The Lifschitz-Slyozov equation with space-diffusion of monomers

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November 24, 2010

Abstract

The Lifschitz–Slyozov system describes the dynamics of mass exchanges between macro–particles and monomers in the theory of coarsening. We consider a variant of the classical model where monomers are subject to space diffusion. We establish the existence–uniqueness of solutions for a wide class of relevant data and kinetic coefficients. We also derive a numerical scheme to simulate the behavior of the solutions.

1 Introduction

The standard Lifschitz-Slyozov system, as introduced in [15, 16], describes the evolution of a solution of polymers. In this model, macro-particles, or polymers, interact with free particles, or monomers. The macro-particles are described by their size distribution function $f(t, \xi)$, with $t \geq 0$ and $\xi \geq 0$ the time and size variables respectively, while the monomers are described by their concentration $c(t)$. The dynamics is governed by the growth rate

$$V(t, \xi) = a(\xi)c(t) - b(\xi)$$

with a, b given non negative functions: these kinetic coefficients represent the rates at which monomers are added to or removed from the macro-particles with size ξ. The precise expression of the coefficient relies on the modeling of the precipitation/dissolution processes; in [16], assuming that mass transfer is based on monomer diffusion, the following expression is proposed

$$a(\xi) = \xi^{1/3}, \qquad b(\xi) = 1.$$

We refer to [23] for other relevant formulae for the kinetic coefficients. In this paper we shall assume the following

Hypothesis 1.1 *The kinetic coefficients a, b are required to satisfy:*

i) $b = 1$,

ii) *a is non decreasing with $a(0) = 0$ and $a(+\infty) = +\infty$,*

iii) *$a \in C^0([0, \infty)) \cap C^1((0, \infty))$ and for any $\xi_0 > 0$ there exists $L_{a,0} > 0$ such that $0 \leq a'(\xi) \leq L_{a,0}$ for $\xi \geq \xi_0 > 0$.*

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As a matter of fact we remark that at any time $t \geq 0$ the size space splits into two parts: $0 \leq \xi \leq \xi_c(t)$ and $\xi \geq \xi_c(t)$ where $\xi_c(t)$ is the unique positive number verifying $a(\xi_c(t)) = 1/c(t)$: accordingly, large particles grow at the expense of the smaller ones, a coarsening phenomenon referred to as Ostwald ripening.

Therefore, the dynamics of the precipitation process is embodied into the transport equation

$$\partial_t f + \partial_\xi (V f) = 0 \tag{1}$$

coupled to the mass conservation relation

$$c(t) + \int_0^\infty \xi f(t,\xi)\,\mathrm{d}\xi = \rho \tag{2}$$

a given positive constant. Eq. (1) is a conservation law for the polymer concentration in size space, while (2) expresses the fact that the total mass is conserved, the solute material being accounted for either as dissolved particles or as macro-particles. Indeed, the quantity $\int_\zeta^{\zeta'} f(t,\xi)\,\mathrm{d}\xi$ is interpreted as the number of polymers having at time t their size between ζ and ζ' while $\int_\zeta^{\zeta'} \xi f(t,\xi)\,\mathrm{d}\xi$ is proportional to the corresponding mass. We point out that for $\xi = 0$, the growth rate $V(t,0) = -1$ is negative so that we do not need a boundary condition. Despite its apparent simplicity the Lifschitz-Slyozov system is quite intriguing for the mathematical analysis. We refer to [6, 12, 14, 18] for existence-uniqueness results in various functional frameworks. The understanding of the large time behavior is highly challenging, definitely far from the asymptotic trend to a universal profile, as derived in [16]. We refer on this aspect to the analysis performed in [8, 19, 20] and the numerical simulations in [2, 25]. The Lifschitz-Slyozov system (1)–(2) has been extended to account for more physical phenomena: the addition of a coagulation operator, as suggested in [16, 23], is considered in [5, 13]. A version with a parabolic correction has been introduced in [11]; it is intended to share more basic features with the discrete Becker-Döring model, in particular concerning selection mechanisms of the large time asymptotics. By the way, the connection between discrete (Becker-Döring) and continuous (Lifschitz-Slyozov) models is discussed in [7]. Another diffusive correction is discussed in [22], based on a deep mean field analysis.

In this paper we wish to discuss another relevant version of the Lifschitz-Slyozov equations by assuming that monomers are also subject to space diffusion. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, with boundary $\partial\Omega$; given $x \in \partial\Omega$ we denote $\nu(x)$ the outward unit normal vector at point x. Then, we are interested in the following variant of (1)–(2), where now the unknowns also depend on the space variable and monomers are subject to diffusion:

$$\left\{ \begin{array}{ll} \partial_t f(t,x,\xi) + \partial_\xi\big((a(\xi)c(t,x) - 1) f(t,x,\xi)\big) = 0 & t \geq 0,\ x \in \Omega,\ \xi \geq 0 \\[2ex] \partial_t \Big(c(t,x) + \displaystyle\int_0^\infty \xi f(t,x,\xi)\,\mathrm{d}\xi \Big) - \Delta_x c(t,x) = 0 & t \geq 0,\ x \in \Omega, \end{array} \right. \tag{3}$$

endowed with homogeneous Neumann boundary condition

$$\partial_\nu c = \nabla c \cdot \nu = 0, \qquad \text{on } \partial\Omega. \tag{4}$$

Finally, the problem is completed by initial conditions

$$c(0,x) = c_{\mathrm{init}}(x) \geq 0, \qquad f(0,x,\xi) = f_{\mathrm{init}}(x,\xi) \geq 0. \tag{5}$$

In view of the physical interpretation it appears quite natural to assume

Hypothesis 1.2 *The data satisfy*
- $c_{\mathrm{init}} \in L^\infty(\Omega)$,
- $f_{\mathrm{init}} \in L^\infty(\Omega; L^1((0,\infty), (1+\xi)\,\mathrm{d}\xi))$.

