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Abstract: We consider a linear quadratic stochastic optimal control problem whith non-negativity control constraints. The latter are penalized with the classical logarithmic barrier. Using a duality argument and the stochastic minimum principle, we provide an error estimate for the solution of the penalized problem which is the natural extension of the well known estimate in the deterministic framework.

Key-words: Stochastic control, linear quadratic problems, non-negativity control constraints, logarithmic barrier, stochastic minimum principle.

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Estimations d'erreur de la méthode de barrière logarithmique pour les problèmes de contrôle optimal stochastique linéaire quadratique

Résumé : Nous considérons un problèmes de contrôle optimal stochastique linéaire quadratique avec contrainte de positivité de la commande. Cette dernière est pénalisée avec la barrière logarithmique classique. Utilisant un argument de dualité et le principe du minimum stochastique, nous obtenons des estimations d'erreur pour la solution du problème pénalisé qui apparaît comme une extension naturelle de celle, bien connue, du cas déterministe.

Mots-clés : Commande stochastique, problèmes linéaire quadratique, contrainte de non-négativité, barrière logarithmique, principe du minimum stochastique

1 Introduction

The study of stochastic linear quadratic (LQ) optimal control problems is an area of active research. In fact, many problems arising in engineering design and mathematical finance can be modeled as stochastil LQ problems. Let us cite, for example, the portfolio selection problem ([22, 16]) and the contingent claim problem ([15]). The stochastic LQ problem, in a finite time horizon $[0,T]$ and without constraints, can be stated as follows:

$$\begin{array}{l} \text{Minimize } \mathbb{E}\left(\int_0^T \left[u(t)^\top R(t)u(t) + y(t)^\top C(t)y(t)\right] \mathrm{d}t + y(T)^\top M y(T)\right) \\ \text{s.t. } \left\{ \begin{array}{rcl} \mathrm{d}y(t) & = & [A_0(t)y(t) + B_0(t)u(t)]\,\mathrm{d}t + [A_1(t)y(t) + B_1(t)u(t)]\,\mathrm{d}W(t), \\ y(0) & = & x_0 \end{array} \right. \end{array}$$

Assuming that $R(t)$ is positive definite, the problem above was extensively investigated in the 1960s and 1970s (see e.g. [21, 17, 6, 7, 13], the surveys in [2] and references therein). In the mid-1990s, using an approach based on a stochastic Riccati equation, Chen-Li-Zhou [11] treated the stochastic LQ problem even when $R(t)$ can be indefinite. See also [12], where the relations between the stochastic LQ problem, the stochastic Pontryagin minimum principle (SPMP) and linear forward-backward stochastic differential equations, are studied.

Even if the unconstrained case is well studied, when control constraints are present the only reference that we know is [14]. In fact, the authors consider a stochastic LQ problem where the control is constrained in a cone. They obtain explicit solutions for the optimal control and the optimal cost via solutions of a system of extended stochastic Riccati equations.

In this work we study a convex stochastic LQ problem with non-negativity control constraints. We consider a family of *logarithmic penalized* problems, parameterized by $\varepsilon > 0$. This means that the cost function is modified by adding a logarithmic barrier function multiplied by ε, which implies that the solution of the new problem is strictly positive. Our aim is to study the convergence, as $\varepsilon \downarrow 0$, of the solution of the penalized problem to the solution of the *initial* one. In fact, we will obtain error estimates for the cost, control, state and adjoint state in the appropriate spaces. This result extend the classical error estimates obtained by Weiser [20] in the deterministic framework.

The article is organized as follows: In section 2 we fix the standard notation and the initial and penalized problems are stated. Using the stochastic Pontryaguin minimum principle (SPMP) (see [3, 4, 5, 6, 8, 19, 10]), first order necessary and sufficient conditions are derived. Our main result is provided in section 3, in which we derive the error estimates. The proof use a simple duality argument and an application of the SPMP.

