From invariants to predicates: example of line transversals to lines

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Abstract

This work explores a method that reduces the design of evaluation strategies for geometric predicates to the computation of polynomial invariants of a group action. We apply it to the classical problem of counting line transversals to lines in $\mathbb{P}^3$ and capture polynomials previously obtained by more pedestrian approaches.

1 Introduction

In computational geometry, algorithms are often designed over the reals but are implemented in floating-point arithmetic, which may lead to inconsistent decisions. To ensure correctness, refinement strategies or exact computations can be used but they may become very time–consuming, depending on the *evaluation strategy* of the decision problem or *predicate*. Consider for example the problem of deciding if four points in the plane are cocyclic. One approach is to compute the circumscribed circle of three points and test if the fourth point lies on it. Another approach consists in testing the vanishing of the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x_4 & y_4 & x_4^2+y_4^2 \end{vmatrix}$$

where (x_i, y_i) are the coordinates of the points. A major question is to find *efficient* and *robust* evaluation strategies for a given predicate.

We are here interested in strategies involving only polynomial evaluations from the inupts of a predicate. Robustness issues are guaranted through exact computation paradigm [10] and efficiency can be improved by using simplest possible polynomials. An immediate approach to find such polynomials consists in translating the problem into equations and extracting polynomial constraints that characterize the solutions of the resulting system. This has to be carried out carefully in order to avoid polynomials of huge degrees. Consider for example the problem of counting line transversals to four lines of $\mathbb{R}^3$ given as pairs of points. There may be 0, 1, 2 or infinitely many ones. Indeed, consider the ruled quadric generated by three input lines: the fourth line intersects it in at most two points or is contained in it. While the naive approach gives polynomial of degree 24 [3], the predicate can be decided with polynomials of degree at most 12 [2]. This gap can become more substantial: for ordering planes through a line ℓ, each containing a line transversal to three lines and ℓ, degree 144 in [3] collapses to degree 36 in [2]. These two *ad-hoc* approaches provide polynomials whose "complexity" strongly depends on the analytical formulation of the problem.

A general approach mainly based on the geometry of the predicate would be more satisfying. From this perspective, Petitjean [8] proposed an invariant–based method he applied to the problem of deciding the real intersection type of two projective planar conics (four simple points, two double points, a quadruple point,*etc* there are altogether 12 different types). What are the symmetries of the problem? Given two conics, observe that their intersection type is left unchanged under any simultaneous projective transformation of the two conics. The same is true when exchanging both conics or, more generally, replacing their equations by linear combinations of them. All of these symmetries are structured in a *group* that *acts* on the set of pairs of conics: any element of the group maps any pair of conics to another pair with the same intersection type. All pairs of conics obtained in this way from a fixed pair form an *orbit* of the group action. Invariant theory provides polynomial *invariants* that discriminate these orbits, and therefore distinguish intersection types.

In this work, we unfold the invariant–based method of [8] on the problem of counting line transversals to four linearly independent lines in $\mathbb{P}^3$. It provides the same polynomial of degree 12 in [2] in a more geometric manner and yields a better understand of the geometry of the problem. In this article, we focus on the construction of an appropriate group action and leave apart the computation of its polynomial invariants. For the latter problem, the interested reader will find an introduction in [6] and two different techniques in [4] and [5] among various existing strategies.

2 Preliminaries

Notations. The general linear group GL_n over $\mathbb{R}$ is the set of real invertible matrices of size n. (By extension, $GL(W)$ is the set of invertible linear trans-

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formations of the vector space W.) We denote by $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ the real projective space of dimension n whose points are represented by homogeneous coordinates $[x_0 : \ldots : x_n]$. (By extension, $\mathbb{P}W$ represents the quotient of the vector space W by nonzero scalings.) A *collineation* (or *projective transformation*) of $\mathbb{P}^n$ is defined by a matrix M of GL_{n+1}: it maps $[x_0 : \ldots : x_n]$ to $[Mx_0 : \ldots : Mx_n]$. The set of hyperplanes of $\mathbb{P}^n$ is denoted by $\mathbb{P}^{n\star}$. The *duality operator* $\star : \mathbb{P}^n \to \mathbb{P}^{n\star}$ maps a point $[x_0 : \ldots : x_n]$ to the hyperplane defined by the equation $\sum_{i=0}^{n} x_i y_i = 0$. A *correlation* of $\mathbb{P}^n$ is the composition of a collineation with the duality operator.