By using the conservation equation for f and integrating by parts, we observe that

$$\partial_t \int_0^\infty \xi f(t,x,\xi)\,\mathrm{d}\xi = \int_0^\infty a(\xi)c(t,x)f(t,x,\xi)\,\mathrm{d}\xi - \int_0^\infty f(t,x,\xi)\,\mathrm{d}\xi. \tag{6}$$

It allows to rewrite the equation for the monomers concentration in the more familiar fashion

$$\partial_t c + c\int_0^\infty a(\xi)f(t,x,\xi)\,\mathrm{d}\xi = \Delta_x c + \int_0^\infty f(t,x,\xi)\,\mathrm{d}\xi. \tag{7}$$

Of course, the system preserves the total mass: we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega\int_0^\infty \xi f(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x + \int_\Omega c(t,x)\,\mathrm{d}x\right] = 0.$$

We point out that a coupling with the stationary diffusion equation is derived in [17] through homogenization arguments, the model being further analyzed in [21]. In this paper we wish to investigate the system (3)–(5). In particular, we shall establish the following well-posedness statement

Theorem 1.1 *Suppose that Hypothesis 1.1 and 1.2 are fulfilled. Then, there exists a weak solution (c,f) of (3)–(5) with, for any $0 < T < \infty$, $c \in L^\infty((0,T)\times\Omega)\cap L^2(0,T;H^1(\Omega))$, $f \in L^\infty((0,T)\times\Omega;L^1((0,\infty),(1+\xi)\,\mathrm{d}\xi))$, $c \in C^0([0,T];L^2(\Omega)-weak)$, $f \in C^0([0,T];L^1(\Omega\times(0,\infty))-weak)$.*

The difficulty of course arises from the non-linear coupling which involves PDEs of different types acting on different variables. This work is organized as follows. In Section 2, we briefly set up the necessary material on transport and diffusion equations. Then, in Section 3 we make use of a fixed point strategy to obtain the existence-uniqueness of solutions associated to bounded initial data when the kinetic coefficients are globally Lipschitz. Section 4 extends the result to more general data. Finally, in Section 5 we introduce a numerical scheme for the simulation of (3)–(5) and we conclude with some commented numerical experiments.

2 Basic results on diffusion and transport equations

In this Section we collect some statements on diffusion and transport equations which will be useful for our purposes. We start with the following claim.

Proposition 2.1 *Let $0 < T < +\infty$. Let A and B be non negative functions in $L^\infty((0,T)\times\Omega)$. Suppose that $0 \le B(t,x) \le C_0 < \infty$ for almost every (t,x). Then, for any $c_{\mathrm{init}} \in L^2(\Omega)$, there exists a unique $c \in L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))$ with $\partial_t c \in L^2(0,T;(H^1(\Omega))')$ solution of*

$$\partial_t c + Ac - \Delta_x c = B \quad \text{in } (0,T)\times\Omega, \qquad \partial_\nu c = 0 \quad \text{on } \partial\Omega,$$

with initial data $c(t=0,x) = c_{\mathrm{init}}(x)$. Furthermore if $c_{\mathrm{init}} \ge 0$ belongs to $L^\infty(\Omega)$, then the solution c satisfies $0 \le c(t,x) \le K_T$ with K_T a constant depending on C_0, $\|c_{\mathrm{init}}\|_\infty$ and T. We also have for $0 \le t \le T < \infty$

$$\int_\Omega |c(t,x)|^2\,\mathrm{d}x \le C_T, \qquad \int_0^t\int_\Omega |\nabla_x c(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \le C_T$$

for some constant C_T depending on C_0, T, Ω and $\|c_{\mathrm{init}}\|_{L^2(\Omega)}$.

Proof. The existence result is a direct consequence of a general statement on parabolic equation. Indeed, the bilinear form

$$\mathbb{A}(t;c,\bar{c}) = \int_\Omega \nabla_x c\cdot\nabla_x\bar{c}\;dx + \int_\Omega A\,c\cdot\bar{c}\,\mathrm{d}x$$

is well defined on $H^1(\Omega) \times H^1(\Omega)$ and it verifies the continuity estimate:

$$\big|\mathbb{A}(t; c, \bar{c})\big| \leq \big(1 + \|A\|_{L^\infty((0,T)\times\Omega)}\big)\ \|c\|_{H^1(\Omega)}\ \|\bar{c}\|_{H^1(\Omega)}.$$

Furthermore, we also have the coercivity property

$$\mathbb{A}(t; c, c) = \int_\Omega |\nabla_x c|^2 \,\mathrm{d}x + \int_\Omega A c^2 \,\mathrm{d}x \geq \|c\|^2_{H^1(\Omega)} - \|c\|^2_{L^2(\Omega)}.$$

We can therefore apply the analog of the Lax-Milgram theorem for parabolic equations, see e. g. [1, Theorem X.9, p. 218], and we get the existence uniqueness statement in Proposition 2.1.

In order to prove the uniform estimate, we proceed as follows. Consider a function $G \in C^1(\mathbb{R}_+)$ such that

- There exists $M_0 > 0$ such that $|G'(s)| \leq M_0$ for any $s \in \mathbb{R}$;
- The function $s \mapsto G(s)$ is increasing on $(0, +\infty)$;
- $G(s) = 0$ on $(-\infty, 0]$.

We start by checking that $c(t, x) \geq 0$. We set

$$s \in \mathbb{R} \longmapsto H(s) = \int_0^s G(\sigma)\,\mathrm{d}\sigma \qquad \text{and} \qquad t \in [0, T] \longmapsto \varphi(t) = \int_\Omega H\big(-c(t,x)\big)\,\mathrm{d}x \geq 0.$$

In particular, we observe that

$$\varphi(0) = \int_\Omega H\big(-c_{\mathrm{init}}(x)\big)\,\mathrm{d}x = 0$$

since $c_{\mathrm{init}}(x) \geq 0$. Next, we compute

$$\begin{aligned}\varphi'(t) = &-\int_\Omega G\big(-c(t,x)\big)B(t,x)\,\mathrm{d}x - \int_\Omega G'\big(-c(t,x)\big)\big|\nabla_x c(t,x)\big|^2\,\mathrm{d}x\\ &+ \int_\Omega G\big(-c(t,x)\big)A(t,x)c(t,x)\,\mathrm{d}x \leq 0,\end{aligned}$$

since $tG(t) \geq 0$ and $G'(t) \geq 0$. We conclude that $\varphi(t) = 0$ and thus $H(-c(t,x)) = 0$ for a.e. (t,x) It implies $c(t,x) \geq 0$ a.e.
Next, we prove the bound from above. To this end, we set

$$K(t) = \|c_{\mathrm{init}}\|_{L^\infty(\Omega)} + C_0 t$$

and

$$H(s) = \int_0^s G(\sigma)\,\mathrm{d}\sigma, \qquad \varphi(t) = \int_\Omega H\big(c(t,x) - K(t)\big)\,\mathrm{d}x \geq 0.$$

We have $\varphi(0) = 0$ and

$$\begin{aligned}\varphi'(t) = &\int_\Omega G\big(c(t,x) - K(t)\big)\ \big(B(t,x) - C_0\big)\,\mathrm{d}x - \int_\Omega G'\big(c(t,x) - K(t)\big)\ \big|\nabla_x c(t,x)\big|^2\,\mathrm{d}x\\ &- \int_\Omega G\big(c(t,x) - K(t)\big)\ A(t,x)c(t,x)\,\mathrm{d}x \leq 0.\end{aligned}$$

