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Abstract: Subspace methods enjoy some popularity, especially in mechanical engineering, where large model orders have to be considered. In the context of detecting changes in the structural properties and the modal parameters linked to them, some subspace based fault detection residual has been recently proposed and applied successfully. However, most works assume that the unmeasured ambient excitation level during measurements of the structure in the reference and possibly damaged condition stays constant, which is not satisfied by any application. This paper addresses the problem of robustness of such fault detection methods. An efficient subspace-based fault detection test is derived that is robust to excitation change but also to numerical instabilities that could arise easily in the computations. Furthermore, the fault detection test is extended to the Unweighted Principal Component subspace algorithm.

Key-words: Fault detection; Subspace methods; Robustness; Linear systems

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Détection robuste de fautes par méthodes des sous espaces

Résumé : Les méthodes des sous espaces jouissent d'une certaine popularité, notamment en ingénierie mécanique, où des modèles de grande taille sont à considérer. Dans l'objectif de détecter des changements dans les propriétés structurelles - ainsi que dans les paramètres modaux associés, un résidu sous espace pour la détection de pannes a été récemment proposé, puis appliqué avec succès. Cependant, généralement, une hypothèse restrictive est présupposée, c'est à dire que les propriétés de l'excitation ambiante et non mesurée restent constantes entre les états de référence et les états possiblement endommagés de la structure. Cette hypothèse n'est pas valide pour la plupart des cas d'étude. Ce travail adresse le problème de la robustesse d'un tel résidu. Un nouveau résidu numériquement plus efficace et plus robuste est proposé. De plus, ce test de détection est adapté à d'autres classes que les méthodes des sous espaces par covariance.

Mots-clés : Détection de fautes; Méthodes des sous espaces; Robustesse; Systèmes linéaires

1 Introduction

In the last ten years, monitoring the integrity of the civil infrastructure has been an active research topic, including in connected areas such as automatic control, for mastering the aging of bridges, or the resistance to seismic events and the protection of the cultural heritage.

Damage detection in the context of mechanical engineering corresponds to detecting changes in the modal parameters. Robustness to non-stationary excitation has been already addressed in [6]. Then, a fault detection algorithm based on a residual associated with an output-only subspace identification algorithm and a χ^2-test built on that residual has been proposed in [2]. This subspace-based residual uses the left null space of a nominal observability matrix of the system in a reference state, which is the same as the corresponding subspace matrix built from the output data. In a possibly damaged state it is then checked, whether the corresponding subspace matrix is still well described by the null space of the reference state, using a χ^2-test.

In practice, this class of tests asks for a robust implementation, dealing with

- data measured under varying ambient excitation,
- highly dimensional observations (many sensors),
- sparse data (short measurements).

This paper addresses these points. A residual function robust to excitation change is considered as in [12]. Section 3 is devoted to the analysis of the associated χ^2-test. In Section 4, an efficient computation formulation for its covariance is proposed, reducing the computational cost due to high dimensional observations.

Furthermore, the computation of the χ^2-test itself is a numerically critical issue, as it involves the inversion of big low-rank matrices. In Section 5 a numerically robust scheme is proposed. Finally, a new residual covariance formulation for the Unweighted Principal Component (UPC) subspace algorithm is formulated in Section 6.

2 Stochastic Subspace Identification (SSI) and Fault Detection

2.1 General SSI Algorithm

Consider the discrete time model in state space form:

$$\left\{ \begin{array}{rcl} X_{k+1} & = & AX_k + W_{k+1} \\ Y_k & = & CX_k \end{array} \right. \tag{1}$$

with the state $X \in \mathbb{R}^n$, the output $Y \in \mathbb{R}^r$, the state transition matrix $A \in \mathbb{R}^{n \times n}$ and the observation matrix $C \in \mathbb{R}^{r \times n}$. The state noise W is unmeasured and assumed to be Gaussian, zero-mean, white, with a covariance Σ_W.

A subset of the r sensors may be used for reducing the size of the matrices in the identification process, see e.g. [7]. These sensors are called projection channels or reference sensors. Let r_0 be the number of reference sensors ($r_0 \leq r$)

and p and q chosen parameters with $(p+1)r \geq qr_0 \geq n$. From the output data $(Y_k)_{k=1,\ldots,N+p+q}$ a matrix $\mathcal{H}_{p+1,q} \in \mathbb{R}^{(p+1)r \times qr_0}$ is built according to a chosen SSI algorithm, see e.g. [4] for an overview. The matrix $\mathcal{H}_{p+1,q}$ will be called "subspace matrix" in the following and enjoys asymptotically the factorization property

$$\mathcal{H}_{p+1,q} = \mathcal{O}_{p+1}\mathcal{Z}_q \tag{2}$$

into the matrix of observability

$$\mathcal{O}_{p+1} \stackrel{\text{def}}{=} \begin{bmatrix} C^T & (CA)^T & \ldots & (CA^p)^T \end{bmatrix}^T \tag{3}$$

and a matrix $\mathcal{Z}_q$ depending on the selected SSI algorithm. Examples of two SSI algorithms are given in Section 6.1.

