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Abstract: In this work we consider a stochastic optimal control problem with either convex control constraints or finitely many equality and inequality constraints over the final state. Using the variational approach, we are able to obtain first and second order expansions for the state and cost function, around a local minimum. This fact allows us to prove general first order necessary condition and, under a geometrical assumption over the constraint set, second order necessary conditions are also established. We end by giving second order optimality conditions for problems with constraints on expectations of the final state.

Key-words: Stochastic optimal control, variational approach, first and second order optimality conditions, polyhedric constraints, final state constraints.

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Conditions d'optimalité du premier et second ordre en commande optimale stochastique

Résumé : Nous considèrons un problème de commande optimale stochastique avec soit des contraintes convexes sur la commande soit un nombre fini de contraintes d'égalité et d'inégalité sur l'état final. L'approche dite *variationnelle* nous permet d'obtenir un dèveloppement au premier et au second ordre pour l'état et la fonction de coût, autour d'un minimum local. Avec ces dèveloppements on peut montrer des conditions générales d'optimalité de premier ordre et, sous une hypothèse géométrique sur l'ensemble des contraintes, des conditions nécessaires du second ordre sont aussi établies. On finit l'article en fournissant des conditions d'optimalité du second ordre pour de problèmes avec de contraintes en espérance sur l'état final

Mots-clés : Commande optimale stochastique, approche variationnelle, conditions d'optimalité de premier et second ordre, contraintes polyédriques, contraintes sur l'état final.

1 Introduction

Let us consider a controlled Itô process satisfying the following stochastic differential equation (SDE)

$$\begin{array}{rcl} \mathrm{d}y(t) & = & f(t,y(t),u(t),\omega)\mathrm{d}t + \sigma(t,y(t),u(t),\omega)\mathrm{d}W(t), \text{ on } [0,T]\times\Omega, \\ y(0) & = & y_0 \in \mathbb{R}^n. \end{array} \tag{1}$$

In the notation above $y(t) \in \mathbb{R}^n$ denotes the *state function* and $u(t) \in \mathbb{R}^m$ the *control*. Associated to a pair *state-control* (y,u) we define its cost $J(y,u)$ by

$$J(y,u) := \mathbb{E}\left(\int_0^T \ell(t,y(t),u(t))\mathrm{d}t + \phi(y(T)) \right). \tag{2}$$

Precise definitions of the suitable spaces for (y,u) and assumptions over the data (f,σ,ℓ,ϕ), will be provided in the next section. For the dynamics (1) and the cost (2), we will study two types of problems. In the first one, we suppose that we are given a nonempty closed and convex subset $\mathcal{U}$ of the space $L^2_{\mathcal{F}}$ of adapted square integrable process, and we analyze the following stochastic optimal control problem with *control constraints*

$$\mathrm{Min}_{(y,u)}\ J(y,u) \ \text{ s.t. (1) holds and } u \in \mathcal{U}. \tag{$\mathcal{SP}$}$$

As in the case of deterministic optimal control problems, there are two main approaches to study problem $(\mathcal{SP})$ when $\mathcal{U}$ is defined by *local constraints*, i.e. for a given nonempty closed and convex subset $U \subseteq \mathbb{R}^n$,

$$\mathcal{U} := \left\{ u \in L^2_{\mathcal{F}} \ ; \ u(t,\omega) \in U, \text{ for almost all } (t,\omega) \in [0,T]\times\Omega \right\}. \tag{3}$$

The first approach is the global one, based on Bellman's dynamic programming principle, which yields that the value function of $(\mathcal{SP})$ is the unique viscosity solution of an associated second order Hamilton-Jacobi-Bellman equation. For a complete account of this point of view, widely used in practical computations, we refer the reader to Lions [21, 22] and to the books [11, 26, 29]. The second approach is the variational one, which consists in the local behavior analysis of the value function under small perturbations of a local minimum. Using this technique, Kushner [18, 19, 20], Bensoussan [1, 2], Bismut [3, 4, 5] and Haussmann [14] obtained natural extensions of Pontryagin maximum principle to the stochastic case, that were generalized by Peng [25] to the case where U is not necessarily convex and by Cadenillas and Karatzas [8] to the non-Markovian case. Relations between the global and variational approach are studied in [30, 31].

Nevertheless, to the best of our knowledge, nothing has been said about second order optimality conditions for $(\mathcal{SP})$. Using the variational technique we are able to obtain first and second order expansions for the cost function, which are expressed in terms of the derivatives of the Hamiltonian of problem $(\mathcal{SP})$. The main tool is a kind of generalization of Gronwall's lemma for the SDEs (proposition 1) obtained by Mou and Yong [24], which allows to expand the cost with respect to directions belonging to a more regular space than the control space. Note that the idea of using a more regular space that the original one was already used [6] in the context of deterministic state constrained optimal control problems. By a density argument, we establish first order optimality conditions, which in the case of local constraints are a consequence of the maximum principles obtained in the above references. However, we can also deal with constraints sets that are not necessarily local. The main novelty of our work is that under a polyhedricity assumption (see [13, 23]) over the set $\mathcal{U}$, we are also able to provide second order necessary conditions which are new for the stochastic case and are natural extensions of their deterministic counterparts.

In the second type of problem we suppose that we are given functions g^i, h^j with $i \in \{1, ..., n_g\}$, $j \in \{1, ..., n_h\}$, and we study the following optimal control problem with *finitely many equality and inequality constraints*

$$\text{Min}_{(y,u)} \; J(y,u) \; \text{ s.t. (1) holds and } \mathbb{E}\left[g^i(y(T))\right] = 0, \; \mathbb{E}\left[h^j(y(T))\right] \leq 0. \quad (\mathcal{SP}')$$

First order optimality conditions for $(\mathcal{SP}')$, in a maximum principle form, have been obtained in [25, 8]. Under a standard qualification condition over g^i, h^j, the techniques employed for $(\mathcal{SP})$ allow us recover particular cases of the results in [25, 8], but in addition we are also able to prove second order necessary conditions for $(\mathcal{SP}')$.

The article is organized as follows: After introducing standard notations and assumptions in section 2, we obtain in section 3 first and second order expansions for the state and cost function. The proof of technical lemmas of this section are provided in the Appendix. In section 4, first and second order necessary conditions are proved for problem $(\mathcal{SP})$ with explicit results for the case of *box constraints* over the control. A discussion about a non gap second order sufficient condition is also provided. Finally, in section 5 first and second order necessary conditions are derived for problem $(\mathcal{SP}')$.

2 Notations, assumptions and problem statement

Let us first fix some standard notation. For x belonging to an Euclidean space we will write x^i for its i-th coordinate and $|x|$ for its Euclidean norm. Let $T > 0$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, on which a d-dimensional ($d \in \mathbb{N}^*$) Brownian motion $W(\cdot)$ is defined. We suppose that $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the natural filtration, augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$, associated to $W(\cdot)$. Let $(X, \|\cdot\|_X)$ be a Banach space and for $\beta \in [1, \infty)$ set

$$\begin{array}{rcl} L^{\beta}(\Omega; X) & := & \left\{ v : \Omega \to X; \; v \text{ is } \mathcal{F}\text{-measurable and } \mathbb{E}\left(\|v(\omega)\|_X^{\beta} \right) < \infty \right\}, \\ L^{\infty}(\Omega; X) & := & \left\{ v : \Omega \to X; \; v \text{ is } \mathcal{F}\text{-measurable and ess sup}_{\omega \in \Omega} \|v(\omega)\|_X < \infty \right\}. \end{array}$$

For $\beta, p \in [1, \infty]$ and $m \in \mathbb{N}$ let us define

$$L_{\mathcal{F}}^{\beta,p} \;\; := \left\{ v \in L^{\beta}\left(\Omega; L^p\left([0,T]; \mathbb{R}^m\right)\right); \; (t,\omega) \to v(t,\omega) := v(\omega)(t) \text{ is } \mathbb{F}\text{-adapted} \right\}.$$

We endow these space with the norms

$$\|v\|_{\beta,p} := \left[\mathbb{E}\left(\|v(\omega)\|_{L^p([0,T];\mathbb{R}^m)}^{\beta} \right) \right]^{\frac{1}{\beta}} \;\text{ and }\; \|v\|_{\infty,p} := \text{ess sup}_{\omega \in \Omega} \|v(\omega)\|_{L^p([0,T];\mathbb{R}^m)}.$$

For the sake of clarity, when the context is clear, the statement "for a.a. $t \in [0,T]$, a.s. $\omega \in \Omega$ ($\mathbb{P}$-a.s.)" will be abbreviated to "for a.a. (t,ω)". We will write $L^p_{\mathcal{F}} := L^{p,p}_{\mathcal{F}}$ and $\|\cdot\|_p := \|\cdot\|_{p,p}$. The spaces $L^{\beta,p}_{\mathcal{F}}$ endowed with the norms $\|\cdot\|_{\beta,p}$ are Banach spaces and for the specific case $p = 2$ the space $L^2_{\mathcal{F}}$ is a Hilbert space. We will denote by $\langle \cdot, \cdot \rangle_2$ the obvious scalar product. Evidently, for $\beta \in [1, \infty]$ and $1 \leq p_1 \leq p \leq p_2 \leq \infty$, there exist positive constants c_{β,p_1}, c_{β,p_2}, $c_{p_1,\beta}$, $c_{p_2,\beta}$ such that

$$c_{\beta,p_1}\|v\|_{\beta,p_1} \leq \|v\|_{\beta,p} \leq c_{\beta,p_2}\|v\|_{\beta,p_2}, \quad c_{p_1,\beta}\|v\|_{p_1,\beta} \leq \|v\|_{p,\beta} \leq c_{p_2,\beta}\|v\|_{p_2,\beta}.$$

For a function $[0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \ni (t,y,u,\omega) \to \psi(t,y,u,\omega) \in \mathbb{R}^n$ such that for a.a. (t,ω) the function $(y,u) \to \psi(t,y,u,\omega)$ is $\mathcal{C}^2$, set $\psi_y(t,y,u,\omega) := D_y\psi(t,y,u,\omega)$ and $\psi_u(t,y,u,\omega) := D_u\psi(t,y,u,\omega)$. *As usual, when the context is clear, we will systematically omit the ω argument in the defined functions.* Now, let $z \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ be variations associated with y and u respectively. The second derivatives of ψ are written in the following form

$$\begin{gathered}
\psi_{yy}(t,y,u)z^2 := D^2_{yy}\psi(t,y,u)(z,z), \quad \psi_{uu}(t,y,u)v^2 := D^2_{uu}\psi(t,y,u)(v,v), \\
\psi_{yu}(t,y,u)zv := D^2_{yu}\psi(t,y,u)(z,v), \\
\psi_{(y,u)^2}(t,y,u)(z,v)^2 := \psi_{yy}(t,y,u)z^2 + 2\psi_{yu}(t,y,u)zv + \psi_{uu}(t,y,u)v^2.
\end{gathered}$$

Consider the maps $f, \sigma^i : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^n$ $(i = 1, ..., d)$. These maps will define the dynamics for our problem. Let us assume that:

(H1) [Assumptions for the dynamics] The maps $\psi = f, \sigma^i$ satisfy:

(i) The maps are $\mathcal{B}([0,T] \times \mathbb{R}^n \times \mathbb{R}^m) \otimes \mathcal{F}_T$-measurable.

(ii) For all $(y,u) \in \mathbb{R}^n \times \mathbb{R}^m$ the process $[0,T] \ni t \to \psi(t,y,u) \in \mathbb{R}^n$ is $\mathbb{F}$-adapted.

(iii) For almost all $(t,\omega) \in [0,T] \times \Omega$ the mapping $(y,u) \to \psi(t,y,u,\omega)$ is C^3. Moreover, we assume that there exists a constant $L_1 > 0$ such that for almost all (t,ω)

$$\left\{ \begin{array}{c}
|\psi(t,y,u,\omega)| \leq L_1 \left(1 + |y| + |u|\right), \\
|\psi_y(t,y,u,\omega)| + |\psi_u(t,y,u,\omega)| \leq L_1, \\
|\psi_{yy}(t,y,u,\omega)| + |\psi_{yu}(t,y,u,\omega)| + |\psi_{uu}(t,y,u,\omega)| \leq L_1 \\
|\psi_{(y,u)^2}(t,y,u,\omega) - \psi_{(y,u)^2}(t,y',u',\omega)| \leq L_1 \left(|y-y'| + |u-u'|\right).
\end{array} \right. \tag{4}$$

Let us define $\sigma(t,y,u) := (\sigma^1(t,y,u), ..., \sigma^d(t,y,u)) \in \mathbb{R}^{n \times d}$. For variations $z \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, associated with y and u, set

$$\begin{array}{rcl}
\sigma_y(t,y,u)z & := & (\sigma^1_y(t,y,u)z, ..., \sigma^d_y(t,y,u)z), \\
\sigma_{yy}(t,y,u)z^2 & := & (\sigma^1_{yy}(t,y,u)z^2, ..., \sigma^d_{yy}(t,y,u)z^2),
\end{array} \tag{5}$$

and $\sigma_u(t,y,u)v$, $\sigma_{yu}(t,y,u)zv$, $\sigma_{uu}(t,y,u)v^2$, $\sigma_{(y,u)^2}(t,y,u)(z,v)^2$ are analogously defined.

For every $\beta \in [1,\infty)$, let us define the space $\mathcal{Y}^\beta$ as

$$\mathcal{Y}^\beta := \left\{ y \in L^\beta\left(\Omega; C([0,T];\mathbb{R}^n)\right); \ (t,\omega) \to y(t,\omega) := y(\omega)(t) \text{ is } \mathbb{F}\text{-adapted} \right\},$$

endowed with the norm $\|\cdot\|_{\beta,\infty}$. Let $y_0 \in \mathbb{R}^n$, under **(H1)** we have that for every $u \in L^{\beta,2}_{\mathcal{F}}$ the SDE

$$\begin{array}{rcl}
\mathrm{d}y(t) & = & f(t,y(t),u(t))\mathrm{d}t + \sigma(t,y(t),u(t))\mathrm{d}W(t), \\
y(0) & = & y_0,
\end{array} \tag{6}$$

is well posed. In fact (see [24, Proposition 2.1]):

Proposition 1. *Suppose that* ***(H1)*** *holds. Then, there exists* $C>0$ *such that for every* $u \in L^{\beta,2}_{\mathcal{F}}$ $(\beta \in [1,\infty))$ *equation* (6) *admits a unique solution* $y \in \mathcal{Y}^{\beta}$ *and*

$$\|y\|^{\beta}_{\beta,\infty} \leq C\left(|y_0|^{\beta} + \|f(\cdot,0,u(\cdot))\|^{\beta}_{\beta,1} + \|\sigma(\cdot,0,u(\cdot))\|^{\beta}_{\beta,2}\right). \tag{7}$$

Remark 1. *Note that by the first condition in* (4)*, the right hand side of* (7) *is finite.*

Now, let us consider maps $\ell : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\Omega \to \mathbb{R}$ and $\phi : \mathbb{R}^n\times\Omega \to \mathbb{R}$. These maps will define the cost function of our problem. We assume:

(H2) [Assumptions for cost] (i) The maps ℓ and ϕ are respectively $\mathcal{B}([0,T]\times\mathbb{R}^n\times\mathbb{R}^m)\otimes\mathcal{F}_T$ and $\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{F}_T$ measurable.

(ii) For all $(y,u)\in\mathbb{R}^n\times\mathbb{R}^m$ the process $[0,T]\ni t \to \ell(t,y,u)\in\mathbb{R}$ is $\mathbb{F}$-adapted.

(iii) For almost all (t,ω) the maps $(y,u)\to\ell(t,y,u,\omega)$ and $y\to\phi(y,\omega)$ are $\mathcal{C}^2$. In addition, there exists $L_2>0$ such that:

$$\left\{\begin{array}{c} |\ell(t,y,u,\omega)| \leq L_2\left(1+|y|+|u|\right)^2,\ \ |\phi(y,\omega)| \leq L_2\left(1+|y|\right)^2, \\ |\ell_y(t,y,u,\omega)| + |\ell_u(t,y,u,\omega)| \leq L_2\left(1+|y|+|u|\right), \\ |\ell_{yy}(t,y,u,\omega)| + |\ell_{yu}(t,y,u,\omega)| + |\ell_{uu}(t,y,u,\omega)| \leq L_2, \\ |\ell_{(y,u)^2}(t,y,u,\omega) - \ell_{(y,u)^2}(t,y',u',\omega)| \leq L_2\left(|y-y'|+|u-u'|\right), \\ |\phi_y(y,\omega)| \leq L_2\left(1+|y|\right) \\ |\phi_{yy}(y,\omega)| \leq L_2, \quad |\phi_{yy}(y,\omega)-\phi_{yy}(y',\omega)| \leq L_2\left(|y-y'|\right). \end{array}\right. \tag{8}$$

Remark 2. *The above assumptions include the important case when the cost function is quadratic in* (y,u).

