Matching of Asymptotic Expansions for a 2-D eigenvalue problem with two cavities linked by a narrow hole

Abderrahmane Bendali, Abdelkader Tizaoui, Sébastien Tordeux

Matching of Asymptotic Expansions for an eigenvalue problem with two cavities linked by a thin hole

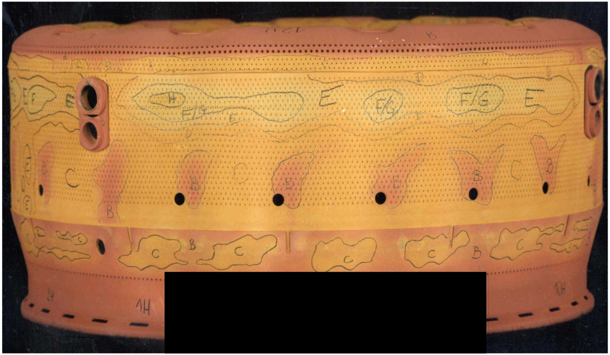
Bendali, Tizaoui, Tordeux

Institut de Mathématiques de Toulouse – INSA Toulouse

WAVES, 15 june 2009

Motivation

- The casing of a combustion chamber of a Turbo engine of Turbo Meca (SAFRAN Group, - Acknowledgment -).



- Goal: Study the effects of small holes on resonance frequencies using **Matching of Asymptotic Expansions** (**MAE**).

A non Exhaustive Bibliography

- Small holes: Rauch and Taylor (75), Tuck (75), Sanchez-Hubert and Sanchez-Palencia (82), Taflov (88), Bonnet-BenDhia, Drissi and Gmati (04), Mendez and Nicoud (08), Gadyl'shin (92).
- Dumbell problems: Beale (73), Jimbo and Morita (92) Brown, Hislop and Martinez (95), Arrieta (95), Anné (95).
- Matching of Asymptotic Expansions: Van Dyke (75), Il'in (92), Joly and Tordeux (06,08)
- quasi-mode and min-max: Bamberger and Bonnet (90), Dauge, Djurdjevic, Faou, and Rössle (99), Bonnaillie-Noël and Dauge (06).

A Toy Problem

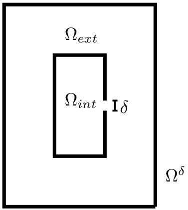
Problem: Find $u_n^\delta \in H_0^1(\Omega^\delta)$ and $\lambda_n^\delta \in \mathbb{R}$ satisfying

$$\begin{cases} -\nabla \cdot (a(x,y)\nabla u_n^\delta)(x,y) = \lambda_n^\delta \, b(x,y) \, u_n^\delta(x,y) \text{ in } \Omega^\delta, \\ u_n^\delta(x,y) = 0 \text{ on } \partial\Omega^\delta, \end{cases} \tag{1}$$

- with a and b two bounded positive regular functions with two sides

$$a_{int}(\mathbf{0}) \neq a_{ext}(\mathbf{0}) \text{ and } a_{int}(\mathbf{0}) \neq a_{ext}(\mathbf{0}) \tag{2}$$

- The small parameter $\delta > 0$ is the width of the hole in the domain Ω^δ.

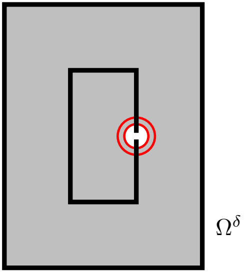


Hypothesis: The eigenvalues of Ω are simple.

Outline

The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping



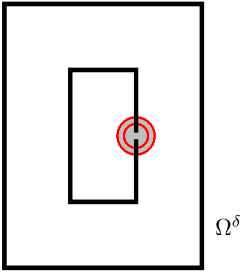
Far-field

The solution is described:

- with a **far**-field.
- with a **near**-field.

The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping



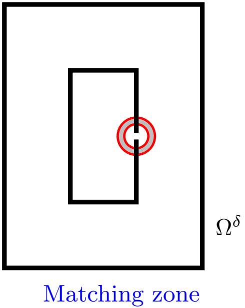
Near field

The solution is described:

- with a **far**-field.
- with a **near**-field.

The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping



The solution is described:

- with a **far**-field.
- with a **near**-field.