The observability matrix $\mathcal{O}_{p+1}$ is obtained from a thin Singular Value Decomposition (SVD) of the matrix $\mathcal{H}_{p+1,q}$ and its truncation at the desired model order n:

$$\begin{aligned} \mathcal{H}_{p+1,q} &= U\Delta V^T \\ &= \begin{bmatrix} U_1 & U_0 \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_0^T \end{bmatrix}, \qquad (4) \\ \mathcal{O}_{p+1} &= U_1\Delta_1^{1/2}, \qquad (5) \end{aligned}$$

with the matrices

$$U_1 = \begin{bmatrix} u_1 & \ldots & u_n \end{bmatrix}, V_1 = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}, \Delta_1 = \text{diag}\{\sigma_1, \ldots, \sigma_n\}$$

containing the first n left and right singular vectors, and singular values. The observation matrix C is then found in the first block-row of the observability matrix $\mathcal{O}_{p+1}$. The state transition matrix A is obtained from the shift invariance property of $\mathcal{O}_{p+1}$, namely as the least squares solution of

$$\mathcal{O}_{p+1}^{\uparrow} A = \mathcal{O}_{p+1}^{\downarrow},$$

where

$$\mathcal{O}_{p+1}^{\uparrow} \stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}, \quad \mathcal{O}_{p+1}^{\downarrow} \stackrel{\text{def}}{=} \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^p \end{bmatrix}.$$

The eigenstructure $(\lambda, \varphi_\lambda)$ of system (1) results from

$$\det(A - \lambda I) = 0, \quad A\phi_\lambda = \lambda\phi_\lambda, \quad \varphi_\lambda = C\phi_\lambda,$$

where λ ranges over the set of eigenvalues of A.

For simplicity, let p and q be given and skip the subscripts related to p and q of $\mathcal{H}_{p+1,q}$, $\mathcal{O}_{p+1}$ and $\mathcal{Z}_q$ in the following. Also, the subscripts of the zero matrix $0_{s,t}$ of size $s \times t$ and identity matrix I_s of size $s \times s$ may be skipped, when their size is obvious.

2.2 Fault Detection Algorithm

In [2] a statistical fault detection method was described, which can be used with subspace algorithms satisfying factorization property (2). This fault detection method consists in comparing characteristics of a reference state with a subspace matrix $\hat{\mathcal{H}}$ computed on a new data sample $(Y_k)_{k=1,\dots,N+p+q}$, corresponding to an unknown, possibly damaged state, assuming that $\hat{\mathcal{H}}$ is a consistent estimate of $\mathcal{H}$.

To compare the states, the left null space matrix S of the observability matrix of the reference state is computed, which is also the left null space of the subspace matrix at the reference state because of factorization property (2). The characteristic property of a system in the reference state then writes $S^T\hat{\mathcal{H}} = 0$ and the residual vector

$$\zeta_1 \stackrel{\text{def}}{=} \sqrt{N}\,\text{vec}(S^T\hat{\mathcal{H}}) \tag{6}$$

describes the difference between the state of matrix $\hat{\mathcal{H}}$ and the reference state.

Let θ be a vector containing a canonical parameterization of the actual state of the system (see [2] for details) and θ_0 the parameterization of the reference state. The damage detection problem is to decide whether the subspace matrix $\hat{\mathcal{H}}$ from the (possibly damaged) system (corresponding to θ) is still well described by the characteristics of the reference state (corresponding to θ_0) or not. This is done by testing between the hypotheses

$$\begin{aligned} \text{H}_0 :\ \theta &= \theta_0 && \text{(reference system)},\\ \text{H}_1 :\ \theta &= \theta_0 + \delta/\sqrt{N} && \text{(faulty system)}, \end{aligned} \tag{7}$$

where δ is unknown but fixed. This is called the local approach, and the following proposition is used to test between both hypotheses.

Proposition 1 ([2]). *The residual ζ_1 is asymptotically Gaussien for large N, and the test between the hypotheses H_0 and H_1 is achieved through the χ^2-test*

$$\chi_1^2 = \zeta_1^T\Sigma_1^{-1}\mathcal{J}_1(\mathcal{J}_1^T\Sigma_1^{-1}\mathcal{J}_1)^{-1}\mathcal{J}_1^T\Sigma_1^{-1}\zeta_1 \tag{8}$$

and comparing it to a threshold, where $\mathcal{J}_1$ and Σ_1 are consistent estimates of the sensitivity and covariance of ζ_1. Both can be estimated in the reference state under the assumption that the covariance Σ_W of the input noise W of the system does not change between the reference state and the possibly damaged state.

The computation of the Jacobian $\mathcal{J}_1$ needs a parameterization of the system, where the eigenvalues and mode shapes of the reference system must be known, and is explained in detail in [2]. In [1] an empirical non-parametric version of the test is proposed, where $\mathcal{J}_1$ is set as the identity matrix.

The computation of the covariance matrix Σ_1 depends on

$$\Sigma_{\mathcal{H}} \stackrel{\text{def}}{=} \lim_N \text{cov}(\sqrt{N}\,\text{vec}\,\hat{\mathcal{H}}), \tag{9}$$

which is dependent on the chosen subspace algorithm. For simplicity, $\Sigma_{\mathcal{H}}$ will be called *covariance of the subspace matrix*. In Section 6, its estimation is explained for covariance-driven SSI and extended to SSI with the Unweighted Principal Component algorithm. Finally, the covariance matrix Σ_1 can be obtained from

$$\Sigma_1 = (I \otimes S^T)\Sigma_{\mathcal{H}}(I \otimes S) \tag{10}$$

due to (6), where $\otimes$ denotes the Kronecker product.

3 Fault Detection Robust to Excitation Change

In Section 2.2 it was assumed that the unmeasured state noise W is stationary and does not change between the reference state and a possibly damaged state of the system. In practice, however, Σ_W may change between different measurements of the system due to different environmental factors (wind, traffic, ...), while the excitation is still assumed to be stationary during one measurement. Now, two modifications of the fault detection algorithm are described, that take a changing excitation into account.