In order to provide second order conditions, which will be natural extensions of well known deterministic results, it will be useful to strengthen **(H1)** and **(H2)**.

[Lipschitz cost] There exists $C_\ell, C_\phi > 0$ such that for almost all $(t,\omega)\in[0,T]\times\Omega$ and for all (y,u), $(y',u')\in\mathbb{R}^n\times\mathbb{R}^m$ we have

$$\begin{array}{rcl} |\ell(t,y,u,\omega)-\ell(t,y',u',\omega)| & \leq & C_\ell\left(|u-u'|+|y-y'|\right), \\ |\phi(y,\omega)-\phi(y',\omega)| & \leq & C_\phi|y-y'|. \end{array} \tag{9}$$

[Affine dynamics] For $\psi = f, \sigma^i$ and for almost all $(t,\omega)\in[0,T]\times\Omega$, we have

$$(y,u)\in\mathbb{R}^n\times\mathbb{R}^m \to \psi(t,y,u,\omega) \quad \text{is affine.} \tag{10}$$

In our main results we will assume:

(H3) At least one of the following assumptions hold:

(H3.i) Condition (9) holds and $\sigma_{uu}\equiv 0$. **(H3.ii)** Condition (10) holds.

For every $u\in L^2_{\mathcal{F}}$ denote by $y_u\in\mathcal{Y}^2$ the solution of (6). Let us define the *cost* function $J : L^2_{\mathcal{F}} \to \mathbb{R}$ by

$$J(u) = \mathbb{E}\left[\int_0^T \ell(t,y_u(t),u(t))\mathrm{d}t + \phi(y_u(T))\right]. \tag{11}$$

Note that, in view of the first condition in (8) and estimate (7) the function J is well defined.

3 Expansions for the state and cost function

From now on we fix $\bar{u} \in L^2_{\mathcal{F}}$ and set $\bar{y} := y_{\bar{u}}$. We also suppose that assumptions **(H1)** and **(H2)** hold. Our aim in this section is to obtain first and second order expansions for $v \in L^\infty_{\mathcal{F}} \to y_{\bar{u}+v} \in \mathcal{Y}^2$ and $v \in L^\infty_{\mathcal{F}} \to J(\bar{u}+v) \in \mathbb{R}$ around $\bar{v} = 0$. The main tool for obtaining the expansion for the state is the following corollary of proposition 1, whose proof is straightforward.

Corollary 2. *Let $A_1, A_2 \in L^\infty_{\mathcal{F}}([0,T];\mathbb{R}^{n\times n})$, $B_i^1 \in L^{\beta,2}_{\mathcal{F}}([0,T];\mathbb{R}^n)$ and $B_i^2 \in L^\infty_{\mathcal{F}}([0,T];\mathbb{R}^{n\times d})$ for $i = 1,2$. Assume that there exists a constant $K > 0$ such that*

$$\|B_1^1\|_{\beta,1} \le K\|B_2^1\|_{\beta,2}. \tag{12}$$

Then, omitting time from function arguments, for every $w \in L^{\beta,2}$, the SDE

$$\begin{array}{rcl} \mathrm{d}z & = & \left[A_1 z + B_1^1 + B_1^2 w\right]\mathrm{d}t + \left[A_2 z + B_2^1 + B_2^2 w\right]\mathrm{d}W(t), \\ z(0) & = & 0, \end{array} \tag{13}$$

has a unique solution in $\mathcal{Y}^\beta$ and

$$\|z\|^\beta_{\beta,\infty} = \left\{ \begin{array}{ll} O\left(\max\left\{\|B_2^1\|^\beta_{\beta,2}, \|w\|^\beta_{\beta,1}\right\}\right) & \text{if } B_2^2 \equiv 0, \\ O\left(\max\left\{\|B_2^1\|^\beta_{\beta,2}, \|w\|^\beta_{\beta,2}\right\}\right) & \text{otherwise.} \end{array}\right.$$

Remark 3. *Note that the estimates given in corollary 2 are sharp. In fact, suppose that $d = 1$ and let $w \in L^2([0,T];\mathbb{R})$ (deterministic). Consider the process $z(t)$ defined by*

$$z(t) := \int_0^t w(s)\mathrm{d}W(s) \quad \text{for all } t \in [0,T].$$

By definition $\|z\|^\beta_{\beta,\infty} \ge \mathbb{E}(|z(T)|^\beta) = \|w\|_2^\beta \mathbb{E}(|Z|^\beta)$, where Z is an standard normal random variable. Since, in this specific case, $\|w\|^\beta_{\beta,2} = \|w\|_2^\beta$, the conclusion follows.

Regarding the expansion for the cost J, the following lemma, which is a consequence of It's lemma for multidimensional It process (see e.g. [17, 29]), will be useful. For the reader convenience we provide the short proof.

Lemma 3. *Let Z_1 and Z_2 be $\mathbb{R}^n$-valued continuous process satisfying*

$$\left\{ \begin{array}{rcl} \mathrm{d}Z_1(t) & = & b_1(t)\mathrm{d}t + \sigma_1(t)\mathrm{d}W(t) \quad \text{for all} \quad t \in [0,T], \\ \mathrm{d}Z_2(t) & = & b_2(t)\mathrm{d}t + \sigma_2(t)\mathrm{d}W(t) \quad \text{for all} \quad t \in [0,T], \end{array}\right. \tag{14}$$

where b_1, $b_2 \in L^2(\Omega, L^2([0,T],\mathbb{R}^n))$ and σ_1, $\sigma_2 \in L^2(\Omega, L^2([0,T],\mathbb{R}^{n\times d}))$ are $\mathbb{F}$-adapted process. Also, let us suppose that $\mathbb{P}$-a.s. we have that $Z_1(0) = 0$. Then

$$\mathbb{E}\left(Z_1(T)\cdot Z_2(T)\right) = \mathbb{E}\left(\int_0^T \left[Z_1(t)\cdot b_2(t) + Z_2(t)\cdot b_1(t) + \sum_{i=1}^d \sigma_1^i(t)\cdot\sigma_2^i(t)\right]\mathrm{d}t\right).$$

Proof. Itô's lemma implies that

$$Z_1(T)\cdot Z_2(T)=\int_0^T\left[Z_1(t)\cdot b_2(t)+Z_2(t)\cdot b_1(t)+\sum_{i=1}^d \sigma_1^i(t)\cdot\sigma_2^i(t)\right]\mathrm{d}t+M(T), \tag{15}$$

where $M(t)$ is a continuous local martingale given by

$$M(t):=\sum_{i=1}^d\int_0^t\left[Z_1(s)\cdot\sigma_2^i(s)+Z_2(s)\cdot\sigma_1^i(s)\right]\mathrm{d}W^i(s).$$

By the *Burkholder-Davis-Gundy* inequality (see e.g [17]) we have the existence of a constant $K>0$ such that for all $i=1,...,d$

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t\left[Z_1(s)\cdot\sigma_2^i(s)+Z_2(s)\cdot\sigma_1^i(s)\right]\mathrm{d}W^i(s)\right|\right)\leq K\left(\left|\left|Z_1\cdot\sigma_2^i+Z_2\cdot\sigma_1^i\right|\right|_{1,2}\right).$$

The Cauchy Schwarz inequality yields that (assuming that $n=1$ for notational convenience)

$$\left|\left|Z_1\cdot\sigma_2^i\right|\right|_{1,2}=\mathbb{E}\left[\left(\int_0^T|Z_1(t)|^2\left|\sigma_2^i(t)\right|^2\mathrm{d}t\right)^{1/2}\right]\leq||Z_1||_{2,\infty}\left|\left|\sigma_2^i\right|\right|_2<+\infty,$$

with an analogous estimate for $\left|\left|Z_2\cdot\sigma_1^i\right|\right|_{1,2}$. Therefore, by [27, Theorem 51], we have that $M(t)$ is a martingale with null expectation. The result follows. □

For $\psi=\ell,f,\sigma$ and $t\in[0,T]$, let us define

$$\begin{gathered}\psi_y(t)=\psi_y(t,\bar{y}(t),\bar{u}(t));\quad\psi_u(t)=\psi_u(t,\bar{y}(t),\bar{u}(t)),\quad\psi_{yu}(t)=\psi_{yu}(t,\bar{y}(t),\bar{u}(t));\\ \psi_{yy}(t)=\psi_{yy}(t,\bar{y}(t),\bar{u}(t));\quad\psi_{uu}(t)=\psi_{uu}(t,\bar{y}(t),\bar{u}(t)),\\ \psi_{(y,u)^2}(t)=\psi_{(y,u)^2}(t,\bar{y}(t),\bar{u}(t)).\end{gathered}$$

As usual in optimal control theory, the expansions for J, with respect to variations of the control variable in the uniform norm $\|\cdot\|_\infty$, will be written in terms of an adjoint state and the derivatives of an associated *Hamiltonian*. Let us define the *adjoint state* $(\bar{p},\bar{q})\in L^2_{\mathcal{F}}([0,T];\mathbb{R}^n)\times(L^2_{\mathcal{F}}([0,T];\mathbb{R}^n))^d$ as the unique solution of the following backward stochastic differential equation (BSDE) (see e.g. [1, 5])

$$\begin{array}{rcl}\mathrm{d}p(t)&=&-\left[\ell_y(t)^\top+f_y(t)^\top p(t)+\displaystyle\sum_{i=1}^m\sigma_y^i(t)^\top q^i(t)\right]\mathrm{d}t+q(t)\mathrm{d}W(t),\\ p(T)&=&\phi_y(\bar{y}(T))^\top.\end{array}\tag{16}$$

In the notation above σ^i and q^i denote respectively the ith column of σ and q. The following estimates hold (see [24, Proposition 3.1]):

Proposition 4. *Assume that **(H1)**, **(H2)** hold and that $\bar{u}\in L^{\beta,2}_{\mathcal{F}}$ ($\beta\in[1,\infty)$). Then there exists $C'>0$ such that*

$$\|\bar{p}\|^\beta_{\beta,\infty}+\sum_{i=1}^d\|\bar{q}^i\|^\beta_{\beta,2}\leq C'\left(1+\|\bar{u}\|^\beta_{\beta,2}\right).$$

The *Hamiltonian* $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \Omega \to \mathbb{R}$ is defined as

$$H(t,y,u,p,q,\omega) := \ell(t,y,u,\omega) + p\cdot f(t,y,u,\omega) + \sum_{i=1}^{d} q^i \cdot \sigma^i(t,y,u,\omega). \quad (17)$$

For notational convenience, omitting the dependence on ω, we set

$$\left\{ \begin{array}{ll} H_u(t) & := H_u(t,\bar{y}(t),\bar{u}(t),\bar{p}(t),\bar{q}(t)), \\ H_{(y,u)^2}(t) & := H_{(y,u)^2}(t,\bar{y}(t),\bar{u}(t),\bar{p}(t),\bar{q}(t)). \end{array} \right. \quad (18)$$

3.1 First order expansions

Let $\beta \in [1,\infty]$ and $v \in L_{\mathcal{F}}^{\beta,2}$. We consider the *linearized* mapping $v \in L_{\mathcal{F}}^{\beta,2} \to y_1[\bar{u},v] \in \mathcal{Y}^{\beta}$, where $y_1[\bar{u},v]$ is the unique solution of

$$\begin{array}{rcl} \mathrm{d}y_1(t) & = & [f_y(t)y_1(t) + f_u(t)v(t)]\mathrm{d}t + [\sigma_y(t)y_1(t) + \sigma_u(t)v(t)]\mathrm{d}W(t), \\ y_1(0) & = & 0. \end{array} \quad (19)$$

The second assumption in (4) and proposition 1 yields that $y_1[\bar{u},v]$ is well defined. If the context is clear, for notational convenience we will write $y_1 = y_1[\bar{u},v]$. Corollary 2 will be the main tool for establishing the following useful estimates:

Lemma 5. *Let $v \in L_{\mathcal{F}}^{2\beta,4}$ with $\beta \in [1,\infty)$ and set*

$$\delta y = \delta y[\bar{u},v] := y_{\bar{u}+v} - \bar{y}, \quad d_1 = d_1[\bar{u},v] := \delta y - y_1.$$

Then, the following estimates hold:

$$\|\delta y\|_{\beta,\infty}^{\beta} + \|y_1\|_{\beta,\infty}^{\beta} = \left\{ \begin{array}{ll} O(\|v\|_{\beta,1}^{\beta}) & \text{if } \sigma_u \equiv 0, \\ O(\|v\|_{\beta,2}^{\beta}) & \text{otherwise.} \end{array} \right. \quad (20)$$

$$\|d_1\|_{\beta,\infty}^{\beta} = \left\{ \begin{array}{ll} O(\|v\|_{2\beta,2}^{2\beta}) & \text{if } \sigma_{uu} \equiv 0, \\ O(\|v\|_{2\beta,4}^{2\beta}) & \text{otherwise.} \end{array} \right. \quad (21)$$

Proof. See the appendix. □

Now we can prove the following proposition.

Proposition 6. *The map $v \in L_{\mathcal{F}}^{\infty} \to \hat{y}(v) := y_{\bar{u}+v} \in \mathcal{Y}^2$ is differentiable with a Lipschitz derivative given by*

$$D\hat{y}(\bar{v})v = y_1[\bar{u}+\bar{v},v] \quad \text{for all} \quad \bar{v}, v \in L_{\mathcal{F}}^{\infty}. \quad (22)$$

Proof. By estimate (21) in lemma 5 we have that $\|\delta y - y_1\|_{2,\infty} = O(\|v\|_{\infty}^2)$, implying that (22) holds. Let us prove that $D\hat{y}$ is Lipschitz. For notational convenience, we assume that $n = m = d = 1$. Let $v_1, v_2 \in L_{\mathcal{F}}^{\infty}$ and write $\delta v := v_1 - v_2$. For $\psi = f, \sigma$, $i = 1,2$ and $v' \in L_{\mathcal{F}}^{\infty}$ with $\|v'\|_{\infty} = 1$, by an abuse of notation we set

$$\begin{array}{l} y^{(i)} := y_{\bar{u}+v_i}, \quad y_1^{(i)} := y_1[\bar{u}+v_i,v'], \quad \delta y := y^{(1)} - y^{(2)}, \quad \delta y_1 := y_1^{(1)} - y_1^{(2)} \\ \psi_y^{(i)}(t) := \psi_y(t,y^{(i)}(t),\bar{u}(t)+v_i(t)), \; \psi_u^{(i)}(t) := \psi_u(t,y^{(i)}(t),\bar{u}(t)+v_i(t)), \\ \delta\psi_y(t) := \psi_y^{(1)}(t) - \psi_y^{(2)}(t), \quad \delta\psi_u(t) := \psi_u^{(1)}(t) - \psi_u^{(2)}(t). \end{array}$$

A straightforward calculation shows that

$$\begin{array}{rcl} \mathrm{d}\delta y_1(t) & = & [f_y^{(2)}(t)\delta y_1(t) + \delta f_y(t) y_1^{(1)}(t) + \delta f_u(t) v'(t)]\mathrm{d}t \\ & & +[\sigma_y^{(2)}(t)\delta y_1(t) + \delta\sigma_y(t) y_1^{(1)}(t) + \delta\sigma_u(t) v'(t)]\mathrm{d}W(t), \\ \delta y_1(0) & = & 0. \end{array} \tag{23}$$

Using **(H1)** and corollary 2 we obtain that

$$\|\delta y_1\|_{2,\infty}^2 = O\left(\|\delta y\|_{2,\infty}^2 + \|\delta v\|_2^2\right),$$

uniformly in $v' \in L_{\mathcal{F}}^\infty$ with $\|v'\|_\infty = 1$. The result follows from (20). □

Now we focus our attention on the cost function J. Let us define $\Upsilon_1 : L_{\mathcal{F}}^2 \to \mathbb{R}$ by

$$\Upsilon_1(v) := \mathbb{E}\left(\int_0^T H_u(t)\ v(t)\mathrm{d}t\right). \tag{24}$$

In view of proposition 4, with $\beta = 2$, Υ_1 is well defined. Lemma 3 yields the following well known alternative expression for Υ_1.

