Sur l'effet de peau en électromagnétisme

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Projet MC2
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27.01.2010

Motivation

Skin effect in a half-space

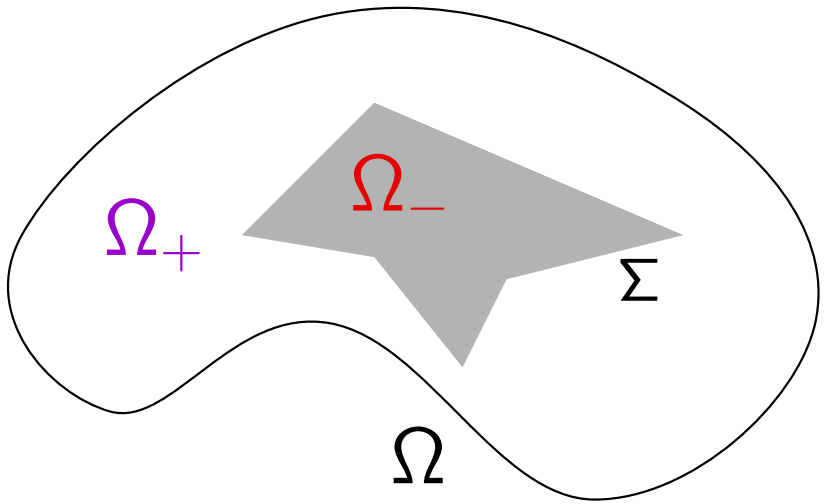
Skin Effect : in $\{z > 0\}$,

$$\|\mathbf{E}_\sigma(z)\| = E_0 \, \mathrm{e}^{-z/\ell(\sigma)}$$

Skin Depth :

$$\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$$

The Model Problem



- Ω_- Highly Conducting body $\subset\subset \Omega$: Conductivity $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ Insulating or Dielectric body: Conductivity $\sigma_+ = 0$

Aim : Understand the Behavior of the Electromagnetic Field as $\sigma \longrightarrow \infty$

Tools for the Asymptotic Analysis

- Tensor Calculus & Differential Geometry
- Asymptotic Expansions: *Scaling* Technique. *Boundary Layer* Expansion
- PDE Analysis : *Uniform Estimates*
- Numerical Simulations : the Asymptotic solution Procedure yields a *Fast and Efficient method*

References



E.P. STEPHAN, R.C. MCCAMY (83-84-85)

Plane Interface and Eddy Current Problems



H. HADDAR, P. JOLY, H.N. NGUYEN (08)

Smooth Interface and Generalized Impedance Boundary Conditions



V. PÉRON (09)

PhD : Smooth and Polyhedral Interfaces



M. DAUGE, E. FAOU, V. PÉRON (10)

Asymptotic Behavior for High Conductivity of the Skin Depth



G. CALOZ, M. DAUGE, V. PÉRON (09)

Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism

Outline

1. Uniform Estimates for Interface Problems
2. 3D Multiscaled Asymptotic Expansion
3. Numerical Simulations

Outline

1 Uniform Estimates for Interface Problems

2 3D Multiscaled Asymptotic Expansion

3 Numerical Simulations

Maxwell Problem

Issue and Framework

$$(\mathbf{P}_{\underline{\sigma}}) \qquad \operatorname{curl}\mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma})\mathbf{E} = \mathbf{j} \quad \underline{\sigma} = (0, \sigma)$$

Issue: *Uniform* L^2 *estimates* for solutions $(\mathbf{E}, \mathbf{H})$ of $(\mathbf{P}_{\underline{\sigma}})$ as $\sigma \to \infty$?

Maxwell Problem

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Hypothesis

Σ *is a bounded Lipschitz surface in* $\mathbb{R}^3$

Hypothesis (*SH*)

The angular frequency ω *is not an eigenfrequency of the problem*

$$\begin{cases} \operatorname{curl}\mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \quad \textit{and} \quad \operatorname{curl}\mathbf{H} + i\omega\varepsilon_0\mathbf{E} = 0 & \textit{in} \quad \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 & \textit{on} \quad \Sigma \\ \textit{B.C.} & \textit{on} \quad \partial\Omega \end{cases}$$

Maxwell Problem

Uniform $\mathrm{L}^2(\Omega)$ Estimates

Boundary Conditions : $\mathbf{E} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega$

$\mathbf{j} \in \mathrm{H}(\mathrm{div}, \Omega) = \{\mathbf{u} \in \mathrm{L}^2(\Omega) \mid \quad \mathrm{div}\, \mathbf{u} \in \mathrm{L}^2(\Omega)\}$

Theorem (CALOZ, DAUGE, P., 09)

Under Hypothesis (SH), there exist σ_0 and $C > 0$, such that for all $\sigma \geqslant \sigma_0$, $(\mathbf{P}_{\underline{\sigma}})$ with B.C. and $\mathbf{j} \in \mathrm{H}(\mathrm{div}, \Omega)$ has a unique solution $(\mathbf{E}, \mathbf{H})$ in $\mathrm{L}^2(\Omega)^2$, and

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma}\, \|\mathbf{E}\|_{0,\Omega_-} \leqslant C\, \|\mathbf{j}\|_{\mathrm{H}(\mathrm{div},\Omega)}$$

Maxwell Problem

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Application: Convergence of asymptotic expansion for high conductivity

Maxwell Problem

Proof of Uniform Estimates

Lemma

Under Hypothesis (SH)*, there are* σ_0 *and* $C_0 > 0$ *such that if* $\sigma \geqslant \sigma_0$ *any solution* $(\mathbf{E}, \mathbf{H}) \in \mathrm{L}^2(\Omega)^2$ *of* $(\mathbf{P}_{\underline{\sigma}})$ *with B.C. and data* $\mathbf{j} \in \mathrm{H}(\mathrm{div}, \Omega)$ *satisfies*

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Maxwell Problem

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$$\|\mathbf{E}\|_{0,\Omega} \leqslant C_0 \|\mathbf{j}\|_{\mathrm{H}(\mathrm{div},\Omega)}$$

Proof by contradiction: keys

1.  C. Amrouche, C. Bernardi, M. Dauge, V. Girault (98)
   Vector Potential technique: $\mathbf{E} =: \mathbf{w} + \nabla\varphi$ and $\mathrm{div}\,\mathbf{w} = 0$

$$\mathrm{curl}\,\mathbf{H} + \underbrace{(i\omega\varepsilon_0 - \underline{\sigma})}_{=:\,\underline{a}}(\mathbf{w} + \nabla\varphi) = \mathbf{j} \longrightarrow \mathrm{div}\,\underline{a}\,\nabla\varphi = \mathrm{div}\,\mathbf{j} - \underbrace{\mathrm{div}\,\underline{a}\,\mathbf{w}}_{=\,[\,\underline{a}\,]_\Sigma\,\mathbf{w}\cdot\mathbf{n}\,\delta_\Sigma}$$

2. Issue: Uniform piecewise regularity for
   $\mathrm{div}\,\underline{a}\,\mathbf{grad}\,\varphi = f + [\,\underline{a}\,]_\Sigma\, g$ as $|a_-/a_+| \to \infty$?

Uniform Estimates

Scalar Problem

$$(\mathbf{P}_{\underline{a}}): \qquad \operatorname{div} \underline{a}\, \mathbf{grad}\, \varphi = f + [\,\underline{a}\,]_\Sigma\, g \quad \text{as} \quad |\rho| := |a_-/a_+| \to \infty$$

Theorem (CALOZ, DAUGE, P., 09)

There exists $\rho_0 > 0$ independent of $a_+ \neq 0$ s. t. for all $a_- \in \{z \in \mathbb{C} \,|\, |z| \geqslant \rho_0 |a_+|\}$, and for all $(f, g) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Sigma)$ s.t. $\int_\Sigma g \,\mathrm{d}s = 0$, $(\mathbf{P}_{\underline{a}})$ has a unique solution $\varphi \in \mathrm{H}^1_0(\Omega)$, piecewise $\mathrm{H}^{3/2}$ and

$$\|\varphi^+\|_{\frac{3}{2},\Omega_+} + \|\varphi^-\|_{\frac{3}{2},\Omega_-} \leqslant C_{\rho_0} \left(|a_+|^{-1} \|f\|_{0,\Omega} + \|g\|_{0,\Sigma} \right)$$

with $C_{\rho_0} > 0$, independent of a_+, a_-, f, and g.

Uniform Estimates

Scalar Problem

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with $C_{\rho_0} > 0$, independent of a_+, a_-, f, and g.