It follows that $\varphi(t) = 0$ and thus $H\big(c(t,x) - K(t)\big) = 0$ for a. e. $t \geq 0$, $x \in \Omega$ which implies $0 \leq c(t,x) \leq K(T)$ a.e. on $(0,T) \times \Omega$. The last estimate follows from standard energy estimates and application of the Grönwall lemma. $\square$

Let us now recall a few facts about transport equations. For the time being we neglect the space variable which appears only as a parameter in the equation for the size density. Thus, we are concerned with the problem

$$\left\{\begin{array}{l} \partial_t f + \partial_\xi(Vf) = 0, \\ f(0,\xi) = f_{\mathrm{init}}(\xi) \end{array}\right. \tag{8}$$

on $t \geq 0$ and $\xi \geq 0$ where the function $(t,\xi) \mapsto V(t,\xi)$ is required to satisfy

Hypothesis 2.1 *We have $V(t,\xi) = a(\xi)c(t) - b(\xi)$ with continuous and non negative functions a, b, c, such that $a(0) = 0$, $b(0) > 0$. We suppose that c is locally bounded while a' and b' belong to $L^\infty(\mathbb{R})$. Accordingly, for any $0 \leq t \leq T < \infty$, there exists M_T such that for any $\xi, \xi' \geq 0$, we have*

- $V(t,0) \leq 0$,
- $V(t,\xi) \leq M_T \xi$ *and* $|V(t,\xi)| \leq M_T(1+\xi)$
- $|V(t,\xi) - V(t,\xi')| \leq M_T|\xi - \xi'|$.

Remark that $V(t,x,\xi) = a(\xi)c(t,x) - 1$ satisfies the requirements in Hypothesis 2.1, uniformly with respect to the parameter $x \in \Omega$, as far as the kinetic coefficient a has a globally bounded derivative (see Hypothesis 3.1 below, $||a'||_\infty \leq L_a$) and satisfies $a(0) = 0$, and the monomers concentration satisfies the L^∞ estimate $0 \leq c(t,x) \leq K_T$ (with $M_T = L_a K_T$).

Owing to Hypothesis 2.1, we can solve (8) by means of integration along characteristics. Indeed, we can define the characteristic curves solutions to the ODE

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}s}\Xi(s;t,\xi) = V\big(s, \Xi(s;t,\xi)\big), \quad s \in \mathbb{R}, \\ \\ \Xi(t;t,\xi) = \xi. \end{array}\right. \tag{9}$$

Then, (8) recasts as

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[f\big(s,\Xi(s;t,\xi)\big)\Big] = -\partial_\xi V\big(s,\Xi(s;t,\xi)\big)\ f\big(s,\Xi(s;t,\xi)\big).$$

It yields

$$f(t,\xi) = f_{\mathrm{init}}(\Xi(0;t,\xi))\ J(0;t,\xi) \tag{10}$$

with

$$J(s;t,\xi) = \partial_\xi \Xi(s;t,\xi) = \exp\left(-\int_s^t \partial_\xi V(\sigma,\Xi(\sigma;t,\xi))\,\mathrm{d}\sigma\right) \geq 0, \tag{11}$$

the Jacobian of the change of variables $\xi \mapsto \zeta = \Xi(s;t,\xi)$. The fundamental properties on the characteristics that are needed for our analysis are summarized in the following claim. (We refer to [6] for similar considerations and details.)

Lemma 2.2 *Let Hypothesis 2.1 be fulfilled. Then, we have*

i) for any $t \geq 0$, $\Xi(0;t,0) \geq 0$,

ii) for any $t \geq 0$, $\lim_{\xi\to\infty}\Xi(0;t,\xi) = \infty$,

iii) for any $0 \leq t \leq T < \infty$ and $\xi \geq 0$, there exists $L_T > 0$ such that $\Xi(t;0,\xi) \leq L_T\ \xi$.

Proof. Derivating with respect to the initial time, we obtain

$$\partial_t \Xi(s;t,\xi) = -V(t,\xi)J(s;t,\xi).$$

Since $J \geq 0$ and $V(t,0) \leq 0$, we deduce that $t \mapsto \Xi(0;t,0)$ is non decreasing and thus i) holds. Next, we have

$$\Xi(s_2;t,\xi) - \Xi(s_1;t,\xi) = \int_{s_1}^{s_2} V\big(\sigma,\Xi(\sigma;t,\xi)\big)\,\mathrm{d}\sigma.$$

(Note that $\Xi(s_2;t,\xi) \geq 0$ for $s_2 \leq t$ owing to the fact that $V(t,0) \leq 0$.) Since $V(t,\xi) \leq M_T\xi$ we obtain for $0 \leq s_1 \leq s_2 \leq t \leq T$

$$0 \leq \Xi(s_2;t,\xi) \leq \Xi(s_1;t,\xi) + M_T \int_{s_1}^{s_2} \Xi(\sigma;t,\xi)\,\mathrm{d}\sigma$$

and the Grönwall lemma yields

$$0 \leq \Xi(s_2;t,\xi) \leq e^{M_T(s_2-s_1)}\Xi(s_1;t,\xi).$$

With $s_2 = t$ we have $e^{M_T(s_1-t)}\ \xi \leq \Xi(s_1;t,\xi)$ which allows to conclude for ii) by letting ξ go to ∞. The third item is a direct consequence of the Grönwall lemma. $\square$

Proposition 2.3 *Let Hypothesis 2.1 be fulfilled. Let f be the solution of (8), as given by (10). Then, the following assertions hold*

i) If $f_{\mathrm{init}} \in L^1((0,\infty))$ with $\xi f_{\mathrm{init}} \in L^1((0,\infty))$, then for any $t \geq 0$, $\xi \mapsto f(t,\xi)$ and $\xi \mapsto \xi f(t,\xi)$ are integrable. More precisely, we have $f \in C^0([0,T];L^1((0,\infty))$ and the following estimates hold for any $t \geq 0$

$$\int_0^\infty |f(t,\xi)|\,\mathrm{d}\xi \leq \int_0^\infty |f_{\mathrm{init}}(\xi)|\,\mathrm{d}\xi, \tag{12}$$

and, for any $0 \leq t \leq T < \infty$

$$\int_0^\infty \xi\ |f(t,\xi)|\,\mathrm{d}\xi \leq L_T \int_0^\infty \xi\ |f_{\mathrm{init}}(\xi)|\,\mathrm{d}\xi, \tag{13}$$

with L_T depending on M_T in Hypothesis 2.1.

ii) If $f_{\mathrm{init}} \geq 0$, then $f(t,\xi) \geq 0$ too.