Lemma 7. *For every $v \in L_{\mathcal{F}}^2$ we have that:*

$$\Upsilon_1(v) = \mathbb{E}\left(\int_0^T [\ell_y(t) y_1(t) + \ell_u(t) v(t)]\,\mathrm{d}t + \phi_y(\bar{y}(T)) y_1(T)\right). \tag{25}$$

Proof. Noting that

$$\phi_y(\bar{y}(T)) y_1(T) = \bar{p}(T) \cdot y_1(T) - \bar{p}(0) \cdot y_1(0),$$

lemma 3, applied to $Z_1 = y_1$ and $Z_2 = \bar{p}$, yields $\mathbb{E}\left(\phi_y(\bar{y}(T)) y_1(T)\right) = I_1 + I_2 + I_3$, where

$$\begin{array}{rcl} I_1 & := & -\mathbb{E}\left(\int_0^T y_1(t) \cdot \left[\ell_y(t)^\top + f_y(t)^\top \bar{p}(t) + \sum_{i=1}^d \sigma_y^i(t)^\top \bar{q}^i(t)\right] \mathrm{d}t\right), \\ I_2 & := & \mathbb{E}\left(\int_0^T \bar{p}(t) \cdot [f_y(t) y_1(t) + f_u(t) v(t)]\,\mathrm{d}t\right), \\ I_3 & := & \sum_{i=1}^d \mathbb{E}\left(\int_0^T \bar{q}^i(t) \cdot \left[\sigma_y^i(t) y_1(t) + \sigma_u^i(t) v(t)\right] \mathrm{d}t\right). \end{array}$$

Plugging the expressions of I_1, I_2 and I_3 introduced above into the right hand side of (25) yields the result. □

The expression above for Υ_1 allows to obtain a *first order expansion* of J around $\bar{u}$.

Proposition 8. *Assume that **(H1)**, **(H2)** hold and let $v \in L_{\mathcal{F}}^4$. Then,*

$$J(\bar{u} + v) = J(\bar{u}) + \Upsilon_1(v) + r_1(v),$$

with

$$\Upsilon_1(v) = O(\|v\|_2); \quad r_1(v) = \left\{ \begin{array}{ll} O\left(\|v\|_{4,2}^2\right) & \textit{if } \sigma_{uu} \equiv 0, \\ O\left(\|v\|_4^2\right) & \textit{otherwise.} \end{array} \right. \tag{26}$$

If in addition (9) *holds, then*

$$\Upsilon_1(v) = \left\{ \begin{array}{ll} O(\|v\|_1) & \textit{if } \sigma_u \equiv 0, \\ O(\|v\|_{1,2}) & \textit{otherwise,} \end{array} \right. ; \quad r_1(v) = \left\{ \begin{array}{ll} O\left(\|v\|_2^2\right) & \textit{if } \sigma_{uu} \equiv 0, \\ O\left(\|v\|_{2,4}^2\right) & \textit{otherwise.} \end{array} \right. \tag{27}$$

Proof. Let us denote $\delta J := J(\bar{u}+v) - J(\bar{u})$. By definition

$$\delta J = \mathbb{E}\left(\int_0^T \left[\ell(t, y_{\bar{u}+v}(t), \bar{u}(t)+v(t)) - \ell(t, \bar{y}(t), \bar{u}(t))\right] \mathrm{d}t + \phi(y_{\bar{u}+v}(T)) - \phi(\bar{y}(T))\right).$$

Using **(H2)**, a Taylor expansion for ℓ and ϕ implies that

$$\delta J = \mathbb{E}\left[\int_0^T \ell_y(t)\delta y(t) + \ell_u(t)v(t)\mathrm{d}t + \phi_y(\bar{y}(T))\delta y(T)\right] + O(\|\delta y\|_{2,\infty}^2 + \|v\|_2^2).$$

Since $\delta y = y_1 + d_1$, lemma 7 implies that $\delta J = \Upsilon_1(v) + r_1(v)$ where $r_1(v) = z_1(v) + z_2(v)$ with

$$z_1(v) := \mathbb{E}\left[\int_0^T \ell_y(t)d_1(t)\mathrm{d}t + \phi_y(\bar{y}(T))d_1(T)\right], \; z_2(v) := O\left(\|\delta y\|_{2,\infty}^2 + \|v\|_2^2\right).$$

Now, we estimate $\Upsilon_1(v)$ using (25). By assumption **(H2)** and the Cauchy Schwarz inequality $\mathbb{E}\left(\int_0^T \ell_u(t)v(t)\mathrm{d}t\right) = O\left(\|v\|_2\right)$. On the other hand, by (20)

$$\mathbb{E}\left(\int_0^T \ell_y(t)y_1(t)\mathrm{d}t + \phi_y(\bar{y}(T))y_1(T)\right) = O\left(\left[\mathbb{E}\left(\sup_{t\in[0,T]} |y_1(t)|^2\right)\right]^{\frac{1}{2}}\right) = O(\|v\|_2).$$

Thus $\Upsilon_1(v) = O(\|v\|_2)$. If (9) holds, then $\mathbb{E}\left(\int_0^T \ell_u(t)v(t)\mathrm{d}t\right) = O\left(\|v\|_1\right)$, and

$$\mathbb{E}\left(\int_0^T \ell_y(t)y_1(t)\mathrm{d}t + \phi_y(\bar{y}(T))y_1(T)\right) = O\left(\mathbb{E}\left[\sup_{t\in[0,T]} |y_1(t)|\right]\right).$$

Thus, estimates for $\Upsilon_1(v)$ in (27) follow from (20) with $\beta = 1$. Let us estimate $r_1(v)$. Assumption **(H2)** and (20) imply that $z_2(v) = O(\|v\|_2^2)$. On the other hand, by **(H2)** and the Cauchy Schwarz inequality

$$z_1(v) = O\left(\left[\mathbb{E}\left(\sup_{t\in[0,T]} |d_1(t)|^2\right)\right]^{\frac{1}{2}}\right).$$

Thus (26) follows from estimates (21) with $\beta = 2$. If in addition (9) holds, then $z_1(v) = O\left(\mathbb{E}\left[\sup_{t\in[0,T]} |d_1(t)|\right]\right)$ and the estimates for $r_1(v)$ in (27) follows from (21) with $\beta = 1$. □

Remark 4. *The above proof shows that the hypotheses for the perturbation v can be weakened. For example, if (9) holds and $\sigma_{uu} = 0$, for all $v \in L^2_{\mathcal{F}}$ we have that $J(\bar{u}+v) = J(\bar{u}) + \Upsilon_1(v) + r_1(v)$ with $\Upsilon_1(v) = O(\|v\|_1)$ and $r_1(v) = O(\|v\|_2^2)$. Analogously, if (10) holds, then $d_1 \equiv 0$ and we have that $J(\bar{u}+v) = J(\bar{u}) + \Upsilon_1(v) + r_1(v)$ with $\Upsilon_1(v) = O(\|v\|_2)$ and $r_1(v) = O(\|v\|_2^2)$. Therefore, if **(H3)** holds, the function J is differentiable at $\bar{u}$.*

The following corollary is an inmediate consequence of the proposition above.

Corollary 9. *Assume that **(H1)**, **(H2)** hold and let $v \in L^\infty_{\mathcal{F}}$. Then, $\Upsilon_1(v) = O(\|v\|_2)$ and $J(\bar{u}+v) = J(\bar{u}) + \Upsilon_1(v) + r_1(v)$ with $r_1(v) = O(\|v\|_\infty^2)$.*

3.2 Second order expansions

In this subsection we obtain second order expansions for the state and the cost function. As in the precedent subsection we begin with the analysis for the state mapping. The *second order* linearization of $v \in L^\infty_{\mathcal{F}} \mapsto \hat{y}(v) = y_{\bar{u}+v} \in \mathcal{Y}^2$ around $\bar{v} \equiv 0$ in the direction $v \in L^\infty_{\mathcal{F}}$ is defined as the unique solution $y_2 = y_2[\bar{u}, v]$ of

$$\begin{array}{rcl} \mathrm{d}y_2(t) & = & \left[f_y(t)y_2(t) + f_{yy}(t)y_1(t)^2 + 2f_{yu}(t)y_1(t)v(t) + f_{uu}(t)v(t)^2\right]\mathrm{d}t \\ & & + \left[\sigma_y(t)y_2(t) + \sigma_{yy}(t)y_1(t)^2 + 2\sigma_{yu}(t)y_1(t)v(t) + \sigma_{uu}(t)v(t)^2\right]\mathrm{d}W(t); \\ y_2(0) & = & 0. \end{array} \tag{28}$$

Note that by the third assumption in (4) and proposition 1, we have that y_2 is well defined. We give now some useful bounds over y_2 and the rest

$$d_2 = d_2[\bar{u}, v] := \delta y - y_1 - \tfrac{1}{2}y_2. \tag{29}$$

As for y_1 and d_1, when the context is clear, we omit the arguments of y_2 and d_2.

Lemma 10. *For $v \in L^\infty_{\mathcal{F}}$ and $\beta \in [1, \infty)$ the following estimates hold:*

$$\|y_2\|^\beta_{\beta,\infty} = \begin{cases} O(\|v\|^{2\beta}_{2\beta,2}) & \text{if } \sigma_{uu} \equiv 0, \\ O(\|v\|^{2\beta}_{2\beta,4}) & \text{otherwise.} \end{cases} \tag{30}$$

$$\|d_2\|^\beta_{\beta,\infty} = \begin{cases} O(\|v\|^\beta_{2\beta,2}\|v\|^{2\beta}_{4\beta,4}) & \text{if } \sigma_{uuu} \equiv 0, \\ O(\|v\|^\beta_{2\beta,2}\|v\|^{2\beta}_{4\beta,4} + \|v\|^{3\beta}_{3\beta,6}) & \text{otherwise.} \end{cases} \tag{31}$$

Proof. See the appendix. □

Now, we study J. Let us define $\Upsilon_2 : L^\infty_{\mathcal{F}} \to \mathbb{R}$ by

$$\Upsilon_2(v) := \mathbb{E}\left(\int_0^T H_{(y,u)^2}(t)(y_1(t), v(t))^2 \mathrm{d}t + \phi_{yy}(\bar{y}(T))(y_1(T))^2\right). \tag{32}$$

As for Υ_1, a useful alternative expression for Υ_2 holds.

Lemma 11. *For every $v \in L^\infty_{\mathcal{F}}$ we have that:*

$$\begin{array}{rcl} \Upsilon_2(v) & = & \mathbb{E}\left(\int_0^T \left[\ell_y(t)y_2(t) + \ell_{(y,u)^2}(t)(y_1(t), v(t))^2\right]\mathrm{d}t\right) \\ & + & \mathbb{E}\left[\phi_y(\bar{y}(T))y_2(T) + \phi_{yy}(\bar{y}(T))(y_1(T))^2\right]. \end{array} \tag{33}$$

Proof. By definition of y_2 and $\bar{p}$, we have that

$$\phi_y(\bar{y}(T))y_2(T) = \bar{p}(T) \cdot y_2(T) - \bar{p}(0) \cdot y_2(0).$$

Lemma 3 yields $\mathbb{E}\left(\phi_y(\bar{y}(T))y_2(T)\right) = I_1' + I_2' + I_3'$, where

$$\begin{array}{rcl} I_1' & := & -\mathbb{E}\left(\int_0^T y_2(t) \cdot \left[\ell_y(t)^\top + f_y(t)^\top \bar{p}(t) + \sum_{i=1}^d \sigma_y^i(t)^\top \bar{q}^i(t)\right]\mathrm{d}t\right), \\ I_2' & := & \mathbb{E}\left(\int_0^T \bar{p}(t) \cdot \left[f_y(t)y_2(t) + f_{(y,u)^2}(t)(y_1(t), v(t))^2\right]\mathrm{d}t\right), \\ I_3' & := & \sum_{i=1}^d \mathbb{E}\left(\int_0^T \bar{q}^i(t) \cdot \left[\sigma_y^i(t)y_2(t) + \sigma^i_{(y,u)^2}(t)(y_1(t), v(t))^2\right]\mathrm{d}t\right). \end{array}$$

Plugging the expressions of I_1', I_2' and I_3' introduced above into the right hand side of (33) yields the result. □

In order to obtain the main result of this section, the following lemma is useful.

Lemma 12. *Assume that* ***(H1)****,* ***(H2)*** *hold and let* $v \in L^\infty_\mathcal{F}$*. Then,*

$$J(\bar{u}+v) = J(\bar{u}) + \Upsilon_1(v) + \tfrac{1}{2}\Upsilon_2(v) + \zeta(v) + z(v), \tag{34}$$

where $z(v) = O\left(\|v\|_\infty \|v\|_2^2\right)$ *and, recalling* (29)*,*

$$\zeta(v) := \mathbb{E}\left(\int_0^T \ell_y(t) d_2(t) \mathrm{d}t + \phi_y(\bar{y}(T)) d_2(T) \right). \tag{35}$$

Proof. See the appendix. □

Now we are able to obtain a *second order expansion* of J around $\bar{u}$.

Proposition 13. *Assume that* ***(H1)****,* ***(H2)*** *hold and let* $v \in L^\infty_\mathcal{F}$*. Then,*

$$J(\bar{u}+v) = J(\bar{u}) + \Upsilon_1(v) + \tfrac{1}{2}\Upsilon_2(v) + r_2(v), \tag{36}$$

and the following estimates hold:

$$\Upsilon_2(v) = \left\{ \begin{array}{ll} O(\|v\|_{4,2}^2) & \textit{if } \sigma_{uu} \equiv 0, \\ O(\|v\|_4^2) & \textit{otherwise,} \end{array} \right. \quad r_2(v) = \left\{ \begin{array}{ll} O(\|v\|_\infty \|v\|_{4,2}^2) & \textit{if } \sigma_{uuu} \equiv 0, \\ O(\|v\|_\infty \|v\|_4^2) & \textit{otherwise.} \end{array} \right. \tag{37}$$

If in addition (9) *holds then*

$$\Upsilon_2(v) = \left\{ \begin{array}{ll} O(\|v\|_2^2) & \textit{if } \sigma_{uu} \equiv 0, \\ O(\|v\|_{2,4}^2) & \textit{otherwise,} \end{array} \right. \quad r_2(v) = \left\{ \begin{array}{ll} O(\|v\|_\infty \|v\|_2^2) & \textit{if } \sigma_{uuu} \equiv 0, \\ O(\|v\|_\infty \|v\|_{2,4}^2) & \textit{otherwise.} \end{array} \right. \tag{38}$$

Proof. Let us first estimate $\Upsilon_2(v)$ by using the expression obtained in lemma 11 and the bounds obtained in lemmas 5 and 10. By (20) with $\beta = 2$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} |y_1(t)|^2 + \int_0^T |v(t)|^2 \mathrm{d}t \right) = O(\|v\|_2^2). \tag{39}$$

In view of assumption **(H2)** and (39) we obtain that

$$\mathbb{E}\left(\int_0^T \ell_{(y,u)^2}(t)(y_1(t), v(t))^2 \mathrm{d}t + \phi_{yy}(\bar{y}(T))(y_1(T))^2 \right) = O(\|v\|_2^2). \tag{40}$$

On the other hand, assumption **(H2)** and the Cauchy Schwarz inequality yield

$$\mathbb{E}\left(\int_0^T \ell_y(t) y_2(t) \mathrm{d}t + \phi_y(\bar{y}(T)) y_2(T) \right) = O\left(\left[\mathbb{E}(\sup_{t\in[0,T]} |y_2|^2) \right]^{\frac{1}{2}} \right), \tag{41}$$

and the estimate for $\Upsilon_2(v)$ in (37) follows from (30). If (9) holds, then

$$\mathbb{E}\left(\int_0^T \ell_y(t) y_2(t) \mathrm{d}t + \phi_{yy}(\bar{y}(T)) y_2(T) \right) = O\left(\mathbb{E}\left[\sup_{t\in[0,T]} |y_2| \right] \right), \tag{42}$$

and the estimate for $\Upsilon_2(v)$ in (38) follows from (30).