2 Problem Statement and Optimality Conditions

Let us first fix some notations. The space $\mathbb{R}^m$ ($m \in \mathbb{N}^*$) is endowed with its standard Euclidean norm denoted by $|\cdot|$. The ith coordinate of a vector x is denoted by x^i. We set $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x^i \geq 0\}$, and $\mathbb{R}^m_{++} := \{x \in \mathbb{R}^m : x^i > 0\}$. Let $T > 0$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, on which a d-dimensional ($d \in \mathbb{N}^*$) Brownian motion $W(\cdot)$ is defined with $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ being its natural filtration, augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$. For $\ell \in \mathbb{N}^*$ let us define

$$
\begin{array}{rcl}
L^2_{\mathcal{F}}\left([0,T];\mathbb{R}^\ell\right) & := & \left\{v:[0,T]\times\Omega\to\mathbb{R}^\ell \ / \ v \text{ is } \mathbb{F}-\text{adapted and } ||v||_2<\infty\right\},\\
L^{2,\infty}_{\mathcal{F}}\left([0,T];\mathbb{R}^\ell\right) & := & \left\{v:[0,T]\times\Omega\to\mathbb{R}^\ell \ / \ v \text{ is } \mathbb{F}-\text{adapted and } ||v||_{2,\infty}<\infty\right\},
\end{array}
$$

where we assume that all the mappings are $\mathcal{B}([0,T])\times\mathcal{F}_T$-measurable and ´

$$
||v||_2 := \left[\mathbb{E}\left(\int_0^T |v(t)|^2 \mathrm{d}t\right)\right]^{\frac{1}{2}}, \quad ||v||_{2,\infty} := \left[\mathbb{E}\left(\sup_{t\in[0,T]} |v(t)|^2\right)\right]^{\frac{1}{2}}.
$$

It is well known that $\left(L^2_{\mathcal{F}}\left([0,T];\mathbb{R}^\ell\right), \langle\cdot,\cdot\rangle_2\right)$ is a Hilbert space, where

$$
\langle u,v\rangle_2 := \sum_{i=1}^{\ell}\mathbb{E}\left(\int_0^T u^i(t)v^i(t)\mathrm{d}t\right). \tag{1}
$$

Let $x_0:\Omega\to\mathbb{R}^n$ be $\mathcal{F}_0$ measurable and such that $\mathbb{E}(|x_0|^2)<\infty$. Consider the following affine stochastic differential equation (SDE)

$$
\begin{array}{rcl}
\mathrm{d}y(t) & = & f(t,\omega,y(t),u(t))\mathrm{d}t + \sum_{i=1}^d \sigma^i(t,\omega,y(t),u(t))\mathrm{d}W(t),\\
y(0) & = & x_0\in\mathbb{R}.
\end{array} \tag{2}
$$

In the notation above $y(t)\in\mathbb{R}^n$ denotes the state function, which is controlled by $u(t)\in\mathbb{R}^m$, and

$$
f:[0,T]\times\Omega\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n, \quad \sigma^i:[0,T]\times\Omega\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^{n\times d}
$$

are defined by

$$
\begin{array}{rcl}
f(t,\omega,y,u) & := & A_0(t,\omega)y+B_0(t,\omega)u+D_0(t,\omega),\\
\sigma^i(t,\omega,y,u) & := & A_i(t,\omega)y+B_i(t,\omega)u+D_i(t,\omega),
\end{array}
$$

where, for $i=0,...,d$, $A_i:[0,T]\times\Omega\to\mathbb{R}^{n\times n}$, $B_i:[0,T]\times\Omega\to\mathbb{R}^{n\times m}$ and $D_i:[0,T]\times\Omega\to\mathbb{R}^n$. We assume that:

(H1) The random matrices A_i, B_i, D_i are progressively measurable with respect to $\mathbb{F}$ and bounded uniformly in $(t,\omega)\in[0,T]$ by a constant $\bar{D}>0$.