Key of the Proof:

$$\varphi^+ = \sum_{n=0}^{\infty} \varphi_n^+\, \rho^{-n} \quad \Omega_+ \qquad \text{and} \qquad \varphi^- = \sum_{n=0}^{\infty} \varphi_n^-\, \rho^{-n} \quad \Omega_-$$

Remarks

1. Similar estimate hold when a_+ and a_- are switched.
   More precise estimate when a_+ and a_- play symmetric roles
2. The assumption that Σ is Lipschitz is necessary
3. For *polyhedral* Lipschitz interface Σ

$$\|\varphi^+\|_{s,\Omega_+} + \|\varphi^-\|_{s,\Omega_-} \leqslant C_{\rho_0}\left(|a_+|^{-1}\|f\|_{s-2,\Omega} + \|g\|_{s-\frac{3}{2},\Sigma}\right)$$

   for $s \leqslant s_\Sigma$ with some $\frac{3}{2} < s_\Sigma \leqslant 2$
4. For *smooth* interface Σ, uniform piecewise H^s estimates for any $s \geqslant 2$

Outline

1 Uniform Estimates for Interface Problems

2 **3D Multiscaled Asymptotic Expansion**

3 Numerical Simulations

References and Notations

Asymptotic Expansion as $\sigma \longrightarrow \infty$ of solutions of $\left(\mathbf{P}_{\underline{\sigma}}\right)$ *when* Σ *is smooth* :


STEPHAN, MCCAMY (83-84)

Plane Interface


HADDAR, JOLY, NGUYEN (08)


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Hypothesis

1. Σ *is a Smooth Surface*
2. ω *satisfies the Spectral Hypothesis (SH)*
3. $\mathbf{j} \in \mathrm{H}(\mathrm{div}, \Omega)$ *and* $\mathbf{j} = 0$ *in* Ω_-
4. *Perfectly Conducting Electric B.C.*

Asymptotic Expansion

$$\delta := \sqrt{\frac{\omega\varepsilon_0}{\sigma}} \longrightarrow 0 \quad \text{as} \quad \sigma \to \infty$$

By Theorem there exists δ_0 s.t. for all $\delta \leqslant \delta_0$, there exists a unique solution $\left(\mathbf{E}_{(\delta)}, \mathbf{H}_{(\delta)}\right)$ to $\left(\mathbf{P}_{\underline{\sigma}}\right)$

$$\mathbf{E}^+_{(\delta)}(\mathbf{x}) \approx \sum_{j\geqslant 0} \delta^j \, \mathbf{E}^+_j(\mathbf{x}), \qquad \mathbf{E}^-_{(\delta)}(\mathbf{x}) \approx \chi(y_3) \sum_{j\geqslant 0} \delta^j \, \mathbf{W}_j(y_\beta, \frac{y_3}{\delta})$$

(y_β, y_3): "*normal coordinates*" to Σ in a tubular region $\mathcal{U}_-$ of Σ in Ω_-

$$\mathbf{E}^+_j \in \mathrm{H}(\mathrm{curl}, \Omega_+) \quad , \quad \mathbf{W}_j \in \mathrm{H}(\mathrm{curl}, \Sigma \times \mathbb{R}_+) \quad \textit{profiles}$$

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Similar Expansion for the Magnetic Field $\mathbf{H}_{(\delta)}$

Validation of the Asymptotic Expansion

Remainders

Aim : proving Estimates for Remainders $\mathbf{R}_{m;\,\delta}$

$$\mathbf{R}_{m;\,\delta} := \mathbf{E}_{(\delta)} - \sum_{j=0}^{m} \delta^j \, \mathbf{E}_j \quad \text{in} \quad \Omega$$

Evaluation of the right hand side when the Maxwell operator is applied to $\mathbf{R}_{m;\,\delta}$

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{R}^+_{m;\,\delta} - \kappa^2\alpha_+\mathbf{R}^+_{m;\,\delta} &= 0 & \text{in} \quad \Omega_+ \\ \operatorname{curl}\operatorname{curl}\mathbf{R}^-_{m;\,\delta} - \kappa^2\alpha_-\mathbf{R}^-_{m;\,\delta} &= \mathbf{j}^-_{m;\,\delta} & \text{in} \quad \Omega_- \\ \left[\mathbf{R}_{m;\,\delta} \times \mathbf{n}\right]_\Sigma &= 0 & \text{on} \quad \Sigma \\ \left[\operatorname{curl}\mathbf{R}_{m;\,\delta} \times \mathbf{n}\right]_\Sigma &= \mathbf{g}_{m;\,\delta} & \text{on} \quad \Sigma \\ \operatorname{curl}\mathbf{R}^+_{m;\,\delta} \times \mathbf{n} &= 0 & \text{on} \quad \partial\Omega \end{cases}$$

with $\alpha_+ = 1$ and $\alpha_- = 1 + \mathrm{i}\,\delta^{-2}$, and $\kappa = \omega\sqrt{\varepsilon_0\mu_0}$

Validation of the asymptotic expansion

Estimates for Remainders

Estimates for the residues $\mathbf{j}^-_{m;\,\delta}$ and $\mathbf{g}_{m;\,\delta}$:

$$\|\mathbf{j}^-_{m;\,\delta}\|_{2,\Omega_-} + \|\mathbf{g}_{m;\,\delta}\|_{\frac{1}{2},\Sigma} + \|\operatorname{curl}_\Sigma \mathbf{g}_{m;\,\delta}\|_{\frac{3}{2},\Sigma} \leqslant C_m\, \delta^{m-1}$$

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Theorem (CALOZ, DAUGE, P., 09)

Under Hypothesis in the framework above, for all $m \in \mathbb{N}$ and $\delta \in (0, \delta_0]$, the remainders $\mathbf{R}_{m;\,\delta}$ satisfy the optimal estimates

$$\|\mathbf{R}^+_{m;\,\delta}\|_{0,\Omega_+} + \|\operatorname{curl} \mathbf{R}^+_{m;\,\delta}\|_{0,\Omega_+} + \delta^{-\frac{1}{2}}\|\mathbf{R}^-_{m;\,\delta}\|_{0,\Omega_-} + \delta^{\frac{1}{2}}\|\operatorname{curl} \mathbf{R}^-_{m;\,\delta}\|_{0,\Omega_-} \leqslant C'_m\, \delta^{m+1}$$

Profiles of the Magnetic Field

$\mathbf{v}_j =: (\mathcal{V}_j^\alpha \; ; \; v_j)$ in coordinates (y_β, Y_3) with $Y_3 = \frac{y_3}{\delta}$

Profiles of the Magnetic Field

$\mathbf{V}_j =: (\mathcal{V}_j^\alpha \,;\, v_j)$ in coordinates (y_β, Y_3) with $Y_3 = \frac{y_3}{\delta}$

$$\mathbf{V}_0(y_\beta, Y_3) = \mathbf{h}_0(y_\beta)\, \mathrm{e}^{-\lambda Y_3}$$

$$\mathcal{V}_1^\alpha(y_\beta, Y_3) = \left[\mathrm{h}_1^\alpha + Y_3\left(\mathcal{H}\,\mathrm{h}_0^\alpha + b_\sigma^\alpha\,\mathrm{h}_0^\sigma\right)\right](y_\beta)\, \mathrm{e}^{-\lambda Y_3}$$

with

$\mathcal{H}$ mean curvature of Σ

$$\mathbf{h}_0(y_\beta) = (\mathbf{n} \times \mathbf{H}_0^+) \times \mathbf{n}(y_\beta, 0) \quad \text{and} \quad \mathrm{h}_j^\alpha(y_\beta) := (\mathbf{H}_j^+)^\alpha(y_\beta, 0)$$

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Comparison with



HADDAR, JOLY, NGUYEN (08)

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Tool: a Normal Parameterization Technique

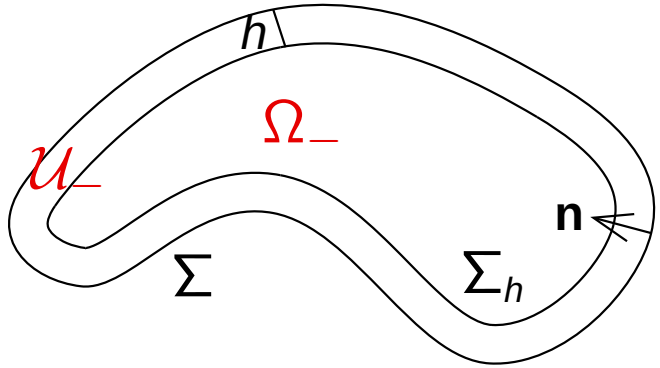


FAOU (02)