In order to conclude the proof, letting $r_2(v) := \zeta(v)+z(v)$, lemma 12 implies that it suffices to estimate $\zeta(v)$. By **(H2)** and the Cauchy Schwarz inequality

$$\zeta(v) = O\left(\left[\mathbb{E}\left(\sup_{t\in[0,T]} |d_2(t)|^2\right)\right]^{\frac{1}{2}}\right).$$

Hence, using (31) with $\beta = 2$,

$$\zeta(v) = \begin{cases} O(\|v\|_{4,2}\|v\|_{8,4}^2) & \text{if } \sigma_{uuu} \equiv 0, \\ O(\|v\|_{4,2}\|v\|_{8,4}^2 + \|v\|_{6,6}^3) & \text{otherwise.} \end{cases}$$

Since $O(\|v\|_{4,2}\|v\|_{8,4}^2) = O(\|v\|_\infty\|v\|_{4,2}^2)$ and $O(\|v\|_{6,6}^3) = O(\|v\|_\infty\|v\|_4^2)$, the estimate for $r_2(v)$ in (37) follows. If in addition assumption (9) holds, then by (31) with $\beta = 1$,

$$\zeta(v) = O\left(\mathbb{E}\left[\sup_{t\in[0,T]} |d_2(t)|\right]\right) = \begin{cases} O(\|v\|_2\|v\|_4^2) & \text{if } \sigma_{uuu} \equiv 0, \\ O(\|v\|_2\|v\|_4^2 + \|v\|_{3,6}^3) & \text{otherwise.} \end{cases}$$

Since $O(\|v\|_2\|v\|_4^2) = O(\|v\|_\infty\|v\|_2^2)$ and $O(\|v\|_{3,6}^3) = O(\|v\|_{2,4}^2)$, the estimate for $r_2(v)$ in (38) follows. $\square$

Remark 5. *The proof of proposition 13 shows that the estimates $\Upsilon_2(v) = O(\|v\|_2^2)$ and $r_2(v) = O(\|v\|_\infty\|v\|_2^2)$ also hold in the case when f and σ are affine mappings, since in this case $y_2 = d_2 = 0$. Therefore if **(H3)** holds, then we find the natural extension of the well known expansion for the deterministic case, i.e.*

$$J(\bar{u}+v) = J(\bar{u}) + \Upsilon_1(v) + \tfrac{1}{2}\Upsilon_2(v) + O(\|v\|_\infty\|v\|_2^2)$$

with $\Upsilon_1(v)$ and $\Upsilon_2(v) = O(\|v\|_2^2)$.

Since Υ_2 is a quadratic form and, for every $\beta, p \in [1,\infty]$, the space $L_{\mathcal{F}}^\infty$ is dense in $L_{\mathcal{F}}^{\beta,p}$, we have that: If $\Upsilon_2(v) = O(\|v\|_{\beta,p})$ then Υ_2 admits a unique continuous extension in $L^{\beta,p}$. This fact, together with the following corollary, will allow us to prove, in the next sections, second order necessary condition for the problems explained in the introduction.

Corollary 14. *Assume that **(H1)**-**(H3)** hold. Then,*

$$J(\bar{u}+v) = J(\bar{u}) + \Upsilon_1(v) + \tfrac{1}{2}\Upsilon_2(v) + r(v) \quad \textit{for all } v \in L_{\mathcal{F}}^\infty, \tag{43}$$

where $\Upsilon_1(v) = O(\|v\|_2)$, $\Upsilon_2(v) = O(\|v\|_2^2)$ and $r(v) = O(\|v\|_\infty\|v\|_2^2)$.

4 Optimality conditions: The case of control constraints

Let $\mathcal{U}$ be a nonempty closed and convex subset of $L_{\mathcal{F}}^2$ and consider the problem

$$\text{Min } J(u) \quad \text{subject to } u \in \mathcal{U}. \tag{$\mathcal{SP}$}$$

The asymptotic expansions obtained for J in section 3 allow us to obtain first and second order necessary conditions at a local optimum $\bar{u} \in L^2_{\mathcal{F}}$ for the control constrained problem $(\mathcal{SP})$. We first obtain first order optimality conditions using the procedure explained in the introduction: According to the regularity of the data of $(\mathcal{SP})$ and the dependence on u of the σ-term, a perturbation in an appropriate space is taken. Then, the results of the previous section yield a positivity condition of Υ_1 over a certain cone which is extended, by a density argument, to a larger one. Similar considerations apply in order to establish second order necessary conditions.

Let us first fix some notations which are standard in optimization theory. Consider a Banach space $(X, \|\cdot\|_X)$ and a nonempty closed convex set $C \subseteq X$. For $x, x' \in X$ define the *segment* $[x, x'] := \{x + \lambda(x' - x) \ ; \lambda \in [0,1]\}$. The radial, the tangent and the normal cone to C at $\bar{x}$ are defined respectively by

$$\begin{array}{rcl} \mathcal{R}_C(\bar{x}) & := & \{h \in X \ ; \ \exists\, \theta > 0 \quad \text{such that} \ \ [\bar{x}, \bar{x} + \theta h] \subseteq C\}, \\ \mathcal{T}_C(\bar{x}) & := & \{h \in X \ ; \ \exists\, x(\theta) = \bar{x} + \theta h + o(\theta) \in C, \ \theta > 0, \ ||o(\theta)/\theta||_X \to 0\}, \\ N_C(\bar{x}) & := & \{x^* \in X^* \ ; \ \langle x^*, x - \bar{x}\rangle_{X^*,X} \leq 0, \ \text{for all } x \in C\}, \end{array} \tag{44}$$

where X^* denotes the dual topological space of X and $\langle \cdot, \cdot \rangle_{X^*,X}$ is the duality product. Recall that, since C is a closed convex set, the tangent cone $\mathcal{T}_C(\bar{x})$ is the closure of the radial cone $\mathcal{R}_C(\bar{x})$ in X, and that $N_C(\bar{x})$ is the polar cone of $\mathcal{T}_C(\bar{x})$, i.e.,

$$N_C(\bar{x}) = \{x^* \in X^* \ ; \ \langle x^*, h\rangle_{X^*,X} \leq 0, \ \text{for all } h \in \mathcal{T}_C(\bar{x})\}. \tag{45}$$

4.1 First order necessary conditions

Consider as in section 3 a fixed $\bar{u} \in L^2_{\mathcal{F}}$. For $\beta, p \in [1, \infty]$ and a subset $A \subseteq L^{\beta,p}_{\mathcal{F}}$ we write $\mathrm{clo}_{\beta,p}(A)$ for the closure of A in $L^{\beta,p}_{\mathcal{F}}$. If $A \subseteq L^{\beta}_{\mathcal{F}}$ we set $\mathrm{clo}_{\beta}(A) := \mathrm{clo}_{\beta,\beta}(A)$.

We have the following first order conditions for $(\mathcal{SP})$.

Lemma 15. *Assume that **(H1)**, **(H2)** hold and let $\bar{u} \in \mathcal{U}$ be a local solution of $(\mathcal{SP})$. Then:*

$$\Upsilon_1(v) \geq 0 \ \ \textit{for all } v \in \left\{ \begin{array}{ll} \mathrm{clo}_2\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{4,2}_{\mathcal{F}}\right) & \textit{if } \sigma_{uu} \equiv 0, \\ \mathrm{clo}_2\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{4}_{\mathcal{F}}\right) & \textit{otherwise.} \end{array} \right. \tag{46}$$

If in addition (9) *holds then*

$$\Upsilon_1(v) \geq 0 \ \ \textit{for all } v \in \left\{ \begin{array}{ll} \mathrm{clo}_1\left(\mathcal{R}_{\mathcal{U}}(\bar{u})\right) & \textit{if } \sigma_{u} \equiv 0, \\ \mathrm{clo}_{1,2}\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{2,4}_{\mathcal{F}}\right) & \textit{otherwise.} \end{array} \right. \tag{47}$$

Proof. Let $v \in \mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^4_{\mathcal{F}}$. Since $\bar{u}$ is a local solution of $(\mathcal{SP})$, proposition 8 implies that, for $\theta > 0$ small enough, we have

$$0 \leq J(\bar{u} + \theta v) - J(\bar{u}) = \theta \Upsilon_1(v) + \|v\|_4^2 O(\theta^2). \tag{48}$$

Thus, dividing by θ in (48) and letting $\theta \downarrow 0$, we have that $\Upsilon_1(v) \geq 0$. Analogously, if $\sigma_{uu} = 0$ we have that $\Upsilon_1(v) \geq 0$ for all $v \in \mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{4,2}_{\mathcal{F}}$. Condition (46) follows from the continuity of Υ_1. The proof of (47) follows in the same manner, with the obvious modifications. □

Note that the results obtained in lemma 15 are rather general, since they include the case of non local constraints. Of course they will give no information if the cones in (46)-(47) reduce to $\{0\}$. This may happen for somewhat pathological examples, as we next show.

Example 1. Let $u_0 \in L^2_{\mathcal{F}}$ and suppose that $u_0 \notin L^{\beta,p}_{\mathcal{F}}$ for any $\beta, p \in (2,\infty]$. The constraint $\mathcal{U} := \{u = \alpha u_0 \ / \ \text{for some } \alpha \in [0,1]\}$ is such that, at $\bar{u} = 0$, the radial cone is given by $\mathcal{R}_{\mathcal{U}}(\bar{u}) = \{\lambda u_0 \ / \ \text{for } \lambda \geq 0\}$, but $\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{\beta,p}_{\mathcal{F}} = \{0\}$.

Thus, we will assume the following assumption over the constraint set $\mathcal{U}$:

(H4) For every $\bar{u} \in \mathcal{U}$ we have that

$$\mathcal{T}_{\mathcal{U}}(\bar{u}) = \mathrm{clo}_2 \left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{\infty}_{\mathcal{F}} \right). \tag{49}$$

We have the following proposition whose proof is straightforward.

Proposition 16. *Assume that **(H1)**, **(H2)**, **(H4)** hold and let $\bar{u}$ be a local solution of $(\mathcal{SP})$. Then*

$$\Upsilon_1(v) \geq 0 \quad \text{for all } v \in \mathcal{T}_{\mathcal{U}}(\bar{u}). \tag{50}$$

Remark 6. *Note that if $J(\cdot)$ is convex, then* (50) *is a sufficient condition for the (global) optimality of $\bar{u}$.*

Clearly, we have that **(H4)** can hold for non local constraints. As an example, it can be checked that (49) holds for $\mathcal{U} = \left\{ u \in L^2_{\mathcal{F}} \ / \ \|u\|_2 \leq 1 \right\}$ and $\bar{u} \in \mathcal{U}$. Now we consider the case when $\mathcal{U}$ is defined by *local constraints.* The next lemma extends a well-known result on *deterministic* local constraints to the case of *adapted* local constraints. We first need to introduce some basic notions of multifunction theory. We say that the application $(t,\omega) \in [0,T] \times \Omega \to U(t,\omega) \in \mathcal{P}(\mathbb{R}^m)$ is a $B([0,T]) \times \mathcal{F}_T$-measurable multifunction if, for any closed set $C \subset \mathbb{R}^m$, we have that $U^{-1}(C) \in B([0,T]) \times \mathcal{F}_T$. In addition, we say that it is $\mathbb{F}$ -adapted if for all $t \in [0,T]$ we have that $U(t,\cdot)^{-1}(C) \in \mathcal{F}_t$. Finally, U is said to be closed-convex-valued if $U(t,\omega)$ is a.a. closed and convex.

Given a $B([0,T]) \times \mathcal{F}_T$-measurable multifunction U, consider the set

$$\mathcal{U} := \left\{ u \in L^2_{\mathcal{F}} \ ; \ u(t,\omega) \in U(t,\omega), \ \text{ a.a. } (t,\omega) \in [0,T] \times \Omega \right\}. \tag{51}$$

Lemma 17. *Let $\bar{u} \in \mathcal{U}$, with U closed-convex-valued and $\mathbb{F}$ -adapted. Then:*

(i) *Assumption* (49) *holds at $\bar{u}$.*

(ii) *The tangent cone is given by*

$$\mathcal{T}_{\mathcal{U}}(\bar{u}) = \left\{ v \in L^2_{\mathcal{F}} \ ; \ v(t,\omega) \in \mathcal{T}_{U(t,\omega)}(\bar{u}(t,\omega)) \ \text{a.a. } (t,\omega) \in [0,T] \times \Omega \right\}. \tag{52}$$

Proof. (i) By a diagonal argument, it suffices to prove that, for every $v \in \mathcal{R}_{\mathcal{U}}(\bar{u})$, there exists a sequence $v_k \in \mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{\infty}_{\mathcal{F}}$ such that $\|v_k - v\|_2 \to 0$. Indeed, set

$$v_k(t,\omega) := \mathbf{1}_{\{|v(t,\omega)| \leq k\}} v(t,\omega). \tag{53}$$

Being a product of adapted functions, v_k is adapted and, since $U(t,\omega)$ is convex, it belongs to $\mathcal{R}_{\mathcal{U}}(\bar{u})$. Finally, since $v_k(t,\omega) \to v(t,\omega)$ for a.a. (t,ω) and $|v_k(t,\omega)| \leq |v(t,\omega)|$, the dominated convergence theorem implies that $v_k \to v$

in $L^2_{\mathcal{F}}$.
(ii) Let $v \in \mathcal{T}_{\mathcal{U}}(\bar{u})$. By definition, for $\theta > 0$ small enough and a.a. (t,ω):

$$\bar{u}(t,\omega) + \theta v(t,\omega) + r_\theta(t,\omega) \in U(t,\omega), \tag{54}$$

where $r_\theta(\cdot,\cdot)/\theta \to 0$ in $L^2_{\mathcal{F}}$ as $\theta \downarrow 0$, and therefore also in $L^2_{\mathcal{F}_T}$. Thus, extracting a subsequence if necessary, we have that $r_\theta(t,\omega)/\theta \to 0$ for a.a. (t,ω) from which we deduce that $v(t,\omega) \in \mathcal{T}_{U(t,\omega)}(\bar{u}(t,\omega))$. Conversely, let v belongs to the r.h.s. of (52) and for $\varepsilon > 0$ set

$$v_\varepsilon := \varepsilon^{-1}\left(P_{\mathcal{U}}(\bar{u} + \varepsilon v) - \bar{u}\right), \tag{55}$$

where $P_{\mathcal{U}}(\cdot)$ denotes the orthogonal projection in $L^2_{\mathcal{F}}$ onto $\mathcal{U}$. By definition of $P_{\mathcal{U}}(\cdot)$ we have that $v_\varepsilon \in \mathcal{R}_{\mathcal{U}}(\bar{u})$. For (t,ω) in $[0,T]\times\Omega$ set $P_{U(t,\omega)}(\cdot)$ for the orthogonal projection in $\mathbb{R}^m$ onto $U(t,\omega)$. In view of lemma 34 in the appendix, we have that for a.a. (t,ω)

$$v_\varepsilon(t,\omega) := \varepsilon^{-1}\left(P_{U(t,\omega)}(\bar{u}(t,\omega) + \varepsilon v(t,\omega)) - \bar{u}(t,\omega)\right). \tag{56}$$

Clearly, $v_\varepsilon(t,\omega) \in \mathcal{R}_{U(t,\omega)}(\bar{u}(t,\omega))$ and for a.a. (t,ω) we have $v_\varepsilon(t,\omega) \to v(t,\omega)$. Since $|v_\varepsilon(t,\omega)| \leq |v(t,\omega)|$, the dominated convergence theorem implies that $v_\varepsilon \to v$ in $L^2_{\mathcal{F}_T}$ and therefore also in the closed subspace $L^2_{\mathcal{F}}$. Using that $v_\varepsilon \in \mathcal{R}_{\mathcal{U}}(\bar{u})$ we obtain that $v \in \mathcal{T}_{\mathcal{U}}(\bar{u})$. □

Let $a, b \in \overline{\mathbb{R}}^m$ with $-\infty \leq a^i < b^i \leq +\infty$ for all $i \in \{1,...,m\}$ and define

$$U_{a,b} := \left\{x \in \mathbb{R}^m \ ; \ a^i \leq x^i \leq b^i\right\}. \tag{57}$$

For $u \in L^2_{\mathcal{F}}$ and every index $i \in \{1,...,m\}$, set

$$\begin{aligned} I^{a^i}(u) &:= \left\{(t,\omega) \in [0,T]\times\Omega \ ; \ u^i(t,\omega) = a^i\right\}, \\ I^{b^i}(u) &:= \left\{(t,\omega) \in [0,T]\times\Omega \ ; \ u^i(t,\omega) = b^i\right\}. \end{aligned}$$

The following corollary is a direct consequence of proposition 16 and lemma 17.