iii) We assume furthermore that $\partial_\xi V(t,\xi) \geq 0$ for any $t,\xi \geq 0$, then if f_{init} belongs to $L^\infty((0,\infty))$, we have $f \in L^\infty((0,\infty)\times(0,\infty))$ with

$$||f||_\infty \leq \|f_{\mathrm{init}}\|_\infty.$$

Proof. We simply integrate (10) and use Lemma 2.2 to obtain

$$\int_0^\infty |f(t,\xi)|\,\mathrm{d}\xi = \int_{\Xi(0;t,0)}^\infty |f_{\mathrm{init}}(\xi)|\,\mathrm{d}\xi \leq \int_0^\infty |f_{\mathrm{init}}(\xi)|\,\mathrm{d}\xi.$$

Similarly, we have

$$\int_0^\infty \xi|f(t,\xi)|\,\mathrm{d}\xi = \int_{\Xi(0;t,0)}^\infty \Xi(t;0,\xi)\ |f_{\mathrm{init}}(\xi)|\,\mathrm{d}\xi \leq L_T \int_0^\infty \xi\ |f_{\mathrm{init}}(\xi)|\,\mathrm{d}\xi.$$

When $\partial_\xi V \geq 0$, we observe that $0 \leq J(s;t,\xi) \leq 1$ holds when $s \leq t$. Therefore we obtain

$$|f(t,\xi)| = \left|f_{\mathrm{init}}\big(\Xi(0;t,\xi)\big)\right|\ J(0;t,\xi) \leq \|f_{\mathrm{init}}\|_\infty$$

for almost every (t,ξ). $\square$

3 Existence-uniqueness for bounded data

In this Section, we restrict to the case where the data are bounded and the coefficients are globally Lipschitz. To be more specific we strengthen Hypothesis 1.1 and 1.2 as follows

Hypothesis 3.1 *Additionally to the requirements in Hypothesis 1.1 and 1.2 we suppose* *a)* $a \in C^1([0,\infty))$ *and there exists a constant* $L_a > 0$ *such that* $0 \le a'(\xi) \le L_a$ *for any* $\xi \ge 0$*,*

b) $f_{\mathrm{init}} \in L^\infty(\Omega \times (0,\infty))$.

We wish to prove the well-posedness of the non homogeneous Lifschitz-Slyozov equation in this framework.

Theorem 3.1 *Suppose that Hypothesis 3.1 is fulfilled. Then, there exists a unique weak solution* (c,f) *of* (3)–(5) *with, for any* $0 < T < \infty$*,*

$$\begin{aligned}
&c \in L^\infty((0,T)\times\Omega) \cap L^2(0,T;H^1(\Omega)),\\
&f \in L^\infty(((0,T)\times\Omega\times(0,\infty)) \cap L^\infty((0,T)\times\Omega; L^1((0,\infty),(1+\xi)\,\mathrm{d}\xi)),\\
&c \in C^0([0,T];L^2(\Omega)-\textit{weak}),\\
&f \in C^0([0,T];L^1(\Omega\times(0,\infty))-\textit{weak}).
\end{aligned}$$

The proof uses the Schauder fixed point theorem, see [10, Corollary 3.6.2]. We set $\mathscr{Q}_T = [0,T]\times\Omega$ for a fixed $0 < T < \infty$. Let us denote

$$C_0 = \sup_{x\in\Omega} \int_0^\infty f_{\mathrm{init}}(x,\xi)\,\mathrm{d}\xi < \infty. \tag{14}$$

We associate to this quantity the constant K_T as defined in the proof of Proposition 2.1, $K_T = \|c_{\mathrm{init}}\|_\infty + C_0 T$. We introduce the set

$$\mathscr{C}_T = \big\{\tilde c \in L^2(\mathscr{Q}_T) \text{ such that } 0 \le \tilde c(t,x) \le K_T\big\}.$$

Then, we define the mapping

$$\begin{array}{rccc}
\mathscr{T}: & \mathscr{C}_T & \longrightarrow & L^2(\mathscr{Q}_T)\\
& \tilde c & \longmapsto & \mathscr{T}(\tilde c) = c,
\end{array}$$

with c solution of

$$\left\{\begin{array}{ll}
\partial_t c(t,x) - \Delta_x c(t,x) + A(t,x)c(t,x) = B(t,x) & \text{for } t \ge 0,\ x\in\Omega,\\
\partial_\nu c = 0 & \text{on } \partial\Omega,\\
c_{|t=0} = c_{\mathrm{init}} & \text{on } \Omega,
\end{array}\right.$$

where the coefficients A, B are given by

$$A(t,x) = \int_0^\infty a(\xi) f(t,x,\xi)\,\mathrm{d}\xi, \qquad B(t,x) = \int_0^\infty f(t,x,\xi)\,\mathrm{d}\xi,$$

f being solution of

$$\left\{\begin{array}{ll}
\partial_t f(t,x,\xi) + \partial_\xi\big((a(\xi)\tilde c(t,x) - 1) f(t,x,\xi)\big) = 0 & \text{for } t \ge 0,\ x\in\Omega,\ \xi \ge 0,\\
f_{|t=0} = f_{\mathrm{init}} & \text{on } \Omega\times(0,\infty).
\end{array}\right.$$

From now on we adopt the convention to denote by $L_T > 0$ a constant that depends on T, C_0, $\|c_{\mathrm{init}}\|_\infty$, and on the Lipschitz constant L_a of a, even if the precise value of the constant might change from a line to another. Conversely, we will denote by C_T a constant which depends only on T, C_0 and $\|c_{\mathrm{init}}\|_\infty$ but not on L_a (like K_T). According to Hypothesis 1.1, for any $\tilde c \in \mathscr{C}_T$, the rate $V(t,x,\xi) = a(\xi)\tilde c(t,x) - 1$, which is now parametrized by $x \in \Omega$, satisfies the estimates required in Hypothesis 2.1, uniformly with respect to $x \in \Omega$. (Namely M_T in Hypothesis 2.1 is $L_a K_T$.) Up to a slight abuse with regularity issues we can therefore appeal to the results established in Section 2. Indeed, within the functional framework adopted here, for fixed $x \in \Omega$, $t \mapsto \tilde c(t,x)$ cannot be considered as a continuous function of the time variable. The classical theory of characteristics with C^1 solutions of the ODE (9) does not apply. The alternative to circumvent the difficulty is as follows. The first option consists in dealing with a less regular notion of characteristics. The standard Picard iteration scheme actually shows that

$$\Xi(s;t,\xi) = \xi + \int_t^s V\big(\sigma, \Xi(\sigma;t,\xi)\big)\,\mathrm{d}\sigma$$