Elasticity on a thin shell

Curl Operator and Levi-Civita Tensor

$$h := y_3$$



Curl Operator in Normal Parameterization:

$$(\nabla \times \mathbf{E})^{\alpha} = \epsilon^{3\beta\alpha}(\partial_3^h E_\beta - \partial_\beta E_3) \quad \text{on} \quad \Sigma_h$$

$$(\nabla \times \mathbf{E})^{3} = \epsilon^{3\alpha\beta} D_\alpha^h E_\beta \quad \text{on} \quad \Sigma_h$$

Levi-Civita Tensor ϵ:

$$\epsilon^{ijk} = \epsilon_0(i,j,k) \Big/ \sqrt{\det a_{\alpha\beta}(h)}$$

with $a_{\alpha\beta}(h) =$ metric on Σ_h and $\epsilon_0(i,j,k) \in \{0,1,-1\}$

Covariant Derivative:

$$D_\alpha^h E_\beta = \partial_\alpha E_\beta - \Gamma_{\alpha\beta}^{\gamma}(h) E_\gamma$$

Application 1

A New Definition of the Skin Depth

3D model : $\mathbf{V}_{(\delta)}(y_\alpha, y_3) := \mathbf{H}^-_{(\delta)}(\mathbf{x}), \quad y_\alpha \in \Sigma, \quad 0 \leq y_3 < h_0$

Definition

Let Σ be a smooth surface, and $\mathbf{j}$ s.t. $\mathbf{V}_{(\delta)}(y_\alpha, 0) \neq 0$. The skin depth is the smallest length $\mathcal{L}(\sigma, y_\alpha)$ defined on Σ s.t.

$$\|\mathbf{V}_{(\delta)}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \|\mathbf{V}_{(\delta)}(y_\alpha, 0)\| \, \mathrm{e}^{-1}$$

Theorem (Dauge, Faou, P., 10)

Let Σ be a regular surface with mean curvature $\mathcal{H}$, and assume $\mathbf{h}_0(y_\alpha) \neq 0$.

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma)\Big(1 + \mathcal{H}(y_\alpha)\,\ell(\sigma) + \mathcal{O}(\sigma^{-1})\Big), \quad \sigma \to \infty$$

$$\|\mathbf{V}_{(\delta)}\|^2 = \Big[\|\mathbf{h}_0\|^2 + 2y_3\mathcal{H}\,\|\mathbf{h}_0\|^2 + 2\delta\,\mathrm{Re}\langle \mathbf{h}_0, \mathbf{h}_1\rangle + \mathcal{O}\big((\delta + y_3)^2\big)\Big]\,\mathrm{e}^{-2y_3/\ell(\sigma)}$$

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3 **Numerical Simulations**

Framework

- Magnetic Field $\mathbf{H}_\sigma$
- Axisymmetric Problem: Ω_- and Ω Axisymmetric Domains
- $\mathrm{curl}\,\mathbf{j}$ Axisymmetric & Orthoradial $\longrightarrow$ $\mathbf{H}_\sigma$ Axisymmetric & Orthoradial

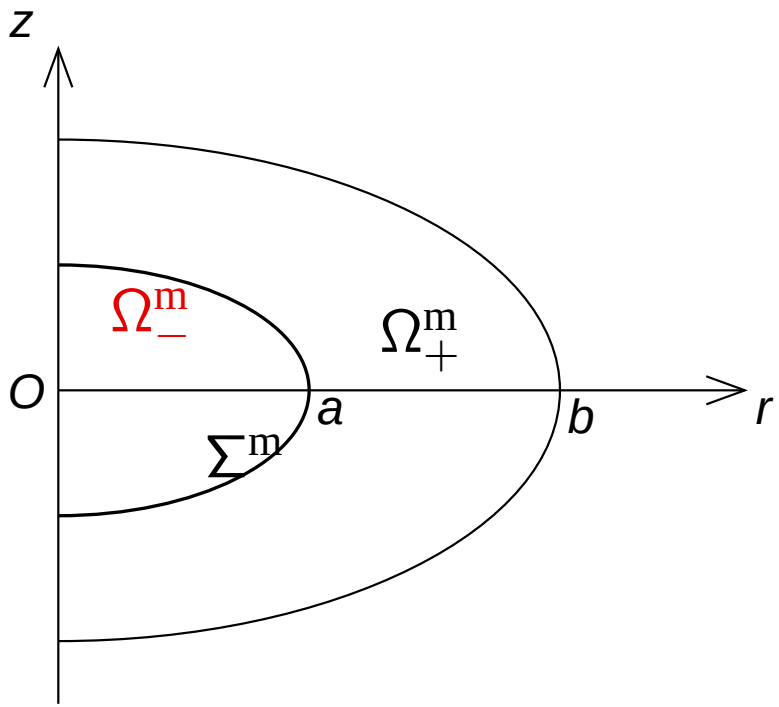
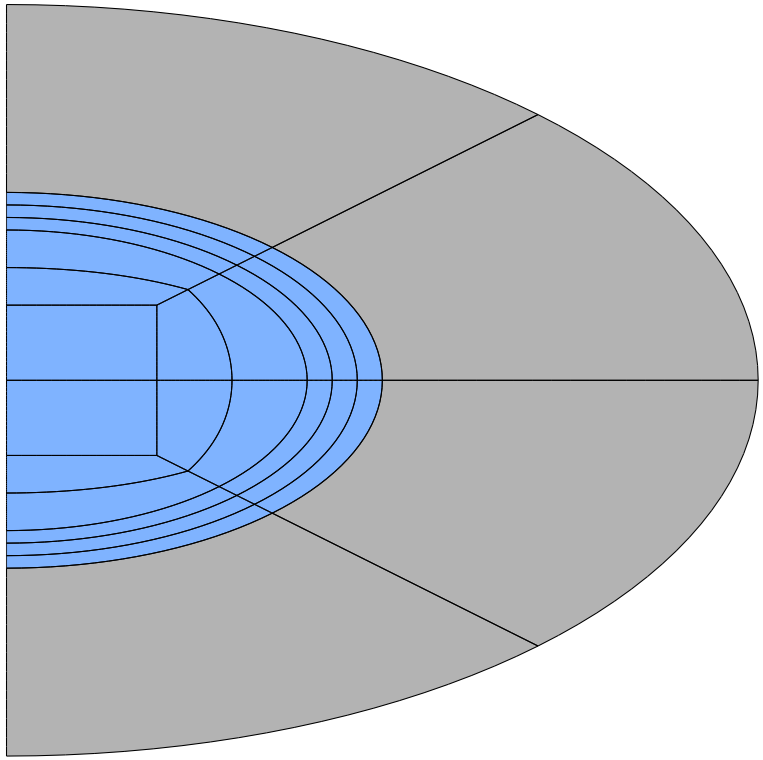


Figure: The meridian domain. Configuration A

- FEM: Finite Element Library Melina

Convergence w.r.t. Interpolation degree p

H_σ^p: the computed solution with an interpolation degree p and a fixed mesh



We define

$$A_\sigma^p := \left\| \mathrm{H}_\sigma^p \right\|_{\mathrm{L}^2_1(\Omega^{\mathrm{m}}_-)} \qquad \left(\text{weight function} = r\right)$$