We take as state and control space, respectively,

$$
\mathcal{Y}:=L^{2,\infty}_{\mathcal{F}}([0,T];\mathbb{R}^m), \ \ \mathcal{U}:=L^2_{\mathcal{F}}([0,T];\mathbb{R}^m). \tag{3}
$$

It is well known that for every $u\in\mathcal{U}$, equation (2) has a unique solution $y_u\in\mathcal{Y}$ and the following estimate hold:

$$
||y||^2_{2,\infty}\le L_1\left(\mathbb{E}(y_0^2)+||u||_2^2+\sum_{i=0}^d ||D_i||_2^2\right), \tag{4}
$$

for some positive constant L_1. Denote respectively by $\mathcal{S}^m_+$ and $\mathcal{S}^m_{++}$ the sets of symmetric positive semidefinite and symmetric positive definite matrices of order m. Now, let us consider the set

$$
\mathcal{U}^+ := \{u\in\mathcal{U} \ / \ u(t,\omega)\ge 0 \ \text{ for a.a. } (t,\omega)\in[0,T]\times\Omega\}, \tag{5}
$$

and the random matrices $R:[0,T]\times\Omega\to\mathcal{S}^m_{++}$, $C:[0,T]\times\Omega\to\mathcal{S}^n_+$, $M:\Omega\to\mathcal{S}^n_+$. We assume:

(H2) The matrices R,C,M are bounded uniformly in $(t,\omega)\in[0,T]$ by a constant $\bar{C}$. In addition, we assume that R is uniformly positive definite, i.e. there exists $\alpha>0$ such that for a.a. $(t,\omega)\in[0,T]\times\Omega$

$$
v^\top R(t,\omega)v\ge\alpha|v|^2 \ \text{ for all } v\in\mathbb{R}^m. \tag{6}
$$

2.1 The initial problem

Let $\bar{y} \in \mathcal{Y}$ be a reference state function and define $g_0 : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as

$$g_0(t,\omega,y,u) := \tfrac{1}{2} u^\top R(t,\omega) u + [y - \bar{y}(t,\omega)]^\top C(t,\omega)[y - \bar{y}(t,\omega)]. \qquad (7)$$

The cost function $J_0 : \mathcal{U} \to \mathbb{R}$ is defined as

$$J_0(u) := \mathbb{E}\left(\tfrac{1}{2} \int_0^T g_0(t, y(t), u(t)) \mathrm{d}t + \tfrac{1}{2} y_u(T)^\top M y_u(T) \right). \qquad (8)$$

We consider the following stochastic optimal control problem:

$$\text{Min } J_0(u) \quad \text{subject to } u \in \mathcal{U}^+. \qquad (\mathcal{CP})_0$$

Assumptions **(H1)**, **(H2)** imply that J_0 is a strongly convex continuous function. Since $\mathcal{U}^+$ is closed and convex, we have that $(\mathcal{CP})_0$ has a unique solution u_0. We denote $y_0 := y_{u_0}$ its associated state.

As usual in optimal control theory, optimality conditions can be expressed in terms of a Hamiltonian and an adjoint state. In fact, let

$$H_0 : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m \to \mathbb{R}$$

be the *Hamiltonian* of problem $(\mathcal{CP})_0$, defined as

$$H_0(t,\omega,y,p,q,u) := g_0(t,\omega,y,u) + p \cdot f(t,\omega,y,u) + \sum_{i=1}^d q^i \cdot \sigma^i(t,\omega,y,u),$$

where q^i denotes the ith column of q. For $u \in \mathcal{U}$ let $(p_u, q_u) \in L^{2,\infty}_{\mathcal{F}}([0,T] \times \mathbb{R}^n) \times L^2_{\mathcal{F}}([0,T] \times \mathbb{R}^{n \times d})$, called the *adjoint state* associated to u, be the unique solution of the following linear backward stochastic differential equation (BSDE)(see [5]) :

$$\begin{array}{rcl} \mathrm{d}p(t) &=& -D_y H_0(t, y_u(t), p(t), q(t), u(t)) \mathrm{d}t + q(t) \mathrm{d}W(t), \\ p(T) &=& M y_u(T). \end{array} \qquad (9)$$

