Geometric predicates as arrangements of hypersurfaces: Application to comparison of algebraic numbers

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Predicates

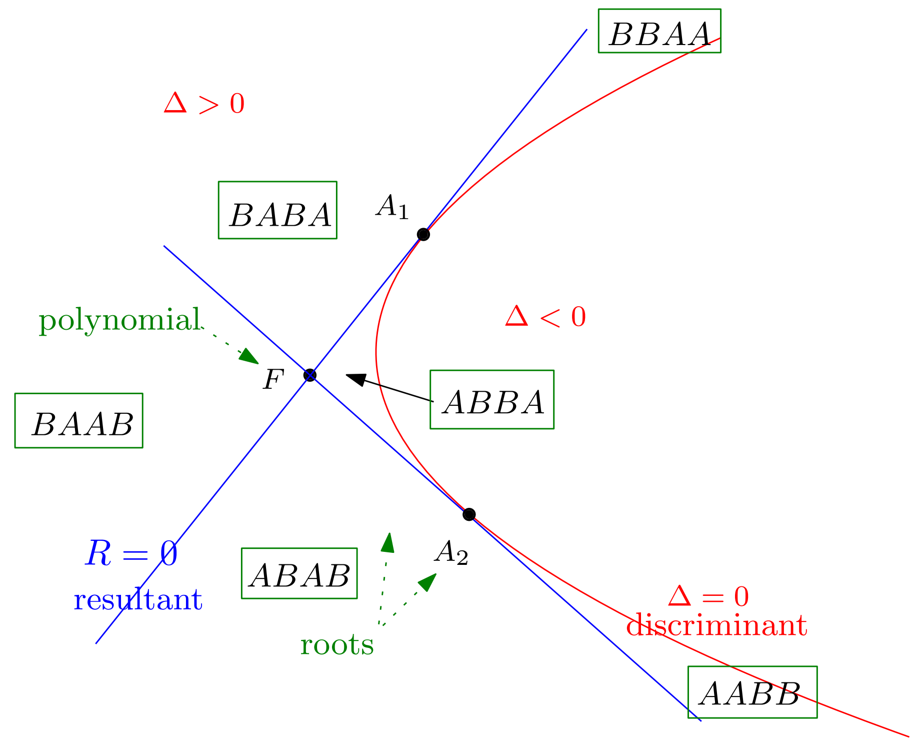
Formulation of a wide range of geometric predicates:

- arrangement of hypersurfaces in $\mathbb{R}^d$
- cells of dimension $\leq d$ (i.e., vertices, edges, faces)
- point location queries

Geometry in the space of coefficients

Given 2 polynomials of deg. 2, determine their root ordering among the 6 non-degenerate cases $AABB$, $ABAB$, $ABBA$, $BAAB$, $BABA$, $BBAA$.

Coefficients	$\longleftrightarrow$	point in affine space
$F(x) = x^2 + Xx + Y$	$\longleftrightarrow$	$(X, Y) \in \mathbb{R}^2$
$A_1^2 + XA_1 + Y = 0$	$\longleftrightarrow$	line for root A_1 of $F(x)$
all polynomials with A_1 as root	$\longleftrightarrow$	points on the line
quadric $F(x)$ has 2 roots	$\longleftrightarrow$	2 lines intersect at a point
Resultant of 2 quadrics	$\longleftrightarrow$	union of 2 lines
Discriminant $X^2 - 4Y$	$\longleftrightarrow$	parabola Δ
roots of F	$\longleftrightarrow$	lines *tangent* to Δ
# roots of F	$\longleftrightarrow$	rel. pos. of F wrt. Δ