admits a continuous solution assuming only integrability of V with respect to the time variable and all the necessary estimates on Ξ hold in this framework (see [4, Theorem 1.1, p. 43] for an existence theorem without regularity in time). The second option consists in replacing $\tilde{c}$ in the convection term by $\zeta_\epsilon \star_{t,x} \tilde{c}$, with ζ_ϵ a convenient sequence of mollifiers. Again, all the necessary estimates are not affected by the regularization process and are uniform with respect to ϵ. Accordingly, the compactness arguments detailed below apply to pass to the limit as ϵ goes to 0. We do not detail further this issue, adopting the slight abuse of working with the characteristics Ξ, parametrized by the space variable x, without any further precision. Hence, we can apply Proposition 2.3: f reads

$$f(t,x,\xi) = f_{\mathrm{init}}(x, \Xi(0;t,x,\xi))\ J(0,t,x,\xi).$$

with Ξ and J defined by the characteristics equation associated to $\tilde{c}$. In particular, we have

$$\left\{\begin{array}{l} 0 \le f(t,x,\xi) \le \|f_{\mathrm{init}}\|_\infty \quad a.e., \\ \sup\limits_{x\in\Omega} \displaystyle\int_0^\infty f(t,x,\xi)\,\mathrm{d}\xi \le \sup\limits_{x\in\Omega} \int_0^\infty f_{\mathrm{init}}(x,\xi)\,\mathrm{d}\xi = C_0 < \infty, \\ \sup\limits_{x\in\Omega} \displaystyle\int_0^\infty \xi\ f(t,x,\xi)\,\mathrm{d}\xi \le L_T\ \sup\limits_{x\in\Omega} \int_0^\infty \xi\ f_{\mathrm{init}}(x,\xi)\,\mathrm{d}\xi < \infty. \end{array}\right. \tag{15}$$

It follows that $A(t,x) \ge 0$ lies in $L^\infty((0,T)\times\Omega)$, and $0 \le B(t,x) \le C_0$. Coming back to Proposition 2.1 we conclude that $\mathscr{T}$ is well defined with $c = \mathscr{T}(\tilde{c}) \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ and furthermore $0 \le c(t,x) \le K_T$. In other words $\mathscr{T}(\mathscr{C}_T) \subset \mathscr{C}_T$.

Let us now show that $\mathscr{T}(\mathscr{C}_T)$ is a compact set in $L^2(\mathscr{Q}_T)$. In fact Proposition 2.1 also shows that

$$c = \mathscr{T}(\tilde{c}) \text{ lies in a bounded set in } L^2(0,T;H^1(\Omega)).$$

The equation satisfied by c finally tells us that

$$\partial_t c \text{ is bounded } L^2(0,T;H^{-1}(\Omega)).$$

Since $H^1(\Omega)$ embeds compactly in $L^2(\Omega)$, we can therefore apply the compactness results in [24, Corollary 4] to conclude that $\mathscr{T}(\mathscr{C}_T)$ is a compact set in $L^2(\mathscr{Q}_T)$.

It remains to establish the continuity of $\mathscr{T}$ in the sense of the $L^2(\mathscr{Q}_T)$ norm. To this end, let us consider a sequence $\big(\tilde{c}_n\big)_{n\in\mathbb{N}}$ in $\mathscr{C}_T$ which converges to some $\tilde{c}$ (strongly) in $L^2(\mathscr{Q}_T)$. Clearly $\tilde{c} \in \mathscr{C}_T$. We define f_n and f as to be the solution of the transport equations

$$\left\{\begin{array}{l} \partial_t f_n(t,x,\xi) + \partial_\xi\big((a(\xi)\tilde{c}_n(t,x) - 1)f_n(t,x,\xi)\big) = 0 \\[2ex] \partial_t f(t,x,\xi) + \partial_\xi\big((a(\xi)\tilde{c}(t,x) - 1)f(t,x,\xi)\big) = 0 \end{array}\right.$$

for $t \ge 0$, $x \in \Omega$ and $\xi \ge 0$, with the common initial data f_{init}. Using the characteristics

$$\left\{\begin{array}{ll} \dfrac{\mathrm{d}}{\mathrm{d}s}\Xi_n(s;t,x,\xi) = a\big(\Xi_n(s;t,x,\xi)\big)\tilde{c}_n(t,x) - 1, & \dfrac{\mathrm{d}}{\mathrm{d}s}\Xi(s;t,x,\xi) = a\big(\Xi(s;t,x,\xi)\big)\tilde{c}(t,x) - 1, \\ \Xi_n(t;t,x,\xi) = \Xi(t;t,x,\xi) = \xi & \end{array}\right.$$

we write

$$\left\{\begin{array}{l} f_n(t,x,\xi) = f_{\mathrm{init}}(x, \Xi_n(0;t,x,\xi))\ J_n(0;t,x,\xi), \\[2ex] f(t,x,\xi) = f_{\mathrm{init}}(x, \Xi(0;t,x,\xi))\ J(0;t,x,\xi) \end{array}\right.$$

with

$$J_n(s;t,x,\xi) = \exp\Big(- \int_s^t a'\big(\Xi(\sigma;t,x,\xi)\big)\ \tilde c_n(\sigma;x)\,\mathrm{d}\sigma \Big),$$
$$J(s;t,x,\xi) = \exp\Big(- \int_s^t a'\big(\Xi(\sigma;t,x,\xi)\big)\ \tilde c(\sigma;x)\,\mathrm{d}\sigma \Big).$$

The first step of the the proof consists in establishing the following claim

Lemma 3.2 *Let us set*

$$A_n(t,x) = \int_0^{+\infty} a(\xi)\ f_n(t,x,\xi)\,\mathrm{d}\xi, \qquad A(t,x) = \int_0^{+\infty} a(\xi)\ f(t,x,\xi)\,\mathrm{d}\xi,$$
$$B_n(t,x) = \int_0^{+\infty} f_n(t,x,\xi)\,\mathrm{d}\xi, \qquad B(t,x) = \int_0^{+\infty} f(t,x,\xi)\,\mathrm{d}\xi.$$

Then, A_n and B_n tend to A and B, respectively, in $L^2(\mathscr{Q}_T)$.

In order to establish this property, we need an estimate on the distance between characteristic curves associated to different rates.

