Semi-automatic Reconstruction of Implicit Surfaces for Medical Applications

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Implicit Surfaces for Semi-Automatic Medical Organs Reconstruction

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Abstract

This paper proposes a new method based on implicit surfaces for medical organs reconstruction. The original data are noisy scattered points of an organ, that may be arranged in any non-uniform repartition. The knowledge of the normal vectors is not needed. Organs of any geometry and topology can be reconstructed, as for instance a vertebra that is characterized by a hole and by several branchings.

The proposed method uses implicit iso-surfaces generated by skeletons, that are a particularly compact way of giving a smooth representation of a surface. Validity of the reconstructed shape is insured, since implicit surfaces bound a well defined volume. As in a previous work [Mur91] the principle is the minimization of an energy that represents the distance between the set of points and the iso-surface. Skeletons generating field functions are automatically subdivided when needed. But the algorithm we use is novel. Firstly, we introduce local control on the reconstructed shape thanks to local and less expensive field functions. Secondly, we propose a much more efficient skeleton subdivision process, based on a notion of "skeleton energy" that gives a robust criterion for choosing the next skeleton to subdivide. Another optimization consists in splitting the data space into several overlapping windows, where only local data points are used for reconstruction. Implemented as a semi-automatic process, the reconstruction module enables the user to take benefits of his initial knowledge of the surface to guide computations. We use an interactive system for visualizing the data, initially positioning some skeletons, defining local reconstruction areas, and visualizing intermediate results. We have successfully applied the method to both non-medical and medical data.

Keywords: Implicit surfaces, medical applications, shape reconstruction

*iMAGIS is a joint project of CNRS, INRIA, Institut National Polytechnique de Grenoble and Université Joseph Fourier.

1 Introduction

The problem of shape reconstruction of an object is a center of interest of many researchers in various fields, such as medical applications, computer vision, computer animation, or CAD. The medical community needs to reconstruct the human organs for diagnostics, as well as for surgery planning or simulation. The information can be contained into images, volumes, or can merely be a set of scattered points. Either coming from a range sensor or being the result of images segmentation, these points are the basis of most of the reconstruction algorithm. The large interest for the problem explains why so many different approaches have been developed:

Boissonnat [Boi84, BG92] proposes to use the Delaunay triangulation of the set of points. Edelsbrunner and Mücke [EM92] introduce a parameter α that enables to tune the degree of detail while deriving a shape from the Delaunay Triangulation. Attali [ABM94] computes the Voronoï graph of the points to build the "skeleton" of the object and reconstruct its shape. An other approach, introduced in [HDDW92] and completed in [HDDH94] by a mesh optimization, consists in using graph traversal techniques to calculate a signed distance function from the points, and build an iso-surface from this function.

A different philosophy consists in deforming a surface or a volume to fit the data points. The introduction of energy-minimizing curves, known as "snakes", is due to Kass, Witkin and Terzopoulos [KWT88]. The idea was then widely used and developed [TWK88], and the active contour turned to an active surface, deformed by internal forces, such as pressure, and by external forces based on the data to fit [CC92]. The implementation relies most of the time on splines, but Miller also proposes a Geometrically Deformed Model based on triangles [MBE$^+$91]. Another option is to use deformable implicit volumes. Sclaroff and Pentland [SS91] fit a super-quadric by use of modal deformations and displacement maps. Terzopoulos and Metaxas choose a physically based approach for local and global deformations of super-quadrics [TM91].

Most of the developed methods only reconstruct surfaces of fixed topological type. However, a few papers describe models that are able to change their topology: Delingette [Del94] introduces the "Simplex Meshes" for which the user can interactively adapt topology. Leitner uses an adaptative spline surface [Lei93], and Lachaud and Bainville a model derived from triangulation [LB94].

Muraki [Mur91] is the first one who uses implicit iso-surfaces generated by skeletons for shape reconstruction. An initial skeleton is positioned at the center of mass of the data points, and divided until the process reaches a correct approximation level. But Muraki does not take a full advantage of the potentialities of implicit surfaces, and the reconstruction process he proposes is very slow.