The Asymptotic Expansions: The Eigenvalue Expansion

- The second order asymptotic expansion reads

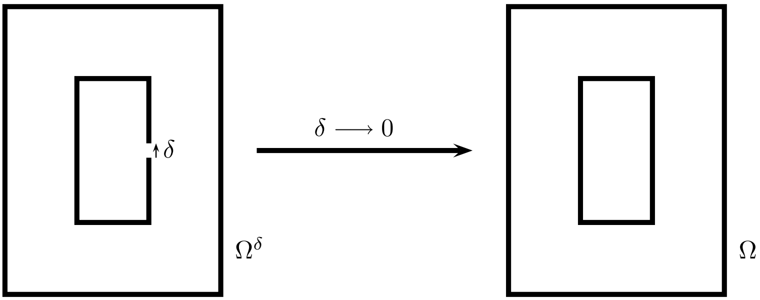
$$\lambda^\delta = \lambda^0 + \delta\lambda^1 + \delta^2\lambda^2 + \underset{\delta\to 0}{o}(\delta^2). \tag{3}$$

- polynomial gauge functions? not trivial: at fourth order

$$\lambda^\delta = \lambda^0 + \delta\lambda^1 + \delta^2\lambda^2 + \delta^3\lambda^3 + \delta^4\lambda^{4,0} + \delta^4 \ \ln\delta \ \lambda^{4,1} + \underset{\delta\to 0}{o}(\delta^4). \tag{4}$$

- The coefficients $\lambda^i \in \mathbb{R}$ and do not depend on δ
- Necessity of a proof!

The Asymptotic Expansions: The Far-field Expansion



Far-field (Asymptotic Expansion):

$$u^\delta = u^0 + \delta u^1 + \delta^2 u^2 + \underset{\delta \to 0}{o}(\delta^2)$$

The coefficients of the far-field asymptotic expansion u^i will be

- defined in the far-field domain Ω: The limit of Ω^δ when $\delta \to 0$,
- independent of δ.

Asymptotic Expansion: The Far-Field Expansion

They are solutions of the following problems

$$\left\{\begin{array}{ll} \text{Find } u^0 : \Omega \to \mathbb{R} \text{ and } \lambda^0 \in \mathbb{R} \text{ such that} & \\ \quad \nabla \cdot (a \nabla u^0) + \lambda^0 \; b \; u^0 = 0, & \text{in } \Omega, \\ \quad u^0 = 0, & \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array}\right. \tag{5}$$

$$\left\{\begin{array}{ll} \text{Find } u^1 : \Omega \to \mathbb{R} \text{ and } \lambda^1 \in \mathbb{R} \text{ such that} & \\ \quad \nabla \cdot (a \nabla u^1) + \lambda^0 \; b \; u^1 = -\lambda^1 \; b \; u^0, & \text{in } \Omega, \\ \quad u^1 = 0, & \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array}\right. \tag{6}$$

$$\left\{\begin{array}{ll} \text{Find } u^2 : \Omega \to \mathbb{R} \text{ and } \lambda^2 \in \mathbb{R} \text{ such that} & \\ \quad \nabla \cdot (a \nabla u^2) + \lambda^0 \; b \; u^2 = -\lambda^2 \; b \; u^0 - \lambda^1 \; b \; u^1, & \text{in } \Omega, \\ \quad u^2 = 0, & \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array}\right. \tag{7}$$

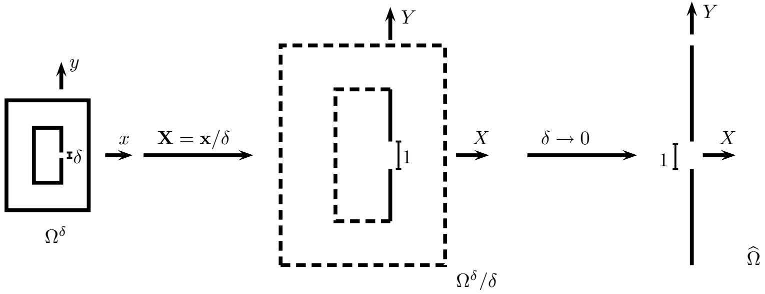
- The problem (5) can be interpreted in the following way:

  $$u^0 \text{ is an eigenfunction in } \Omega \text{ associated to } \lambda^0.$$

- The coefficients u^1 and u^2 are possibly singular at $\mathbf{x} = 0$.
- The problems (6) and (7) do not uniquely define u^1 and u^2.

