

We only need $n+1$ alternating points. Hence this theorem is proved. □

3. Examples

Figure 1 shows an example of the best approximation of $f_1(x)$ of degree 4 on the interval $[1,9]$, where $a=1$, $b=9$, $c=10$, $n=5$, and

$$f_1(x)=\frac{22x-211}{9(x-10)^2}.$$

From Theorem 1, we obtain

$$t=\frac{2\cdot 10-(1+9)-2\cdot\sqrt{(10-1)\cdot(10-9)}}{9-1}=\frac{1}{2},$$

and

$$f_1(x)=\frac{1}{(x-10)^2}-\frac{4\cdot(\frac{1}{2})\cdot(5\cdot(\frac{1}{2})^2-3\cdot(\frac{1}{2})^2-5-1)}{(9-1)\cdot(1-(\frac{1}{2})^2)^2\cdot(x-10)}.$$

Figure 2 shows an example of the best approximation of $f_2(x)$ of degree 3 on the interval $[-2,3]$, where $a=-2$, $b=3$, $c=-6$, $n=4$, and

$$f_2(x)=\frac{-150-31x}{36(x+6)^2}.$$

From Theorem 1, we obtain

$$t=\frac{2\cdot(-6)-(-2+3)+2\cdot\sqrt{((-6)-(-2))\cdot((-6)-3)}}{3-(-2)}=\frac{-1}{5},$$

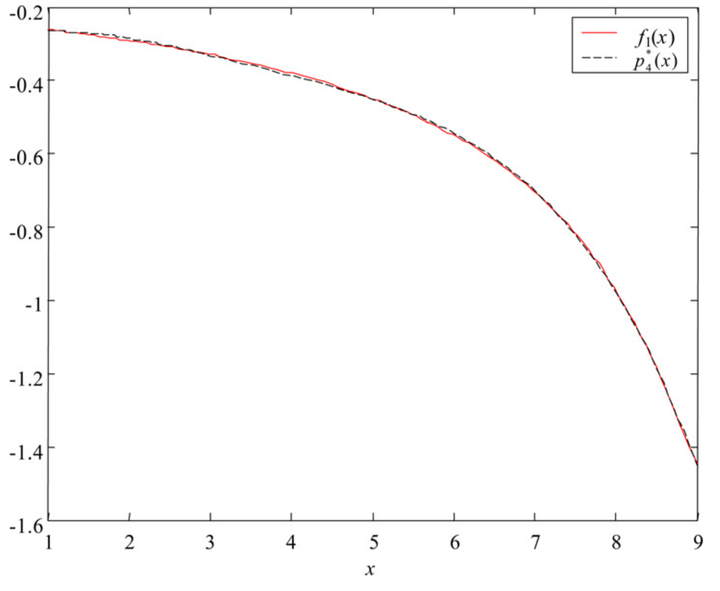

Fig. 1. The best approximation of $f_1(x) = (22x - 211)/(9(x - 10)^2)$.

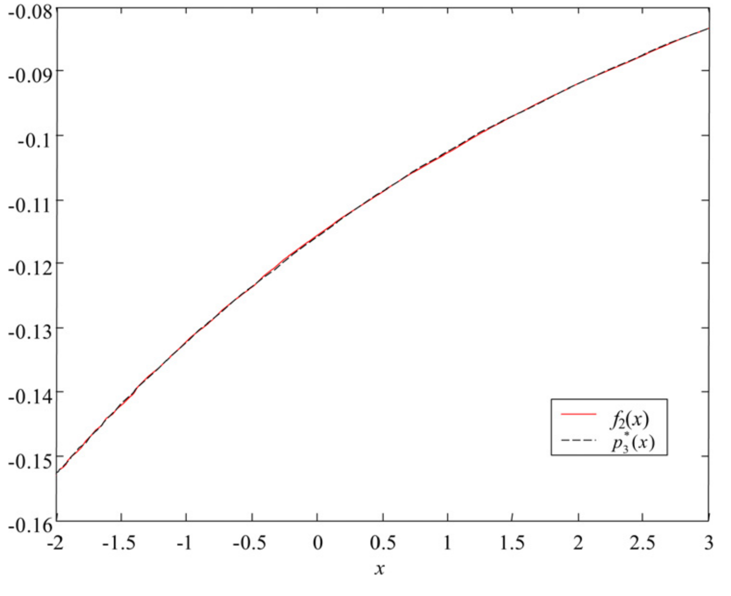

Fig. 2. The best approximation of $f_2(x) = -(150 + 31x)/(36(x + 6)^2)$.