This paper presents a new method for reconstruction with implicit surfaces generated by skeletons. We introduce local control on the reconstructed shape thanks to a local field function, that enables the definition of local energy terms associated with each skeleton. This leads to a much more efficient skeleton subdivision process, since we get a robust criterion telling which skeleton should be divided next. Unlike in Muraki's work, the knowledge of the normal vectors at the data points is not needed. The method works as a semi-automatic process: the user can visualize the data, initially position some skeletons thanks to an interactive implicit surfaces editor, and further optimize the process by specifying several "reconstruction windows", that slightly overlap, and where surface reconstruction follows a local criterion. If needed, different precisions of reconstruction can be defined in each window. The shapes to reconstruct can be of any topology and geometry, and for instance include holes and branchings. We show reconstruction experiments from noisy medical data, for which scattered points are arranged in non-uniform repartition.

The remainder of this paper develops as follows: Section 2 introduces implicit surfaces generated by skeletons and discusses Muraki's reconstruction method. Section 3 details the amelioration we propose to the original algorithm, that offer both local control on the reconstructed shape and a robust heuristic for choosing the skeleton to split. The user interface we offers for semi-automatic reconstruction is described in Section 4, and results are presented in Section 5. Section 6 concludes and discusses work in progress.

2 Shape reconstruction with implicit surfaces

2.1 Implicit surfaces generated by skeletons

Developed up to now as a tool for free form modeling [WMW86, WW89, BW90, BS91], implicit surfaces generated by skeletons constitute a good alternative to traditional implicit surfaces directly given by an analytical equation. Indeed, manipulating simple geometric primitives such as skeletons facilitates the design of complex shapes.

An implicit surface S generated by a set of skeletons $S_i (i = 1..n)$ with associated "field functions" f_i is defined by:

$$S = \{P \in I\!R^3 \;/\; f(P) = iso\} \text{ where } f(P) = \sum_{i=1}^{n} f_i(P)$$

- iso is a given scalar iso-value.
- The skeletons S_i can be any geometric primitive admitting a well defined distance function.
- The field functions f_i are monotonically decreasing functions of the distance to the associated skeleton. They can be defined, for instance, by exponential functions [Bli82], by pieces of polynomials [NHK$^+$85, WMW86], or by parametrized procedural functions [KAW91].

The surface surrounds a volume defined by $f(P) \geq iso$, and normal vectors are directed along the gradient of f.

2.2 Muraki's reconstruction method

Muraki [Mur91] presents an automatic method for generating an implicit shape description from range data that include both points and normal vectors.

The surface used for reconstruction is defined by Blinn's "blobby model" [Bli82]. Only point-skeletons are used, and fields are expressed as exponential functions of the distance $d(P, S_i)$:

$$f_i(P) = b_i\, e^{-a_i\, d(P,S_i)}$$

where b_i can be either positive of negative[1].

The principle of the reconstruction algorithm consists in minimizing an energy that represents the "distance" between the current implicit surface and the data. Muraki proposes the use of three terms:

- E_{value} fits the surface to the data points:

$$E_{\text{value}} = \sum_{j=1}^{M} (f\,(P_j) - iso)^2$$

- E_{normal} uses normal vectors to insure that the external side of the iso-surface corresponds to the external side of the object:

$$E_{\text{normal}} = \sum_{j=1}^{M} \frac{|nj - N(Pi)|}{||N(Pi)||^2}$$

- E_{shrink} expresses constraints on the relative values of the parameters a_i and b_i of the n skeletons that avoid surface degeneration in low-sampled areas:

$$E_{\text{shrink}} = (\sum_{i=1}^{n} a_i{}^{-3/2}|bi|)^2$$

The total energy E is given by the expression:

$$E = \frac{1}{M}\,(E_{\text{value}} + \alpha\, E_{\text{normal}}) + \beta\, E_{\text{shrink}}$$

where M is the number of data points.

Directly using n skeletons to reconstruct the surface would be a very computer intensive process, since it would require solving a non-linear minimization problem with $5n$ unknowns (the 5 parameters a_i, b_i, x_i, y_i, z_i for each skeleton). Muraki performs instead a progressive refinement process. The main steps of the algorithm are:

1. Position a single skeleton at the barycenter of the data points.
2. Make trials, independently for each skeleton S_i:
   - Split S_i into two skeletons S'_i and $S"_i$:
   - Minimize energy by modifying only S'_i and $S"_i$'s parameters ;
   - Store the final energy.
3. Keep the modification of step 2 that gave the lowest energy (all the other trials are not used anymore).
4. If the energy is sufficiently low, stop the process ; else, go back to step 2.