It is well known (see e.g. [18, Proposition 3.1]) that there exists $L_2 > 0$, such that

$$||p_u||^2_{2,\infty} + ||q_u||^2_2 \leq L_2 \left(\mathbb{E}(y_u(T)^2) + ||u||_2^2 \right). \qquad (10)$$

Let us set $p_0 := p_{u_0}$ and $q_0 := q_{u_0}$. Since $g_0(t,\omega,y,\cdot)$ is strictly convex, the stochastic Pontryagin minimum principle (SPMP) for linear convex optimal control with random coefficients [10, Theorem 3.2], yields that u_0 is a solution of $(\mathcal{CP})_0$ if and only if for a.a. $(t,\omega) \in [0,T] \times \Omega$,

$$u_0(t,\omega) = \operatorname{argmin}_{w \in \mathbb{R}^m_+} H_0(t,\omega, y_0(t,\omega), p_0(t,\omega), q_0(t,\omega), w). \qquad (11)$$

A straightforward computation (see [1, Section 2.1]) yields that

$$u_0(t,\omega) = \pi_0\left(R(t,\omega), z_0(t,\omega)\right) \quad \text{for a.a. } (t,\omega) \in [0,T] \times \Omega, \qquad (12)$$

where

$$z_0(t,\omega) := -R(t,\omega)^{-1} \left[B_0(t,\omega)^\top p_0(t,\omega) + \sum_{i=1}^d B_i^\top(t,\omega)^\top q_0^i(t,\omega) \right]$$

and for $(R,z) \in \mathcal{S}^m_{++} \times \mathbb{R}^m$ the map $\pi_0(R,z)$ is defined as the unique solution of

$$\text{Min } \tfrac{1}{2}(x-z)^\top R (x-z), \quad \text{s.t.} \quad x \in \mathbb{R}^m_+.$$

2.2 The penalized problem

For $\varepsilon > 0$ define the function $J_\varepsilon : \mathcal{U}^+ \to \mathbb{R} \cup \{+\infty\}$ by

$$J_\varepsilon(u) := \mathbb{E}\left(\frac{1}{2} \int_0^T \left[g_0(t, y_u(t), u(t)) + \varepsilon \hat{L}(u(t)) \right] \mathrm{d}t + y_u(T)^\top \frac{1}{2} M y_u(T) \right), \tag{13}$$

where $\hat{L} : \mathbb{R}^m_+ \to \mathbb{R} \cup \{+\infty\}$ is defined as $\hat{L}(u) := -\sum_{i=1}^m \log u^i$. Let us consider the *penalized* problem

$$\text{Min } J_\varepsilon(u) \quad \text{subject to } u \in \mathcal{U}^+. \tag{$(\mathcal{CP})_\varepsilon$}$$

Using the arguments of [1, Lemma 1], we have that

$$u \in \mathcal{U}^+ \to \mathbb{E}\left(\int_0^T \hat{L}(u(t)) \mathrm{d}t \right) \in \mathbb{R} \cup \{+\infty\}$$

is convex lower-semicontinuous (l.s.c), hence J_ε is a strongly convex l.s.c. function. Therefore, $(\mathcal{CP})_\varepsilon$ has a unique solution u_ε with associated state $y_\varepsilon := y_{u_\varepsilon}$. The Hamiltonian for $(\mathcal{CP})_\varepsilon$

$$H_\varepsilon : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^m_+ \to \mathbb{R} \cup \{+\infty\}$$

is defined as

$$H_\varepsilon(t, \omega, y, p, q, u) := H_0(t, \omega, y, p, q, u) + \varepsilon \hat{L}(u).$$

We set $(p_\varepsilon, q_\varepsilon) := (p_{u_\varepsilon}, q_{u_\varepsilon})$ for the unique solution of the following BSDE:

$$\begin{array}{rcl} \mathrm{d}p(t) & = & -D_y H_\varepsilon(t, y_\varepsilon(t), p(t), q(t), u_\varepsilon(t))\mathrm{d}t + q(t)\mathrm{d}W(t), \\ p(T) & = & M y_\varepsilon(T). \end{array} \tag{14}$$