Asymptotic Expansion: The Near-Field Expansion

Let $X = \dfrac{x}{\delta}$, $Y = \dfrac{y}{\delta}$, and we put $\Pi^\delta(X,Y) = u^\delta(\delta X, \delta Y)$.



Near-field (Asymptotic Expansion):

$$\Pi^\delta(X,Y) = \Pi^0(X,Y) + \delta\,\Pi^1(X,Y) + \delta^2\,\Pi^2(X,Y) + \underset{\delta\to 0}{o}(\delta^2). \qquad (8)$$

These functions will be

- defined on the near-field domain $\widehat{\Omega}$.
- independent of δ.

Asymptotic Expansion: The near-field expansion

They are solutions of elliptic problems

- with coefficients of the principal symbol that are two sides constants
- on the near-field domain $\widehat{\Omega}$

$$\widehat{\Omega} := \mathbb{R}^2 \setminus \left(\{0\} \times \left(\left] -\infty, -\frac{1}{2} \right[\cup \left] \frac{1}{2}, +\infty \right[\right) \right). \tag{9}$$

- The problems do not uniquely define Π^0, Π^1, and Π^2.

The Matched Asymptotic Expansions Method: Theoretical context

In order to ensure the uniqueness of u^1, u^2, Π^0, Π^1, and Π^2 we derive additional matching conditions. We use the following procedure to obtain these extra conditions.

1. We consider the far-field approximation of order m written with $\mathbf{x} = \delta \mathbf{X}$

$$\sum_{i=0}^{m} \delta^i u^i(\delta \mathbf{X}). \tag{10}$$

2. Then this sum is expanded up to $o(\delta^m)$. This defines the U^i_m in the $\mathbf{X}$ coordinates

$$\sum_{i=0}^{m} \delta^i u^i(\delta \mathbf{X}) = \sum_{i=0}^{m} \delta^i U^i_m(\mathbf{X}) + \underset{\delta \to 0}{o}(\delta^m). \tag{11}$$

3. The matching conditions are the following

$$\Pi^i - U^i_m = \underset{R \to +\infty}{o}\left(\frac{1}{R^{m-i}}\right) \quad \forall i \in [\![0, m]\!]. \tag{12}$$

The Limits

Using the **matching condition** , we get the problem defining u^0, λ^0, and Π^0.
Far-field. The function u^0 is an eigenfunction of the Dirichlet Laplace:

$$\left\{\begin{array}{ll} \text{Find } u^0 \in H^1(\Omega) \text{ such that} & \\ \nabla\cdot(a\nabla u^0) + \lambda^0\; b\; u^0 = 0, & \text{in } \Omega, \\ u^0 = 0, & \text{on } \partial\Omega. \end{array}\right. \tag{13}$$

To simplify the presentation $u^0 \equiv 0$ in Ω_{ext}

The First Order Asymptotic Expansion

Using the **matching condition** , we get the problem defining u^1, λ^1, and Π^1.

Far-field.

$$\left\{\begin{array}{ll} \text{Find } u^1 \in H^1(\Omega) \text{ and } \lambda^1 \in \mathbb{R} \text{ such that} & \\ \quad \nabla \cdot \Big(a \, \nabla u^1 \Big) + \lambda^0 \, b \, u^1 = -\lambda^1 u^0, & \text{in } \Omega, \\ \quad u^1 = 0, & \text{on } \partial\Omega. \end{array}\right. \tag{14}$$

Note that u^1 is regular. Due to the Fredholm alternative, the second member has to be orthogonal to u^0

$$\lambda^1 \int_\Omega b(x,y) \, \big(u^0(x,y)\big)^2 \, dx \, dy \; = \; 0. \tag{15}$$

We obtain $\lambda^1 \; = \; 0.$

The function u^1 is still defined up to its u^0-component, which is classical for eigenvalue problems. Then

$$u^1 = \gamma \, u^0 \text{ in } \Omega_{int} \text{ and } u^1 = 0 \text{ in } \Omega_{ext}, \qquad \text{with } \gamma \in \mathbb{R}. \tag{16}$$

We add the condition $\displaystyle\int_\Omega b(x,y) \, u^1(x,y) \, u^0(x,y) \, dx \, dy = \; 0 \Rightarrow u^1 \equiv 0.$