Corollary 18. *Assume that **(H1)**, **(H2)** hold suppose that $\mathcal{U}$ is in the form* (51) *with U closed-convex-valued and $\mathbb{F}$ -adapted. Let $\bar{u} \in \mathcal{U}$ be a local solution of $(\mathcal{SP})$, then*

$$H_u(t,\omega)v(t,\omega) \geq 0 \quad \textit{for all } v \in \mathcal{T}_{U(t,\omega)}(\bar{u}(t,\omega)), \ \textit{a.e.}. \tag{58}$$

In particular, if $U(t,\omega) \equiv U_{a,b}$ (defined in (57)*), then for every $i \in \{1,...,m\}$, a.e.:*

$$H^i_u(t,\omega) = \begin{cases} \geq 0 & \textit{if } (t,\omega) \in I^{a^i}(\bar{u}), \\ \leq 0 & \textit{if } (t,\omega) \in I^{b^i}(\bar{u}), \\ = 0 & \textit{elsewhere.} \end{cases} \tag{59}$$

Remark 7. *Since* (58) *is equivalent to* (50) *when $\mathcal{U}$ is in the form* (51)*, we have that if $J(\cdot)$ is convex then* (58) *is a sufficient condition for the (global) optimality of $\bar{u}$.*

4.2 Second order necessary conditions

In order to obtain second order necessary conditions for $(\mathcal{SP})$ we proceed as in the previous section, i.e. we prove a general result and after, under some standard assumptions, we yield a more precise characterization for the important case of local constraints. Let us define

$$\Upsilon_1^{\perp} := \left\{ v \in L^2_{\mathcal{F}} \; ; \; \Upsilon_1(v) = 0 \right\}. \tag{60}$$

We have the following general second order necessary conditions.

Proposition 19. *Assume that* ***(H1)***, ***(H2)*** *hold and suppose that* $\bar{u} \in \mathcal{U}$ *is a local solution of* $(\mathcal{SP})$. *Then, the following second order necessary condition holds:*

$$\Upsilon_2(v) \geq 0 \quad \textit{for all } v \in \begin{cases} \operatorname{clo}_{4,2}\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^\infty_{\mathcal{F}} \cap \Upsilon_1^{\perp}\right) & \textit{if } \sigma_{uu} \equiv 0, \\ \operatorname{clo}_{4}\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^\infty_{\mathcal{F}} \cap \Upsilon_1^{\perp}\right) & \textit{otherwise.} \end{cases} \tag{61}$$

If in addition ***(H3)*** *holds, then*

$$\Upsilon_2(v) \geq 0 \quad \textit{for all } v \in \begin{cases} \operatorname{clo}_{2}\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^\infty_{\mathcal{F}} \cap \Upsilon_1^{\perp}\right) & \textit{if } \sigma_{uu} \equiv 0, \\ \operatorname{clo}_{2,4}\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^\infty_{\mathcal{F}} \cap \Upsilon_1^{\perp}\right) & \textit{otherwise.} \end{cases} \tag{62}$$

Proof. If $v \in \mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^\infty_{\mathcal{F}} \cap \Upsilon_1^{\perp}$, proposition 13 implies that for θ small enough

$$0 \leq J(\bar{u} + \theta v) - J(\bar{u}) = \frac{\theta^2}{2}\Upsilon_2(v) + \theta^3 O(\|v\|_\infty^3).$$

Dividing the above equation by θ and letting $\theta \downarrow 0$ yields $\Upsilon_2(v) \geq 0$ and the result follows from the bounds in proposition 13 for the quadratic form Υ_2. □

The critical cone to $\mathcal{U}$ at $\bar{u}$ are defined by

$$C(\bar{u}) \quad := \quad \{ v \in \mathcal{T}_{\mathcal{U}}(\bar{u}) \; / \; \Upsilon_1(v) \leq 0 \}. \tag{63}$$

In order to obtain more precise second order necessary conditions, we suppose standard assumptions in the second order analysis of problems with convex constraints. The first one is a natural extension of **(H4)** to the second order case.

(H5) For every $\bar{u} \in \mathcal{U}$ and $v^* \in N_{\mathcal{U}}(\bar{u})$ (recall (44)), we have that

$$\operatorname{clo}_2\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^\infty_{\mathcal{F}} \cap (v^*)^{\perp}\right) = \operatorname{clo}_2\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap (v^*)^{\perp}\right). \tag{64}$$

For our second assumption, we need the following notion of polyhedricity (see [13, 23]). The set $\mathcal{U}$ is said to be *polyhedric* at $\bar{u} \in \mathcal{U}$ if for all $v^* \in N_{\mathcal{U}}(\bar{u})$, the set $\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap (v^*)^{\perp}$ is dense in $\mathcal{T}_{\mathcal{U}}(\bar{u}) \cap (v^*)^{\perp}$ with respect to the $\|\cdot\|_2$ norm. If $\mathcal{U}$ is polyhedric at each $u \in \mathcal{U}$ we say that $\mathcal{U}$ is *polyhedric.*

Remark 8. *Note that, if* ***(H1)***, ***(H2)*** *and* ***(H4)*** *hold, proposition 16 yields that, at a local minimum,* $-\Upsilon_1 \in N_{\mathcal{U}}(\bar{u})$ *and* $C(\bar{u}) = \mathcal{T}_{\mathcal{U}}(\bar{u}) \cap \Upsilon_1^{\perp}$. *Thus, if* $\mathcal{U}$ *is polyhedric and in addition* ***(H5)*** *holds,*

$$\operatorname{clo}_2\left(\mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^\infty_{\mathcal{F}} \cap \Upsilon_1^{\perp}\right) = C(\bar{u}). \tag{65}$$

We state a second order necessary condition which is a natural extension of the deterministic counterpart.

Theorem 20. *Let $\bar{u}$ be a local solution of $(\mathcal{SP})$ and assume that*
(i) *Assumptions **(H1)-(H5)** hold.*
(ii) *The constraint set $\mathcal{U}$ is polyhedric.*
Then, the following second order necessary condition hold at $\bar{u}$:

$$\Upsilon_2(v) \geq 0 \quad \textit{for all } v \in C(\bar{u}). \tag{66}$$

Proof. As in the proof of proposition 19 we have that $\Upsilon_2(v) \geq 0$ for all $v \in \mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^\infty_{\mathcal{F}} \cap \Upsilon_1^\perp$. The result follows as in the proof of proposition 19 since under our assumptions $\Upsilon_2(v) = O(\|v\|_2^2)$ and (65) holds. □

Now, let us focus our attention in local constraints, i.e. when $\mathcal{U}$ is defined by (51).

Lemma 21. *Let $\mathcal{U}$ be defined by* (51) *with U closed-convex-valued and $\mathbb{F}$ -adapted. Let $\bar{u} \in \mathcal{U}$, it holds that*
(i) *The normal cone $N_{\mathcal{U}}(\bar{u})$ is given by*

$$N_{\mathcal{U}}(\bar{u}) = \left\{ v^* \in L^2_{\mathcal{F}} \ / \ v^*(t,\omega) \in N_{U(t,\omega)}(\bar{u}(t,\omega)), \quad a.a. \ (t,\omega) \in [0,T] \times \Omega \right\}. \tag{67}$$

(ii) *For every $v^* \in N_{\mathcal{U}}(\bar{u})$ we have that*

$$\mathcal{T}_{\mathcal{U}}(\bar{u}) \cap (v^*)^\perp = \left\{ v \in \mathcal{T}_{\mathcal{U}}(\bar{u}) \ / \ v^*(t,\omega) \cdot v(t,\omega) = 0, \quad a.a. \ (t,\omega) \in [0,T] \times \Omega \right\}. \tag{68}$$

Proof. Since (ii) follows directly from (i) and lemma 17 (ii), it is enough to show (i). By lemma 17(ii) and since $N_{U(t,\omega)}(\bar{u}(t,\omega))$ is the polar cone of $\mathcal{T}_{U(t,\omega)}(\bar{u}(t,\omega))$, the r.h.s. of (67) is included in $N_{\mathcal{U}}(\bar{u})$. To prove the other inclusion, let $v^* \in N_{\mathcal{U}}(\bar{u})$ and consider $v'(t,\omega) \in \mathcal{T}_{U(t,\omega)}(\bar{u}(t,\omega))$, $v''(t,\omega) \in N_{U(t,\omega)}(\bar{u}(t,\omega))$ such that

$$v^*(t,\omega) = v'(t,\omega) + v''(t,\omega), \qquad v'(t,\omega) \cdot v''(t,\omega) = 0.$$

We know that the above decomposition exists and it is unique. By lemma 17(ii) we have that $v' \in \mathcal{T}_{\mathcal{U}}$, and thus $0 \geq \mathbb{E}\left(\int_0^T v^*(t,\omega) \cdot v'(t,\omega) \mathrm{d}t\right) = \|v'\|_2^2$ proving that $v' = 0$. The result follows. □

In order to verify the polyhedricity assumption in the case of local constraints, we will need in fact to assume that for a.a. (t,ω) the set $U(t,\omega)$ is a polyhedron. More precisely, let $q \in \mathbb{N}$ and suppose that there exist mappings $\Sigma : [0,T] \times \Omega \to \mathcal{P}(\{1,...,q\})$, $a_i : [0,T] \times \Omega \to \mathbb{R}^m$, $b_i : [0,T] \times \Omega \to \mathbb{R}^m$, where $i \in \{1,...,q\}$, such that Σ, a_i and b_i are $\mathcal{B}([0,T]) \times \mathcal{F}_T$ measurable and for each t we have that $\Sigma(t,\cdot)$, $a_i(t,\cdot)$ and $b_i(t,\cdot)$ are $\mathcal{F}_t$ measurable. We suppose that

$$U(t,\omega) = \{ x \in \mathbb{R}^m \ / \ \langle a_i(t,\omega), x \rangle \leq b_i(t,\omega), \text{ for } i \in \Sigma(t,\omega) \}. \tag{69}$$

We have

Lemma 22. *The set of local constraints $\mathcal{U}$ defined in* (51)*, with $U(t,\omega)$ given by* (69)*, is polyhedric and satisfies* (64)*.*

Proof. Let $\bar{u} \in \mathcal{U}$ and $v^* \in N_{\mathcal{U}}(\bar{u})$. For $v \in T_{\mathcal{U}}(\bar{u}) \cap (v^*)^{\perp}$ and $k \geq 0$ set

$$\widehat{v}_k(t,\omega) := \begin{cases} v(t,\omega) & \text{if } |v(t,\omega)| \leq k \text{ and } \bar{u}(t,\omega) + \frac{1}{k}v(t,\omega) \in U(t,\omega), \\ 0 & \text{otherwise.} \end{cases} \tag{70}$$

Lemma 21(ii) implies that $\widehat{v}_k \in \mathcal{R}_{\mathcal{U}}(\bar{u}) \cap L^{\infty}_{\mathcal{F}} \cap (v^*)^{\perp}$. On the other hand, since $U(t,\omega)$ is a polyhedron, lemma 17 (ii) implies that $v(t,\omega) \in T_{\mathcal{U}}(\bar{u}(t,\omega)) = \mathcal{R}_{\mathcal{U}}(\bar{u}(t,\omega))$. Thus, as $k \uparrow \infty$, we have that $\widehat{v}_k \to v(t,\omega)$ for a.a. (t,ω). The dominated convergence theorem, yields that $\widehat{v}_k \to v$ in $L^2_{\mathcal{F}}$, hence $\mathcal{U}$ is polyhedric and (64) holds. □

The following corollary is a consequence of theorem 20 and lemmas 21, 22.

Corollary 23. *Assume that **(H1)** - **(H3)** hold and suppose that $\bar{u}$ is a local solution of $(\mathcal{SP})$ where $\mathcal{U}$ is defined in* (51)*, with $U(t,\omega)$ given by* (69)*. Then, the following second order necessary conditions holds:*

$\Upsilon_2(v) \geq 0$, *for all $v \in T_{\mathcal{U}}(\bar{u})$ such that $H_u(t)v(t,\omega) = 0$ for a.a.* $(t,\omega) \in [0,T] \times \Omega$.

4.3 On the second order sufficient condition

In this section we give a second order sufficient condition for the unconstrained case and we briefly discuss the difficulties arising in the constrained case.

When $\mathcal{U} = L^2_{\mathcal{F}}$, **(H4)** is trivially satisfied and for every $\bar{u} \in \mathcal{U}$ it holds that $T_{\mathcal{U}}(\bar{u}) = L^2_{\mathcal{F}}$. The following proposition is a consequence of corollary 14.

Proposition 24. *Assume that **(H1)**-**(H3)** hold and that $\mathcal{U} = L^2_{\mathcal{F}}$. Suppose that there exist $\alpha > 0$ such that $\bar{u} \in L^2_{\mathcal{F}}$ satisfies:*

$$\Upsilon_1(v) = 0, \text{ and } \Upsilon_2(v) \geq \alpha \|v\|_2^2 \quad \text{for all } v \in L^2_{\mathcal{F}}. \tag{71}$$

Then, there exists $\delta > 0$ such that for all $v' \in L^{\infty}_{\mathcal{F}}$ with $\|v'\|_{\infty} \leq \delta$, we have

$$J(\bar{u} + v') \geq J(\bar{u}) + \tfrac{1}{2}\alpha\|v'\|_2^2. \tag{72}$$

Only very partial results are obtained when $\mathcal{U} \neq L^2_{\mathcal{F}}$. Let us recall (see [15]) that a quadratic form $Q : H \to \mathbb{R}$, where H is a Hilbert space, is a Legendre form if it is weakly lower semi-continuous (w.l.s.c.) quadratic form over H, such that, if $h_k \to h$ weakly in H and $Q(h_k) \to Q(h)$, then $h_k \to h$ strongly. We have the following proposition, whose proof follows the lines of the parallel deterministic result (see [7, Section 3.3]):

Proposition 25. *Assume that **(H1)**- **(H3)** hold. In addition, assume that at $\bar{u} \in \mathcal{U}$ the quadratic form Υ_2 is a Legendre form and there exist $\alpha > 0$ such that*

$$\Upsilon_1(v) = 0, \text{ and } \Upsilon_2(v) \geq \alpha \|v\|_2^2 \quad \text{for all } v \in C(\bar{u}). \tag{73}$$

Then, there exists $\delta > 0$ such that for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{\infty} \leq \delta$, we have

$$J(u) \geq J(\bar{u}) + \tfrac{1}{2}\alpha\|u - \bar{u}\|_2^2. \tag{74}$$

In the deterministic case it is well known that the application $u \in L^2([0,T];\mathbb{R}^m) \to y_1(u)(T) \in \mathbb{R}^n$ is weakly continuous. This allows to verify that the associated quadratic form is a Legendre form iff some form of the strong Legendre condition holds. We show with two examples that $u \in L^2_{\mathcal{F}} \to y_1(u)(T) \in L^2_{\mathcal{F}_T}(\mathbb{R}^n)$ is not weakly continuous.