Lemma 3.3 *We assume that Hypothesis 3.1 is fulfilled. Let c_1 and c_2 in $\mathscr{C}_T$ and set $V_i(t,x,\xi) = a(\xi)c_i(t,x) - 1$, $i = 1,2$. We denote by Ξ_1 and Ξ_2 the associated characteristics. Then, we have for any $0 \le s,t \le T < \infty$*

$$|\Xi_1 - \Xi_2|(s;t,x,\xi) \le L_T\ (1+\xi)\ \Big(\int_s^t |c_1 - c_2|^2(\sigma,x)\,\mathrm{d}\sigma \Big)^{1/2}. \tag{16}$$

Proof. We detail the proof for the case $0 \le s \le t \le T$, the other situation follows by the same argument. By using the equation for the characteristics, we arrive at

$$\begin{aligned}
\big|\Xi_1(s;t,x,\xi) - \Xi_2(s;t,x,\xi)\big| &= \Big| \int_s^t \Big[a\big(\Xi_1(\sigma;t,x,\xi)\big)c_1(\sigma,x) - a\big(\Xi_2(\sigma;t,x,\xi)\big)c_2(\sigma,x) \Big]\,\mathrm{d}\sigma \Big| \\
&\le \int_s^t c_1(\sigma,x)\Big| a\big(\Xi_1(\sigma;t,x,\xi)\big) - a\big(\Xi_2(\sigma;t,x,\xi)\big) \Big|\,\mathrm{d}\sigma \\
&\quad + \int_s^t a\big(\Xi_2(\sigma;t,x,\xi)\big)\ \big|c_1 - c_2\big|(\sigma,x)\,\mathrm{d}\sigma \\
&\le K_T L_a \int_s^t \big|\Xi_1(\sigma;t,x,\xi) - \Xi_2(\sigma;t,x,\xi)\big|\,\mathrm{d}\sigma \\
&\quad + \Big(\int_s^t \Big| a\big(\Xi_2(\sigma;t,x,\xi)\big) \Big|^2 \mathrm{d}\sigma \Big)^{1/2} \Big(\int_s^t |c_1(\sigma,x) - c_2(\sigma,x)|\,\mathrm{d}\sigma \Big)^{1/2}.
\end{aligned}$$

On the one hand, since $a(0) = 0$, we have

$$|a(\Xi)| = \Big| a(0) + \int_0^\Xi a'(\zeta)\,\mathrm{d}\zeta \Big| \le L_a|\Xi|.$$

On the other hand, we remark that

$$\begin{aligned}
\big|\Xi_2(s;t,x,\xi)\big| &= \Big| \xi + \int_t^s \Big(a\big(\Xi_2(\sigma;t,x,\xi)\big)c_2(\sigma,x) - 1 \Big)\,\mathrm{d}\sigma \Big| \\
&\le \xi + \int_s^t \Big(1 + K_T L_a \big|\Xi_2(\sigma;t,x,\xi)\big| \Big)\,\mathrm{d}\sigma
\end{aligned}$$

holds. The Grönwall lemma then yields the estimate

$$\big|\Xi_2(s;t,x,\xi)\big| \le L_T(1+\xi).$$

It follows that

$$\big|a(\Xi_2(s;t,x,\xi))\big| \le L_T(1+\xi)$$

holds. Therefore, we obtain

$$\begin{aligned}&\big|\Xi_1(s;t,x,\xi)-\Xi_2(s;t,x,\xi)\big|\\&\quad\le L_T\left(\int_s^t\big|\Xi_1(\sigma;t,x,\xi)-\Xi_2(\sigma;t,x,\xi)\big|\,\mathrm{d}\sigma+(1+\xi)\left(\int_s^t|c_1(\sigma,x)-c_2(\sigma,x)|\,\mathrm{d}\sigma\right)^{1/2}\right).\end{aligned}$$

Applying the Grönwall lemma again leads to (16). $\square$

Proof of Lemma 3.2. By using the characteristics, we write

$$\begin{aligned}(B_n-B)(t,x) &= \int_0^\infty f_{\text{init}}(x,\Xi_n(0;t,x,\xi))J_n(0;t,x,\xi)\,\mathrm{d}\xi\\&\qquad-\int_0^{+\infty} f_{\text{init}}(x,\Xi(0;t,x,\xi))J(0;t,x,\xi)\,\mathrm{d}\xi\\&= \int_{\Xi_n(0;t,x,0)}^{+\infty} f_{\text{init}}(x,y)\,\mathrm{d}y-\int_{\Xi(0;t,x,0)}^{+\infty} f_{\text{init}}(x,y)\,\mathrm{d}y.\end{aligned}$$

It follows that

$$|B_n-B|(t,x)\le\left|\int_{\Xi(0;t,x,0)}^{\Xi_n(0;t,x,0)}|f_{\text{init}}(x,y)|\,\mathrm{d}y\right|\le\|f_{\text{init}}\|_{L^\infty(\Omega\times\mathbb{R}_+)}\big|\Xi_n-\Xi\big|(0;t,x,0),$$

and integrating over $x\in\Omega$ it yields

$$\|(B_n-B)(t,\cdot)\|^2_{L^2(\Omega)}\le\|f_{\text{init}}\|^2_{L^\infty(\Omega\times\mathbb{R}_+)}\int_\Omega\big|\Xi_n-\Xi\big|^2(0;t,x,0)\,\mathrm{d}x.$$

Hence, using Lemma 3.3, we get

$$\|(B_n-B)(t,\cdot)\|^2_{L^2(\Omega)}\le L_T\ \|f_{\text{init}}\|^2_{L^\infty(\Omega\times\mathbb{R}_+)}\int_0^t\int_\Omega\big|\tilde c_n-\tilde c\big|^2(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma.$$

We apply similar manipulations to evaluate

$$\begin{aligned}A_n(t,x)-A(t,x)&=\int_0^\infty a(\xi)f_n(t,x,\xi)d\xi-\int_0^\infty a(\xi)f(t,x,\xi)\,\mathrm{d}\xi\\&=\int_{\Xi_n(0;t,x,0)}^\infty a(\Xi_n(t;0,x,y))f_{\text{init}}(x,y)\,\mathrm{d}y\\&\qquad-\int_{\Xi(0;t,x,0)}^\infty a(\Xi(t;0,x,y))f_{\text{init}}(x,y)\,\mathrm{d}y.\end{aligned}$$

Indeed, we have

$$\begin{aligned}\big|A_n-A\big|(t,x)&\le\Big|\int_{\Xi_n(0;t,x,0)}^{\Xi(0;t,x,0)}a(\Xi_n(t;0,x,y))\ f_{\text{init}}(x,y)\,\mathrm{d}y\Big|\\&\qquad+\int_{\Xi(0;t,x,0)}^{+\infty}\big|a\big(\Xi_n(t;0,x,y)\big)-a\big(\Xi(t;0,x,y)\big)\big|\ f_{\text{init}}(x,y)\,\mathrm{d}y\\&\le L_a\left(\Big|\int_{\Xi_n(0;t,x,0)}^{\Xi(0;t,x,0)}|\Xi_n(t;0,x,y)|\ f_{\text{init}}(x,y)\,\mathrm{d}y\Big|\right.\\&\qquad\left.+\int_{\Xi(0;t,x,0)}^{+\infty}\big|\Xi_n(t;0,x,y)-\Xi(t;0,x,y)\big|\ f_{\text{init}}(x,y)\,\mathrm{d}y\right).\end{aligned}$$