The Second Order Asymptotic Expansion

The **matching procedure** leads to the following problem

$$\left\{\begin{array}{l}\text{Find } u^2 : \Omega \to \mathbb{R} \text{ and } \lambda^2 \in \mathbb{R} \text{ such that}\\ \nabla \cdot (a\, \nabla u^2) + \lambda^0\, b\, u^2 = -\lambda^2\, b\, u^0, \text{in } \Omega,\\ u^2 = 0, \text{on } \partial\Omega \setminus \{\mathbf{0}\}.\\ u^2_{int}(\mathbf{x}) - \partial_x u^0_{int}(\mathbf{0})\, \dfrac{1}{8}\, \dfrac{a_{int}(\mathbf{0})}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})}\, \dfrac{\sin(\theta)}{r} = \underset{r\to 0}{O}(1),\\ u^2_{ext}(\mathbf{x}) + \partial_x u^0_{int}(\mathbf{0})\, \dfrac{1}{8}\, \dfrac{a_{ext}(\mathbf{0})}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})}\, \dfrac{\sin(\theta)}{r} = \underset{r\to 0}{O}(1),\end{array}\right. \tag{17}$$

- Note that u^2 is singular $u^2 \notin H^1(\Omega)$.
- Due to the singularity of u^2, the Fredholm alternative theory can not be directly applied to obtain λ^2.

Proposition: The problem (17) has solutions. Moreover if (u^2, λ^2) and (u^2_*, λ^2_*) are solutions, one has

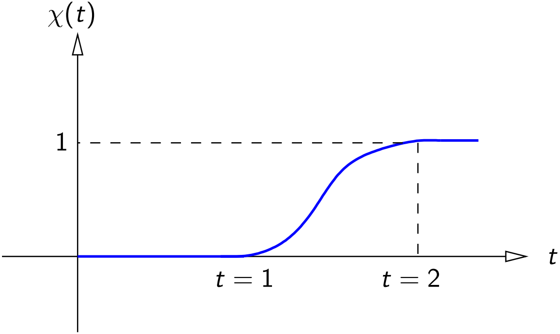
$$\lambda^2 = \lambda^2_* = -\frac{\pi}{8}\, \frac{(a_{int}(\mathbf{0}))^2}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})}\, \frac{|\partial_x u^0_{int}(\mathbf{0})|^2}{\|u^0\|_0^2} \quad \text{and} \quad \exists \gamma \in \mathbb{R} : u^2_* - u^2 = \gamma u^0.$$

The Second Order Asymptotic Expansion

Sketch of proof: We introduce the auxiliary function ω^2

$$\omega^2_{int,ext} = u^2_{int,ext} - \chi(r)\,\partial_x u^0_{int,ext}(\mathbf{0})\,\alpha_{int,ext}\,\frac{\sin(\theta)}{r} \in H^1(\Omega_{int,ext}), \qquad (18)$$

with χ the regular cut-off function:



Applying Fredholm alternative to ω^2 which is regular, we obtain λ^2.

Asymptotic Expansion of the Eigenvalues

Theorem:

Let λ^0 be a simple eigenvalue of Ω. For all $\delta > 0$, there exists an eigenvalue λ^δ of the Dirichlet laplacian in Ω^δ, see (1), satisfying

$$\left|\lambda^\delta - (\lambda^0 + \delta^2 \lambda^2)\right| \leq C\, \delta^3\, |\ln(\delta)| \qquad (19)$$

with λ^2 given by

$$\left\{ \begin{array}{ll} \lambda^2 = -\dfrac{\pi}{8} \dfrac{(a_{int}(\mathbf{0}))^2}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})} \dfrac{|\partial_x u^0_{int}(\mathbf{0})|^2}{\int_\Omega b(u^0)^2}, & \text{if } u^0_{ext} = 0, \\ \lambda^2 = -\dfrac{\pi}{8} \dfrac{(a_{ext}(\mathbf{0}))^2}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})} \dfrac{|\partial_x u^0_{ext}(\mathbf{0})|^2}{\int_\Omega b(u^0)^2}, & \text{if } u^0_{int} = 0. \end{array} \right. \qquad (20)$$

Generalization of the result of Gadyl'shin (92) that has looked to the case of constant coefficients.

