Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux, Patrick Joly

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P. Joly[†], **S. Tordeux**[‡,*]
[†]UMR POems, INRIA-Rocquencourt, 78153 Le Chesnay Cedex, France,
[‡]SAM, ETHZ, 8092 Zurich, Switzerland,
[*]Email: tordeux@math.ethz.ch

Abstract

In this talk we will use the matching of asymptotic expansion to derive new slot models.This models will be mathematically validated via some error estimates.

Introduction

In practical applications concerning both electromagnetic or acoustic waves, many wave propagation problems involve the presence of structures whose at least one space dimension of characteristic length, denoted ε, is small with respect to the wave length λ: we think for instance to highly oscillating coefficients, thin layers, thin screens, wires or thin slots, which is the topic we address in this work. For the numerical simulation of such wave propagation problems, it is natural to look for approximate or "effective" models that should permit to avoid to mesh the computational domain, even locally, at the scale of ε. One meets a similar situation for the treatment of thin slots. Let us take the example of a thin slot in a 2D context. The geometry of the problem is represented by figure 1. A first application is the microwave shielding of

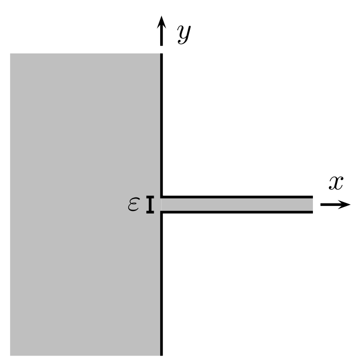

Figure 1: Geometry of the propagation domain.

thin slots. From the physical point of view, the domain of validity is typically given by:

$$\begin{cases} \lambda/1000 \ < \ \varepsilon \ < \ \lambda/10, \\ \varepsilon \ < \ L/10, \\ \lambda/10 \ < \ L \ < \ 10\,\lambda, \end{cases} \tag{1}$$

where λ is the wave length, ε is the width of the slot and L the length of the slot. One can refer to [5] for examples. A second application in electromagnetism corresponds to the so-called flanged waveguide antenna which corresponds to a semi-infinite straight slot. This case, that has been studied in [1] in a non asymptotic context, will be analyzed in detail in this paper.

An intuitive idea is to try to derive an approximate "1D - 2D" model: a 1D model, posed on the curve that materializes the limit of the slot when ε tends to 0, for the propagation inside the slot and a 2D model for the rest of the computational domain. Such models have been designed in the engineering literature (see [2], [3], [4], [5]) and are commonly used in various computational codes. One can find a mathematical analysis of a related model in [6].

An alternative way to obtain some approximate model comes from the analytical work by the "british and russian school" of applied mathematics on this type of problem [7], [8], [9]. The aim they pursue a priori is not to derive approximate models for numerical approximation. They wish to describe as accurately as possible the behaviour of the solution with the help of analytical formulas or at least solutions of simple canonical problems. The technique that is used is the method of matched asymptotics expansions which consists in separating the domain of propagation in several parts in which one makes different ansatz on the solution. For instance, for the slot problem, it would consist in considering three zones : the inner slot, the exterior region and a neighborhood of the end points of the slot. Such techniques permit for instance to take correctly into account the singularities of the solution. This type of technique is bere used to derive effective models for numerical computation.

One can refer to [6] to see this technique in a more complicated case where there is an obstacle on the half space.

The exact model

In this work, we shall consider the simplest possible propagation model, namely the 2D scalar wave equation, that can be used in acoustics of course, but also in electromagnetism if one separates the transverse electric or transverse magnetic polarizations:

$$\frac{\partial^2 U^\varepsilon}{\partial t^2} - \Delta U^\varepsilon = F, \tag{2}$$

where we have assumed, for simplicity, that the propagation velocity of waves is equal to 1. We shall look at the time harmonic regime, i.e. consider a source term of the form:

$$F(x,t) = f(x) \cdot \exp(-i\,\omega\, t). \tag{3}$$

where the pulsation $\omega > 0$ is a data of the problem, related to the wavelength λ by:

$$\omega\,\lambda = 2\,\pi. \tag{4}$$

In (3), f denotes a compactly supported function whose support does not intersect the slot. We then look for solutions of the same form:

$$U^\varepsilon(x,t) = u^\varepsilon(x) \cdot \exp(-i\,\omega\, t), \tag{5}$$

which leads to the Helmholtz equation for the complex value function u:

$$-\Delta u^\varepsilon - \omega^2\, u^\varepsilon = f. \tag{6}$$

A typical source in application is a regularized δ-function an approximation to a point source.

For the boundary condition, we shall consider the homogeneous Neumann condition:

$$\frac{\partial u^\varepsilon}{\partial n} = 0. \tag{7}$$

In some sense, it is the "only" boundary condition that permits the propagation of waves inside the slot. This this due to the fact that the slot behaves as a waveguide whose first cut-offl frequency is 0.

The results

The matching of asymptotic expansions consists in approximating the exact solution with series in diffecrent zones. These series have to match in some transition zones (where two expansions are valid). Here one has to use three zones.

The first zone is a far field zone. This zone consists in the $(x < 0, y)$ far away (with respect to ε) from the end of the slot.

The second zone is a slot zone. It consists in the point in the slot which are far away (with respect to ε) from the end of the slot.