and

$$f_2(x) = \frac{1}{(x - (-6))^2} - \frac{4 \cdot (\frac{-1}{5}) \cdot (4 \cdot (\frac{-1}{5})^2 - 3 \cdot (\frac{-1}{5})^2 - 4 - 1)}{(3 - (-2)) \cdot (1 - (\frac{-1}{5})^2)^2 \cdot (x - (-6))}.$$

Figures 1 and 2 show that both target functions and their best approximating polynomials are very close to each other.

4. On the best uniform approximation of Chebyshev expansions

In [7], Jokar and Mehri stated the following conjecture:

If $f(x)=\sum_{j=0}^{\infty} a_j T_j^*(x)$, then the best approximation of f out of P_n on $[a,b]$ is of the form $p_n^*(x)=\sum_{j=0}^{n-1} a_j T_j^*(x) - c a_n T_n^*(x)$ for some $c \in R$.

If such a representation holds for a certain class of target functions, it facilitates to find the best uniformly approximating polynomial for this class with the phase angle method. However, we will show that this conjecture does not hold in general. Moreover, Jokar and Mehri studied the example

$$|x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{4j^2-1}T_{2j}(x) \quad \text{on } -1 \leqslant x \leqslant 1.$$

Now we want to show that the conjecture is not true. If the conjecture was true, then the best approximation to $|x|$ out of P_{2n} on $[-1,1]$ would be of the form

$$p_{2n}^*(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{n-1}\frac{(-1)^{j+1}}{4j^2-1}T_{2j}(x) - c\frac{4}{\pi}\frac{(-1)^{n+1}}{4n^2-1}T_{2n}(x) \quad \text{for some } c \in R \quad \bigl(\text{see [7]}\bigr).$$

From [1], we have $p_2^*(x) = x^2 + \frac{1}{8}$ on $-1 \leqslant x \leqslant 1$. Therefore, we obtain

$$p_2^*(x) = \frac{2}{\pi} - c\frac{4}{\pi}\frac{(-1)^{1+1}}{4\cdot 1^2 - 1}T_2(x) = \frac{-8c}{3\pi}x^2 + \frac{2}{\pi} + \frac{4c}{3\pi} = x^2 + \frac{1}{8}.$$

Thus, we have

$$\begin{cases} \frac{-8c}{3\pi} = 1, \\ \frac{2}{\pi} + \frac{4c}{3\pi} = \frac{1}{8}. \end{cases} \tag{5}$$

System (5) has no solution, so the conjecture in [7] does not hold in this example. Finally, we formulate an ansatz to find the best approximating polynomial as follows.

Ansatz 1. *If*

$$g(x) = \sum_{j=0}^{\infty} a_j T_j^*(x),$$

there exists an integer variable $1 \leqslant k \leqslant n+1$ such that the best uniform approximation to $g(x)$ out of P_n on $[a,b]$ is of the form

$$p_n^*(x) = \sum_{j=0}^{n-1} a_j T_j^*(x) - c_1 T_n^*(x) - c_2 T_{n-1}^*(x) - \cdots - c_k T_{n-k+1}^*(x),$$

where c_v is a real number for $v = 1, 2, \ldots, k$.

Remark 2. Hence, we start with an ansatz for $p_n^*(x)$ with a small k and try to determine the coefficients of the best approximating polynomial by the procedure explained below. If this fails,

k has to be increased. The functions considered by Bernstein [4] possess a best approximating polynomial with $k=1$ and $c_1=-a_n$ in the ansatz formulated above. A general choice of $k=1$ leads to Jokar and Mehri's conjecture [7], which has been disproven above. This shows that in general a more flexible ansatz is necessary. When $k=n+1$, it is always true because any continuous function has a best approximating polynomial of degree n. For example, the best approximation to $|x|$ out of P_2 on $[-1,1]$ is given by