3.1 Covariance Estimation Robust to Excitation Change

The property, that $\Sigma_{\mathcal{H}}$ is (asymptotically) equal under both hypotheses H_0 and H_1, is not true anymore in the case of a change of Σ_W. A simple measure to evade this problem is the computation of $\Sigma_{\mathcal{H}}$ under H_1. This was already applied successfully, e.g. in [9]. However, the computation of $\Sigma_{\mathcal{H}}$ requires many samples and hence it would be favorable to compute it in the reference state. This is possible with a residual which is robust to excitation change in the next section.

3.2 χ^2-test Robust to Excitation Change

A new possibility to compensate a change in the excitation level or excitation geometry, is the use of a residual function that is robust to these changes. In this section, a χ^2-test on such a residual is derived.

Using the factorization property (4), the orthonormal matrix U_1, satisfying $S^T U_1 = 0$ in the reference state, relates to $\mathcal{O}$ and (3) through the invertible matrix Δ_1, but is independent of the excitation. Then, a residual that is robust to excitation change can be defined as

$$\zeta_2 = \sqrt{N}\,\mathrm{vec}(S^T \hat{U}_1), \tag{11}$$

where $\hat{U}_1$ is computed on new data in the possibly damaged state from the SVD

$$\hat{\mathcal{H}} = \begin{bmatrix} \hat{U}_1 & \hat{U}_0 \end{bmatrix} \begin{bmatrix} \hat{\Delta}_1 & 0 \\ 0 & \hat{\Delta}_0 \end{bmatrix} \begin{bmatrix} \hat{V}_1^T \\ \hat{V}_0^T \end{bmatrix}.$$

Note that the SVD is a continuous but in general non-unique decomposition. Hence, the choice of basis of $\hat{U}_1$ has to be fixed by using a unique SVD.[1]

Proposition 2. *The χ^2-test between the hypotheses H_0 and H_1 defined in (7) using the robust residual ζ_2 is achieved through*

$$\chi_2^2 = \zeta_2^T \Sigma_2^{-1} \mathcal{J}_2 (\mathcal{J}_2^T \Sigma_2^{-1} \mathcal{J}_2)^{-1} \mathcal{J}_2^T \Sigma_2^{-1} \zeta_2 \tag{12}$$

and comparing it to a threshold, where $\mathcal{J}_2$ and Σ_2 are consistent estimates of the sensitivity and covariance of ζ_2.

[1] This can e.g. be done by multiplying the appropriate columns of $\hat{U}_1$ and $\hat{V}_1$ with -1, if the first value of the column of $\hat{U}_1$ is negative.

Proof. Following the lines of [6] and [5], $\sqrt{N}\ \mathrm{vec}\left(\hat{U}_1 - U_1\right)$ converges in distribution to a Gaussian $\mathcal{N}(0, \Sigma_{U_1})$, where

$$\Sigma_{U_1} = \lim_N \mathrm{cov}(\sqrt{N}\,\mathrm{vec}\,\hat{U}_1) = (\Delta_1^{-1} V_1^T \otimes I)\,\Sigma_{\mathcal{H}}\,(V_1 \Delta_1^{-1} \otimes I).$$

It follows that the the residual is asymptotically Gaussian, with a covariance satisfying $\Sigma_2 = (I \otimes S^T)\,\Sigma_{U_1}\,(I \otimes S)$. The proof finishes as in Proposition 1. □

The computation of the sensitivity $\mathcal{J}_2$ is analogous to the computation of the sensitivity $\mathcal{J}_1$ of ζ_1 in [2], where $\mathcal{H}$ has to be replaced by U_1. An efficient computation of Σ_2 is addressed in the following section.

4 Efficient computation of Σ_2

The covariance Σ_2 of the robust residual ζ_2 defined in (11) depends on the covariance of vec U_1 and hence on the first n singular vectors of $\mathcal{H}$, which can be linked to the covariance of the subspace matrix $\mathcal{H}$ by a sensitivity analysis through

$$\mathrm{cov}(\mathrm{vec}\ \hat{U}_1) = \mathcal{J}_{U_1} \mathrm{cov}(\mathrm{vec}\,\hat{\mathcal{H}}) \mathcal{J}_{U_1}^T, \tag{13}$$

where $\mathcal{J}_{U_1}$ is the sensitivity of the left singular vectors vec U_1 with respect to vec $\mathcal{H}$. The computation of $\mathcal{J}_{U_1}$ is numerically costly and was proposed in [10] (see Proposition 13 in Appendix A). A more efficient computation of $\mathcal{J}_{U_1}$ is derived in this section.

Definition 3. *Let $E_{k_1,k_2}^{l_1,l_2} \in \{0,1\}^{l_1 \times l_2}$ be matrices whose entries are zeros, except the entry at position (k_1, k_2) which is one, then the permutation matrix*

$$\mathcal{P} \stackrel{\mathrm{def}}{=} \sum_{k_1=1}^{(p+1)r} \sum_{k_2=1}^{qr_0} E_{k_1,k_2}^{(p+1)r,qr_0} \otimes E_{k_2,k_1}^{qr_0,(p+1)r}$$

is defined. Furthermore, let define for $j = 1, \ldots, n$:

$$K_j \stackrel{\mathrm{def}}{=} -\frac{\mathcal{H}}{\sigma_j}\left(I_{qr_0} + \begin{bmatrix} 0_{qr_0-1,qr_0} \\ 2v_j^T \end{bmatrix} - \frac{\mathcal{H}^T\mathcal{H}}{\sigma_j^2}\right)^{-1},$$

$$\hat{E}_j \stackrel{\mathrm{def}}{=} \left[I_{(p+1)r} + K_j\left(-\frac{\mathcal{H}^T}{\sigma_j} + \begin{bmatrix} 0_{qr_0-1,(p+1)r} \\ u_j^T \end{bmatrix}\right) \quad -K_j\right], \tag{14}$$

$$F_j \stackrel{\mathrm{def}}{=} \frac{1}{\sigma_j}\begin{bmatrix} v_j^T \otimes (I_{(p+1)r_0} - u_j u_j^T) \\ (u_j^T \otimes (I_{qr_0} - v_j v_j^T))\mathcal{P} \end{bmatrix}. \tag{15}$$