As for the initial problem, the SPMP implies that u_ε is the solution of $(\mathcal{CP})_\varepsilon$ if and only if for a.a. $(t, \omega) \in [0,T] \times \Omega$

$$u_\varepsilon(t,\omega) = \mathrm{argmin}_{w \in \mathbb{R}^m_+} H_\varepsilon(t, \omega, y_\varepsilon(t,\omega), p_\varepsilon(t,\omega), q_\varepsilon(t,\omega), w), \tag{15}$$

Since $H_\varepsilon(t, \omega, \cdot)$ is convex and differentiable in u, condition (15) is satisfied if and only if for a.a. $(t, \omega) \in [0,T] \times \Omega$,

$$D_u H_0(t, \omega, y_\varepsilon(t,\omega), p_\varepsilon(t,\omega), q_\varepsilon(t,\omega), u_\varepsilon(t,\omega)) - \varepsilon \frac{1}{u_\varepsilon(t,\omega)} = 0, \tag{16}$$

where $1/u_\varepsilon(t,\omega) \in \mathbb{R}^m$ denotes the vector whose ith component is $1/u^i_\varepsilon(t,\omega)$. Equation (16) implies that (see [1, Section 2.2])

$$u_\varepsilon(t,\omega) = \pi_\varepsilon\left(R(t,\omega), z_\varepsilon(t,\omega)\right) \quad \text{for a.a. } (t,\omega) \in [0,T] \times \Omega, \tag{17}$$

where

$$z_\varepsilon(t,\omega) := -R(t,\omega)^{-1} \left[B_0(t,\omega)^\top p_\varepsilon(t) + \sum_{i=1}^d B_i^\top(t,\omega)^\top q^i_\varepsilon(t) \right]$$

and for $(R, z) \in \mathcal{S}^m_{++} \times \mathbb{R}^m$ the map $\pi_\varepsilon(R, z)$ is defined as the unique solution of

$$\text{Min } \tfrac{1}{2}(x - z)^\top R(x - z) + \varepsilon \hat{L}(x), \quad \text{s.t.} \quad x \in \mathbb{R}^m_+.$$

3 Main Result

In this section we provide error estimates for the cost, control, state and adjoint state of the penalized problem. We denote by $1/u_\varepsilon : [0,T] \times \Omega \to \mathbb{R}^m$ the mapping $(1/u_\varepsilon(t,\omega))^i := 1/u_\varepsilon^i(t,\omega)$.

Lemma 1. *For every $\varepsilon > 0$ we have that $1/u_\varepsilon \in \mathcal{U}^+$.*

Proof. The proof is based on (15). For notational convenience we assume that $n = m = d = 1$. The proof for the general case can be easily adapted. First, note that integrability problem comes when $u_\varepsilon(t,\omega)$ is small. Thus, fix $K_0 > 0$ and set

$$\Omega_{K_0} := \{(t,\omega) \in [0,T] \times \Omega \ / \ u_\varepsilon(t,\omega) \le K_0\}.$$

Now, let $\eta \in (0, K_0)$ and set

$$\hat{H}_\varepsilon(t,\omega,w) := H_\varepsilon(t,\omega,y_\varepsilon(t,\omega),p_\varepsilon(t,\omega),q_\varepsilon(t,\omega),w).$$

If $u_\varepsilon(t,\omega) \le \eta/2$ we have for a.a. $(t,\omega) \in \Omega_{K_0}$, omitting the (t,ω) argument,

$$\begin{array}{rcl} \hat{H}_\varepsilon(\eta) - \hat{H}_\varepsilon(u_\varepsilon) & = & \frac{1}{2}R(\eta + u_\varepsilon)(\eta - u_\varepsilon) + [B_0 p_\varepsilon + B_1 q_\varepsilon](\eta - u_\varepsilon) \\ & & +\varepsilon\left[\log(u_\varepsilon) - \log(\eta)\right] \end{array}$$