We observe that

$$\begin{aligned}
\big|\Xi_n(t;0,x,y)\big| &= \big|\Xi_n(t;0,x,y) - \Xi_n(t;0,x,\Xi_n(0;t,x,0))\big| \\
&= \Big|\int_y^{\Xi_n(0;t,x,0)} \partial_\xi \Xi_n(t;0,x,\xi)\,\mathrm{d}\xi\Big| \\
&= \Big|\int_y^{\Xi_n(0;t,x,0)} \exp\Big(-\int_t^0 a'\big(\Xi_n(\sigma;t,x,\xi)\big)\ \tilde{c}_n(\sigma,x)\,\mathrm{d}\sigma\Big)\,\mathrm{d}\xi\Big| \\
&\leq L_T\big|y - \Xi_n(0;t,x,0)\big|
\end{aligned}$$

by using Hypothesis 1.1. Since we are concerned with y restricted to the interval defined by $\Xi_n(0;t,x,0)$ and $\Xi(0;t,x,0)$ we have

$$\big|\Xi_n(t;0,x,y)\big| \leq L_T\ \big|\Xi - \Xi_n\big|(0;t,x,0).$$

It yields

$$\Big|\int_{\Xi_n(0;t,x,0)}^{\Xi(0;t,x,0)} a\big(\Xi_n(t;0,x,y)\big)\ f_{\mathrm{init}}(x,y)\,\mathrm{d}y\Big|\ \leq L_T\ \big|\Xi - \Xi_n\big|(0;t,x,0)\ \int_0^{+\infty} f_{\mathrm{init}}(x,y)\,\mathrm{d}y. \tag{17}$$

Moreover, Lemma 3.3 allows to estimate

$$\big|\Xi_n - \Xi\big|(t;0,x,y) \leq L_T\ (1+y)\ \Big(\int_0^t |\tilde{c}_n - \tilde{c}|^2(\sigma,x)\,\mathrm{d}\sigma\Big)^{1/2}. \tag{18}$$

Combining (17) et (18) we are led to

$$\begin{aligned}
\big|(A_n - A)(t,x)\big| \leq L_T\Bigg(&\big|\Xi_n - \Xi\big|(0;t,x,0)\int_0^{+\infty} f_{\mathrm{init}}(x,y)\,\mathrm{d}y \\
&+\Big(\int_0^t |\tilde{c}_n - \tilde{c}|^2(\sigma,x)\,\mathrm{d}\sigma\Big)^{1/2}\int_0^{+\infty}(1+y)f_{\mathrm{init}}(x,y)\,\mathrm{d}y\Bigg).
\end{aligned}$$

Therefore, we deduce that

$$\big\|A_n - A\big\|_{L^2(\mathscr{Q}_T)} \leq L_T\big\|\tilde{c}_n - \tilde{c}\big\|_{L^2(\mathscr{Q}_T)}.$$

It finishes the proof of Lemma 3.2. $\square$

We are left with the task of proving that $c_n = \mathscr{T}(\tilde{c}_n)$ converges to $c = \mathscr{T}(\tilde{c})$ in $L^2(\mathscr{Q}_T)$. We remind that c_n and c are the solutions of

$$\begin{cases}
\partial_t c_n(t,x) - \Delta_x c_n(t,x) + A_n(t,x)c_n(t,x) = B_n(t,x), \\
\partial_t c(t,x) - \Delta_x c(t,x) + A(t,x)c(t,x) = B(t,x), \\
\partial_\nu c_n = 0, \qquad \partial_\nu c = 0 \qquad \text{on } \partial\Omega, \\
c_n(0,x) = c(0,x) = c_{\mathrm{init}}(x).
\end{cases}$$

We obtain the following energy estimate

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (c_n - c)^2(t,x)\,\mathrm{d}x + \int_\Omega \big|\nabla_x(c_n - c)\big|^2(t,x)\,\mathrm{d}x \\
&\quad = -\int_\Omega (c_n - c)(c_n A_n - cA)(t,x)\,\mathrm{d}x + \int_\Omega (c_n - c)(B_n - B)(t,x)\,\mathrm{d}x.
\end{aligned}$$

It can be recast as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (c_n-c)^2(t,x)\,\mathrm{d}x + \int_\Omega \left|\nabla_x(c_n-c)\right|^2(t,x)\,\mathrm{d}x + \int_\Omega A(c_n-c)^2(t,x)\,\mathrm{d}x$$

$$= -\int_\Omega c_n\ (c_n-c)\ (A_n-A)(t,x)\,\mathrm{d}x + \int_\Omega (c_n-c)\ (B_n-B)(t,x)\,\mathrm{d}x.$$

We make use of the Cauchy-Schwarz and Young inequalities, together with the fact that $c_n \in \mathscr{C}_T$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (c_n-c)^2(t,x)\,\mathrm{d}x + \int_\Omega \left|\nabla_x(c_n-c)\right|^2(t,x)\,\mathrm{d}x + \int_\Omega A(c_n-c)^2(t,x)\,\mathrm{d}x$$

$$\le \int_\Omega (c_n-c)^2(t,x)\,\mathrm{d}x + \frac{K_T^2}{2}\int_\Omega (A_n-A)^2(t,x)\,\mathrm{d}x + \frac{1}{2}\int_\Omega (B_n-B)^2(t,x)\,\mathrm{d}x.$$

Eventually, an application of the Grönwall lemma yields

$$\int_\Omega (c_n-c)^2(t,x)\,\mathrm{d}x \le C_T\left(\int_{\mathscr{Q}_T} |B_n-B|^2(s,x)\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathscr{Q}_T} |A_n-A|^2(s,x)\,\mathrm{d}x\,\mathrm{d}s\right)$$

on $0 \le t \le T < \infty$ where C_T depends only on T, C_0 and $\|c_{\mathrm{init}}\|_\infty$. Coming back to Lemma 3.2 we conclude that c_n tends to c in $L^2(\mathscr{Q}_T)$. □

Having established the properties of the mapping $\mathscr{T}$, we can apply the Schauder theorem which proves the existence of a fixed point $c = \mathscr{T}(c) \in \mathscr{C}_T$. The fixed point c then satisfies

$$\partial_t c + Ac = \Delta_x c + B \qquad \text{on } (0,T)\times\Omega,$$

endowed with $\partial_\nu c = 0$ on $\partial\Omega$ and the initial data $c_{|t=0} = c_{\mathrm{init}}$, where

$$A(t,x) = \int_0^\infty a(\xi) f(t,x,\xi)\,\mathrm{d}\xi, \qquad B(t,x) = \int_0^\infty f(t,x,\xi)\,\mathrm{d}\xi,$$

and

$$\partial_t f + \partial_\xi\big((a(\xi)c(t,x)-1)f\big) = 0 \text{ on } (0,T)\times\Omega\times(0,\infty),$$

with initial data $f_{|t=0} = f_{\mathrm{init}}$. This ends the proof of the existence of solution to the system (3)–(5).