Proposition 4. *With $\hat{E}_j$, F_j, $j = 1, \ldots, n$, defined in* (14) *and* (15), *$\mathcal{J}_{U_1}$ holds*

$$\mathcal{J}_{U_1} = \left[(\hat{E}_1 F_1)^T \quad \ldots \quad (\hat{E}_n F_n)^T\right]^T, \tag{16}$$

altogether involving n matrix inversions of size qr_0.

Proof. See Appendix A. □

Remark 5. *The computation of $\mathcal{J}_{U_1}$ in (16) is less costly than computing n pseudoinverses of square matrices of size $(p+1)r + qr_0$ that are necessary for the computation in [10] as stated in Proposition 13 in Appendix A.*

With (11) and (13), the covariance computation of the robust residual ζ_2 finishes with

$$\Sigma_2 = (I \otimes S^T)\, \mathcal{J}_{U_1}\, \Sigma_{\mathcal{H}}\, \mathcal{J}_{U_1}^T\, (I \otimes S). \tag{17}$$

As ζ_2 is independent of the excitation, both $\mathcal{J}_2$ and Σ_2 can be estimated in the reference state.

5 Numerical Robustness

In this section, special care is taken of numerical aspects of the computation of the χ^2-tests in (8) or (12), as matrix inversions of big and sometimes rank deficient matrices are involved. This applies to the fault detection tests of Sections 2.2 and 3.2 and hence the subscripts of ζ, Σ, $\mathcal{J}$ and χ^2 are skipped.

First, results from [13] for a numerical stable computation of the χ^2-test are extended and rank conditions on the involved matrices are derived. Second, an efficient computation of the half inverse of the covariance matrix using sampled data is derived, whose computation takes a key role in the robust χ^2-test.

Lemma 6. *Let $\Sigma^{-1/2}$ be a half inverse of the covariance matrix, such that $\Sigma^{-1} = (\Sigma^{-1/2})^T \Sigma^{-1/2}$. If*

$$\Sigma^{-1/2}\mathcal{J} \quad \textit{is full column rank}, \tag{18}$$

the χ^2-test writes as

$$\chi^2 = \xi^T \xi \quad \textit{with} \quad \xi = Q^T \Sigma^{-1/2} \zeta, \tag{19}$$

where Q is obtained from the thin QR decomposition of $\Sigma^{-1/2}\mathcal{J} = QR$.

Note that condition (18) is weaker than asking for Σ being positive definite in [13].

Corollary 7. *Let c be the number of columns of $\mathcal{J}$. If*

$$c \geq \operatorname{rank}(\Sigma^{-1/2}\mathcal{J}), \tag{20}$$

condition (18) is violated and the χ^2-test boils down to

$$\chi^2 = \zeta^T \Sigma^{-1} \zeta = \xi^T \xi \quad \textit{with} \quad \xi = \Sigma^{-1/2} \zeta. \tag{21}$$

Proof. With condition (20), the matrix $\mathcal{J}^T \Sigma^{-1} \mathcal{J}$ in the χ^2-test is not invertible anymore. Taking its pseudoinverse instead, leads to $\chi^2 = \zeta^T \Sigma^{-1} \zeta$ by plugging into (8) or (12) the thin QR decomposition of $\mathcal{J}^T (\Sigma^{-1/2})^T$. □

When an estimate of $\Sigma_{\mathcal{H}}$ is used in the computation of Σ, using samples h_k, $k = 1, \ldots, n_b$, obtained from instances of the subspace matrix $\mathcal{H}$ by cutting the sensor data into n_b independent blocks (see Section 6 for some explicit formulae), condition (20) is fulfilled, if $n_b \leq c$.

In the χ^2-test in (19) or (21), a computation of $\widehat{\Sigma}^{-1/2}$ from the estimated covariance matrix $\widehat{\Sigma}$ is needed, which can pose a numerical problem when n_b is insufficient for assuring full rank of $\widehat{\Sigma}$. In this case, the pseudoinverse of $\widehat{\Sigma}$ is used instead of its inverse, and $\widehat{\Sigma}^{-1/2}$ is computed, such that $\widehat{\Sigma}^{\dagger} = (\widehat{\Sigma}^{-1/2})^T\widehat{\Sigma}^{-1/2}$.

A factorization property for an estimate of $\Sigma_{\mathcal{H}}$ will now be used for an efficient computation of $\Sigma^{-1/2}$.

Lemma 8. *Let $\widehat{\Sigma}_h$ be an estimate of a covariance Σ_h with*

$$\widehat{\Sigma}_h = \frac{1}{n_b - 1}\sum_{k=1}^{n_b}(h_k - \bar{h})(h_k - \bar{h})^T,$$

using samples h_k, $k = 1, \ldots, n_b$, $\bar{h} \stackrel{\text{def}}{=} \frac{1}{n_b}\sum_{k=1}^{n_b} h_k$, and

$$\mathcal{K} \stackrel{\text{def}}{=} \frac{1}{\sqrt{n_b - 1}}\begin{bmatrix}\tilde{h}_1 & \tilde{h}_2 & \ldots & \tilde{h}_{n_b}\end{bmatrix} \quad \textit{with} \quad \tilde{h}_k \stackrel{\text{def}}{=} h_k - \bar{h}.$$

Then, $\widehat{\Sigma}_h = \mathcal{K}\mathcal{K}^T$.