Interpolation degree p

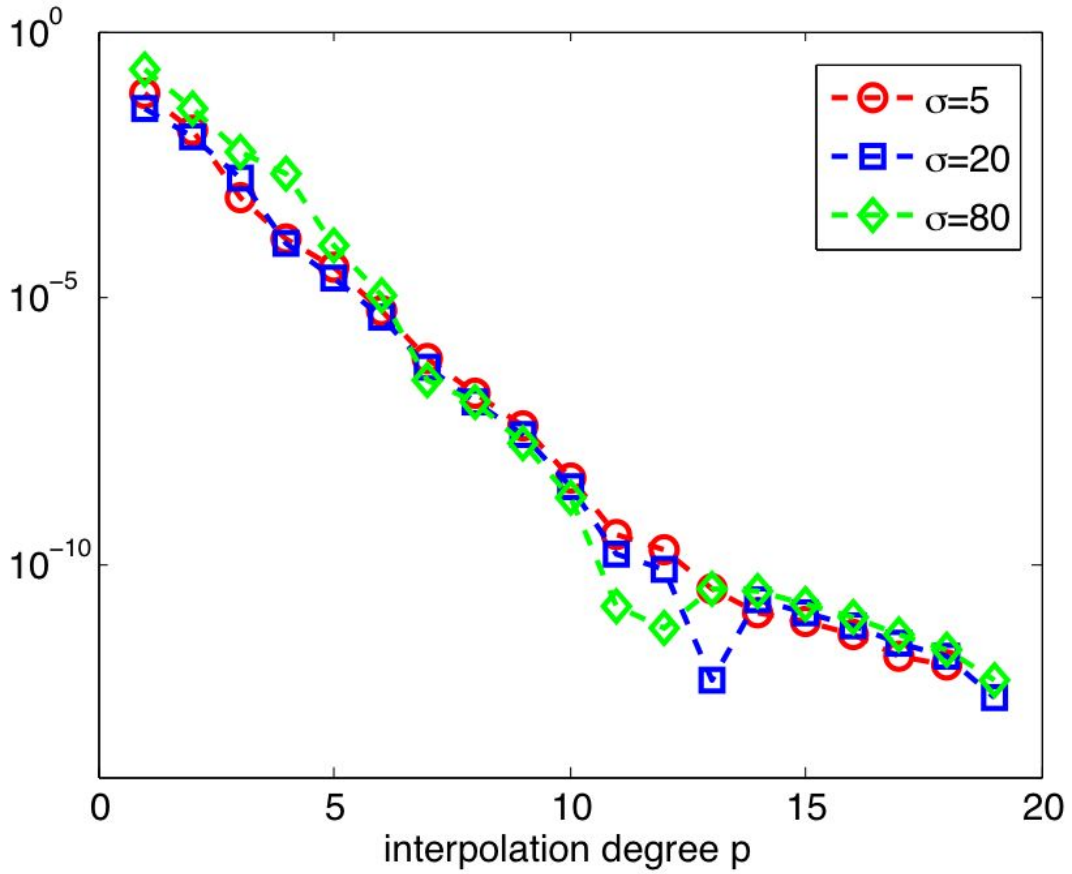
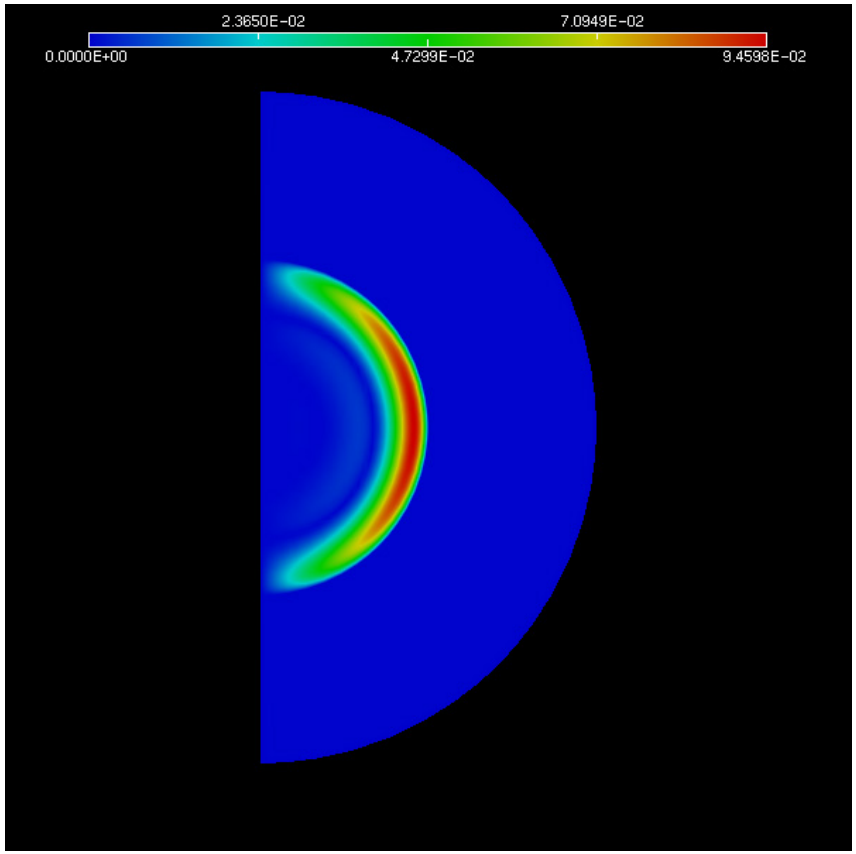


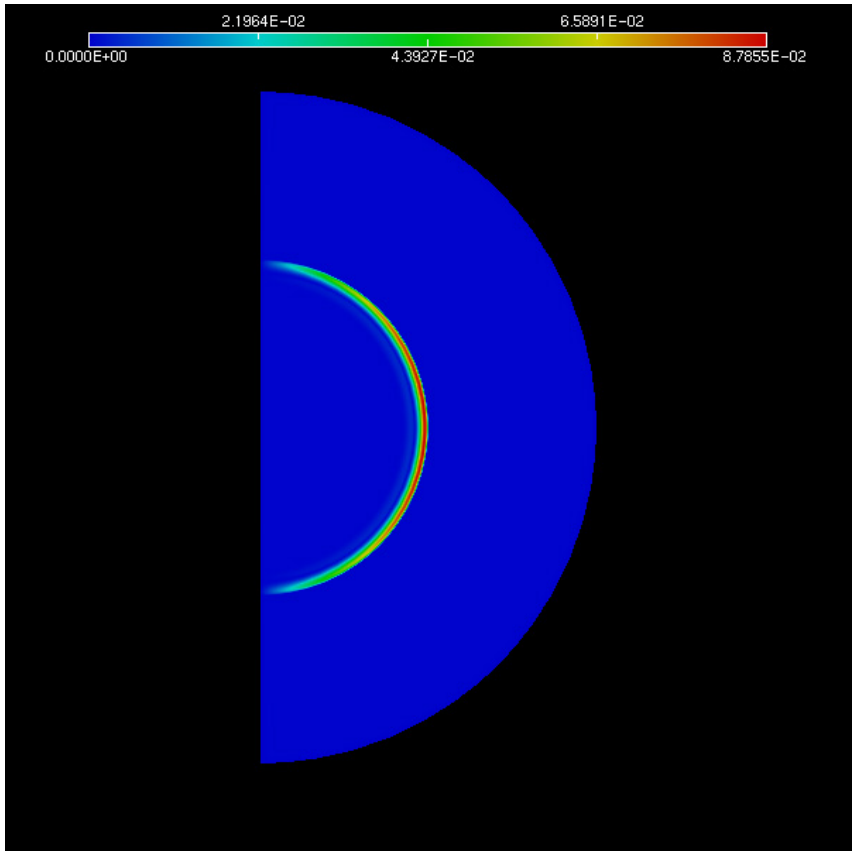
Figure: $\left|A_\sigma^p - A_\sigma^{20}\right|$ when $p = 1, \cdots, 19$ in semilog scale

Spherical Geometry

$|\mathrm{Im}\,\mathrm{H}_\sigma|$. Configuration A



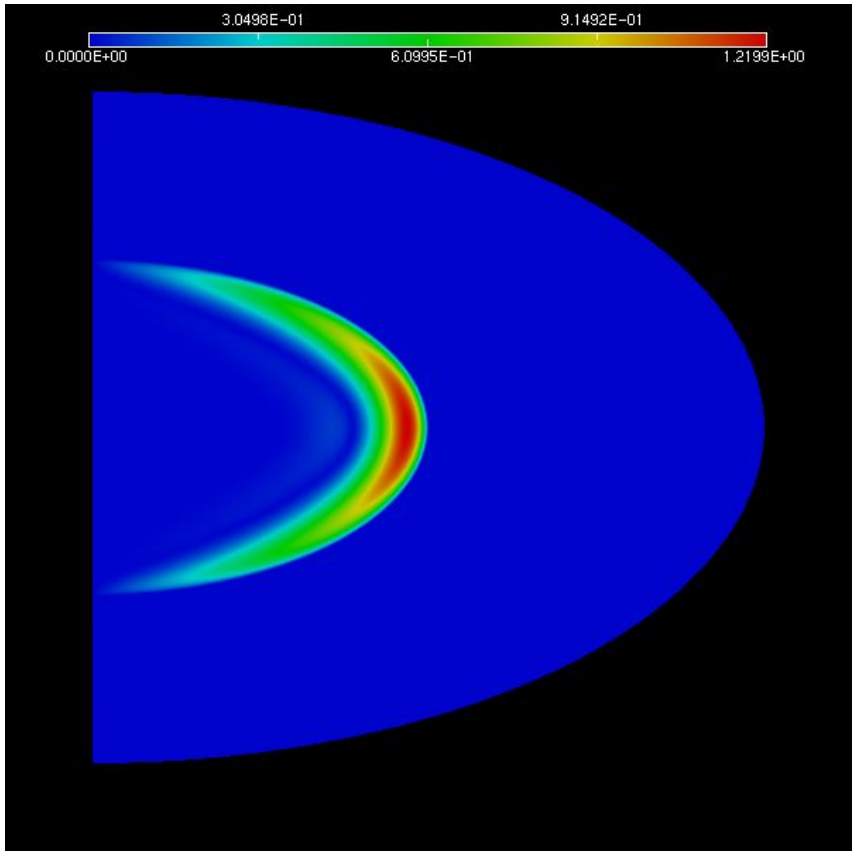
$|\mathrm{Im}\,\mathrm{H}_\sigma|$ as $\sigma = 5S.m^{-1}$