[1]Using a "negative skeleton" ie. a skeleton with a negative field reduces the implicit volume, while positive skeletons extend it.

2.3 Discussion

Using implicit surfaces generated by skeletons for reconstructing smooth shapes seems to be a very good idea, since these surfaces are particularly convenient for defining an energy that gives a current "distance" to the data, and since they can be stored in a very compact way (position and field parameters for each skeleton).

However, several problems can be identified, that seem to forbid the direct use of Muraki's method for our goal of medical organs reconstruction:

- The choice of an exponential field function doesn't give any locality on the reconstructed shape. Every data point is influenced by all the skeletons, even those that are distant. In consequence, each refinement changes the shape everywhere, even in areas where the reconstruction was already sufficient.
- The initialization is performed with a single skeleton, positioned at the barycenter of the data-points. This gives quite good results for reconstructing heads (the only examples given in the paper), for which the progressive refinement of an implicit sphere is well adapted. But one can wonder how the process would work for arbitrary shapes, with holes and branchings, such as a vertebra. In particular, the initial skeleton could be located inside a hole, probably perturbating the reconstruction.
- The very important computational time is really limitative: the given examples, that are said to take "a few days", correspond to 2893 data points. Direct use of the method on the echography of an organ (about 25000 scattered points) would be intractable.

  The most expensive process seems to be the way the next skeleton to split is selected. The author admits that trying every skeleton for just keeping the solution that decreases the energy the most is very slow, and should be changed. In practice, some of the examples in [Mur91] are computed by merely splitting each skeleton successively, in the arbitrary order given by the current list of skeletons. As this method doesn't seem very satisfactory as well, better heuristics need to be found.

3 A novel method for reconstruction

This paper proposes a more efficient method for shape reconstruction with implicit surfaces, that we are experimenting on medical data. This section describes our main contributions for improving Muraki's technique. The main idea we are developing is to take benefits of locally defined field functions to optimize the reconstruction process. This is done by defining a new energy criterion telling which skeleton must be splitted next, and by offering the possibility of processing reconstruction in a more local way, through the use of several "reconstruction windows".

3.1 Choice of a local field function

Local field function, that become zero with their derivatives at a certain radius of influence R have been introduced in the literature for optimizing field computations during the design of complex objects [WMW86]. Another benefits is that these fields offer local control on the implicit surface, which is particularly important for our reconstruction process: splitting and optimizing skeletons in an area should not modify a good reconstruction of the data already obtained in another area.

Contrary to Muraki that uses both positive and negative field functions, we want to reconstruct objects with positive fields only. Indeed, for medical applications that may include a future physically-based simulation of organs deformations, the exclusive use of positive skeletons seems more appropriate.

The chosen field function will be used in a computer-intensive minimization process. So we need a model that is both controlled by very few parameters (in order to limit the dimension of the search space), and that provides efficient computations. In consequence, we are currently experimenting field contributions f_i that are composed of a linear and a quadratic curve segments, and that are parametrized by two parameters (see Figure 1):

- The radius e_i of the sphere created by a skeleton S_i alone (e_i is characterized by $f_i(e_i) = iso$) ;
- The stiffness k_i in e_i, that defines the blending properties of the surface.

The function is expressed by:

$$
\begin{aligned}
f_i(P) &= -k_i r + k_i e_i + 1 && \text{if } r \in [0, e_i] \\
f_i(P) &= \frac{k_i e_i(e_i - R_i) + 3e_i - R_i}{(e_i - R_i)^3}(r - R_i)^2 && \text{if } r \in [e_i, R_i] \\
f_i(P) &= 0 && \text{elsewhere}
\end{aligned}
$$

Where $r = d(P, S_i)$ and $R_i = (e_i - \frac{2}{k_i})$ is the radius of influence.