The third zone is a near field zone (or boundary layer zone). It consists in the point wich are closed (with respect to 1 from the end of the slot)

Far field expansion. Firstly, one can expand the exact solution in the half-space with respect to the coordinates x and y using a formal series of functions independant of ε:

$$u^\varepsilon(x,y) \;=\; u^0 \;+\; \sum_{i=1}^{+\infty}\sum_{k=0}^{i-1} \varepsilon^i\, \log^k \varepsilon\;\, u_i^k \tag{8}$$

where u^0 is the outgoing solution of the slotless problem:

$$\begin{cases} \Delta u^0 \;+\; \omega^2 u^0 \;=\; -f, & \text{for } x < 0 \\ \dfrac{\partial u^0}{\partial x} = 0, & \text{for } x = 0. \end{cases} \tag{9}$$

The u_i^k's are defined in the half-space. They satisfy Helmholtz condition with Neumann boundary condition but are singular at the origin (u^0 is regular):

$$\begin{cases} \Delta u_i^k \;+\; \omega^2\; u_i^k \;=\; 0, & \text{for } x < 0, \\ \dfrac{\partial u_i^k}{\partial n} \;=\; 0, & \text{for } x = 0 \text{ if } y \neq 0 \end{cases} \tag{10}$$

For example, u_1^0 is given by:

$$u_1^0 \;=\; -\frac{\omega}{2}\; u^0(0)\; H_0^{(1)}(\omega r) \tag{11}$$

The following error estimates permit to understand the meaning of the formal expansion (8):

$$\begin{cases} \|u^\varepsilon - u^0 \;+\; \displaystyle\sum_{i=1}^{n}\sum_{k=0}^{i-1} \varepsilon^i\; \log \varepsilon^k u_i^k\|_{H^1(K)} \\ \qquad\qquad \leq\; C\; \varepsilon^{n+1}\; \log^n \varepsilon\; \|f\|_{L^2(\Omega)} \end{cases} \tag{12}$$

where K is any compact included in the closed half space $(x \leq 0)$ which do not contain the origin.

Slot field expansion. Secondly, one can expand the solution inside the slot with respect to the coordinates:

$$x \text{ and } Y \;=\; y/\varepsilon \tag{13}$$

The exact solution admits the following formal expansion:

$$u^\varepsilon(x, \varepsilon Y) \;=\; \sum_{i=0}^{+\infty}\sum_{k=0}^{i} \varepsilon^i\; \log^k \varepsilon\; U_i^k(x,Y) \tag{14}$$

where the U_i^k are 1-D solutions of the Helmholtz equation:

$$U_i^k(x,Y) \;=\; U_i^k(0)\; \exp i\omega x \tag{15}$$

For example, one can write U_0^0 as:

$$U_0^0(x) \;=\; u^0(0)\; \exp i\omega x \tag{16}$$

The following error estimates allows us to understand the formal serie (14):

$$\left\{ \begin{array}{l} \left\| u^\varepsilon(x,\varepsilon Y) - \displaystyle\sum_{i=0}^{n}\sum_{k=0}^{i} \varepsilon^i \;\log^k \varepsilon\; U_i^k \right\|_{H^1(K)} \\ \qquad \leq\; C\; \varepsilon^{n+1}\; |\log\varepsilon|^{n+1}\; \|f\|_{L^2(\Omega)} \end{array} \right. \tag{17}$$

where K is any compact such that:

$$K \subset]0;+\infty[_x \times [-\frac{1}{2};\frac{1}{2}]_Y \tag{18}$$

Near field expansion. Finally, one can expand the solution in a neighborhood of the end of the slot with respect to the coordinates:

$$X \;=\; x/\varepsilon \quad \text{and} \quad Y = y/\varepsilon \tag{19}$$

This introduces the canonical domain

$$T =]-\infty;0]_X \times \mathbb{R}_Y \cup [0;+\infty[_X \times [-\frac{1}{2};\frac{1}{2}]_Y. \tag{20}$$

The exact solution admits the following formal expansion:

$$u^\varepsilon(\varepsilon X,\varepsilon Y) \;=\; \sum_{i=0}^{+\infty}\sum_{k=0}^{i} \varepsilon^i \;\log^k \varepsilon\; \mathcal{U}_i^k(X,Y) \tag{21}$$

where the $\mathcal{U}_i^k$'s are defined on T. They are solutions of the Laplace equation with Neumann boundary condition and are growing at infinity:

$$\left\{ \begin{array}{l} \Delta\mathcal{U}_i^k \;=\; 0, \text{ in } T, \text{ for } i = k \text{ or } k+1, \\ \Delta\mathcal{U}_i^k \;+\; \omega^2\, \mathcal{U}_{i-2}^k \;=\; 0, \text{ in } T \text{ for } i > k+1, \\ \dfrac{\partial \mathcal{U}_i^k}{\partial n} \;=\; 0, \text{ on } \partial T \end{array} \right. \tag{22}$$

One can use the following error estimate to understand the meaning of (21):

$$\left\{ \begin{array}{l} \left\| u^\varepsilon(\varepsilon X,\varepsilon Y) - \displaystyle\sum_{i=0}^{n}\sum_{k=0}^{i} \varepsilon^i \log^k \varepsilon\, \mathcal{U}_i^k(X,Y) \right\|_{H^1(K)} \\ \qquad \leq\; C\; \varepsilon^{n+1}\; \log^{n+1}\varepsilon \quad \|f\|_{L^2(\Omega)} \end{array} \right. \tag{23}$$

For numerical computations. All the u_i^k's, U_i^k's and $\mathcal{U}_i^k$'s are solutions of PDE which can easily be solved numerically. As the terms of the asymptotic expansions do not depends on ε, one can use a mesh which does not also depends on ε. For example, to have an approximation of order 2 of the exact solution in the far field zone, one has just to compute

$$u^0,\; u_1^0,\; u_2^0 \text{ and } u_2^1 \tag{24}$$

and to constuct the truncated serie:

$$u^0 \;+\; \varepsilon\, u_1^0 \;+\; \varepsilon^2\, u_2^0 \;+\; \varepsilon^2\; \log\varepsilon\, u_2^1. \tag{25}$$

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