2.1 Invariants of group actions

A *transformation group* is a subset of GL_n containing the identity matrix and closed by multiplication. In what follows, G will denote an abstract group but it is sufficient to restrict to transformation groups for the sake of understanding.

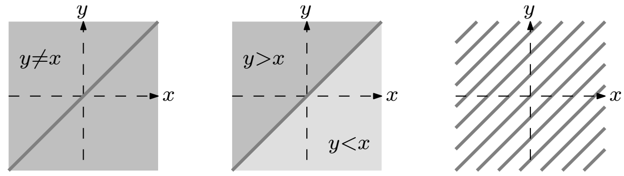
Group action. The *action* ρ of a group G on a set X is denoted by $\rho : G \circlearrowleft X$ and is defined as follows: all $\rho(g)$ with $g \in G$ are bijections of X such that $\rho(1)$ is the identity map on X and $\rho(gg') = \rho(g) \circ \rho(g')$ for any $g, g' \in G$. For example, the group of isometries preserving a cube acts on the set of diagonals of that cube: applying two successive isometries on the cube induces a composition of two permutations of its diagonals. A *linear group action* of G on a vector space W is a group action ρ of G on W where the bijections $\rho(g)$ on W are linear[1] (*i.e.* elements of $GL(W)$). We denote it by $\rho : G \to GL(W)$.

Consider a fixed element x in X and form the set of all $y \in Y$ that can be obtained from x by a map $\rho(g)$ (with $g \in G$): this defines an *orbit* of ρ. These orbits form a partition of X. In the previous example, there is just one orbit because any diagonal of the cube can be mapped to any other one by an isometry preserving the cube. Let us give two another examples.

Example 1. Let G be the group of affine motions[2] of the real line $\mathbb{R}$ and $\rho : G \circlearrowleft \mathbb{R}^2$ its action on pairs of points defined by $\rho(g)(x, y) = (g(x), g(y))$. Figure 1 represents the orbits of ρ where we restrict to different subgroup of G. We observe that the smaller (for inclusion) the group, the larger the number of orbits.

Example 2. Consider the action S_2^2 of GL_2 on the space $S^2(\mathbb{R}^2)$ of binary quadratic forms, defined by:

$$S_2^2 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} (ax^2 + 2bxy + cy^2) = \bar{a}x^2 + 2\bar{b}xy + \bar{c}y^2$$



(a) affine motions (b) non-reversing affine motions (c) rigid motions

Figure 1: Orbits of ρ from Example 1.

$$\text{where} \quad \begin{cases} \bar{a} = \alpha^2 a + 2\alpha\gamma b + \gamma^2 c \\ \bar{b} = \alpha\beta a + (\alpha\delta + \beta\gamma) b + \gamma\delta c \\ \bar{c} = \beta^2 a + 2\beta\delta b + \delta^2 c \end{cases} .$$

It simply consists of a change of coordinates induced by $g \in GL_2$ on the quadratic form. This action has three orbits, depending on the number of distinct factors in which a quadratic form can be factored.

Invariant. Let $\rho : G \to GL(W)$ be a linear group action of a transformation group $G \subset GL_n$. A homogeneous polynomial P on W is a *(relative) invariant* for ρ if there exists $\lambda \in \mathbb{Z}$ such that

$$\forall (g, w) \in G \times W \quad P(\rho(g)(w)) = (\det g)^\lambda \, P(w). \quad (1)$$

Some properties of an invariant remains unchanged on each orbit, as the previous two examples illustrate.

In Example 1, the polynomial $P(x, y) = x - y$ is invariant for ρ (with $\lambda = 1$) since $g(x) - g(y) = (\det g)(x - y)$. If we restrict G to rigid motions (for which $\det g = 1$), the invariant P is constant on each orbit and its value discriminates the orbits.