Example 2 (σ dependent on u)**.** Let us take $m = n = 1$ and let us consider the dynamics

$$\mathrm{d}y_1(t) = u(t)\mathrm{d}W(t) \text{ for } t \in [0,T]; \quad y_1(0) = 0.$$

Let u_n be a (deterministic) orthonormal base of $L^2([0,T];\mathbb{R})$ and denote $y_n := y_1[u_n]$. By the dominated convergence theorem it is easy to check that u_n converges weakly to 0 in $L^2_{\mathcal{F}}$, but

$$\mathbb{E}\left[y_n(T)^2\right] = \mathbb{E}\left[\left(\int_0^T u_n(t)\mathrm{d}W(t)\right)^2\right] = \int_0^T u_n^2(t)\mathrm{d}t = 1.$$

Example 3 (σ independent on u)**.** We take $m = n = 1$ and $T = 2$. Let us consider the dynamics

$$\mathrm{d}y_1(t) = u(t)\mathrm{d}t \text{ for } t \in [0,T]; \quad y_1(0) = 0.$$

Let ϕ_n be an orthonormal base of the Hilbert space $L^2(\mathbb{R})$ endowed with the scalar product

$$\langle g, h\rangle_* := \int_{-\infty}^{+\infty} g(x)h(x)e^{\frac{-x^2}{2}}\mathrm{d}x.$$

As a classical example (see e.g [16]) we can take $\phi_n(x) = h_n(x)/\sqrt{2\pi n!}$, where h_n is the nth Hermite polynomial. Consider the sequence $u_n \in L^2_{\mathcal{F}}$ defined by $u_n(t) := \phi_n(W(1))\mathbb{I}_{(1,2]}(t)$ and set $y_n := y_1[u_n]$. For every $f \in L^2_{\mathcal{F}}$, we have

$$\begin{aligned}
\mathbb{E}\left(\int_0^2 f(t)u_n(t)\ \mathrm{d}t\right) &= \mathbb{E}\left(\phi_n(W(1))\int_1^2 f(t)\mathrm{d}t\right), \\
&= \mathbb{E}\left[\phi_n(W(1))\mathbb{E}\left(\int_1^2 f(t)\mathrm{d}t | W(1)\right)\right] \to 0,
\end{aligned}$$

by definition of ϕ_n. Thus, u_n converges weakly to 0 in $L^2_{\mathcal{F}}$. On the other hand,

$$\mathbb{E}\left(y_n(T)^2\right) = \mathbb{E}\left(\left[\int_0^2 u_n\mathrm{d}t\right]^2\right) = \mathbb{E}\left(\phi_n(W(1))^2\right) = 1.$$

5 Optimality conditions: The case of final state constraints

In this section we suppose that $\mathcal{U} = L^2_{\mathcal{F}}$ and we consider the problem

$$\mathrm{Min}_{u\in\mathcal{U}}\ J(u) \ \text{ subject to } \ \mathbb{E}\left[g^i(y_u(T))\right] = 0, \ \ \mathbb{E}\left[h^j(y_u(T))\right] \le 0, \qquad (\mathcal{SP}')$$

for all $i \in \{1,...,n_g\}$, $j \in \{1,...,n_h\}$. In the notation above, $g^i : \mathbb{R}^n \times \Omega \to \mathbb{R}$ and $h^j : \mathbb{R}^n \times \Omega \to \mathbb{R}$. We make the following assumption:

(H6) The maps g^i and h^j satisfy the same assumptions made for ϕ in **(H2)**.

In order to provide first and second order necessary conditions for $(\mathcal{SP}')$ we will need the following lemma:

Lemma 26. *Let $\bar{u} \in \mathcal{U}$ and consider a function $c : \mathbb{R}^n \times \Omega \to \mathbb{R}$ satisfying the assumptions for ϕ in **(H2)**. Then, the map*

$$v \in L^\infty_{\mathcal{F}} \to \hat{c}(v) := \mathbb{E}\left[c(y_{\bar{u}+v}(T))\right] \in \mathbb{R}$$

is differentiable with a Lipschitz derivative and admits the following second order expansion at $\bar{v} = 0$

$$\hat{c}(v) = \hat{c}(0) + \Upsilon_1^c(v) + \tfrac{1}{2}\Upsilon_2^c(v) + O(\|v\|_\infty^3), \tag{75}$$

where

$$\begin{aligned} \Upsilon_1^c(v) &:= \mathbb{E}\left[c_y(y_{\bar{u}}(T))y_1[\bar{u},v](T)\right] = O(\|v\|_2), \\ \Upsilon_2^c(v) &:= \mathbb{E}\left[c_y(y_{\bar{u}}(T))y_2[\bar{u},v](T) + c_{yy}(y_{\bar{u}}(T))(y_1[\bar{u},v](T))^2\right] = O(\|v\|_\infty^2). \end{aligned} \tag{76}$$

*If in addition c satisfies the assumptions for ϕ in **(H3)**, then*

$$\Upsilon_2^c(v) := O(\|v\|_2^2). \tag{77}$$

Proof. Letting $\ell \equiv 0$ and $\phi = c$ in proposition 13 yields (75) as well as the estimates in (76) and (77). It remains to show that the derivative of $\hat{c}(v)$ is Lipschitz. In view of proposition 6 and the chain rule, it suffices to show that $y \in \mathcal{Y}^2 \to \tilde{c}(y) := \mathbb{E}\left[c(y(T))\right]$ is differentiable with a Lipschitz derivative given by

$$D\tilde{c}(\bar{y})(y) = \mathbb{E}\left[c_y(\bar{y}(T))y(T)\right] \quad \text{for all } \bar{y}, y \in \mathcal{Y}^2. \tag{78}$$

For any $\bar{y}, y \in \mathcal{Y}^2$, we have, for a.a. ω:

$$c(\bar{y} + y) = c(\bar{y}) + c_y(\bar{y})y + \int_0^1 \left[c_y(\bar{y} + \theta y) - c_y(\bar{y})\right] y \mathrm{d}\theta. \tag{79}$$

Using **(H2)** and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} &\mathbb{E}\left[c(\bar{y}(T) + y(T))\right] = \mathbb{E}\left[c(\bar{y}(T)) + c_y(\bar{y}(T))\left(y(T) - \bar{y}(T)\right)\right] + O\left(||y - \bar{y}||_{2,\infty}^2\right), \\ &\mathbb{E}\left[c_y(\bar{y}(T))\left(y(T) - \bar{y}(T)\right)\right] = O\left(||y - \bar{y}||_{2,\infty}\right). \end{aligned}$$

Thus, expression (78) holds. We next prove that the derivative is Lipschitz. For $\bar{y}$ and $\tilde{y}$ in $\mathcal{Y}^2$, define

$$\Delta(\bar{y}, \tilde{y}) := \max_{\|y\|_{2,\infty} \leq 1} \left|I\!\!E\left(\left[c_y(\bar{y}) - c_y(\tilde{y})\right] y\right)\right| \tag{80}$$

By **(H2)** $Dc(\cdot, \omega)$ is Lipschitz, hence the Cauchy-Schwarz inequality implies that

$$\Delta(\bar{y}, \tilde{y}) \leq L_2 I\!\!E\left(|\bar{y} - \tilde{y}|\,|y|\right) \leq L_2\|\bar{y} - \tilde{y}\|_{2,\infty}. \tag{81}$$

The result follows. $\square$

Remark 9. *If $\Upsilon_1^c(\cdot)$ is surjective then Graves theorem [12] (see also Dontchev [10]) implies that $\hat{c}$ enjoys the metric regularity property. In other words, if $\hat{c}(0) = 0$ and $\|v\|_\infty$ is small enough, then there exists $v' \in L^\infty$ such that $\hat{c}(v') = 0$ and $\|v' - v\|_\infty = O(|\hat{c}(v)|)$.*

5.1 First order necessary condition

Let $\bar{u} \in L^2_{\mathcal{F}}$ be a local solution of $(\mathcal{SP}')$ and denote by $\bar{y}$ its associated state. Lemma 26 implies that the maps

$$\begin{array}{rcl} v \in L^\infty_{\mathcal{F}} \to \tilde{g}^i(v) & := & \mathbb{E}\left[g^i(y_{\bar{u}+v}(T))\right] \in \mathbb{R}, \\ v \in L^\infty_{\mathcal{F}} \to \tilde{h}^j(v) & := & \mathbb{E}\left[h^j(y_{\bar{u}+v}(T))\right] \in \mathbb{R}, \end{array} \tag{82}$$

are C^1 with derivatives at $\bar{v} = 0$ given respectively by

$$\begin{array}{rcl} \Upsilon^g_{1i}(\cdot) & := & \mathbb{E}\left[g^i_y(y_{\bar{u}}(T))y_1[\bar{u}, \cdot](T)\right], \\ \Upsilon^h_{1j}(\cdot) & := & \mathbb{E}\left[h^j_y(y_{\bar{u}}(T))y_1[\bar{u}, \cdot](T)\right]. \end{array} \tag{83}$$

For notational convenience we write $\tilde{g} := (\tilde{g}_1, ..., \tilde{g}_{n_g})$ and $\Upsilon^g_1 := (\Upsilon^g_{11}, ..., \Upsilon^g_{1n_g})$. The set $I^1(\bar{u})$ of *active* constraints is defined as

$$I^1(\bar{u}) := \left\{ j \in \{1, ..., n_h\} \;/\; \mathbb{E}\left[h^j(\bar{y}(T))\right] = 0 \right\}. \tag{84}$$

We assume the following constraint qualification condition:

(H7) The following assertions hold true:

(i) The application $\Upsilon^g_1 : L^\infty_{\mathcal{F}} \to \mathbb{R}^{n_g}$ is surjective.
(ii) There exists $\bar{v} \in (\Upsilon^g_1)^\perp \cap L^\infty_{\mathcal{F}}$ such that $\Upsilon^h_{1j}(\bar{v}) < 0$ for all $j \in I^1(\bar{u})$.

We will need the following density lemma, proved in [9, Lemma 1].

Lemma 27. *Let X be a normed vector space. Given a_i, $i = 1$ to q, in X^*, and $b \in \mathbb{R}^q$, define*

$$K := \{x \in X \;/\; \langle a_i, x\rangle \leq b_i,\ i = 1, ..., q\}.$$

If Y is a dense subspace of X, then $K \cap Y$ is a dense subset of K.

Let us define the cones $T_2(\bar{u})$, $T_\infty(\bar{u})$ as

$$\begin{array}{rcl} T_2(\bar{u}) & := & \left\{ v \in L^2_{\mathcal{F}}\,;\, \Upsilon^g_1(v) = 0,\ \Upsilon^h_{1j}(v) \leq 0,\ j \in I(\bar{u}) \right\}, \\ T_\infty(\bar{u}) & := & T_2(\bar{u}) \cap L^\infty_{\mathcal{F}}. \end{array} \tag{85}$$

Lemma 28. *Under **(H1)**,**(H2)**,**(H6)**,**(H7)**, problem*

$$\mathrm{Min}_{v \in \mathcal{U}} \quad \Upsilon_1(v) \quad \text{subject to } v \in T_2(\bar{u}) \tag{86}$$

admits $v = 0$ as a solution.

Proof. Let $v \in T_\infty(\bar{u})$ and consider $\varepsilon > 0$. Assumption **(H7)** (ii) implies that for $\theta > 0$:

$$\tilde{g}(\theta(v + \varepsilon\bar{v})) = \tilde{g}(0) + \theta\Upsilon^g_1(v + \varepsilon\bar{v}) + o(\theta) = o(\theta).$$

Remark 9 implies that there exists $o_\infty(\theta) \in L^\infty_{\mathcal{F}}$, with $\|o_\infty(\theta)\|_\infty/\theta \to 0$ as $\theta \downarrow 0$, such that $\tilde{g}(\theta(v + \varepsilon\bar{v}) + o_\infty(\theta)) = 0$. Therefore, setting $\hat{u}(\theta) := \bar{u} + \theta(v + \varepsilon\bar{v}) + o_\infty(\theta)$, we obtain that $\mathbb{E}\left(g(y_{\hat{u}(\theta)}(T))\right) = 0$ for $\theta > 0$ small enough. On the other hand, for every $j \in \{1, ..., n_h\}$,

$$\mathbb{E}\left(h^j(\hat{u}(\theta))\right) = \tilde{h}^j(\theta(v + \varepsilon\bar{v}) + o_\infty(\theta)) = \tilde{h}(0) + \theta\Upsilon^h_{1j}(v + \varepsilon\bar{v}) + o(\theta).$$

Thus, if the jth constraint is active at $\bar{u}$, then by **(H7)**(ii), we have that

$$\mathbb{E}(h^j(\hat{u}(\theta))) = \varepsilon\theta\Upsilon_{1j}^h(\bar{v}) + o(\theta) < 0 \quad \text{for all } \theta \text{ small enough.}$$

If the constraint is not active the same conclusion trivially holds. Therefore, there exists θ_ε such that for $\theta \in [0, \theta_\varepsilon)$, $\hat{u}(\theta)$ is feasible. Thus,

$$0 \leq J(\hat{u}(\theta)) - J(\bar{u}) = \theta\Upsilon_1(v + \varepsilon\bar{v}) + o(\theta),$$

which implies that $\Upsilon_1(v + \varepsilon\bar{v}) \geq 0$. Since ε is arbitrary, we have $\Upsilon_1(v) \geq 0$. Since $L^\infty_{\mathcal{F}}$ is dense in $L^2_{\mathcal{F}}$ and Υ_1 is continuous, the result follows from lemma 27. □

For $\lambda = (\lambda_g, \lambda_h) \in \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$, let us set $(p_\lambda, q_\lambda) \in \mathcal{Y}^2 \times L^2_{\mathcal{F}}$ for the unique solution of

$$\begin{aligned} \mathrm{d}p(t) &= -\left[\ell_y(t)^\top + f_y(t)^\top p(t) + \sum_{i=1}^m \sigma_y^i(t)^\top q^i(t)\right]\mathrm{d}t + q(t)\mathrm{d}W(t), \\ p(T) &= \phi_y(\bar{y}(T))^\top + g_y(\bar{y}(T))^\top\lambda_g + h_y(\bar{y}(T))^\top\lambda_h. \end{aligned} \tag{87}$$

We say that $\lambda := (\lambda_g, \lambda_h) \in \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$ is a Lagrange multiplier at $\bar{u}$ if the following conditions hold

$$H_u(t, \bar{y}(t), \bar{u}(t), p_\lambda(t), q_\lambda(t), \omega) = 0 \quad \text{for a.a. } (t, \omega), \tag{88}$$

$$\lambda_h^j \mathbb{E}\left(h^j(\bar{y}(T))\right) = 0 \quad \text{ and } \quad \lambda_h^j \geq 0 \ \text{ for all } j \in \{1, ..., n_h\}. \tag{89}$$

We denote by $\Lambda(\bar{u})$ for the set of Lagrange multipliers at $\bar{u}$.

Proposition 29. *Under **(H1),(H2),(H6),(H7)** the set $\Lambda(\bar{u})$ is a nonempty compact subset of $\mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$.*

Proof. Lemma 28 implies that the linear program (86) has value 0. By a standard duality result for linear programs (affine cost function and constraints) in a Banach space setting (see e.g. [7, Thm 2.202]), the set of $(\lambda_g, \lambda_h) \in \mathbb{R}^{n_g} \times \mathbb{R}_+^{I^1(\bar{u})}$ such that for all $v \in L^2_{\mathcal{F}}$

$$\Upsilon_1(v) + \sum_{i=1}^{n_g} \lambda_g^i \Upsilon_{1i}^g(v) + \sum_{j \in I^1(\bar{u})} \lambda_h^j \Upsilon_{1j}^h(v) = 0 \tag{90}$$

is a nonempty compact subset of $\mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$. Letting $\lambda_h^j = 0$ if $j \notin I^1(\bar{u})$ relation (89) trivially holds, while equation (88) follows from (90) and lemma 3, along the lines of the proof of lemma 7. □

Now we treat the so-called *non qualified* case. For $\alpha \in \mathbb{R}$ let us define the *generalized* Hamiltonian $\mathcal{H} : \mathbb{R} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \Omega \to \mathbb{R}$ by

$$\mathcal{H}(\alpha, t, y, u, p, q, \omega) := \alpha\ell(t, y, u, \omega) + p \cdot f(t, y, u, \omega) + \sum_{i=1}^d q^i \cdot \sigma^i(t, y, u, \omega).$$

We say that $\lambda := (\alpha, \lambda_g, \lambda_h) \in \mathbb{R} \times \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$ is a *generalized* Lagrange multiplier at $\bar{u}$ if $\lambda \neq 0$ and the following conditions hold

$$\mathcal{H}_u(\alpha, t, \bar{y}(t), \bar{u}(t), p_\lambda(t), q_\lambda(t), \omega) = 0 \quad \text{for a.a. } (t, \omega), \tag{91}$$

$$\lambda_h^j \mathbb{E}\left(h^j(\bar{y}(T))\right) = 0, \quad \lambda_h^j \geq 0 \ \text{ for all } j \in \{1, ..., n_h\} \ \text{ and } \alpha \geq 0. \tag{92}$$

We denote by $\Lambda^{gen}(\bar{u})$ for the set of generalized Lagrange multipliers at $\bar{u}$.

