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# Grundy number on $P_4$ -classes<sup>1</sup>

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## Abstract

In this article, we define a new class of graphs, the fat-extended  $P_4$ -laden graphs, and we show a polynomial time algorithm to determine the Grundy number of the graphs in this class. This result implies that the Grundy number can be found in polynomial time for any graph of the following classes:  $P_4$ -reducible, extended  $P_4$ -reducible,  $P_4$ -sparse, extended  $P_4$ -sparse,  $P_4$ -extendible,  $P_4$ -lite,  $P_4$ -tidy,  $P_4$ -laden and extended  $P_4$ -laden, which are all strictly contained in the fat-extended  $P_4$ -laden class.

*Keywords:* Graph Theory, Grundy number,  $P_4$ -classes, Modular decomposition

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## 1 Introduction

Given a graph  $G = (V, E)$ , a  $k$ -coloring of  $G$  is an assignment of  $k$  colors to the vertices of  $G$  in such a way that adjacent vertices have distinct colors. The chromatic number of  $G$ ,  $\chi(G)$ , is the minimum integer  $k$  such that  $G$  admits a  $k$ -coloring. The problem of determining  $\chi(G)$  is NP-hard [6]. Therefore, the evaluation of the performance of fast vertex coloring algorithms is a relevant problem. The greedy coloring algorithm is a linear vertex coloring algorithm. Given an order  $\theta = v_1, \dots, v_n$  over  $V$ , the greedy algorithm to color the vertices of  $G$  assigns to  $v_i$  the minimum positive integer that was not already assigned to its neighbors in the set  $\{v_1, \dots, v_{i-1}\}$ . A coloring obtained by an execution of this algorithm is usually called as a greedy coloring.

The maximum number of colors of a greedy coloring of a graph  $G$ , over all the orders  $\theta$  of  $V(G)$ , is the *Grundy number* of  $G$  and it is denoted by  $\Gamma(G)$ .

Determining the Grundy number is NP-complete even for complements of bipartite graphs [10]. In fact, given a graph  $G$  and an integer  $r$  it is a coNP-complete problem to decide if  $\Gamma(G) \leq \chi(G) + r$  [10] or if  $\Gamma(G) \leq r \times \chi(G)$  or if  $\Gamma(G) \leq c \times \omega(G)$  [2], where  $c$  is a constant and  $\omega(G)$  is the size of a maximum clique of  $G$ . However, there are polynomial time algorithms

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to calculate the Grundy number of the following classes of graphs: cographs [4], trees [5] and  $k$ -partial trees [9].

## 2 Fat extended $P_4$ -laden graphs

We start this section by introducing some definitions. Let  $G = (V, E)$  be a graph and  $S$  a subset of  $V(G)$ . We denote by  $G[S]$  the subgraph of  $G$  induced by  $S$ . We say that  $M$  is a *module* of a graph  $G$  if, for every vertex  $w$  in  $V \setminus M$  and every pair of vertices  $x$  and  $y$  in  $M$ , either  $w$  is adjacent to both  $x$  and  $y$  or  $w$  is not adjacent to both  $x$  and  $y$ . The sets  $V$  and  $\{x\}$ , for every  $x \in V$ , are *trivial modules*, the last one being called as a *singleton module*. A graph is *prime* if all its modules are trivial. We say that  $M$  is a *strong module* of  $G$  if, for every module  $M'$  of  $G$ , either  $M' \cap M = \emptyset$  or  $M \subset M'$  or  $M' \subset M$ . The modular decomposition is a form of decomposition of a graph  $G$  that associates with  $G$  a unique *modular decomposition tree*  $T(G)$ . The leaves of  $T(G)$  are the vertices of  $G$  and a set of leaves of  $T(G)$  having the same least common ancestor in  $T(G)$  is a strong module of  $G$ . Let  $r$  be an internal node of  $T(G)$ ,  $M(r)$  be the set of leaves of the subtree of  $T(G)$  rooted on  $r$ , and  $V(r) = \{r_1, \dots, r_k\}$  be the set of children of  $r$  in  $T(G)$ . If  $G[M(r)]$  is disconnected, then  $r$  is called a *parallel node* and  $G[M(r_1)], \dots, G[M(r_k)]$  are its components. If  $\bar{G}[M(r)]$  is disconnected then  $r$  is called a *series node* and  $\bar{G}[M(r_1)], \dots, \bar{G}[M(r_k)]$  are the components of  $\bar{G}[M(r)]$ . Finally, if both graphs  $G[M(r)]$  and  $\bar{G}[M(r)]$  are connected, then  $r$  is called a *neighborhood node* and  $\{M(r_1), \dots, M(r_k)\}$  is the unique set of maximal strong proper submodules of  $M(r)$ . The *quotient graph* of  $r$ , denoted by  $G(r)$ , is  $G[\{v_1, \dots, v_k\}]$ , where  $v_i \in M(r_i)$ , for  $1 \leq i \leq k$ . We say that  $r$  is a *fat node*, if  $M(r)$  is not a singleton module.