Proposition 9. *With $\Sigma = \mathcal{A}\Sigma_{\mathcal{H}}\mathcal{A}^{\mathcal{T}}$ corresponding to* (10) *or* (17) *and the appropriate matrix $\mathcal{A}$, and using the notation of Lemma 8, a half (pseudo-)inverse $\widehat{\Sigma}^{-1/2}$ of the estimated $\widehat{\Sigma}$ is obtained from*

$$\widehat{\Sigma}^{-1/2} = (\mathcal{A}\mathcal{K})^{\dagger}, \tag{22}$$

where $\mathcal{K}$ is related to the chosen SSI algorithm.

Note that Proposition 9 provides an efficient way to compute the half (pseudo-)inverse of the covariance matrix in case of few available samples n_b. In (22), the pseudoinverse of matrix $\mathcal{A}\mathcal{K}$ is computed, which is of size $\dim\zeta \times n_b$. If $n_b < \dim\zeta$, the computation of $\widehat{\Sigma}^{-1/2}$ in (22) is less costly than computing $\widehat{\Sigma}^{-1/2}$ directly from $\widehat{\Sigma}$.

Moreover, the computation of the (pseudo-)inverse of matrix $\mathcal{A}\mathcal{K}$ is numerically more stable than the half (pseudo-)inverse of the squared matrix $\widehat{\Sigma} = \mathcal{A}\mathcal{K}(\mathcal{A}\mathcal{K})^T$. Hence, using (22) may be favorable, even if $n_b \geq \dim\zeta$.

6 Covariance Estimation of Subspace Matrix for two SSI algorithms

In this section, two SSI algorithms, namely the covariance-driven SSI and the data-driven Unweighted Principal Component (UPC) SSI, are introduced to obtain subspace matrix estimators $\hat{\mathcal{H}}$ from observed data. Then, the covariance estimate $\widehat{\Sigma}_{\mathcal{H}}$ is stated for the covariance-driven and extended to the UPC subspace matrix, as needed in the presented fault detection algorithms.

6.1 Covariance and Data-Driven SSI

From the output data, "future" and "past" data matrices

$$\mathcal{Y}^+ = \begin{bmatrix} Y_{q+1} & Y_{q+2} & \vdots & Y_{N+q} \\ Y_{q+2} & Y_{q+3} & \vdots & Y_{N+q+1} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{q+p+1} & Y_{q+p+2} & \vdots & Y_{N+q+p} \end{bmatrix}, \tag{23}$$

$$\mathcal{Y}^- = \begin{bmatrix} Y_q^{(\mathrm{ref})} & Y_{q+1}^{(\mathrm{ref})} & \vdots & Y_{N+q-1}^{(\mathrm{ref})} \\ Y_{q-1}^{(\mathrm{ref})} & Y_q^{(\mathrm{ref})} & \vdots & Y_{N+q-2}^{(\mathrm{ref})} \\ \vdots & \vdots & \vdots & \vdots \\ Y_1^{(\mathrm{ref})} & Y_2^{(\mathrm{ref})} & \vdots & Y_N^{(\mathrm{ref})} \end{bmatrix} \tag{24}$$

are built with the parameters p and q introduced in Section 2.1, where for all samples k, the vector $Y_k^{(\mathrm{ref})} \in \mathbb{R}^{r_0}$ contains the reference sensor data, which is a subset of Y_k. These data matrices are normalized with respect to their number of columns to

$$\tilde{\mathcal{Y}}^+ \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{N}}\mathcal{Y}^+, \quad \tilde{\mathcal{Y}}^- \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{N}}\mathcal{Y}^-. \tag{25}$$

For the *covariance-driven* SSI (see also [3], [7]), the subspace matrix

$$\mathcal{H}^{\mathrm{cov}} \stackrel{\mathrm{def}}{=} \tilde{\mathcal{Y}}^+ \, \tilde{\mathcal{Y}}^{-\,T} \tag{26}$$

with $\mathcal{H}^{\mathrm{cov}} \in \mathbb{R}^{(p+1)r \times qr_0}$ is built, being an estimate of $\mathcal{H}$, which enjoys the factorization property (2) where $\mathcal{Z}$ is the controllability matrix.

For the *data-driven* SSI with the Unweighted Principal Component (UPC) algorithm (see also [11], [7]), the matrix

$$\mathcal{H}^{\mathrm{dat}} \stackrel{\mathrm{def}}{=} \tilde{\mathcal{Y}}^+ \, \tilde{\mathcal{Y}}^{-\,T} (\tilde{\mathcal{Y}}^- \, \tilde{\mathcal{Y}}^{-\,T})^{-1} \tilde{\mathcal{Y}}^- \tag{27}$$

with $\mathcal{H}^{\mathrm{dat}} \in \mathbb{R}^{(p+1)r \times N}$ is defined, which enjoys the factorization property (2) where $\mathcal{Z}$ is the Kalman filter state matrix. $\mathcal{H}^{\mathrm{dat}}$ can be a very large matrix when lots of samples are available. In practice, the numerically stable thin RQ decomposition

$$\begin{bmatrix} \tilde{\mathcal{Y}}^- \\ \tilde{\mathcal{Y}}^+ \end{bmatrix} = RQ = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \tag{28}$$

is done at first, where R and Q are partitioned as stated in (28). Then, with (27) it follows $\mathcal{H}^{\mathrm{dat}} = R_{21}Q_1$. As Q_1 is a matrix with orthogonal rows, an SVD of R_{21} leads to the same observability matrix (up to a change of the modal basis) as an SVD of $\mathcal{H}^{\mathrm{dat}}$ in (5), see also [11] for details. Hence, the subspace matrix estimate

$$\mathcal{H}^{\mathrm{dat,R}} \stackrel{\mathrm{def}}{=} R_{21} \tag{29}$$

is defined, with $\mathcal{H}^{\mathrm{dat,R}} \in \mathbb{R}^{(p+1)r \times qr_0}$ and R_{21} from (28).