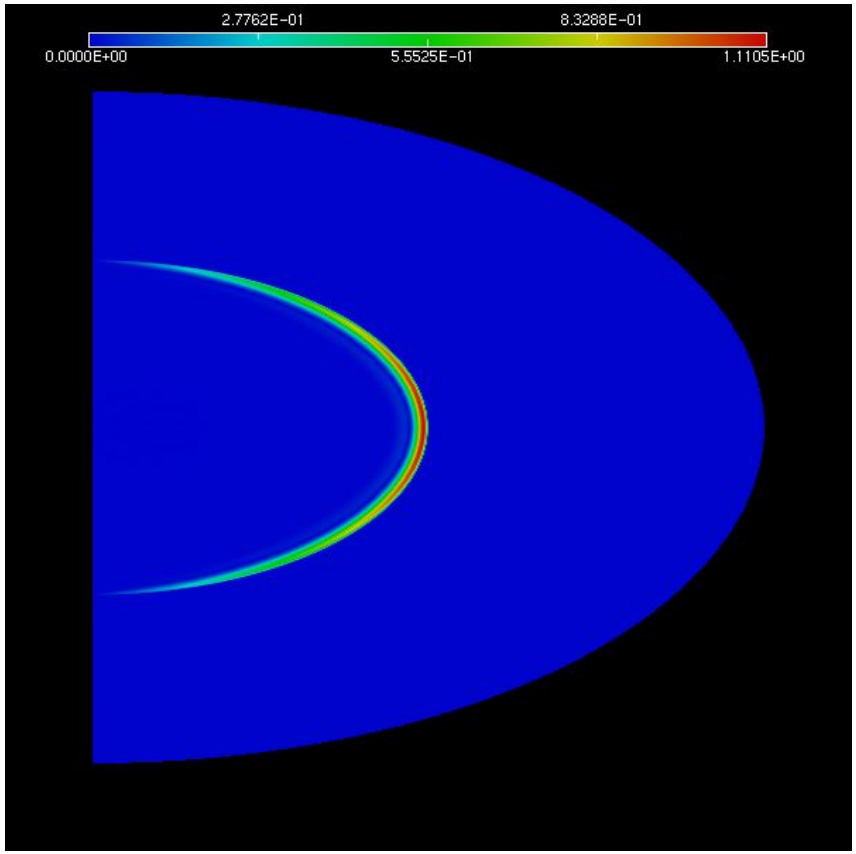
$|\mathrm{Im}\,\mathrm{H}_\sigma|$ as $\sigma = 80S.m^{-1}$

Spheroidal Geometry

$|\mathrm{Im}\,\mathrm{H}_\sigma|$. Configuration A



$|\mathrm{Im}\,\mathrm{H}_\sigma|$ as $\sigma = 5S.m^{-1}$



$|\mathrm{Im}\,\mathrm{H}_\sigma|$ as $\sigma = 80S.m^{-1}$

Influence of the Mean Curvature

Configuration B

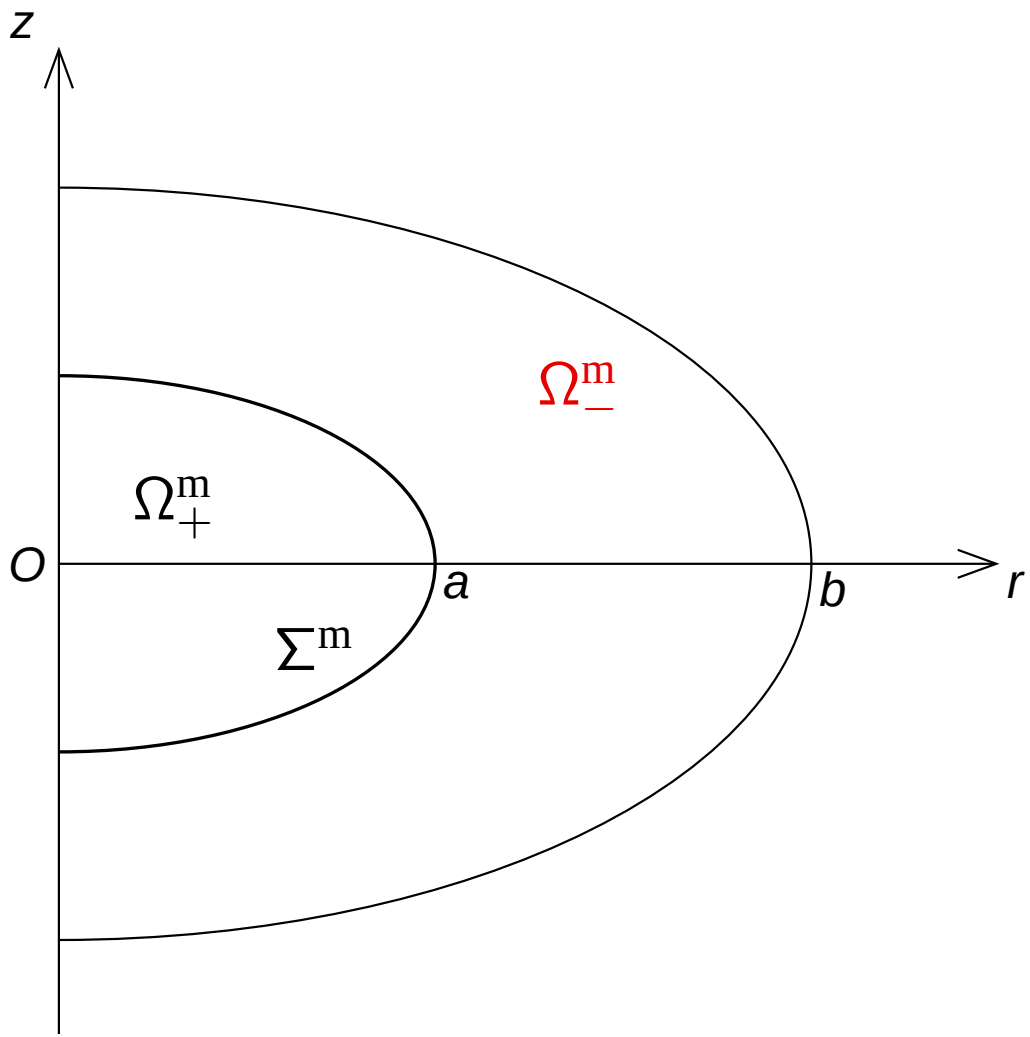
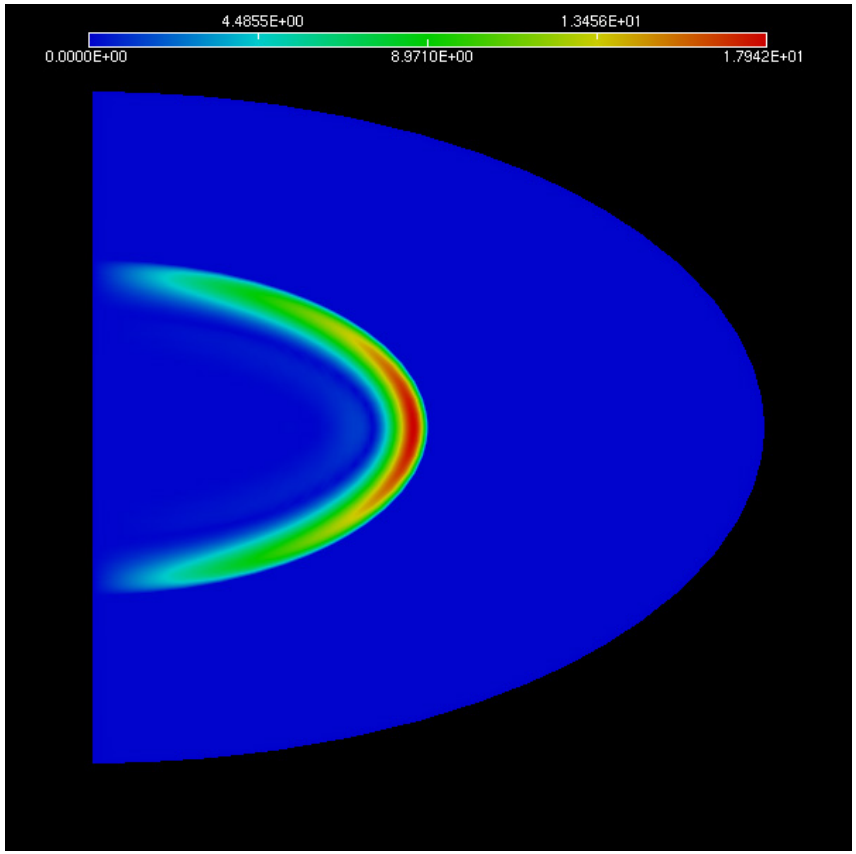