Sketch of the Proof: A Quasi-Mode Approximation

- A mode $(u^\delta, \lambda^\delta) \in H^1_0(\Omega^\delta) \times \mathbb{R}$ satisfies $u^\delta \neq 0$ and

$$\int_{\Omega^\delta} \nabla u^\delta \nabla v - \lambda^\delta \int_{\Omega^\delta} u^\delta v = 0, \quad \forall v \in H^1_0(\Omega^\delta).$$

- An ε-quasi-mode $(w^\delta, \mu^\delta) \in H^1_0(\Omega^\delta) \times \mathbb{R}$ satisifies $w^\delta \neq 0$ and

$$\left| \int_{\Omega^\delta} \nabla w^\delta \nabla v - \mu^\delta \int_{\Omega^\delta} w^\delta v \right| \leq \varepsilon \|w^\delta\|_{L^2(\Omega^\delta)} \|v\|_{L^2(\Omega^\delta)}, \quad \forall v \in H^1_0(\Omega^\delta).$$

Lemma:

Let (w^δ, μ^δ) be an ε-quasi-mode. There exists a mode $(u^\delta, \lambda^\delta)$:

$$\left|\lambda^\delta - \mu^\delta\right| \leq \varepsilon$$

Sketch of the Proof: The Uniformly Valid Approximation

In this proof we consider the quasi-mode $(\widetilde{w}^\delta, \lambda^0 + \delta^2\lambda^2)$ defined by

$$\widetilde{w}^\delta = \chi^\delta \left(u^0 + \delta\, u^1 + \delta^2\, u^2\right) + \Psi\left(\widehat{\Pi}^0 + \delta\, \widehat{\Pi}^1 + \delta^2\, \widehat{\Pi}^2\right) - \chi^\delta \Psi\left(\widehat{U}_2^0 + \delta\, \widehat{U}_2^1 + \delta^2\, \widehat{U}_2^2\right). \quad (21)$$

with

- the cut-off function χ^δ defined by $\chi^\delta(r,\theta) := \chi(r/\delta)$.
- the cut-off function Ψ defined by $\Psi(r,\theta) = 1 - \chi(r)$;
- $\widehat{\Pi}^i$ the near-field terms expressed in the $\mathbf{x}$-coordinates, $\widehat{\Pi}^i(\mathbf{x}) = \Pi^i(\mathbf{x}/\delta)$.
- $\widehat{U}^i$ the near-field behaviors expressed in the $\mathbf{x}$-coordinates, $\widehat{U}^i(\mathbf{x}) = U^i(\mathbf{x}/\delta)$.

Sketch of the Proof: Final Estimate

We obtain

$$\left| a(\widetilde{w}^\delta, v) - \left(\lambda^0 + \delta^2\,\lambda^2\right)\,(\widetilde{w}^\delta, v)_0 \right| \leq C\,\delta^2\,\|\widetilde{w}^\delta\|_0\|v\|_0, \quad \forall v \in H_0^1(\Omega^\delta).$$

$\Rightarrow$ Existence of an eigenvalue λ^δ of the Dirichlet laplacian in Ω^δ, satisfying

$$\lambda^\delta \;=\; \lambda^0 + \delta^2\lambda^2 + \underset{\delta\to 0}{O}\left(\delta^2\right).$$

That is a suboptimal result.

To obtain an optimal result: use the third order asymptotic expansion

$$\lambda^\delta \;=\; \lambda^0 + \delta^2\lambda^2 + \underset{\delta\to 0}{O}\left(\delta^3 \ln\delta\right)$$

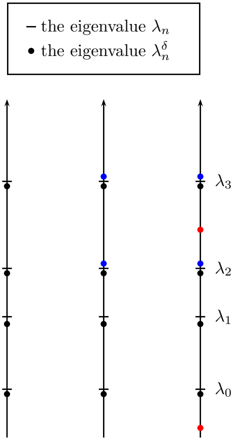
Do we have missed some eigenvalues?

The last result reveals:

- There exists a λ_n^δ in a small neighborhood of each λ_n
- $0 < \lambda_{\sigma(n)}^\delta \leq \lambda_n$ for δ small enough.

Some questions

- only one λ_n^δ in the neighoborhood of λ_n?
- other λ_n^δ?