In Example 2, a straightforward computation shows that the discriminant is a polynomial invariant on $S^2(\mathbb{R}^2)$ (with $\lambda = 2$):

$$\bar{b}^2 - \bar{a}\bar{c} = (\alpha\delta - \beta\gamma)^2 (b^2 - ac).$$

We observe that the sign $(+, -, 0)$ of this invariant is constant on each orbit and it entirely characterizes the orbits.

Covariant. A *covariant* for $\rho : G \to GL(W)$ is a polynomial invariant C for some action $\rho' : G \to GL(W \times (\mathbb{R}^n)^m)$ defined by $\rho'(g)(w, x_1, \ldots, x_m) = (\rho(g)(w), g(x_1), \ldots, g(x_m))$. We write $C \sim_w 0$ for $w \in W$ if $C(w, x_1, \ldots, x_m) = 0$ for all $(x_1, \ldots, x_m) \in (\mathbb{R}^n)^m$. By definition, either $C \sim 0$ or $C \nsim 0$ on a whole orbit.

2.2 Line geometry

Plücker quadric. A line ℓ of $\mathbb{P}^3$ can be represented by its (homogeneous) Plücker coordinates $\xi = [\xi_0 : \cdots : \xi_5]$ that fullfill the quadratic Plücker relation

$$q(\xi) = \xi_0\xi_3 + \xi_1\xi_4 + \xi_2\xi_5 = 0. \quad (2)$$

[1] W is a *representation* of the group G in other words.

[2] An affine motion g of $\mathbb{R}^n$ is represented by a matrix of GL_{n+1} in the form $\left(\begin{smallmatrix} 1 & 0 \\ \vec{t} & M \end{smallmatrix}\right)$ where $\vec{t}$ is the translation vector of g and $M \in GL_n$ its vector part. We define $\det g = \det M$.

It is the equation of a quadric $\mathbb{G}$ of $\mathbb{P}^5$ called the *Plücker quadric*. We denote by $\gamma(\ell)$ the Plücker coordinates of a line ℓ. For a line ℓ in $\mathbb{R}^3$, $\vec{v} = (\xi_0, \xi_1, \xi_2)$ is a direction of ℓ and (ξ_3, ξ_4, ξ_5) the moment of $\vec{v}$ with respect to the origin of $\mathbb{R}^3$. A complete presentation can be found in [9].

Span of lines. For a subset $H \subset \mathbb{P}^5$, $\operatorname{span} H$ is the minimal (for inclusion) projective subspace of $\mathbb{P}^5$ containing H. We define the *span* of a family of lines as the span of their Plücker coordinates. A family of k lines is said *linearly independent* if its span has dimension $k-1$. Any set $\mathcal{L}$ of lines contains a family $\mathcal{L}'$ with at most six linearly independent lines and any line of $\mathcal{L}$ is linearly dependent of those of $\mathcal{L}'$.

Conjugation. The quadric q defines a bilinear form $\odot$ called *side-operator*. We can observe that two lines ℓ and ℓ' meet if and only if $\gamma(\ell) \odot \gamma(\ell') = 0$ ([9]). Given a set H of $\mathbb{P}^5$, we define its *conjugate* as

$$H^\circ = \{x \in \mathbb{P}^5 \mid \forall h \in H \quad x \odot h = 0\}.$$

Geometry of quadratic forms [1, 13.3] shows that H° is a subspace of $\mathbb{P}^5$ of codimension $\dim(\operatorname{span} H)$ and it satisfies $(H^\circ)^\circ = \operatorname{span} H$. In terms of transversality, we immediately have

Observation 1 *$\gamma^{-1}(H^\circ \cap \mathbb{G})$ is the set of line transversals to all of the lines $\gamma^{-1}(H \cap \mathbb{G})$.*

Transformations preserving $\mathbb{G}$. We here consider $\mathbb{G}$ as a homogeneous subset of $\mathbb{R}^6$. A transformation $M \in GL_6$ globally preserves $\mathbb{G}$ if and only if there is $\mu \in \mathbb{R}^*$ such that $q(Mx) = \mu q(x)$ for any $x \in \mathbb{R}^6$ ([7, V.7]). Such transformations form a group $GO_6(q)$ called the *similarity group of* q. The subgroup of similarities M such that $\mu = 1$ and $\det M = 1$ is called the *rotation group of* q and is denoted by $SO_6(q)$.