6.2 Covariance of the Subspace Matrix

For the derivation of the covariance $\Sigma_{\mathcal{H}}$, the data matrices $\mathcal{Y}^+$ and $\mathcal{Y}^-$ from (23) and (24) are split into n_b blocks

$$\mathcal{Y}^+ = \begin{bmatrix} \mathcal{Y}_1^+ & \dots & \mathcal{Y}_{n_b}^+ \end{bmatrix}, \quad \mathcal{Y}^- = \begin{bmatrix} \mathcal{Y}_1^- & \dots & \mathcal{Y}_{n_b}^- \end{bmatrix}. \tag{30}$$

For simplicity, each block $\mathcal{Y}_j^+$ and $\mathcal{Y}_j^-$ may have the same length N_b, such that $n_b \cdot N_b = N$. Each block may be long enough to assume statistical independence between the blocks. They are normalized with respect to their length to

$$\tilde{\mathcal{Y}}_j^+ \stackrel{\text{def}}{=} \frac{1}{\sqrt{N_b}} \mathcal{Y}_j^+, \quad \tilde{\mathcal{Y}}_j^- \stackrel{\text{def}}{=} \frac{1}{\sqrt{N_b}} \mathcal{Y}_j^-, \quad j = 1, \dots, n_b. \tag{31}$$

6.2.1 Covariance-Driven Case

The covariance of the subspace matrix in the covariance-driven case follows easily from the covariance of the sample mean and was used e.g. in [10]. On each normalized data block from (31) a subspace matrix estimate $\mathcal{H}_j^{\text{cov}}$ is built with

$$\mathcal{H}_j^{\text{cov}} \stackrel{\text{def}}{=} \tilde{\mathcal{Y}}_j^+ \, \tilde{\mathcal{Y}}_j^{-\,T}. \tag{32}$$

Proposition 10. *A covariance estimate of the covariance-driven subspace matrix for $\Sigma_{\mathcal{H}}$ in (9) writes as*

$$\widehat{\Sigma}_{\mathcal{H}^{\text{cov}}} = \frac{N}{n_b(n_b-1)} \sum_{j=1}^{n_b} \left(\operatorname{vec} \mathcal{H}_j^{\text{cov}} - \operatorname{vec} \mathcal{H}^{\text{cov}}\right) \left(\operatorname{vec} \mathcal{H}_j^{\text{cov}} - \operatorname{vec} \mathcal{H}^{\text{cov}}\right)^T.$$

Proof. See Appendix B. □

6.2.2 Data-Driven Case

Now, the computation of the covariance of the subspace matrix $\mathcal{H}^{\text{dat,R}}$ (see (28) – (29)) is derived. On each normalized data block from (31) a subspace matrix $\mathcal{H}_j^{\text{dat,R}}$ is constructed by the thin RQ decomposition of

$$\begin{bmatrix} \tilde{\mathcal{Y}}_j^- \\ \tilde{\mathcal{Y}}_j^+ \end{bmatrix} = R^{(j)} Q^{(j)}, \tag{33}$$

where $R^{(j)}$ and $Q^{(j)}$ are partitioned into

$$R^{(j)} = \begin{bmatrix} R_{11}^{(j)} & 0 \\ R_{21}^{(j)} & R_{22}^{(j)} \end{bmatrix}, \quad Q^{(j)} = \begin{bmatrix} Q_1^{(j)} \\ Q_2^{(j)} \end{bmatrix}. \tag{34}$$

Then, a subspace matrix estimate on each data block is

$$\mathcal{H}_j^{\text{dat,R}} \stackrel{\text{def}}{=} R_{21}^{(j)}. \tag{35}$$

Proposition 11. *Let $\breve{Q}_{11}^{(j)}$, $j = 1, \dots, n_b$, be defined from partitioning the Q matrix of the thin RQ decomposition*

$$\begin{bmatrix} R_{11}^{(1)} & \dots & R_{11}^{(n_b)} \end{bmatrix} = \breve{R}_{11} \begin{bmatrix} \breve{Q}_{11}^{(1)} & \dots & \breve{Q}_{11}^{(n_b)} \end{bmatrix}. \tag{36}$$

Then, a covariance estimate of the UPC subspace matrix for $\Sigma_{\mathcal{H}}$ *in* (9) *writes as*

$$\widehat{\Sigma}_{\mathcal{H}^{\mathrm{dat,R}}} = \frac{N}{n_b - 1} \sum_{j=1}^{n_b} \left(\mathrm{vec} \left(\mathcal{H}_j^{\mathrm{dat,R}} \breve{Q}_{11}^{(j)\,T} \right) - \bar{M} \right) \cdot \left(\mathrm{vec} \left(\mathcal{H}_j^{\mathrm{dat,R}} \breve{Q}_{11}^{(j)\,T} \right) - \bar{M} \right)^T$$

with

$$\bar{M} \stackrel{\mathrm{def}}{=} \frac{1}{n_b} \sum_{j=1}^{n_b} \mathrm{vec} \left(\mathcal{H}_j^{\mathrm{dat,R}} \breve{Q}_{11}^{(j)\,T} \right).$$