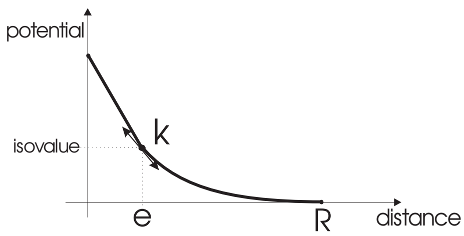


Figure 1: Local field function with two parameters.

As in Muraki's work, there are 5 parameters to optimize for each skeleton: e_i, k_i, x_i, y_i, z_i.

3.2 Basics for semi-automatic reconstruction

The idea is to keep the main principles described in Section 2.2, but with several modifications.

First of all, most medical data do not include any description of normal vectors, that were used by Muraki to insure that the internal volume was reconstructed. However, we are defining a semi-automatic reconstruction process, where the user takes benefits of his knowledge by interactively positioning the initial skeletons with respect to the data. If initial skeletons are positioned inside the volume to reconstruct, the method will work without the need of normal vectors.

In consequence, we redefine the energy to minimize by:

$$E = \frac{1}{M}\left(\sum_{j=1}^{M}(f(P_j) - iso)^2\right) + \frac{1}{n}\left(\alpha_1 \sum_{i=1}^{n} e^{-\beta_1 e_i} + \alpha_2 \sum_{i=1}^{n} e^{-\beta_2 k_i}\right)$$

The two last terms are basically equivalent to E_{shrink} in Muraki's method: they prevent the skeletons from degenerating to negative radius or stiffness (n is the current number of skeletons).

As stressed in Section 2.2, the main problem with Muraki's method is the very intensive computation process, due to the method used for selecting the skeleton to divide next: every solution is tested (which requires running the full minimization) and only the best one is used. The next section describes the heuristic we have developed for accelerating this process.

3.3 An efficient subdivision algorithm

Selecting the skeleton to divide can be done without any extra minimization, by using the new local properties of the field functions. Indeed, the best candidate for splitting is the skeleton whose area of influence corresponds to the zone of the surface were the reconstruction is the worst. See Figure 2.

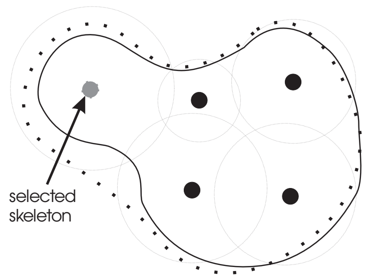


Figure 2: Selection of the next skeleton to subdivide

So we define the splitting criteria as:

$$C_i = \frac{1}{m_i}\left(\sum_{j=1}^{m_i}(f(P_j) - iso)^2\right)$$

where the sum is made only on the m_i point that are inside S_i's sphere of influence.

At each iteration of the algorithm, the skeleton with greatest C_i is replaced by two skeletons, whose 10 parameters are first optimized while keeping the other skeletons and fields constant. Then, we process a few steps of global optimization before going to the next skeleton subdivision (this global optimization step reduces the final number of skeletons: in some cases, just adjusting all the current skeletons at the same time all is sufficient for decreasing the energy to the desired threshold).

3.4 Reconstruction windows

For large scattered points databases, using all the points for every step of energy minimization can be really intractable, since the energy includes a sum of field terms over the points. On an other hand, many objects to reconstruct are easy to decompose into several specific zones, called "reconstruction windows", where reconstruction can be processed separately. For instance, 3 different windows can be defined for the object in Figure 3. The energy to minimize in a 3D window will be calculated by only using the data-points located in this window, and the global optimization step after a refinement will only be performed on the skeletons affected to this window. Processing local reconstructions is justified by our use of local field functions: when we move a skeleton, only local points are affected, so recomputing the entire energy is not really useful. Local windows optimize computations since less points are considered in energy terms, and minimization is performed in a space of smaller dimension.