On the other hand, using that $\log(\cdot)$ is concave,

$$\log(u_\varepsilon) - \log(\eta) \le \frac{1}{\eta}(u_\varepsilon - \eta) \le \frac{1}{\eta}\frac{-\eta}{2} = -\tfrac{1}{2}.$$

Therefore, by optimality of u_ε,

$$0 \le \hat{H}_\varepsilon(\eta) - \hat{H}_\varepsilon(u_\varepsilon) \le \bar{C}K_0\eta + \bar{D}(|p_\varepsilon| + |q_\varepsilon|)\eta - \frac{\varepsilon}{2} \le \eta K_1(1 + |p_\varepsilon| + |q_\varepsilon|) - \tfrac{1}{2}\varepsilon,$$

where $K_1 := \max\left\{\bar{C}K_0, \bar{D}\right\}$. Thus, we conclude that

$$u_\varepsilon \le \tfrac{1}{2}\eta \quad \Rightarrow \quad \eta \ge \frac{\varepsilon}{2K_1\left(1 + |p_\varepsilon| + |q_\varepsilon|\right)}.$$

Henceforth, for a.a. $(t,\omega) \in \Omega_{K_0}$,

$$u_\varepsilon \ge \frac{\varepsilon}{4K_1\left(1 + |p_\varepsilon| + |q_\varepsilon|\right)} \quad \text{and thus} \quad \frac{1}{u_\varepsilon} \le \frac{4K_1\left(1 + |p_\varepsilon| + |q_\varepsilon|\right)}{\varepsilon}. \tag{18}$$

The result follows from (10) using that $u_\varepsilon \in \mathcal{U}$ and that $y_\varepsilon \in \mathcal{Y}$ is almost surely continuous. $\square$

Remark. Estimate (18) generalizes [9, Theorem 1] obtained in the deterministic framework. In the deterministic case we have that u_ε is uniformly positive, whereas in our setting we can prove only (18).

Consider the Lagrangian $\mathcal{L} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$, associated to problem $(\mathcal{CP})_0$, defined by

$$\mathcal{L}(u,\lambda) := J_0(u) - \langle \lambda, u \rangle_2, \tag{19}$$

where we recall that $\langle \cdot, \cdot \rangle_2$ is defined in (1). Define the *dual function* $d : \mathcal{U}^+ \to \mathbb{R}$ by $d(\lambda) := \inf_{u \in \mathcal{U}} \mathcal{L}(u,\lambda)$. We have:

Lemma 2. *For every $\varepsilon > 0$,*

$$d\left(\varepsilon \frac{1}{u^{\varepsilon}}\right) = J_0(u_\varepsilon) - \varepsilon m T.$$

Proof. Consider the following auxiliary problem

$$\text{Min } J_0(u) - \varepsilon \langle 1/u_\varepsilon, u \rangle_2 \;\; \text{subject to } u \in \mathcal{U}. \qquad (\mathcal{CP})_{aux}$$

Lemma 1 implies that the above problem is well-defined. Since the cost function is strongly convex and continuous, problem $(\mathcal{CP})_{aux}$ admits a unique solution u_{aux}, with associated state $y_{aux} := y_{u_{aux}}$. The Hamiltonian H_{aux} of problem $(\mathcal{CP})_{aux}$ is defined as

$$H_{aux}(t, \omega, y, p, q, u) = H_0(t, \omega, y, p, q, u) - \varepsilon \sum_{i=1}^{m} \frac{1}{u_\varepsilon^i(t, \omega)} u^i.$$

We let (p_{aux}, q_{aux}) be the unique solution of the following BSDE:

$$\begin{array}{rcl} \mathrm{d}p(t) & = & -D_y H_{aux}(t, y_{aux}(t), p(t), q(t), u_{aux}(t))\mathrm{d}t + q(t)\mathrm{d}W(t), \\ p(T) & = & M y_{u_{aux}}(T). \end{array} \qquad (20)$$

Define $\hat{H}_{aux} : [0, T] \times \Omega \times \mathbb{R}^m \to \mathbb{R}$ as

$$\hat{H}_{aux}(t, \omega, u) := H_{aux}(t, \omega, y_{u_{aux}}(t, \omega), p_{u_{aux}}(t, \omega), q_{u_{aux}}(t, \omega), u).$$