Proof. See Appendix B. □

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Appendices

A Efficient Computation for Left Singular Vector Sensitivities

Proposition 12 ([8]). *With*

$$E_j \stackrel{\text{def}}{=} \begin{bmatrix} I_{(p+1)r} & -\frac{\mathcal{H}}{\sigma_j} \\ -\frac{\mathcal{H}^T}{\sigma_j} & I_{qr_0} \end{bmatrix}$$

and F_j defined in (15)*, the sensitivity of the concatenated j-th left and right singular vector is a solution of*

$$E_j \begin{bmatrix} \Delta u_j \\ \Delta v_j \end{bmatrix} = F_j \, \Delta(\operatorname{vec} \mathcal{H}). \tag{37}$$

As E_j is a singular matrix (having rank $(p+1)r + qr_0 - 1$), one possible solution of (37) is $E_j^\dagger F_j$:

Proposition 13 ([10]). *Following Proposition 12, the sensitivity $\mathcal{J}_{U_1}$ of* vec U_1 *with respect to* vec $\mathcal{H}$ *writes as*

$$\mathcal{J}_{U_1} = S_1 \begin{bmatrix} (E_1^\dagger F_1)^T & \dots & (E_n^\dagger F_n)^T \end{bmatrix}^T.$$

with selection matrix $S_1 = I_n \otimes \begin{bmatrix} I_{(p+1)r} & 0_{(p+1)r,qr_0} \end{bmatrix}$.

Proof. (Proposition 4). Another possible solution of (37) is achieved by adding the condition $u_j^T \Delta u_j + v_j^T \Delta v_j = 0$ (resulting from the orthogonality of the singular vectors) to the system of equations (37), which was also suggested in [8], leading to a system of full column rank. Without loss of generality, this condition can be added to the last row of the matrices in (37), satisfying

$$\tilde{E}_j \begin{bmatrix} \Delta u_j \\ \Delta v_j \end{bmatrix} = F_j \, \Delta(\operatorname{vec} \mathcal{H}) \tag{38}$$

with

$$\tilde{E}_j \stackrel{\text{def}}{=} E_j + \begin{bmatrix} 0_{m,(p+1)r} & 0_{m,qr_0} \\ u_j^T & v_j^T \end{bmatrix}$$

and $m \stackrel{\text{def}}{=} (p+1)r + qr_0 - 1$. Then, Δu_j consists of the first $(p+1)r$ entries of the solution of (38) and writes as

$$\Delta u_j = \begin{bmatrix} I_{(p+1)r} & 0_{(p+1)r,qr_0} \end{bmatrix} \tilde{E}_j^{-1} \tilde{F}_j \, \Delta(\operatorname{vec} \mathcal{H}). \tag{39}$$

With the block matrix inversion formula for $\tilde{E}_j$

$$\begin{bmatrix} O & P \\ Q & R \end{bmatrix}^{-1} = \begin{bmatrix} O^{-1} + O^{-1} P S_O^{-1} Q O^{-1} & -O^{-1} P S_O^{-1} \\ -S_O^{-1} Q O^{-1} & S_O^{-1} \end{bmatrix},$$

where $S_O \stackrel{\text{def}}{=} R - QO^{-1}P$, and

$$O \stackrel{\text{def}}{=} I_{(p+1)r}, \; P \stackrel{\text{def}}{=} -\frac{\mathcal{H}}{\sigma_j}, \; Q \stackrel{\text{def}}{=} -\frac{\mathcal{H}^T}{\sigma_j} + \begin{bmatrix} 0_{qr_0-1,(p+1)r} \\ u_j^T \end{bmatrix},$$

$$R \stackrel{\text{def}}{=} I_{qr_0} + \begin{bmatrix} 0_{qr_0-1,(p+1)r} \\ u_j^T \end{bmatrix},$$

the first block row of $\tilde{E}_j^{-1}$ effectively is

$$\begin{bmatrix} I_{(p+1)r} & 0_{(p+1)r,qr_0} \end{bmatrix} \tilde{E}_j^{-1} = \hat{E}_j$$

with $\hat{E}_j$ as defined in (14). Hence, together with (39) for $j = 1, \ldots, n$ the assertion follows. □

B Covariance Estimation of Subspace Matrices

Proof. (Proposition 10). The matrices $\mathcal{H}_j^{cov}$, $j = 1, \ldots, n_b$, are independent and identically distributed (i.i.d.) random variables as the underlying data are independent. The same holds obviously for $\operatorname{vec} \mathcal{H}_j^{\text{cov}}$, $j = 1, \ldots, n_b$. From (25), (26), (31) and (32) follows

$$\mathcal{H}^{\text{cov}} = \frac{1}{n_b} \sum_{j=1}^{n_b} \mathcal{H}_j^{\text{cov}}. \tag{40}$$

As the $\operatorname{vec} \mathcal{H}_j^{\text{cov}}$, $j = 1, \ldots, n_b$, are i.i.d., they have the same covariance and it follows together with (9) and (40)

$$\widehat{\Sigma}_{\mathcal{H}^{\text{cov}}} = \frac{N}{n_b^2} \sum_{j=1}^{n_b} \operatorname{cov}\left(\operatorname{vec} \mathcal{H}_j^{\text{cov}}\right) = \frac{N}{n_b} \operatorname{cov}\left(\operatorname{vec} \mathcal{H}_1^{\text{cov}}\right). \tag{41}$$