A graph is a *spider* if its vertex set can be partitioned into three sets  $S$ ,  $K$  and  $R$  in such a way that  $S$  is a stable set,  $K$  is a clique, all the vertices of  $R$  are adjacent to all the vertices of  $K$  and to none of the vertices of  $S$  and there is a bijection  $f : S \rightarrow K$  such that, for all  $s \in S$ , either  $N(s) = f(s)$  (and it is a *thin spider*) or  $N(s) = K - f(s)$  (and it is a *fat spider*).

A graph  $G$  is split if and only if it is  $\{C_5, C_4, \bar{C}_4\}$ -free, which is equivalent to say that  $V(G)$  can be partitioned in two sets  $S$  and  $K$  such that  $S$  is a stable set and  $K$  is a clique. A *pseudo-split* graph is defined as a  $\{C_4, \bar{C}_4\}$ -free graph. Moreover, given a split graph  $G = (S \cup K, E)$ , its vertex set can be partitioned into three disjoint sets  $S(G)$ ,  $K(G)$  and  $R(G)$  such that  $S(G)$  is composed by all the vertices of  $S$  which are not adjacent to at least one vertex in  $K$ ,  $K(G)$  is the neighborhood of the vertices in  $S(G)$  and  $R(G) = V(G) \setminus \{S(G) \cup K(G)\}$ .

Giakoumakis [3] defined a graph  $G$  as *extended  $P_4$ -laden graphs* if, for all  $H \subseteq G$  such that  $|V(H)| \leq 6$ , then the following statement is true: if  $H$  contains more than two induced  $P_4$ 's, then  $H$  is a pseudo-split graph. An extended  $P_4$ -laden graph can be completely characterized by its modular decomposition tree, as follows:

**Theorem 2.1** ([3]) *Let  $G = (V, E)$  be a graph,  $T(G)$  be its modular decomposition tree and  $r$  be any neighborhood node of  $T(G)$ , with children  $r_1, \dots, r_k$ . Then  $G$  is extended  $P_4$ -laden if and only if  $G(r)$  is isomorphic to:*

- (i) a  $P_5$  or a  $\bar{P}_5$  or a  $C_5$ , and each  $M(r_i)$  is a singleton module; or
- (ii) a spider  $H = (S \cup K \cup R, E)$  and each  $M(r_i)$  is a singleton module, except the one corresponding to  $R$  and eventually another one which may have exactly two vertices; or
- (iii) a split graph  $H$ , whose modules corresponding to the vertices of  $S(H)$  are independent sets and the ones corresponding to the vertices of  $K(H)$  are cliques.

We say that a graph is **fat-extended  $P_4$ -laden** if its modular decomposition satisfies the Theorem 2.1, except in the first case, where  $G(r)$  is isomorphic to a  $P_5$  or a  $\bar{P}_5$  or a  $C_5$ , but the maximal strong modules  $M(r_i)$ ,  $1 \leq i \leq 5$ , of  $M(r)$  are not necessarily singleton modules.

### 3 Grundy number on fat extended $P_4$ -laden graphs

From now, let  $G = (V, E)$  be a fat-extended  $P_4$ -laden graph and  $T(G)$  its modular decomposition tree. Since  $T(G)$  can be found in linear time [8], we propose an algorithm to calculate  $\Gamma(G)$  that uses a bottom-up strategy. We know that the Grundy number of the leaves of  $T(G)$  is equal to one and we show in this section how to determine the Grundy number of  $G[M(v)]$ , for every inner node  $v$  of  $T(G)$ , based on the Grundy number of its children.