$$\begin{aligned}
p_2^*(x) &= -\frac{2}{\pi}+\frac{4}{\pi}\cdot\frac{(-1)^{0+1}}{4\cdot 0^2-1}\cdot T_0(x)-c_1\cdot\frac{4}{\pi}\cdot\frac{(-1)^{1+1}}{4\cdot 1^2-1}\cdot T_2(x)\\
&\quad -c_2\cdot\frac{4}{\pi}\cdot\frac{(-1)^{0+1}}{4\cdot 0^2-1}\cdot T_0(x)\\
&=\frac{2}{\pi}-\frac{4c_1}{3\pi}\left(2x^2-1\right)-\frac{4c_2}{\pi}=x^2+\frac{1}{8},
\end{aligned}$$

$$\begin{cases}\dfrac{-8c_1}{3\pi}=1,\\[2mm] \dfrac{2}{\pi}+\dfrac{4c_1}{3\pi}-\dfrac{4c_2}{\pi}=\dfrac{1}{8}.\end{cases}\tag{6}$$

System (6) has exactly one solution.

Moreover, this ansatz can be used to find the best uniform approximation to some more functions combined with the explicit determination of a phase angle η in the representation of the approximation error by a trigonometric function. The following process leads to the result of Theorem 1:

The form of the approximation error is $g(x)-p_n^*(x)=b_n\cos(n\theta+\eta)$.

After expanding the left-hand side of the above equation by the Chebyshev expansion of g, we obtain a formula of the form

$$g(x)-p_n^*(x)=g_1(c_1,\dots,c_k,X)\cos n\theta-g_2(c_1,\dots,c_k,X)\sin n\theta$$

using some equations, such as

$$\begin{aligned}
&g(x)-p_n^*(x)=\sum_{j=n}^{\infty}a_jT_j^*(x)+c_1T_n^*(x)+c_2T_{n-1}^*(x)+\cdots+c_kT_{n-k+1}^*(x),\\
&\sum_{j=n}^{\infty}t^jT_j^*(x)=t^n\frac{\cos n\theta(1-t\cos\theta)-t\sin n\theta\sin\theta}{1+t^2-2t\cos\theta}\\
&\qquad\qquad\qquad =t^n\frac{\cos n\theta(1-tX)-t\sin n\theta\sqrt{1-X^2}}{1+t^2-2tX},\\
&T_{n-1}^*(x)=\cos(n-1)\theta=\cos\theta\cos n\theta+\sin\theta\sin n\theta\\
&\qquad\qquad =X\cos n\theta+\sqrt{1-X^2}\sin n\theta,\quad \text{and so on,}
\end{aligned}$$

where $X=\frac{2x-(b+a)}{b-a}$, $\cos\theta=X$, and $\sin\theta=\sqrt{1-X^2}$. Then we explicitly determine the values of the variables $c_1,\dots,c_k$ by the condition

$$\bigl(g_1(c_1,\dots,c_k,X)\bigr)^2+\bigl(g_2(c_1,\dots,c_k,X)\bigr)^2=1\quad\text{for } X\in[-1,1],$$

and examine whether the range of $n\theta + \eta$ is wide enough, where

$$\cos\eta = g_1(c_1, \dots, c_k, X) \quad \text{and} \quad \sin\eta = g_2(c_1, \dots, c_k, X).$$

If the first s coefficients $-c_1$, $a_{n-1} - c_2, \dots, a_{n-s+1} - c_s$ of the best uniformly approximating polynomial are zero, its degree is $n - s$. We calculate the above values with k varying from 1 to $n + 1$, because the process is terminated if the best uniformly approximating polynomial is found.

5. Conclusions and future work

We have shown that the best approximation to a class of rational functions of the form $1/(x-c)^2 + K/(x-c)$ on the bounded interval $[a, b]$ is of the form given by Eq. (4). Moreover, we formulated a heuristic strategy to determine the best approximating polynomial to a function represented by its Chebyshev expansion. Combined with the phase angle method, this ansatz can be used to find the best uniform approximation to some more functions.

One issue of future work is to characterize the best approximating polynomial to functions of the form $\sum_{k=0}^{\infty} kt^k T_k^*(x)$. An other task is to search for a method to check if a given polynomial satisfies the Chebyshev alternation condition, which is computationally easier to handle than the phase angle method.

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