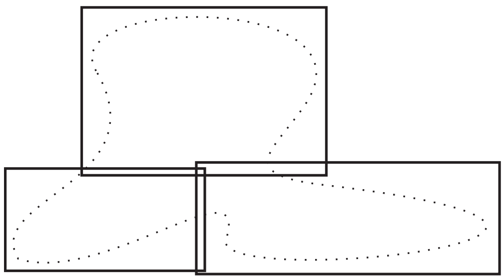


Figure 3: Defining several windows for local shape reconstruction

The algorithm we use for reconstructing data when several reconstruction windows are defined develops as follows. At each reconstruction step:

1. Select the window were:
$$W_k = 1/w_k \, (\sum_{j=1}^{w_k} (f\,(P_j) - iso)^2$$
is the highest (w_k is the number of points in this window).
2. Use the criterion of Section 3.3 for selecting a skeleton to divide in this window.
3. Split the skeleton and optimize its two "children", everything else being kept constant. The energy to minimize is computed from the w_k points

and the n_k skeletons of the window:

$$E_k = \frac{1}{m_k}\left(\sum_{j=1}^{m_k}(f(P_j) - iso)^2\right) + \frac{1}{n_k}\left(\alpha_1 \sum_{i=1}^{n_k} e^{-\beta_1 e_i} + \alpha_2 \sum_{i=1}^{n_k} e^{-\beta_2 k_i}\right)$$

4. Process a few steps of optimization in the window in order to further decrease E_k (and W_k).

5. If the global value defined by: $W_{\text{global}} = \sum W_i$ is smaller than the desired threshold, stop the process. Else, go back to step 1.

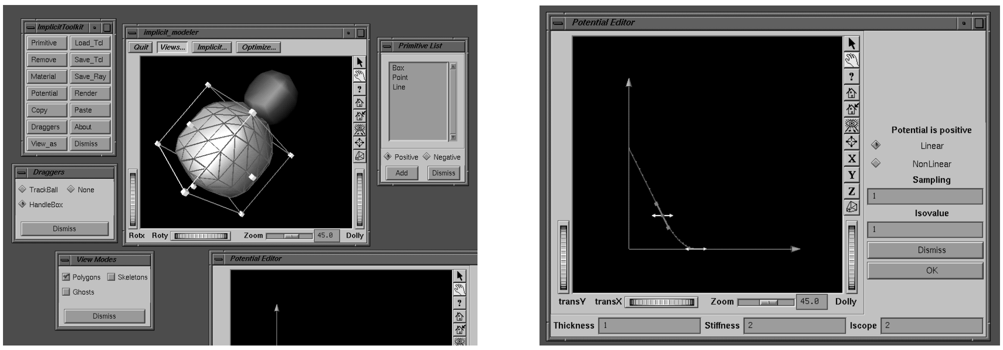
In practice, the list of skeletons and points included into each window are maintained to accelerate the computations.

The user is provided with an interactive interface for visualizing the data, defining reconstruction windows and positioning some initial skeletons. We describe it in the next section.

4 Interface for Semi-Automatic reconstruction

The semi-automatic reconstruction process we have presented is implemented within the system *Fabule*, a platform for interactive modeling and animation that we are developing at *iMAGIS* [Gas94]. It includes an object-oriented library, implemented in C++ and built on *Open-Inventor*, and an interface with the interpreted language *Tcl* and *OSF-Motif* for the easy specification of demonstrators and interfaces.

The user visualizes the data and creates reconstruction windows thanks to 3D widgets provided by *Inventor*. He also interactively specifies skeletons and associated field functions that will initialize reconstruction. Intermediate reconstruction results are visualized within the same interactive system.