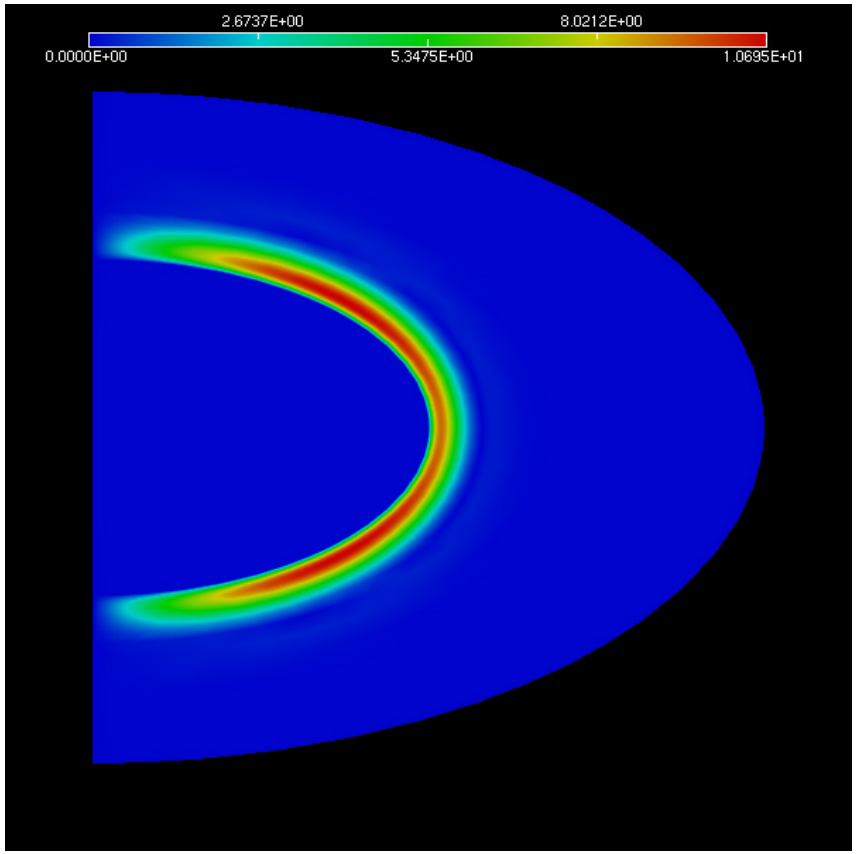
Figure: Configuration B. $\mathcal{H} < 0$

Influence of the Mean Curvature

$|\operatorname{Im} \mathrm{H}_\sigma|$ **when** $\sigma = 5 S.m^{-1}$



Configuration A ($\mathcal{H} > 0$)



Configuration B ($\mathcal{H} < 0$)

Numerical Post-Treatment

Linear Regression. Configuration A

Extract $\left|\mathrm{H}_\sigma(h)\right|$ along edges of the mesh in Ω_-^{m} when $z = 0$: $h = 2 - r$

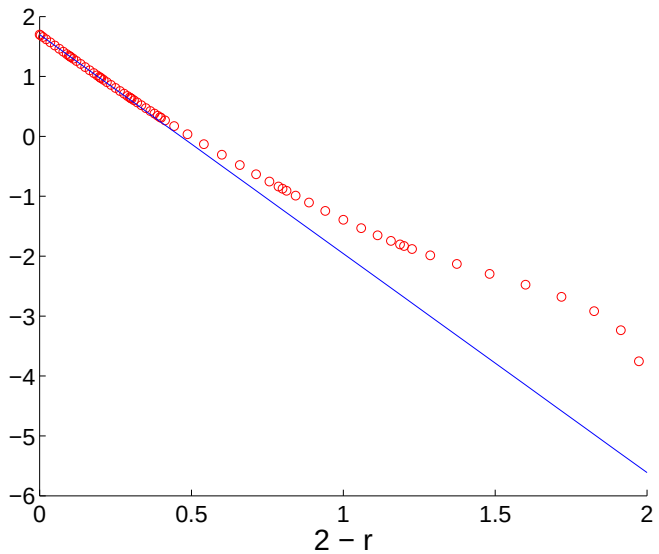
$$\log_{10}\left|\mathrm{H}_\sigma(h)\right| = -s(\sigma)h + \alpha$$

Numerical Post-Treatment

Linear Regression. Configuration A

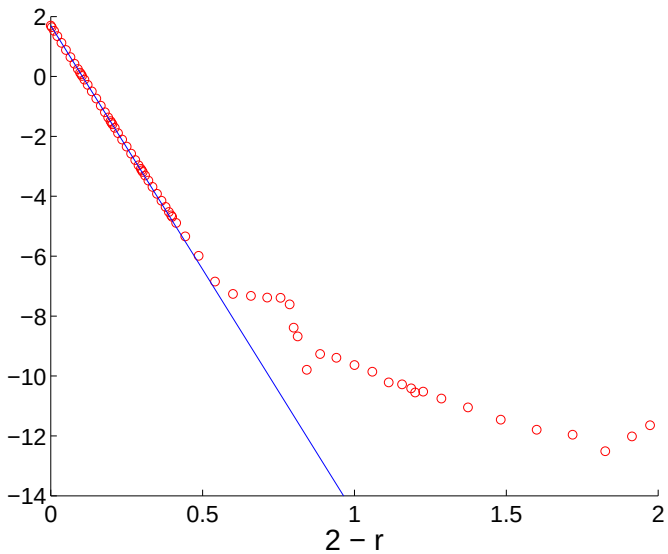
Extract $\left|\mathrm{H}_\sigma(h)\right|$ along edges of the mesh in Ω_-^{m} when $z = 0$: $h = 2 - r$

$$\log_{10}\left|\mathrm{H}_\sigma(h)\right| = -s(\sigma)h + \alpha$$



$\circ$: $\log_{10}\left|\mathrm{H}_\sigma(h)\right|$ when $\sigma = 5$

$n(\sigma) = 7$, $s(5) = 3,65542$



$\circ$: $\log_{10}\left|\mathrm{H}_\sigma(h)\right|$ when $\sigma = 80$

$n(\sigma) = 3$, $s(80) = 16,279162$

Post-Treatment

Accurary of Asymptotics

$$\log_{10}\|\mathbf{V}_{(\delta)}(y_\alpha,h)\| = \underbrace{-\frac{1}{\ln 10}\left(\frac{1}{\ell(\sigma)}-\mathcal{H}(y_\alpha)\right)}_{=:\,s(y_\alpha,\sigma)} h+\beta+\mathcal{O}(\delta)+\mathcal{O}\left((\delta+h)^2\right)$$

Relative Error between slopes:

$$error(\sigma) := \left|\frac{S(\sigma)-s(\sigma)}{S(\sigma)}\right|$$

Post-Treatment

Accurary of Asymptotics

$$\log_{10} \|\mathbf{V}_{(\delta)}(y_\alpha, h)\| = \underbrace{-\frac{1}{\ln 10}\left(\frac{1}{\ell(\sigma)} - \mathcal{H}(y_\alpha)\right)}_{=: s(y_\alpha,\sigma)} h + \beta + \mathcal{O}(\delta) + \mathcal{O}\left((\delta+h)^2\right)$$