The SPMP yields that u_{aux} is a solution of $(\mathcal{CP})_{aux}$ if and only if

$$u_{aux}(t, \omega) = \text{argmin}_{w \in \mathbb{R}^m} \hat{H}_{aux}(t, \omega, w). \quad \text{for a.a. } (t, \omega) \in [0, T] \times \Omega. \qquad (21)$$

Using that $\hat{H}_{aux}(t, \omega, \cdot)$ is convex and differentiable, (21) is satisfied if and only if

$$D_u H_0(t, \omega, y_{u_{aux}}(t, \omega), p_{u_{aux}}(t, \omega), q_{u_{aux}}(t, \omega), u) - \varepsilon \frac{1}{u_\varepsilon(t, \omega)} = 0. \qquad (22)$$

Therefore, noting that (ommiting the (t, ω) argument)

$$D_y H_{aux}(t, \omega, y_{aux}, p_{aux}, q_{aux}, u_{aux}) = D_y H_\varepsilon(t, \omega, y_{aux}, p_{aux}, q_{aux}, u_{aux}),$$

equations (14), (16) imply that $(y_\varepsilon, p_\varepsilon, q_\varepsilon, u_\varepsilon)$ satisfies (20)-(22). Therefore, $u_{aux} = u_\varepsilon$ solves $(\mathcal{CP})_{aux}$. Finally,

$$\text{Min }_{u \in \mathcal{U}} \; J_0(u) - \varepsilon \langle 1/u_\varepsilon, u \rangle = J_0(u_\varepsilon) - \varepsilon \langle 1/u_\varepsilon, u_\varepsilon \rangle = J_0(u_\varepsilon) - \varepsilon m T.$$

□

Now, we can prove our main result, which yields error bounds for $(y_\varepsilon, p_\varepsilon, q_\varepsilon, u_\varepsilon)$, usually referred as the *central path*. In particular, we obtain the convergence of $(y_\varepsilon, p_\varepsilon, q_\varepsilon, u_\varepsilon)$ to (y_0, p_0, q_0, u_0) in the appropriate spaces.

Theorem 3. *Assume that **(H1)** and **(H2)** hold. Then for every $\varepsilon > 0$, the following estimates hold*

$$\begin{array}{rcll} J_0(u_\varepsilon) - J_0(u_0) & \leq & \varepsilon m T & (23) \\ ||u_\varepsilon - u_0||_2^2 + ||y_\varepsilon - y_0||_{2,\infty}^2 + ||p_\varepsilon - p_0||_{2,\infty}^2 + ||q_\varepsilon - q_0||_2^2 & \leq & O(\varepsilon) & (24) \end{array}$$

Proof. By lemma 2, we have

$$J_0(u_\varepsilon) - \varepsilon m T \leq \max_{\lambda \in \mathcal{U}^+} \min_{u \in \mathcal{U}} \mathcal{L}(u, \lambda) \leq \min_{u \in \mathcal{U}} \max_{\lambda \in \mathcal{U}^+} \mathcal{L}(u, \lambda) = \min_{u \in \mathcal{U}^+} J_0(u) = J_0(u_0),$$

from which (23) follows. The strong convexity of $J_0(\cdot)$ implies that

$$||u_\varepsilon - u_0||_2^2 = O(\varepsilon).$$

Taking $u = u_\varepsilon - u_0$ in (4) yields that

$$||y_\varepsilon - y_0||_{2,\infty}^2 = O(\varepsilon).$$

Finally, using the estimates above and that $y_\varepsilon - y_0$ is almost surely continuous, estimate (10) implies that

$$||p_\varepsilon - p_0||_{2,\infty}^2 + ||q_\varepsilon - q_0||_2^2 \leq L_2 \left(\mathbb{E}\left[(y_\varepsilon(T) - y_0(T))^2 \right] + ||u_\varepsilon - u_0||_2^2 \right) = O(\varepsilon).$$

□

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