First, observe that for every series node (resp. parallel node)  $v$  of  $T(G)$ , the Grundy number of  $G[M(v)]$  is equal to the sum of the Grundy number of its children (resp. the maximum Grundy number of its children) [4]. Thus, we only need to prove that the Grundy number of  $G[M(v)]$  can be found in polynomial time when  $v$  is a neighborhood node of  $T(G)$ .

The following result is a simple generalization of a result due to Asté et al. [2] for the Grundy number of lexicographic product of graphs:

**Proposition 3.1** *Let  $G, H_1, \dots, H_n$  be disjoint graphs such that  $n = |V(G)|$  and let  $V(G) = \{v_1, \dots, v_n\}$ . Let  $G'$  be the graph obtained by replacing  $v_i \in V(G)$  by  $H_i$ , in such a way that there exist all the edges between the vertices of  $H_i$  and  $H_j$ ,  $i \neq j$ , if and only if  $v_i v_j \in E(G)$ . Then for every greedy coloring of  $G'$  at most  $\Gamma(H_i)$  colors contain vertices of the induced subgraph  $G'[H_i] \subseteq G'$ , for all  $i \in \{1, \dots, n\}$ .*

Before presenting the next lemma, observe that a greedy  $k$ -coloring of  $G$  can be viewed as a partition  $\mathcal{S} = \{S_1, \dots, S_k\}$  of  $V(G)$  in such a way that every vertex in  $S_j$  has at least one neighbor in the color class  $S_i$ , for all  $j > i$ ,  $i, j \in \{1, \dots, k\}$ .

**Lemma 3.2** *Let  $v$  be a neighborhood node of  $T(G)$  such that  $G(v)$  is isomorphic to a  $P_5$  or a  $C_5$  or a  $\bar{C}_5$ ,  $v_1, \dots, v_5$  be the children of  $v$  and  $\Gamma_i$  be the Grundy number of  $G[M(v_i)]$ ,  $1 \leq i \leq 5$ . Then  $\Gamma(G[M(v)])$  can be found in constant time.*

**Proof (Sketch)** Without loss of generality, suppose that  $v_1, \dots, v_5$  label the children of  $v$  as depicted in Figure 1 and  $\Gamma_i = \Gamma(G[M(v_i)])$ . In order to simplify the notation, denote by  $\theta_i$  an ordering over  $M(v_i)$  that induces a greedy coloring with  $\Gamma_i$  colors,  $1 \leq i \leq 5$ .

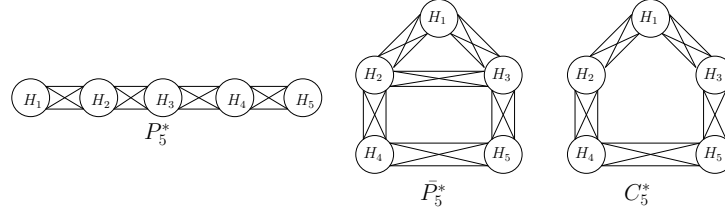
We calculate  $\Gamma(G[M(v)])$  by verifying all the possible configurations for a greedy  $\Gamma(G[M(v)])$ -coloring and by returning the greater value found between all the cases. Suppose that  $G(v)$  is isomorphic to a  $P_5$ . Let  $\mathcal{S} = \{S_1, \dots, S_k\}$  be a greedy  $\Gamma(G[M(v)])$ -coloring of  $G[M(v)]$ .

We claim that if there exists a vertex  $u \in V(H_1)$  colored by  $S_k$ , then  $\Gamma(G[M(v)]) = \Gamma_1 + \Gamma_2$ . This fact holds because combining the observation that  $u$  has at least one vertex colored by  $S_i$ , for all  $i \in \{1, \dots, k-1\}$ , with the Proposition 3.1, we conclude that  $\Gamma(G[M(v)]) \leq \Gamma_1 + \Gamma_2$ . On the other hand, if we consider any ordering  $\theta$  over  $G[M(v)]$  that has starts with  $\theta_1$  and  $\theta_2$ , we see that the first-fit algorithm over this order will produce a greedy coloring with at least  $\Gamma_1 + \Gamma_2$  colors. Using the symmetry, we can also prove that if  $u \in V(H_5)$ , then  $\Gamma(G[M(v)]) = \Gamma_4 + \Gamma_5$ .