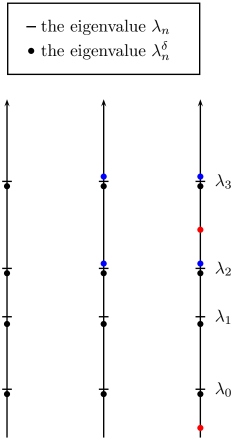
Do we have missed some eigenvalues?

The last result reveals:

- There exists a λ_n^δ in a small neighborhood of each λ_n
- $0 < \lambda_{\sigma(n)}^\delta \leq \lambda_n$ for δ small enough

Some answers (with min-max)

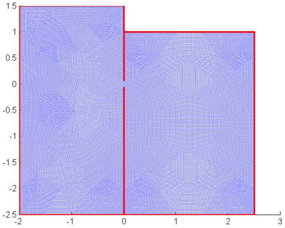
- only one λ_n^δ in the neighoborhood of λ_n? no.
- other λ_n^δ? no.



Numerical Simulations: an Experiment

Let Ω^δ be the domain defined by the following figure with

$$\Omega_{int} =]-2,0[\times]-2.5,1.5[\quad \text{and} \quad \Omega_{ext} =]0,2.5[\times]-1.5,1[. \qquad (22)$$

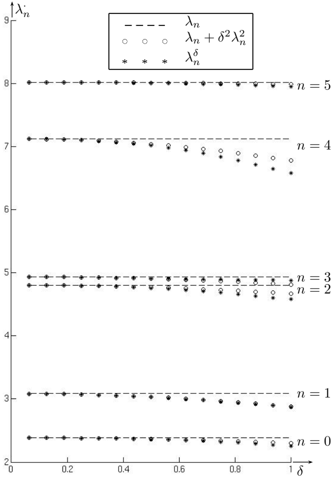


A computational mesh.

We recall that the eigenmodes of the limit problem in a domain $[0,a]\times[0,b]$ are

$$\lambda_{nm} = \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}, \quad U_{nm}(x,y) = \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right). \qquad (23)$$

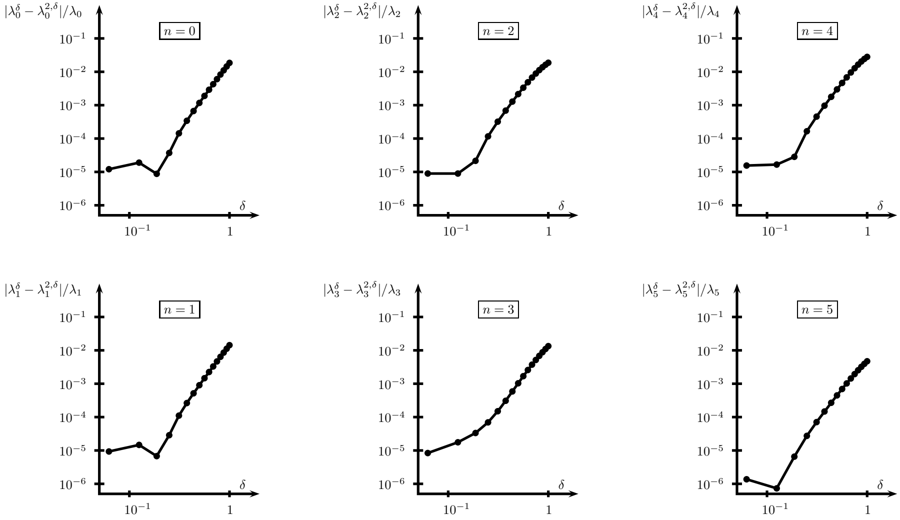
Numerical Simulations: First Experiment



- the $\lambda_n^{2,\delta}$ are numerically computed
- The λ_n and λ_n^2 are analatycally computed

n	λ_n	λ_n^2
0	2.38	−0.087
1	3.08	−0.207
2	4.80	−0.135
3	4.93	−0.121
4	7.12	−0.347
5	8.02	−0.036

The Relative Errors



Conclusion

Main results: an approximation of eigenvalues with

- a theoretical background
- no mesh refinement required

Some publications:

- Bendali, Tizaoui, Tordeux, Vila, SAM Research Report (2009)
- Bendali, Huard, Tizaoui, Tordeux and Vila, CRAS (submitted)

Acknowledgment: This work has been supported by the French National Agency (ANR) in the frame of its programm Systèmes Complexes et Modélisation: project APAM (Acoustique et Paroi Multi-perforée).

Thank you!