Proposition 30. *Under* ***(H1),(H2)*** *the set* $\Lambda^{gen}(\bar{u})$ *is nonempty.*

Proof. If **(H6),(H7)** hold then proposition 29 yields that $\emptyset \neq \Lambda(\bar{u}) \subseteq \Lambda^{gen}(\bar{u})$. If **(H6)** does not hold there exists $\lambda_g \neq 0$ orthogonal to the image of Υ_1^g. Therefore, lemma 3 yields that $(0, \lambda_g, 0)$ is a generalized Lagrange multiplier at $\bar{u}$. If **(H6)** hold but **(H7)** does not, then the pair $(0,0) \in \mathbb{R} \times L_{\mathcal{F}}^{\infty}$ is a solution of

$$\text{Min } z \quad \text{s.t. } \Upsilon_1^g(v) = 0, \ \Upsilon_{1j}^h(v) \leq z \quad \text{for } j \in I^1(\bar{u}).$$

By considering the dual problem, we obtain the existence of a not null $(\lambda_g, \tilde{\lambda}_h) \in \mathbb{R}^{n_g} \times \mathbb{R}_+^{I^1(\bar{u})}$ such that for all $v \in L_{\mathcal{F}}^{\infty}$

$$\sum_{i=1}^{n_g} \lambda_g^i \Upsilon_{1i}^g(v) + \sum_{j \in I^1(\bar{u})} \tilde{\lambda}_h^j \Upsilon_{1j}^h(v) = 0. \tag{93}$$

Define $\lambda_h \in \mathbb{R}^{n_h}$ as $\lambda_h^j = \tilde{\lambda}_h^j$ if $j \in I^1(\bar{u})$ and 0 otherwhise. The result follows, since lemma 3 implies that $(0, \lambda_g, \lambda_h) \in \Lambda^{gen}(\bar{u})$. $\square$

Remark 10. *Proposition 30 is a particular case of the Pontryagin maximum principle in [25].*

5.2 Second order necessary condition

Lemma 26 implies that the maps $\tilde{g}_i$, $\tilde{h}_j$ defined in (82) admit a second order expansion at $\bar{v} = 0$. Let us denote respectively by Υ_{2i}^g, Υ_{2j}^h the associated quadratic forms. As for Υ_{1i}^g, we set $\Upsilon_2^g := (\Upsilon_{21}^g, ..., \Upsilon_{21}^g)$. Define the cones $C_2(\bar{u})$, $C_\infty(\bar{u})$ by

$$\begin{array}{rcl} C_2(\bar{u}) & := & \{v \in T_2(\bar{u}) \ ; \ \Upsilon_1(v) = 0\}, \\ C_\infty(\bar{u}) & := & C_2(\bar{u}) \cap L_{\mathcal{F}}^{\infty}. \end{array} \tag{94}$$

Note that $C_2(\bar{u}) = \text{clo}_2\left(C_\infty(\bar{u})\right)$ by lemma 27. For $v \in C_\infty(\bar{u})$ let us set

$$I^2(\bar{u}, v) := \left\{ j \in I(\bar{u}) \ ; \ \Upsilon_{1j}^h(v) = 0 \right\}.$$

Lemma 31. *Under* ***(H1),(H2),(H6),(H7)****, for every* $v \in C_\infty(\bar{u})$*, the problem*

$$\begin{array}{rl} \text{Min }_{w \in L_{\mathcal{F}}^{\infty}} & \Upsilon_1(w) + \Upsilon_2(v) \\ \text{s.t.} & \Upsilon_1^g(w) + \Upsilon_2^g(v) = 0, \\ & \Upsilon_{1j}^h(w) + \Upsilon_{2j}^h(v) \leq 0 \quad \text{for } j \in I^2(\bar{u}, v) \end{array} \tag{$\mathcal{QP}_v$}$$

has a nonempty feasible set and a non-negative value.

Proof. By **(H7)** the feasible set of $(\mathcal{QP}_v)$ is nonempty. Let $w \in L_{\mathcal{F}}^{\infty}$ be feasible and $\bar{v}$ satisfy **(H7)** (ii). Fix $\varepsilon > 0$ and for $\theta > 0$ set

$$u(\theta) = \bar{u} + \theta v + \tfrac{1}{2}\theta^2 (w + \varepsilon \bar{v}). \tag{95}$$

By lemma 12 applied to the equality constraints, we have that

$$\mathbb{E}[g(y_{u(\theta)}(T))] = \tfrac{1}{2}\theta^2 \left[\Upsilon_1^g(w) + \Upsilon_2^g(v)\right] + o(\theta^2) = o(\theta^2). \tag{96}$$

Remark 9 implies that there exists a path of the form $\tilde{u}(\theta) = u(\theta) + o_\infty(\theta^2) \in L^\infty_{\mathcal{F}}$, where $\|o_\infty(\theta^2)\|_\infty/\theta^2 \to 0$, such that $I\!\!E[g(y_{\tilde{u}(\theta)}(T))] = 0$. At the same time,

$$I\!\!E[h^j(y_{\tilde{u}(\theta)}(T))] = \tfrac{1}{2}\theta^2 \left[\Upsilon^h_{1j}(w+\varepsilon\bar{v}) + \Upsilon^h_{2j}(v)\right] + o(\theta^2) \quad \text{for all } j \in I^2(\bar{u}, v). \tag{97}$$

It follows that, for $\theta > 0$ small enough, $\tilde{u}(\theta)$ is feasible and therefore by lemma 12

$$0 \leq \lim_{\theta \downarrow 0} \frac{J(\tilde{u}(\theta)) - J(\bar{u})}{\frac{1}{2}\theta^2} = \Upsilon_1(w) + \Upsilon_2(v) \tag{98}$$

as was to be proved. □

For $\lambda = (\lambda_g, \lambda_h) \in \mathbb{R}^{n_g} \times \mathbb{R}^{n_h}$ and $z \in \mathbb{R}^n$ let us set

$$\begin{array}{rcl} H_{(y,u)^2}[\lambda](t) & := & H_{(y,u)^2}(t, \bar{y}(t), \bar{u}(t), p_\lambda(t), q_\lambda(t)), \\ \Psi_{yy}[\lambda](z)^2 & := & \phi_{yy}(z)^2 + \sum_{i=1}^{n_g} \lambda_g^i g^i_{yy}(z)^2 + \sum_{j=1}^{n_h} \lambda_h^j h^j_{yy}(z)^2, \end{array} \tag{99}$$

where (p_λ, q_λ) is defined by (87). Now we are able to prove a second order necessary condition for the local minimum $\bar{u}$ of $(\mathcal{SP}')$.

Theorem 32. *Assume that **(H1)-(H3),(H6),(H7)** hold and that $\tilde{g}_i$, $\tilde{h}_j$ satisfy the assumptions for ϕ in **(H3)**. Then for every $v \in C_2(\bar{u})$ we have that*

$$\max_{\lambda \in \Lambda(\bar{u})} \mathbb{E}\left(\int_0^T H_{(y,u)^2}[\lambda](t)(y_1(t), v(t))^2 \mathrm{d}t + \Psi_{yy}[\lambda](\bar{y}(T))(y_1(T))^2 \right) \geq 0. \tag{100}$$

Proof. Let $v \in C_\infty(\bar{u})$. Using lemma 31 and considering the dual problem associated to $(\mathcal{QP}_v)$, we obtain that

$$\max_{(\lambda_g,\lambda_h) \in \tilde{\Lambda}(\bar{u})} \Upsilon_2(v) + \sum_{i=1}^{n_g} \lambda_g^i \Upsilon^g_{2i}(v) + \sum_{j \in I^2(\bar{u},v)} \lambda_h^j \Upsilon^h_{2j}(v) \geq 0, \tag{101}$$

where

$$\tilde{\Lambda}(\bar{u}) := \left\{ (\lambda_g, \lambda_h) \in \mathbb{R}^{n_g} \times \mathbb{R}_+^{I^2(\bar{u},v)} \; ; \; \Upsilon_1(\cdot) + \sum_{i=1}^{n_g} \lambda_g^i \Upsilon^g_{1i}(\cdot) + \sum_{j \in I^2(\bar{u},v)} \lambda_h^j \Upsilon^h_{1j}(\cdot) = 0 \right\}.$$

On the other hand, since $v \in C_\infty(\bar{u})$, for every $\lambda = (\lambda_g, \lambda_h) \in \Lambda(\bar{u})$ we have that

$$\sum_{j \in I^1(\bar{u}) \setminus I^2(\bar{u},v)} \lambda_h^j \Upsilon^h_{1j}(v) = 0.$$

Since $\lambda_h \geq 0$ and $v \in C_\infty(\bar{u})$, each term in the above sum is nonpositive, and hence equal to zero since the sum is null. Thus, by definition of $I^2(\bar{u}, v)$, we have that $\lambda_h^j = 0$ for all $j \in I^1(\bar{u}) \setminus I^2(\bar{u}, v)$. Therefore, by (101)

$$\max_{\lambda \in \Lambda(\bar{u})} \Upsilon_2(v) + \sum_{i=1}^{n_g} \lambda_g^i \Upsilon^g_{2i}(v) + \sum_{j=1}^{n_h} \lambda_h^j \Upsilon^h_{2j}(v) \geq 0. \tag{102}$$

Inequality (100) for every $v \in C_\infty(\bar{u})$ then follows from lemma 3, along the lines of the proof of lemma 11. Using that $C_\infty(\bar{u})$ is dense in $C_2(\bar{u})$, estimate (77) and that $\Lambda(\bar{u})$ is compact, we obtain the result for all $v \in C_2(\bar{u})$. □

6 Appendix

6.1 Technical estimates in the expansion of solution of SDEs

Here we prove some technical estimates stated in section 3.

Proof of lemma 5 For notational convenience we will suppose that $m = n = d = 1$. We have

$$\begin{array}{rcl} \mathrm{d}\delta y(t) &=& \left[\tilde{f}_y(t)\delta y(t) + \tilde{f}_u(t)v(t)\right]\mathrm{d}t + \left[\tilde{\sigma}_y(t)\delta y(t) + \tilde{\sigma}_u(t)v(t)\right]\mathrm{d}W(t), \\ \delta y(0) &=& 0. \end{array} \tag{103}$$

where, for $\psi = f, \sigma$,

$$\begin{array}{rcl} \tilde{\psi}_y(t) &:=& \int_0^1 \psi_y\left(\bar{y}(t) + \theta\delta y(t), \bar{u}(t) + \theta v(t)\right)\mathrm{d}\theta, \\ \tilde{\psi}_u(t) &:=& \int_0^1 \psi_u\left(\bar{y}(t) + \theta\delta y(t), \bar{u}(t) + \theta v(t)\right)\mathrm{d}\theta. \end{array}$$

Using the second assumption in (4), estimates (20), (20) follow from corollary 2 applied to (103) and (19) respectively.

We next prove (21). We have that

$$\begin{array}{rcl} \mathrm{d}d_1(t) &=& \left[\tilde{f}_y(t)\delta y(t) - f_y(t)y_1(t) + \left(\tilde{f}_u(t) - f_u(t)\right)v(t)\right]\mathrm{d}t + \\ && \left[\tilde{\sigma}_y(t)\delta y(t) - \sigma_y(t)y_1(t) + \left(\tilde{\sigma}_u(t) - \sigma_u(t)\right)v(t)\right]\mathrm{d}W(t), \\ d_1(0) &=& 0. \end{array}$$

For $\psi = f, \sigma$, we have that $\left[\tilde{\psi}_y(t) - \psi_y(t)\right]y_1(t) = O\left(\left[\ |\delta y(t)| + |v(t)|\ \right]|y_1(t)|\right)$. Also,

$$\left[\tilde{\sigma}_u(t) - \sigma_u(t)\right]v(t) = \left\{ \begin{array}{ll} O\left(|\delta y(t)||v(t)|\right) & \text{if } \sigma_{uu} \equiv 0, \\ O\left(\left[\ |\delta y(t)| + |v(t)|\ \right]|v(t)|\right) & \text{otherwise.} \end{array} \right.$$

Therefore, the following equation holds for d_1:

$$\begin{array}{rcl} \mathrm{d}d_1(t) &=& \left[\tilde{f}_y(t)d_1(t) + O\left([|\delta y(t)| + |v(t)|][|y_1(t)| + |v(t)|]\right)\right]\mathrm{d}t + \\ && \left[\tilde{\sigma}_y(t)d_1(t) + O\left(D(\delta y, y_1, v)\right)\right]\mathrm{d}W(t), \end{array}$$

where

$$D(\delta y(t), y_1(t), v(t)) = \left\{ \begin{array}{ll} \left[|\delta y(t)| + |v(t)|\right]\left[|y_1(t)| + |v(t)|\right] - |v(t)|^2 & \text{if } \sigma_{uu} \equiv 0, \\ \left[|\delta y(t)| + |v(t)|\right]\left[|y_1(t)| + |v(t)|\right] & \text{otherwise.} \end{array} \right.$$

By (20) and the Cauchy Schwarz inequality

$$\begin{array}{rcl} \|\ |\delta y||y_1|\ \|_{\beta,2}^{\beta} &=& \mathbb{E}\left[\left(\int_0^T |\delta y(t)|^2|y_1(t)|^2\mathrm{d}t\right)^{\frac{\beta}{2}}\right] \\ &=& O\left[\mathbb{E}\left(\sup|\delta y(t)|^{\beta}|y_1(t)|^{\beta}\right)\right] \\ &=& O\left(\left[\mathbb{E}\left(\sup|\delta y(t)|^{2\beta}\right)\right]^{\frac{1}{2}}\left[\mathbb{E}\left(\sup|y_1(t)|^{2\beta}\right)\right]^{\frac{1}{2}}\right) \\ &=& O(\|v\|_{2\beta,2}^{2\beta}). \end{array} \tag{104}$$

By similar arguments,

$$\begin{array}{rcl} \| \, |y_1||v| \, \|_{\beta,2}^{\beta} = \mathbb{E}\left[\left(\int_0^T |y_1(t)|^2 |v(t)|^2 \mathrm{d}t \right)^{\frac{\beta}{2}} \right] & = & O(\|v\|_{2\beta,2}^{2\beta}), \\ \| \, |\delta y||v| \, \|_{\beta,2}^{\beta} = \mathbb{E}\left[\left(\int_0^T |\delta y(t)|^2 |v(t)|^2 \mathrm{d}t \right)^{\frac{\beta}{2}} \right] & = & O(\|v\|_{2\beta,2}^{2\beta}), \end{array}$$

and (21) follows by corollary 2, since $\|v^2\|_{\beta,1}^{\beta} = \|v\|_{2\beta,2}^{2\beta}$ and $\|v^2\|_{\beta,2}^{\beta} = \|v\|_{2\beta,4}^{2\beta}$.

Proof of lemma 10. As in the proof of lemma 5 we suppose that $m = n = d = 1$. We will use repeatedly that for every $\beta, p, q \in [1, \infty)$, we have

$$\| \, |v|^q \|_{\beta,p}^{\beta} = \|v\|_{q\beta,qp}^{q\beta} \quad \text{for all } v \in L_{\mathcal{F}}^{q\beta,qp}.$$

Proof of (30)*:* Recall that, by **(H1)**, for $\psi = f, \sigma$ we assume that ψ_{yy}, ψ_{yu} and ψ_{uu} are bounded. Using (20),

$$\|y_1^2\|_{\beta,2}^{\beta} = \mathbb{E}\left[\left(\int_0^T |y_1(t)|^4 \mathrm{d}t \right)^{\frac{\beta}{2}} \right] = O\left[\mathbb{E}\left(\sup |y_1(t)|^{2\beta} \right) \right] = O\left(\|v\|_{2\beta,2}^{2\beta} \right). \tag{105}$$

Analogously, the estimates associated with the term $y_1 v$ is of order $\|v\|_{2\beta,2}^{2\beta}$. Estimate (30) follows from corollary 2 since $\|v^2\|_{\beta,1}^{\beta} = \|v\|_{2\beta,2}^{2\beta}$ and $\|v^2\|_{\beta,2}^{\beta} = \|v\|_{2\beta,4}^{2\beta}$.