All the other cases use similar arguments, that is, by finding an upper bound based on the position of a vertex colored  $S_k$  and a lower bound based in an ordering over  $M(v)$ . The cases

where  $G(v)$  is isomorphic to  $C_5$  or  $\bar{P}_5$  are also proved by using similar arguments.  $\square$

Fig. 1. Fat neighborhood nodes.



**Lemma 3.3** *Let  $v$  be a neighborhood node of  $T(G)$  such that  $G(v)$  is isomorphic to a spider  $H = (S \cup K \cup R, E)$ ,  $f_r$  be its child corresponding to  $R$ ,  $f_2$  be its child corresponding to the module which has eventually two vertices and  $\Gamma(R)$  be the Grundy number of  $G[M(f_r)]$ . Then  $\Gamma(G[M(v)])$  can be found in  $\mathcal{O}(V(G[M(v)]))$ .*

**Proof (Sketch)** If  $M(f_2)$  is singleton module, then  $G[M(v)]$  is a spider. In this case, we cannot have two colors  $S_i$  and  $S_j$ ,  $j > i$ , such that both contain only vertices of  $S$ . For otherwise, since  $S$  is a stable set, the vertices colored  $S_j$  would not any neighbor colored  $S_i$ , a contradiction. Thus,  $\Gamma(G[M(v)]) \leq 1 + |K| + \Gamma(R)$ . If  $R = \emptyset$ , then an ordering over  $M(v)$  such that all the vertices of  $S$  come before the vertices of  $K$  induces a greedy coloring with  $\Gamma(G[M(v)]) = 1 + |K|$  colors. If  $R \neq \emptyset$ , we will prove that  $\Gamma(G[M(v)]) \leq |K| + \Gamma(R)$ . Observe first that there is at least one color  $S_i$  occurring in  $R$ . Consequently,  $S_i$  does not occur in  $K$ . Thus, there is no order over  $M(v)$  whose greedy coloring returns a color  $S_j$  containing only vertices of  $S$ , because a vertex of  $S$  colored  $S_j$  would not be adjacent to a vertex colored  $S_i$ . On the other hand, if  $\theta_R$  is an ordering that induces a greedy  $\Gamma(R)$ -coloring of  $R$ , then any ordering over  $M(v)$  starting by  $\theta_R$  induces a greedy coloring with at least  $|K| + \Gamma(R)$  colors. The case where  $M(f_2)$  is not a singleton module is proved using similar arguments.  $\square$

If  $G(v)$  is a split graph  $H$  and the factors corresponding to vertices of  $S(H)$  are independent sets and the ones corresponding to vertices of  $K(H)$  are cliques, then we can use the same arguments of Lemma 3.3 observing that  $S$ ,  $K$  and  $R$  correspond to  $S(H)$ ,  $K(H)$  and  $R(H)$ , respectively.

**Theorem 3.4** *If  $G = (V, E)$  is a fat-extended  $P_4$ -laden graph and  $|V| = n$ , then  $\Gamma(G)$  can be found in  $\mathcal{O}(n^3)$ .*

**Proof.** The algorithm calculates  $\Gamma(G)$  by traversing the modular decomposition tree of  $G$  in a postorder way and determining the Grundy of each inner node of  $T(G)$  based on the Grundy number of the leaves. The modular decomposition tree can be found in linear time, the postorder traversal can be done in  $\mathcal{O}(n^2)$ -time and the Grundy number of each inner node can be found in linear time on the number of vertices of the corresponding module, because of Lemmas 3.2 and 3.3 and the results of Gyarfas and J. Lehel [4] for cographs.  $\square$

**Corollary 3.5** *Let  $G$  be a graph that belongs to one of the following classes:  $P_4$ -reducible, extended  $P_4$ -reducible,  $P_4$ -sparse, extended  $P_4$ -sparse,  $P_4$ -lite,  $P_4$ -extendible,  $P_4$ -tidy,  $P_4$ -laden and extended  $P_4$ -laden. Then,  $\Gamma(G)$  can be found in polynomial time.*

**Proof.** According to definition of these classes [7], they are all strictly contained in the fat-extended  $P_4$ -laden graphs and so the corollary follows.  $\square$

The complete proofs of the results in this paper can be found in [1].

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