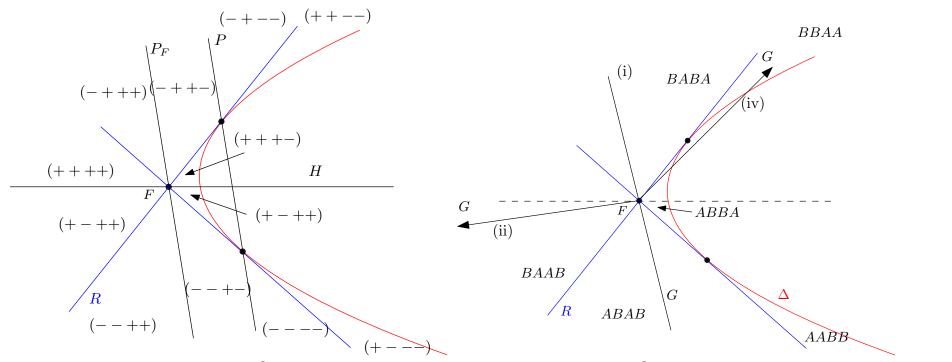
Arrangement consisting of the following hypersurfaces:

- Resultant (pair of lines)
- Discriminant (parabola)

points in same cell	$\longleftrightarrow$	same root ordering
crossing of resultant	$\longleftrightarrow$	roots of 2 polys equal
crossing of discriminant	$\longleftrightarrow$	roots of same poly equal

Decomposition

- exploiting conic properties of pole and polar (duality) or intersecting rays from F with Δ



- When $F(x) = a_2x^2 + a_1x + a_0$ and $G(x) = b_2x^2 + b_1x + b_0$ the tested quanties are:

$$
\begin{aligned}
R &= {b_0}^2{a_2}^2 - 2\,b_0a_0b_2a_2 + {a_0}^2{b_2}^2 - a_1b_1b_0a_2 - a_1b_1a_0b_2 + {a_1}^2b_0b_2 + a_0{b_1}^2a_2 \\
&\quad \text{(resultant)} \\
H &= b_1a_2 - a_1b_2 \quad \text{(horizontal line)} \\
P &= -2\,b_0a_2 + a_1b_1 - 2\,a_0b_2 \quad \text{(polar)} \\
P_F &= a_1b_1a_2 - {a_1}^2b_2 - 2\,b_0{a_2}^2 + 2\,a_0a_2b_2 \quad \text{(polar translated)}
\end{aligned}
$$

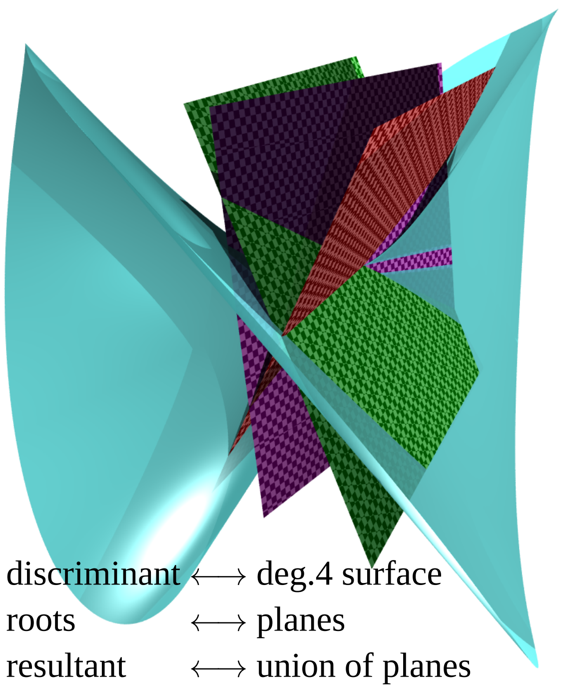
To compare algebraic numbers of degree 2, we have to consider quantities of at most algebraic degree 4 in the worst case. This bound is tight and therefore optimal.

Proof

- walls of arrangement defined by resultant (deg.4) and discriminant (deg.2)
- both irreducible, separating full-dimensional cells
- $\Rightarrow$ they are *necessary* to define the arrangement
- all subsidiary equations have alg. deg. ≤ 4

Cubics

Given 2 polynomials of deg. 3, determine their root ordering among the $\binom{6}{3} + 2\binom{4}{1} + 2 = 30$ non-degenerate cases.



discriminant	$\longleftrightarrow$	deg.4 surface
roots	$\longleftrightarrow$	planes
resultant	$\longleftrightarrow$	union of planes

- quantities of alg. degree at most 6
- optimal, but geometric interpretation not yet known

Higher degree

- difficult to visualise
- *open problem:* optimal decomposition of the arrangement