Proof of (31): Recall that $d_2 = \delta y - y_1 - \frac{1}{2} y_2$. We have, omitting time from the arguments,

$$\begin{array}{l} \mathrm{d}d_2(t) = \left[f_y d_2 + \frac{1}{2} f_{yy} [\delta y]^2 - \frac{1}{2} f_{yy} [y_1]^2 + f_{yu} \delta y v - f_{yu} y_1 v + r_t(f)(\delta y, v)^2 \right] \mathrm{d}t + \\ \qquad \left[\sigma_y(t) d_2 + \frac{1}{2} \sigma_{yy} [\delta y]^2 - \sigma_{yy} [y_1]^2 + \sigma_{yu} \delta y v - \sigma_{yu} y_1 v + r_t(\sigma)(\delta y, v)^2 \right] \mathrm{d}W(t). \end{array}$$

where for $\psi = f, \sigma$ the map $r_t(\psi)$ is defined by

$$r_t(\psi) := \int_0^1 (1 - \theta) \left[\psi_{yy}(\bar{y}(t) + \theta \delta y(t), \bar{u}(t) + \theta v(t)) - \psi_{yy}(\bar{y}(t), \bar{u}(t)) \right] \mathrm{d}\theta.$$

Recall that if Q is a quadratic form and a is the associated symmetric bilinear form, we have the identity $Q(y) - Q(x) = a(y + x, y - x)$. Thus, since $D\psi$ is Lipschitz, we obtain

$$\begin{array}{rcl} \mathrm{d}d_2(t) & = & \left[f_y d_2 + O\left(|d_1| \left\{ |\delta y| + |y_1| \right\} + |d_1||v| + \alpha_t(f) \right) \right] \mathrm{d}t + \\ & & \left[\sigma_y d_2 + O\left(|d_1| \left\{ |\delta y| + |y_1| \right\} + |d_1||v| + \alpha_t(\sigma) \right) \right] \mathrm{d}W(t) \end{array} \tag{106}$$

where, for $\psi = f, \sigma$,

$$\alpha_t(\psi) := \begin{cases} |\delta y(t)|^3 + |v(t)|^3 & \text{if } \psi_{uuu} \neq 0, \\ |\delta y(t)|^3 + |\delta y(t)| \, |v(t)|^2 & \text{if } \psi_{uuu} \equiv 0. \end{cases}$$

Now, let us estimate the terms in the $\mathrm{d}W(t)$ part of (106),

$$\begin{array}{rcl} \| \, |d_1||\delta y| \, \|_{\beta,2}^{\beta} & = & \mathbb{E}\left[\left(\int_0^T |d_1(t)|^2 |\delta y(t)|^2 \mathrm{d}t \right)^{\frac{\beta}{2}} \right] = O\left[\mathbb{E}\left(\sup |d_1(t)|^{\beta} |\delta y(t)|^{\beta} \right) \right] \\ & = & O(\|v\|_{2\beta,2}^{\beta} \|v\|_{4\beta,4}^{2\beta}), \end{array}$$

by (20) and (21). Analogously, estimates for the terms $d_1 y_1$ and $d_1 v$ are of the same order. Let us estimate the terms appearing in $\alpha_t(\sigma)$. Using (20),

$$\| \, |\delta y|^3 \|_{\beta,2}^{\beta} = \mathbb{E}\left[\left(\int_0^T |\delta y(t)|^6 \mathrm{d}t \right)^{\frac{\beta}{2}} \right] = O\left[\mathbb{E}\left(\sup |\delta y(t)|^{3\beta} \right) \right] = O(\|v\|_{3\beta,2}^{3\beta}). \tag{107}$$

By (20), we obtain

$$\begin{array}{rcl} \| \, |\delta y| |v|^2 \, \|_{\beta,2}^{\beta} & = & \mathbb{E}\left[\left(\int_0^T |\delta y(t)|^2 |v(t)|^4 \mathrm{d}t \right)^{\frac{\beta}{2}} \right] \\ & = & O\left(\mathbb{E}\left[\sup |\delta y(t)|^{\beta} \left(\int_0^T |v(t)|^4 \mathrm{d}t \right)^{\frac{\beta}{2}} \right] \right) = O(\|v\|_{2\beta,2}^{\beta} \|v\|_{4\beta,4}^{2\beta}). \end{array}$$

Also, we have that $\|v^3\|_{\beta,1}^{\beta} = \|v\|_{3\beta,3}^{3\beta}$ and $\|v^3\|_{\beta,2}^{\beta} = \|v\|_{3\beta,6}^{3\beta}$. By the Cauchy Schwarz inequality,

$$\|v\|_{3\beta,3}^{3\beta} = \mathbb{E}\left[\left(\int_0^T |v(t)|^3 \mathrm{d}t \right)^{\beta} \right] \leq \mathbb{E}\left[\left(\int_0^T |v(t)|^2 \mathrm{d}t \right)^{\frac{\beta}{2}} \left(\int_0^T |v(t)|^4 \mathrm{d}t \right)^{\frac{\beta}{2}} \right].$$

Using the Cauchy Schwarz inequality again, we get $\|v\|_{3\beta,3}^{3\beta} = O(\|v\|_{2\beta,2}^{\beta} \|v\|_{4\beta,4}^{2\beta})$. Therefore, estimate (31) follows from corollary 2.

Proof of lemma 12 As in the proof or proposition 8 we denote $\delta J := J(\bar{u} + v) - J(\bar{u})$. By definition,

$$\delta J = \mathbb{E}\left(\int_0^T \left[\ell(y_{\bar{u}+v}, \bar{u} + v) - \ell(\bar{y}, \bar{u}) \right] \mathrm{d}t + \phi(y_{\bar{u}+v}(T)) - \phi(\bar{y}(T)) \right) = I_1 + I_2,$$

where, omitting the time argument in the integral,

$$\begin{array}{rcl} I_1 & := & \mathbb{E}\left(\int_0^T \left[\ell_y \delta y + \ell_u v + \frac{1}{2} \ell_{(y,u)^2} (\delta y, v)^2 + r_\ell (\delta y, v)^2 \right] \mathrm{d}t \right), \\ I_2 & := & \mathbb{E}\left[\phi_y(\bar{y}(T)) \delta y(T) + \frac{1}{2} \phi_{yy}(\bar{y}(T)) (\delta y(T))^2 + r_\phi(\bar{y}(T)) (\delta y(T))^2 \right]. \end{array} \tag{108}$$

Recalling that $\delta y = y_1 + d_1 = y_1 + \frac{1}{2} y_2 + d_2$, assumption (8) in **(H2)** yields

$$\begin{array}{rcl} I_1 & = & \mathbb{E}\left(\int_0^T \ell_y(t)(y_1 + \frac{1}{2} y_2) + \ell_u(t) v + \frac{1}{2} D^2 \ell(t)(y_1, v)^2 \mathrm{d}t \right) + \mathbb{E}\left(\int_0^T \ell_y d_2 \mathrm{d}t \right) \\ & & + O(z_1(v)), \end{array}$$

where, omitting time from function arguments,

$$z_1(v) := \mathbb{E}\left(\sup \left[|d_1|^2 + |d_1(t)||y_1| + |\delta y|^3 \right] \right) + \|v\|_1 \mathbb{E}\left(\sup |d_1| \right) + \|v\|_3^3.$$

On the other hand,

$$\begin{array}{rcl} I_2 & = & \mathbb{E}\left[\phi_y(\bar{y}(T)) \left(y_1(T) + \frac{1}{2} y_2(T) \right) + \frac{1}{2} \phi_{yy}(\bar{y}(T)) \left(y_1(T) \right)^2 \right] \\ & & + \mathbb{E}\left[\phi_y(\bar{y}(T)) d_2(T) \right] + O(z_2(v)), \end{array}$$

where

$$z_2(v) := \mathbb{E}\left(|\delta y(T)|^3 + |y_1(T)||d_1(T)| + |d_1(T)|^2 \right).$$

Denoting $z(v) := z_1(v) + z_2(v)$ we get that

$$\begin{array}{rcl}\delta J & = & \mathbb{E}\left(\int_0^T \left[\ell_y(t)(y_1(t) + \frac{1}{2}y_2(t)) + \ell_u(t)v(t) + \frac{1}{2}\ell_{(y,u)^2}(t)(y_1(t), v(t))^2 \right] \mathrm{d}t \right) \\ & & + \mathbb{E}\left[\phi_y(\bar{y}(T))(y_1(T) + \frac{1}{2}y_2(T)) + \frac{1}{2}\phi_{yy}(\bar{y}(T))\,(y_1(T))^2 \right] + \zeta(v) + z(v),\end{array}$$

Therefore, using (25) and (33), we get (34). Now, we proceed to estimate $z(v)$. By (21) we have that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |d_1(t)|^2 \right) = O(\|v\|_4^4) = O(\|v\|_\infty^2 \|v\|_2^2).$$

Estimates (20), (21) and the Cauchy Schwarz inequality yield

$$\mathbb{E}\left(\sup_{t\in[0,T]} |d_1(t)||y_1(t)| \right) = O\left(\|v\|_4^2 \|v\|_2 \right) = O\left(\|v\|_\infty \|v\|_2^2 \right).$$

Analogously, using (21), we have

$$\mathbb{E}\left(\|v\|_1 \sup_{t\in[0,T]} |d_1(t)| \right) = O\left(\|v\|_4^2 \|v\|_{2,1} \right) = O\left(\|v\|_\infty \|v\|_2^2 \right).$$

Estimate (20) yields $\mathbb{E}\left(\sup_{t\in[0,T]} |\delta y(t)|^3 \right) = O(\|v\|_{3,2}^3)$. But

$$\|v\|_{3,2}^3 = \mathbb{E}\left(\left[\int_0^T |v(t)|^2 \mathrm{d}t \right]^{\frac{3}{2}} \right) = O\left(\|v\|_\infty \|v\|_2^2 \right),$$

and$\|v\|_3^3 = O\left(\|v\|_\infty \|v\|_2^2 \right)$. Thus, $z(v) = O\left(\|v\|_\infty \|v\|_2^2 \right)$.

6.2 Adapted projections

This subsection discusses projections in $L_{\mathcal{F}}^2$ in the case of local constraints. In order to give the expression of the tangent cone to $\mathcal{U}$, when $\mathcal{U}$ is defined by (51), we need a characterization of measurable multifunctions with closed values. We call *Castaing representation* of a $(\mathcal{B}[0,T] \times \mathcal{F}_T)$-measurable multifunction $U : [0,T] \times \Omega \to \mathcal{P}(\mathbb{R}^m)$ a countable family of $(\mathcal{B}[0,T] \times \mathcal{F}_T)/\mathcal{B}(\mathbb{R}^m)$-measurable functions $w_k : [0,T] \times \Omega \to \mathbb{R}^m$ such that $U(t,\omega) = \mathrm{clo}\{w_k(t,\omega),\ k \in \mathbb{N}\}$. We say that the Castaing representation is adapted if each process $(w_k(t))_{t\in[0,T]}$ is adapted. By a result due to C. Castaing (see e.g. [28, Thm 1B, p. 161]), any multifunction with closed values that is measurable in $L_{\mathcal{F}_T}^2$ has a Castaing representation in the same space. We next extend the result to the adapted case.

Proposition 33. *Let $U : [0,T] \times \Omega \to \mathcal{P}(\mathbb{R}^m)$ be an $(\mathcal{B}[0,T] \times \mathcal{F}_T)$-measurable, $\mathbb{F}$-adapted closed-convex-valued multifunction. Then:*

(i) *For any $a \in \mathbb{R}^m$ the map $(t,\omega) \in [0,T] \times \Omega \to P_a(t,\omega) := P_{U(t,\omega)}(a) \in \mathbb{R}^m$ is $(\mathcal{B}[0,T] \times \mathcal{F}_T)/\mathcal{B}(\mathbb{R}^m)$-measurable and $\mathbb{F}$-adapted.*

(ii) *For any countable dense subset $\{z_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}^m$ we have that $\{P_{U(t,\omega)}(z_k),\ k \in \mathbb{N}\}$ is an $\mathbb{F}$-adapted Castaing representation of U.*

Proof. First note that (ii) follows directly from (i). In order to prove (i) we essentially reproduce the proof in [28] to obtain that $P_a(t,\omega)$ is $(\mathcal{B}[0,T]\times\mathcal{F}_T)/\mathcal{B}(\mathbb{R}^m)$-measurable and, using that U is $\mathbb{F}$-adapted, we prove that $P_a(t,\omega)$ is $\mathbb{F}$-adapted. Let us fix $a\in\mathbb{R}^m$ and consider the sequence of multifunctions

$$U_k(t,\omega) := \{v\in\mathbb{R}^m;\ \mathrm{dist}(v,U(t,\omega)) < k^{-1};\ |v-a| < \mathrm{dist}(a,U(t,\omega)) + k^{-1}\}. \tag{109}$$

Let C be a closed subset of $\mathbb{R}^m$. Then $P_a(t,\omega)\in C$ iff $C\cap U_k(t,\omega)\neq\emptyset$ for all k, thus

$$P_a^{-1}(C) = \cap_k U_k^{-1}(C). \tag{110}$$

Next, let D be a countable dense subset of C, which always exists. We claim that

$$U_k^{-1}(C) = U_k^{-1}(D) = \cup_{d\in D} U_k^{-1}(d). \tag{111}$$

The second equality is obvious and since D is a dense subset of C, in order to establish the first equality it suffices to check that if $c\in C$ and $(t_0,\omega_0)\in U_k^{-1}(c)$, then for c' close enough to c we have that $(t_0,\omega_0)\in U_k^{-1}(c')$. But this follows directly from the definition of $U_k(t,\omega)$ in (109). Our claim follows.

On the other hand, for any $v\in\mathbb{R}^m$ and $\alpha\geq 0$ we have

$$\{(t,\omega)\in[0,T]\times\Omega\ ;\ \mathrm{dist}(v,U(t,\omega))\leq\alpha\} = U^{-1}(v+\alpha\bar{B}), \tag{112}$$

where $\bar{B}$ is the unit ball in $\mathbb{R}^m$. Thus, since U is $\mathcal{B}[0,T]\times\mathcal{F}_T$-measurable, so is the process $\mathrm{dist}(v,U(t))$. Similarly, for a.a. $t\in[0,T]$, we have that

$$\{\omega\in\Omega\ ;\ \mathrm{dist}(v,U(t,\omega))\leq\alpha\} = U^{-1}(v+\alpha\bar{B}). \tag{113}$$

Since U is $\mathbb{F}$-adapted, it follows that so is $\mathrm{dist}(v,U(t))$. Therefore, from the definition (109), for any $(t,d)\in[0,T]\times\in\mathbb{R}^m$ we have that $U_k^{-1}(d)\in\mathcal{B}[0,T]\times\mathcal{F}_T$ and $U_k^{-1}(t,\cdot)(d)\in\mathcal{F}_t$. Using (110)-(111) we finally obtain that P_a has the desired properties. □

Lemma 34. *Let $\mathcal{U}$ be defined by* (51) *with U being a $(\mathcal{B}[0,T]\times\mathcal{F}_T)/\mathcal{B}(\mathbb{R}^m)$-measurable, $\mathbb{F}$-adapted closed-convex-valued multifunction. For $u\in L^2_{\mathcal{F}}$ we have that $w=P_{\mathcal{U}}(u)$ iff $w(t,\omega)=P_{U(t,\omega)}(u(t,\omega))$ for a.a. $(t,\omega)\in[0,T]\times\Omega$.*

Proof. By the definition of a projection, w is characterized as the solution of the minimization problem

$$\underset{w\in L^2_{\mathcal{F}}}{\mathrm{Min}}\ \mathbb{E}\left[\int_0^T |w(t,\omega)-u(t,\omega)|^2\mathrm{d}t\right];\quad w\in\mathcal{U}. \tag{114}$$

Let u_k $(k\geq 1)$ be a $\mathbb{F}$-adapted Castaing representation of U and let the sequence $u'_k\in L^2_{\mathcal{F}}$ be defined as follows: u'_0 is an arbitrary element of $\mathcal{U}$, and for $k\geq 0$ set

$$u'_{k+1}(t,\omega) = \left\{\begin{array}{ll} u_{k+1}(t,\omega) & \text{if } |u_{k+1}(t,\omega)-u(t,\omega)| < |u'_k(t,\omega)-u(t,\omega)|, \\ u'_k(t,\omega) & \text{otherwise.}\end{array}\right. \tag{115}$$

Then u'_k is a sequence of measurable and adapted functions, and since u_k is a Castaing representation, $|u'_k(t,\omega)-u(t,\omega)|\to\inf_{a\in U(t,\omega)}|u(t,\omega)-a|$ fort a.a. (t,ω). Therefore, since $U(t,\omega)$ is closed and convex, we obtain that

$u_k'(t,\omega) \to P_{U(t,\omega)}u(t,\omega)$ for a.a. (t,ω). Using the dominated convergence theorem, we deduce that u_k' has a limit in $L^2_{\mathcal{F}}$ equal to $P_{U(t,\omega)}(u(t,\omega))$ for a.a. (t,ω) . It follows that the value of problem (114) is less or equal than $\mathbb{E}\left[\int_0^T |P_{U(t,\omega)}u(t,\omega) - u(t,\omega)|^2 \mathrm{d}t\right]$. Since obviously it cannot be less, and the projection problem has a unique solution, the conclusion follows. $\square$

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