(a) General view of the interface

(b) Field function editor

Figure 4: Interface for editing skeletons and field functions

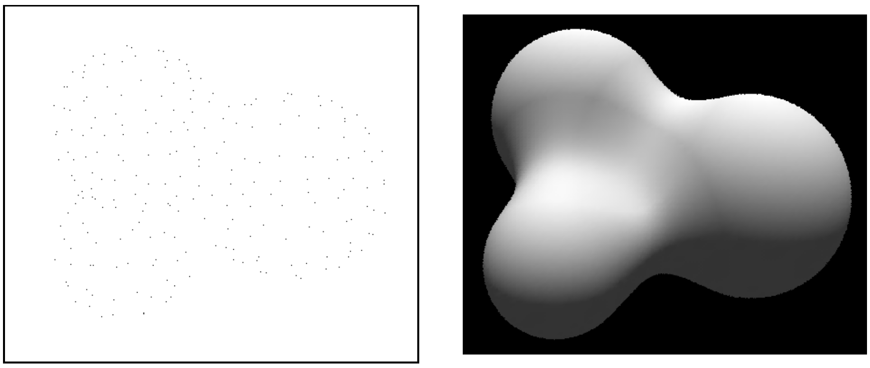
We offer an interactive graphic interface for modeling skeletons, visualizing implicit surfaces, and editing the field function parameters (see Figure 4). Implicit surfaces are visualized in real time thanks to a new sampling technique based on seeds-migration, that adapts at interactive rates to any surface deformation. Sample points are not directly used for display: the user can choose between displaying "scales" laying in each sample point tangent plane, or a piece-wise polygonization of the surface, where a polygon set is associated with each skeleton (this display mode has been used in Figure 8). The full system for interactive modeling and visualization of implicit surfaces is described in [TG94].

5 Experimental results

5.1 Validation of the algorithm on implicitly-defined objects

We have first validated the reconstruction algorithm on objets that had initially been created with implicit surfaces.

The object depicted in Figure 5 has been reconstructed with a single initial skeleton (thus validating the automatic subdivision process), and a single reconstruction window, including all the scene. Results show that the original skeletons and fields are correctly reconstructed by the process, without the creation of any extra skeleton.



(a) Data points

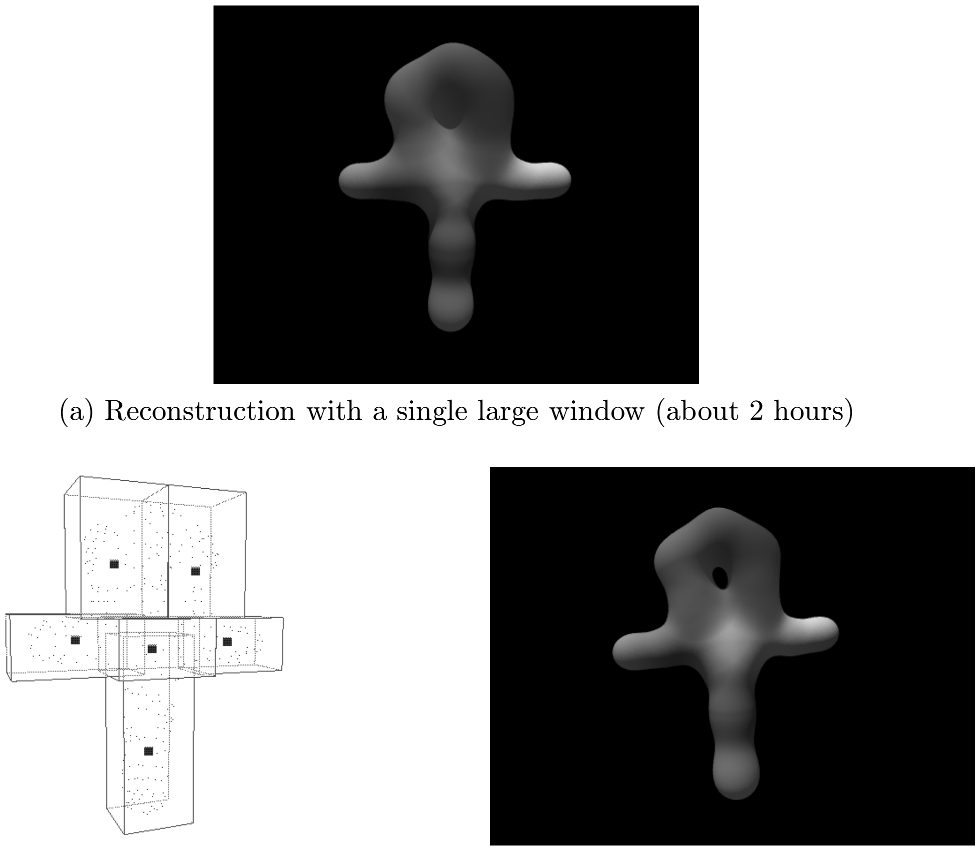
(b) Reconstructed object

Figure 5: Reconstruction of an iso-surface initially created with 3 punctual skeletons

5.2 The use of reconstruction windows

The example depicted in Figure 6 illustrates the use of reconstruction windows on an object with hole and branches, also created with implicit surfaces.