The estimator of the sample covariance of $\operatorname{vec} \mathcal{H}_j^{\text{cov}}$, $j = 1, \ldots, n_b$, is

$$\frac{1}{n_b - 1} \sum_{j=1}^{n_b} \left(\operatorname{vec} \mathcal{H}_j^{\text{cov}} - \operatorname{vec} \bar{\mathcal{H}}^{\text{cov}}\right) \left(\operatorname{vec} \mathcal{H}_j^{\text{cov}} - \operatorname{vec} \bar{\mathcal{H}}^{\text{cov}}\right)^T$$

with the sample mean $\operatorname{vec}\bar{\mathcal{H}}^{\text{cov}}$ with

$$\bar{\mathcal{H}}^{\text{cov}} = \frac{1}{n_b}\sum_{j=1}^{n_b}\mathcal{H}_j^{\text{cov}} = \mathcal{H}^{\text{cov}},$$

and the assertion follows. □

Proof. (Proposition 11). From (25), (30) and (31) follows

$$\begin{bmatrix}\tilde{\mathcal{Y}}^-\\ \tilde{\mathcal{Y}}^+\end{bmatrix} = \frac{1}{\sqrt{n_b}}\begin{bmatrix}\tilde{\mathcal{Y}}_1^- & \dots & \tilde{\mathcal{Y}}_{n_b}^-\\ \tilde{\mathcal{Y}}_1^+ & \dots & \tilde{\mathcal{Y}}_{n_b}^+\end{bmatrix}.$$

Plugging in (28) on the left side and (33) on the right side, leads to

$$RQ = \frac{1}{\sqrt{n_b}}\begin{bmatrix}R^{(1)} & \dots & R^{(n_b)}\end{bmatrix}\begin{bmatrix}Q^{(1)} & & \\ & \ddots & \\ & & Q^{(n_b)}\end{bmatrix}$$

and by enlarging the thin RQ decomposition (36) to

$$\begin{bmatrix}R^{(1)} & \dots & R^{(n_b)}\end{bmatrix} = \breve{R}\breve{Q}, \tag{42}$$

it can be assumed that it holds

$$R = \frac{1}{\sqrt{n_b}}\breve{R}, \quad Q = \breve{Q}\begin{bmatrix}Q^{(1)} & & \\ & \ddots & \\ & & Q^{(n_b)}\end{bmatrix} \tag{43}$$

as the thin RQ decomposition is unique up to a sign change (uniqueness can be enforced by constraining the diagonal elements of the R part to positive values). Let $\breve{R}$ and $\breve{Q}$ be partitioned into

$$\breve{R} = \begin{bmatrix}\breve{R}_{11} & 0\\ \breve{R}_{21} & \breve{R}_{22}\end{bmatrix}, \quad \breve{Q} = \begin{bmatrix}\breve{Q}_1 & \dots & \breve{Q}_{n_b}\end{bmatrix}, \tag{44}$$

where $\breve{R}$ is partitioned analogously to R, and the $\breve{Q}_j$ are of the same size as the $R^{(j)}$ in (42). As $\breve{R}$ and the $R^{(j)}$ are lower triangular in (42), the matrices $\breve{Q}_j$ are also lower triangular and can be partitioned accordingly into

$$\breve{Q}_j = \begin{bmatrix}\breve{Q}_{11}^{(j)} & 0\\ \breve{Q}_{21}^{(j)} & \breve{Q}_{22}^{(j)}\end{bmatrix}, \quad j = 1,\dots,n_b. \tag{45}$$

Multiplying (42) with $\breve{Q}^T$ and replacing $\breve{Q}$ by its partition from (44) leads to $\breve{R} = \sum_{j=1}^{n_b} R^{(j)}\breve{Q}_j^T$. Now, replacing $\breve{R}$, $R^{(j)}$ and Q_j with their partitions from (44), (34) and (45), respectively, leads to

$$\begin{bmatrix}\breve{R}_{11} & 0\\ \breve{R}_{21} & \breve{R}_{22}\end{bmatrix} = \sum_{j=1}^{n_b}\begin{bmatrix}R_{11}^{(j)} & 0\\ R_{21}^{(j)} & R_{22}^{(j)}\end{bmatrix}\begin{bmatrix}\breve{Q}_{11}^{(j)\,T} & \breve{Q}_{21}^{(j)\,T}\\ 0 & \breve{Q}_{22}^{(j)\,T}\end{bmatrix}$$

and hence

$$\breve{R}_{21} = \sum_{j=1}^{n_b} R_{21}^{(j)}\,\breve{Q}_{11}^{(j)\,T}.$$

Then, together with (29), (35), (43) and (44) it follows

$$\mathcal{H}^{\mathrm{dat,R}} = \frac{1}{\sqrt{n_b}} \sum_{j=1}^{n_b} \mathcal{H}_j^{\mathrm{dat,R}} \breve{Q}_{11}^{(j)\,T}. \tag{46}$$

Now, regarding the $\mathrm{vec}(\mathcal{H}_j^{\mathrm{dat,R}} \breve{Q}_{11}^{(j)\,T})$, $j = 1, \ldots, n_b$, as i.i.d., the relation

$$\widehat{\Sigma}_{\mathcal{H}^{\mathrm{dat,R}}} = N \,\mathrm{cov}\left(\mathrm{vec}\left(\mathcal{H}_1^{\mathrm{dat,R}} \breve{Q}_{11}^{(1)\,T}\right)\right)$$

follows analogously to (41) and the assertion follows. □

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