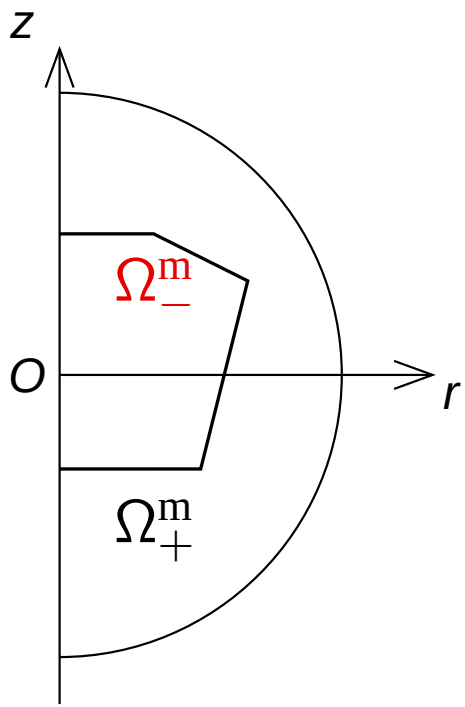
Relative Error between slopes:

$$error(\sigma) := \left|\frac{S(\sigma) - s(\sigma)}{S(\sigma)}\right|$$

σ	5	20
$\ell(\sigma)$ (cm)	10.3	5.15
$s(\sigma)$	$3,65542$	$7,883903$
$S(\sigma)$	$3,673319$	$7,889506$
$error(\sigma)$ (%)	$0,48$	$0,07$
$n(\sigma)$	7	5

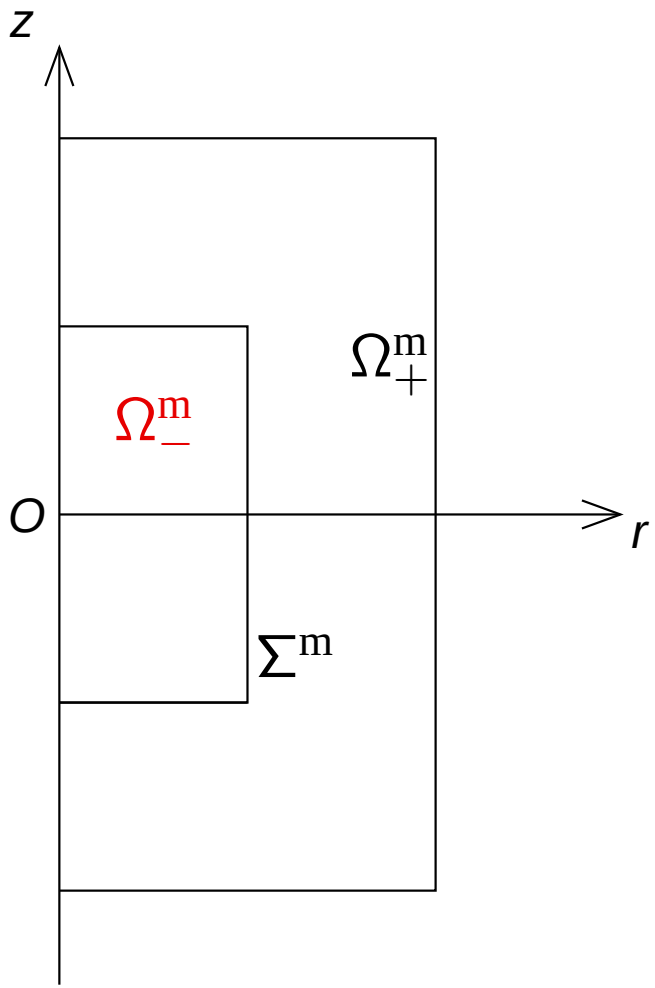
Conclusion

- Efficiency Methods $\longrightarrow$ Skin Effect
- Solution Structure & Optimal Estimates of Remainders
- A Singular Transmission Problem: with C. Poignard and M. Dauge