We have performed two different reconstructions based on the same initial skeleton: one with a single large window surrounding the whole scene (a), and the other with several reconstruction windows (c). Figure 6 (a) shows the first result: an approximation with 16 skeletons obtained after more than two hours of computations on an Indigo 2, R4400, 150 MHz workstation. Figure 6 (b), computed much more efficiently thanks to the local windows, is an approximation with 19 skeletons obtained in half an hour.



(a) Reconstruction with a single large window (about 2 hours)

(b) Interactive positioning of 6 local windows in the data space.

(c) Reconstruction with local reconstruction windows (about 30 mn).

Figure 6: The use of several windows for accelerating reconstruction

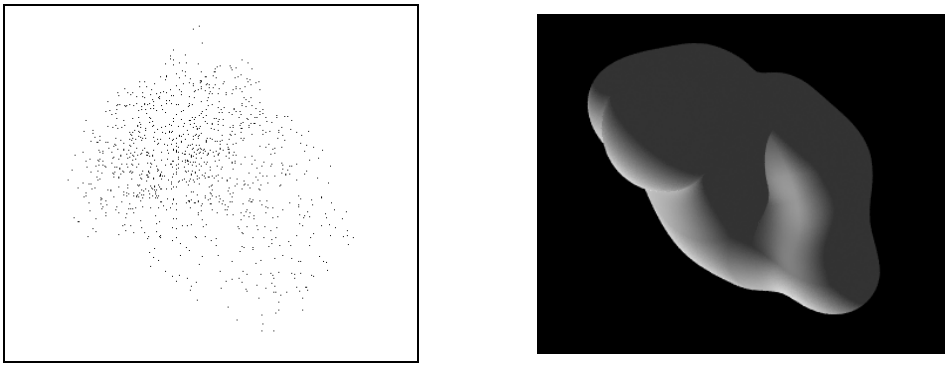
5.3 Reconstruction of medical organs

A kidney

The difficulty for reconstructing this kidney was due to a very noisy data-base, obtained from an echography. Figure 7 shows the data-points, arranged in a non-uniform repartition, and the reconstructed shape we get.

A vertebra (work in progress)

A vertebra is a challenging example for validating the reconstruction process, due to it its very complicated geometry (a central hole and several small branchings). Figure 8 shows an intermediate step of the reconstruction.



(a) Data points

(b) Reconstructed shape

Figure 7: Reconstruction of a kidney from a very noisy database

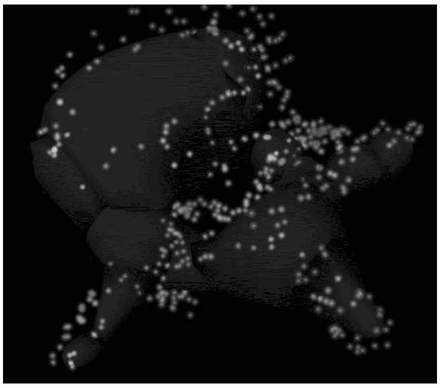


Figure 8: An intermediate step during the reconstruction of a vertebra

6 Conclusion

We have presented a semi-automatic reconstruction method, that can be used on noisy scattered points of a medical organ. The method is based on implicit iso-surfaces generated by skeletons, that provides a smooth and compact representation of the surface. The basics is the minimization of an energy, that represents the distance between the data and the surface, during a series of skeleton refinements. We propose an efficient automatic way of selecting the skeleton to subdivide next, and a new method for optimizing reconstruction through the use of “reconstruction windows”. The method is validated on both medical and non-medical data. The user can guide the reconstruction by initializing some skeletons and their reconstruction windows, thus taking benefits of his initial knowledge of the data.

An easy extension would be to enable the use of other kinds of skeletons generating field functions (such as segments) during the reconstruction process.

For some applications, an automatic generation of the initial skeletons may be more convenient that the interactive initialization process we have described. We are currently studying an approach based on “medial axis” for the automatic positioning of initial skeletons and reconstruction windows.

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