Since a projective transformation g maps lines to lines and preserves incidences, it naturally induces a bijection of $\mathbb{G}$. The same is true for correlations. In fact ([9, Theorem 2.2.1]), such a bijection extends to a projective transformation $\wedge_4^2 g$ of $\mathbb{P}^5$ where $\wedge_n^k M$ is the k^{th} *compound matrix* of the matrix M of size n whose entries are the minors of size k of M.

Lemma 1 [9, Theorem 2.1.10] *$\mathbb{P}GO_6(q)$ is exactly the set of transformations of $\mathbb{P}^5$ induced through $\wedge_4^2$ by collineations and correlations of $\mathbb{P}^3$.*

3 Invariant–based method step by step

In this section, we unfold the invariant–based method for the following predicate: given the Plücker coordinates of four linearly independent lines, how many lines intersect all of them? We denote by X the set of inputs of a predicate.

Step 1: *Find all symmetries of any kind on the inputs X that leave invariant the outputs of the predicate and model them by a group G acting on X by $\psi : G \circlearrowleft X$.*

Here, the inputs of the predicate are quadruplets $(\xi_1, \dots, \xi_4)$ of linearly independent lines (X is an open subset of $\mathbb{G}^4$). Observe first that the order in which the input lines are considered does not matter, hence we can consider the action ψ_1 of the permutation group $\mathfrak{S}_4$ on X defined by

$$\psi_1(\sigma)(\xi_1, \dots, \xi_4) = (\xi_{\sigma(1)}, \dots, \xi_{\sigma(4)}).$$

Since a projective transformation preserves incidences between lines, the action of $\mathbb{P}GL_4$ on X defined by $\wedge_4^2$ leaves the output of the predicate invariant on an orbit. In other words, any change of coordinates does not change the number of line transversals to the input lines. By this process, lines are considered as *intrinsic* geometric objects. In the same way, we can consider the action of correlations that also preserves incidences between lines. According to Lemma 1, the action of collineations and correlations writes as $\psi_2 : \mathbb{P}GO_6(q) \circlearrowleft X$ defined by

$$\psi_2(g)(\xi_1, \dots, \xi_4) = (g(\xi_1), \dots, g(\xi_4)).$$

Altogether, we construct $G = \mathfrak{S}_4 \times \mathbb{P}GO_6(q)$ and $\psi : G \circlearrowleft X$ defined by $\psi(\sigma, g) = \psi_1(\sigma) \circ \psi_2(g)$. In the point of view of Erlangen's program, ψ encodes the geometry of "*sets* of four lines", that is, we identify two *ordered* families of line coordinates if they represent the same set of lines. At this step, our method differs from other approaches based on manipulations of coordinates, here only geometry matters.

Step 2: *Construct an encoding $\pi : X \to Y$ and a group action $\rho : G \circlearrowleft Y$ with finitely many orbits in $\pi(X)$ and "simulating" ψ on Y, i.e.*

$$\forall (g, x) \in G \times X \quad \rho(g)(\pi(x)) = \pi(\psi(g)(x)).$$

Hence the predicate has the same output on x and x' if $\pi(x)$ and $\pi(x')$ are in the same orbit of ρ.

According to Observation 1, the line transversals to an input line family $x \in X$ are exactly those of the span H of x, that is, $H^\circ \cap \mathbb{G}$. Since the four lines of x are linearly independent, H has dimension 3, thus H° has dimension one: it is a line of $\mathbb{P}^5$. As $\mathbb{G}$ is a quadric in $\mathbb{P}^5$, either H° is contained in $\mathbb{G}$ or H° intersects $\mathbb{G}$ in at most two points. The corresponding quadrics $H \cap \mathbb{G}$ are listed in Table 1. We observe that the type of $H \cap \mathbb{G}$ entirely characterizes the number of line transversals to the family x. So we consider the encoding $\pi : x \mapsto H$ that maps a line family x to its span and Y the set $\mathbb{G}_{4,6}$ of 3-dimensional subspaces of $\mathbb{P}^5$. We can show that $\pi(X) = Y$.