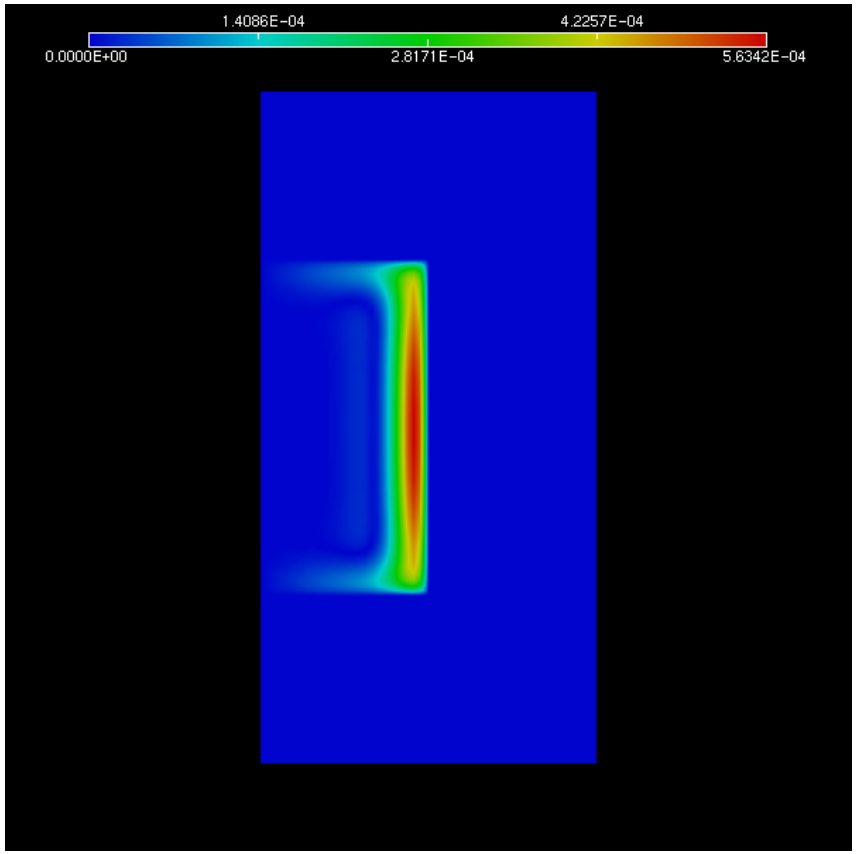
Cylindrical Geometry

Meridian domain Ω^{m}

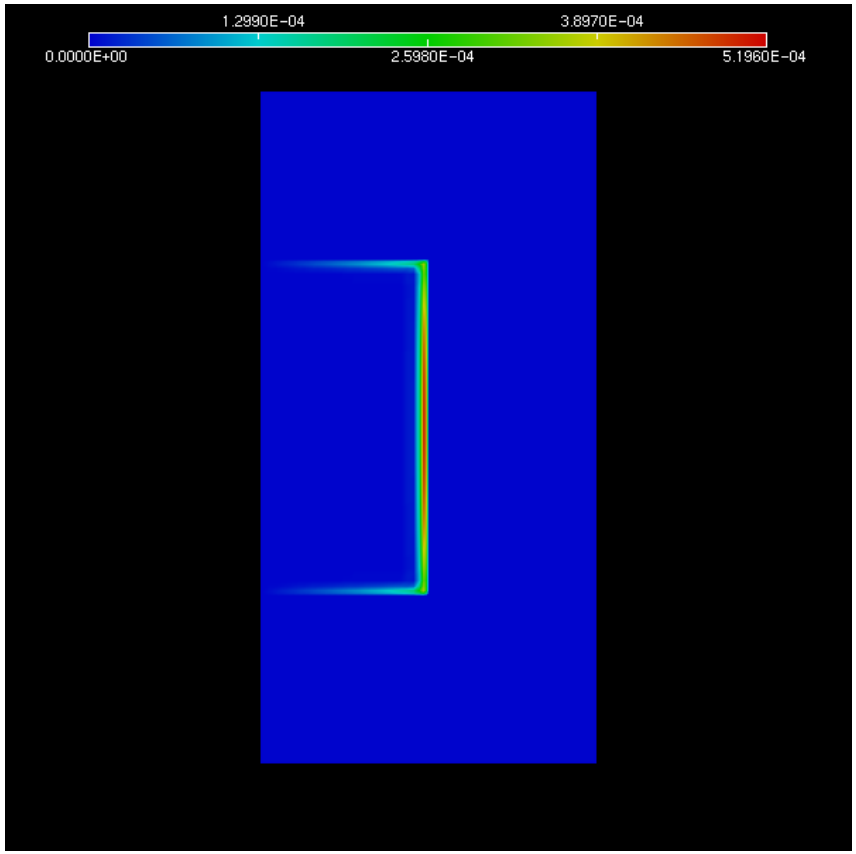


Cylindrical Geometry

$|\operatorname{Im} \mathrm{H}_\sigma|$



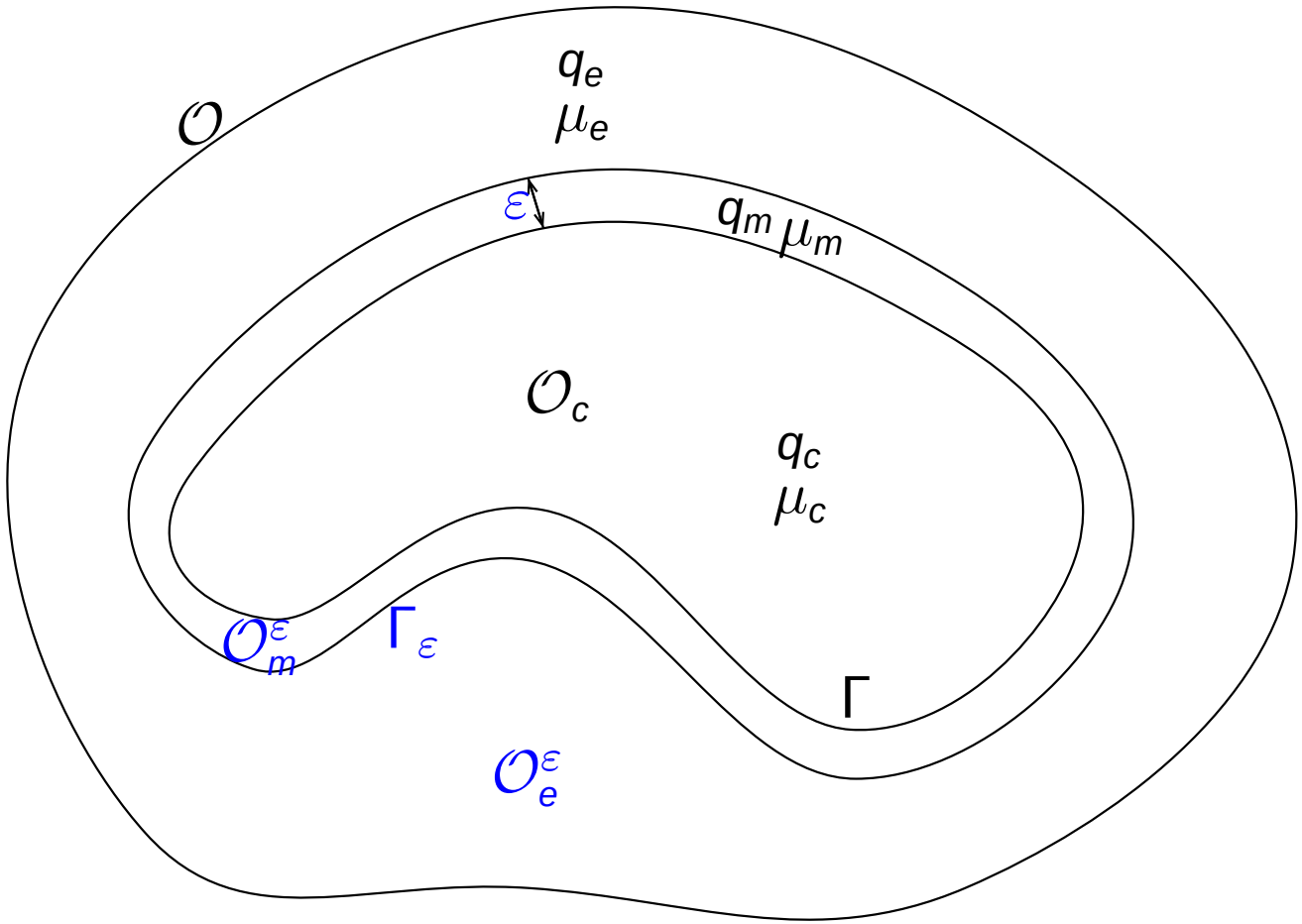
$|\operatorname{Im} \mathrm{H}_\sigma|$ as $\sigma = 5S.m^{-1}$



$|\operatorname{Im} \mathrm{H}_\sigma|$ as $\sigma = 80S.m^{-1}$

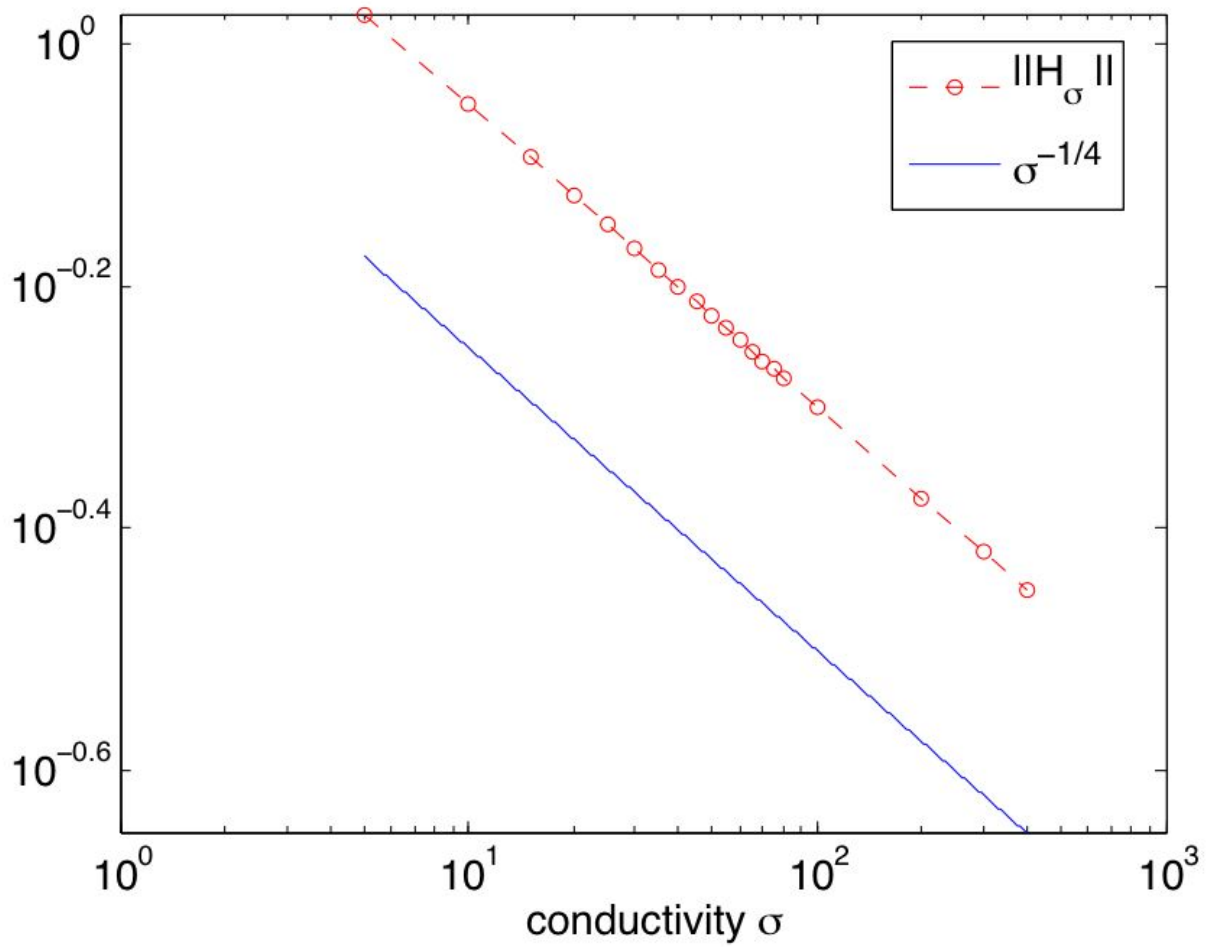
Application 2

Approximate Transmission Conditions [POIGNARD, P., 08]



Thanks for your attention !

Weighted Norm of H^{16}_{σ}



Behavior consistent with the asymptotic expansion ($\sqrt{\delta} \sim \sigma^{-1/4}$)