$H^\circ \cap \mathbb{G}$	$H \cap \mathbb{G}$	$q_{\mid H}$
2 points	hyperboloid	$(2,2)$
0 point	ellipsoid	$(3,1)$ or $(1,3)$
1 point	cone	$(2,1)$ or $(1,2)$
a line	two planes sharing a line	$(1,1)$

Table 1: Types of spans of four linearly independent lines. The third column denotes the inertia of the restriction to H of the quadratic form q given in (2).

Let us "simulate" ψ on Y. Since $\pi(\psi_1(\sigma)(x)) = \pi(x)$ for any $x \in X$, the action of $\mathfrak{S}_4$ has no effect on Y thus we can remove this group from G. We construct $\rho : \mathbb{P}GO_6(q) \circlearrowleft \mathbb{G}_{4,6}$ defined by $\rho(g) = \wedge_6^4 g$. By Witt's Theorem [1, 13.7.1 and 13.7.9], the orbits of ρ restricted to the group $\mathbb{P}SO_6(q)$ (Figure 2b) are characterized by the inertia of the quadric defined by $H \cap \mathbb{G}$ (see Table 1). Since a similarity with negative multiplicator μ change the sign of q (Equation (2)), the orbits of ρ (Figure 2a) are obtained by merging the previous orbits with the same *unsigned* inertia.

Step 3: *Use appropriate techniques to compute some polynomial invariants of ρ.*

Here, we consider $\rho' : SO_6(q) \to GL(\mathbb{R}^{15})$ (Y is an homogenous subset of $\mathbb{R}^{15}$) defined by $\rho'(g) = \wedge_6^4 g$. Using the symbolic method of [4], we obtain[3] a polynomial invariant of degree 2:

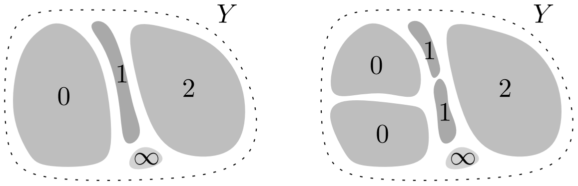
$$\Delta = y_4^2 + y_8^2 + y_{13}^2 - 2y_1y_{10} - 2y_{14}y_2 - 2y_3y_{15} + 2y_7y_5 + 2y_{11}y_6 + 2y_{12}y_9$$

and a covariant $Cov(y, x, x')$ defined on $Y \times (\mathbb{R}^6)^2$ with 21 distinct coefficients in x, x' of degree 2. Since Δ is a homogenous polynomial of degree 2, its sign remains unchanged up to nonzero scalings, thus is invariant on each orbit of $\rho : \mathbb{P}SO_6(q) \circlearrowleft Y$. Since Cov is homogeneous, Cov is a covariant of ρ.

Step 4: *Evaluate the previous polynomials on some representative of each orbit and observe if geometric situations are discriminated.*

Finally, we obtain the following algorithm for counting line transversals to a family x of four linearly independent lines. We compute $y = \pi(x)$. If $\Delta(y) > 0$, there are 2 line transversals. If $\Delta(y) < 0$, there is no transversal. Otherwise, if $Cov \sim_y 0$, then there are infinitely many transversals, else there is only one.

[3] In a symbolic form, Δ is written as the bracket polynomial $[\alpha^{(4)}ab][\beta^{(4)}ab]$ and Cov as $[\alpha^{(4)}au][\beta^{(4)}av]$ where α, β are letters representing $\mathbb{R}^{15} = \Lambda^4\mathbb{R}^6$, a, b representing $S^2\mathbb{R}^6$ (it simulates $SO_6(q) \subset GL_6$) and u, v representing $\mathbb{R}^6$.



(a) $G = \mathbb{P}GO_6(q)$ (b) $G = \mathbb{P}SO_6(q)$

Figure 2: Orbits of ρ.

4 Conclusion

For counting line transversals to four linearly independent lines, our invariant–based method provides the same polynomial Δ as [2] but polynomials of higher degrees than in [2] to discriminate the degenerate cases. The same technique applies for five lines and gives rise to the same polynomial as [2]. Finally, some polynomials involved in predicates appear as invariants of group actions, that is they originate from the geometry of the problem. They might be essential in any evaluation strategy for a predicate, based on polynomials. This point of view seems to be a promising approach to tackle optimality questions on predicates.

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