What we did can be used to justify the uniqueness of the solution as well. Indeed let us assume that (c_1,f_1) and (c_2,f_2) are solutions of (3)–(5) for the same initial data $(c_{\mathrm{init}}, f_{\mathrm{init}})$. Reproducing the arguments for proving the continuity of $\mathscr{T}$, we arrive at

$$\int_\Omega (c_1-c_2)^2(t,x)\,\mathrm{d}x \le C_T\left(\int_0^t\int_\Omega \Big(|B_1-B_2|^2 + |A_1-A_2|^2\Big)(s,x)\,\mathrm{d}x\,\mathrm{d}s\right).$$

Now, coming back to the proof of Lemma 3.2, we can estimate the right hand side so that

$$\int_\Omega (c_1-c_2)^2(t,x)\,\mathrm{d}x \le L_T\int_0^t\int_\Omega |c_1-c_2|^2(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

The Grönwall lemma then implies that $c_1 = c_2$. □

As a concluding remark of this section, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \xi f(t,x,\xi)\,\mathrm{d}\xi = \int_0^\infty \big(a(\xi)c(t,x)-1\big) f(t,x,\xi)\,\mathrm{d}\xi = A(t,x)c(t,x) - B(t,x)$$

holds. (It follows by integrating by parts, we refer to [6, Lemma 3] for details.) Thus, the obtained solution satisfies the mass conservation relation

$$\int_\Omega c(t,x)\,\mathrm{d}x + \int_\Omega\int_0^\infty \xi f(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \int_\Omega c_{\mathrm{init}}(x)\,\mathrm{d}x + \int_\Omega\int_0^\infty \xi f_{\mathrm{init}}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x.$$

4 Existence-uniqueness for general data

In this section we wish to relax the assumptions on the initial data, requiring only

$$f_{\text{init}} \in L^\infty(\Omega; L^1((0,\infty),(1+\xi)\,\mathrm{d}\xi))$$

and removing the finiteness of the uniform norm of f_{init} which could be physically questionable.

4.1 Existence

To justify the existence of solution in this framework, we appeal to approximation and compactness arguments. To this end, we consider a sequence f^n_{init} made of bounded functions which converge to f_{init} in $L^1(\Omega \times \mathbb{R}_+, (1+\xi)\,\mathrm{d}\xi\,\mathrm{d}x)$:

$$\begin{aligned}
&0 \le f^n_{\text{init}}(x,\xi) \le C_n, \qquad 0 \le f^n_{\text{init}}(x,\xi) \le f_{\text{init}}(x,\xi),\\
&\int_\Omega \int_0^\infty (1+\xi)|f^n_{\text{init}} - f|(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \xrightarrow[n\to\infty]{} 0,\\
&\int_0^\infty f^n_{\text{init}}(x,\xi)\,\mathrm{d}\xi \le \int_0^\infty f_{\text{init}}(x,\xi)\,\mathrm{d}\xi \le C_0, \qquad \int_0^\infty \xi f^n_{\text{init}}(x,\xi)\,\mathrm{d}\xi \le \int_0^\infty \xi f_{\text{init}}(x,\xi)\,\mathrm{d}\xi.
\end{aligned}$$

(with C_n possibly tending to $+\infty$; for instance we can set $f^n_{\text{init}}(x,\xi) = \mathbf{1}_{0 \le f_{\text{init}}(x,\xi) \le n} f_{\text{init}}(x,\xi)$). According to the previous Section we can associate to f^n_{init} the solution of the system

$$\left\{\begin{array}{ll}
\partial_t f^n(t,x,\xi) + \partial_\xi((a(\xi)c^n(t,x)-1)f^n(t,x,\xi)) = 0 & t \ge 0,\ x \in \Omega,\ \xi \ge 0,\\[2ex]
\partial_t c^n(t,x) - \Delta_x c^n(t,x) + A^n(t,x)c^n(t,x) = B^n(t,x) & t \ge 0,\ x \in \Omega,\\
\partial_\nu c^n = 0 & \text{on } \partial\Omega,\\
A^n(t,x) = \displaystyle\int_0^\infty a(\xi) f^n(t,x\xi)\,\mathrm{d}\xi, \quad B^n(t,x) = \int_0^\infty f^n(t,x\xi)\,\mathrm{d}\xi, &\\
f^n_{|t=0} = f^n_{\text{init}}, \qquad c^n_{|t=0} = c_{\text{init}}. &
\end{array}\right. \tag{19}$$

We can collect the following estimates, on $0 \le t \le T < \infty$

$$\begin{aligned}
&0 \le c^n(t,x) \le K_T(= \|c_{\text{init}}\|_\infty + C_0 T),\\
&\int_0^\infty f^n(t,x,\xi)\,\mathrm{d}\xi \le C_0,\\
&\int_\Omega |c^n|^2(t,x)\,\mathrm{d}x + \int_0^t \int_\Omega |\nabla_x c^n|^2(s,x)\,\mathrm{d}x\,\mathrm{d}s \le C_T < \infty\\
&\int_\Omega \int_0^\infty \xi f^n\,\mathrm{d}\xi\,\mathrm{d}x \le \int_\Omega c^n(t,x)\,\mathrm{d}x + \int_\Omega \int_0^\infty \xi f^n(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\
&\qquad\qquad \le \int_\Omega c_{\text{init}}(x)\,\mathrm{d}x + \int_\Omega \int_0^\infty \xi f^n_{\text{init}}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \le \int_\Omega c_{\text{init}}(x)\,\mathrm{d}x + \int_\Omega \int_0^\infty \xi f_{\text{init}}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x,
\end{aligned}$$

with C_T a finite constant depending on $\|c_{\text{init}}\|_{L^2(\Omega)}$, C_0 and T. Accordingly,

$$A^n \text{ and } B^n \text{ are bounded in } L^\infty(\mathscr{Q}_T).$$

Therefore, $\partial_t c^n$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. We can apply the compactness statement in [24] which implies that, possibly at the price of extracting a subsequence,

$$c^n \to c \text{ strongly in } L^2(\mathscr{Q}_T) \text{ and a. e..}$$

We can also show that c^n converges to c in $C^0([0,T]; L^2(\Omega) - \text{weak})$.

Next, we discuss further estimates on f^n. From the uniform integrability of $(f^n)_n$ and by using De La Vallée Poussin's lemma, see [9, p. 38], there exists a non negative function Φ satisfying

$$\Phi(0) = 0, \qquad \lim_{\tau\to+\infty} \frac{\Phi(\tau)}{\tau} = +\infty, \qquad \Phi \text{ is convex},$$

and such that

$$\sup_{n\in\mathbb{N}}\int_\Omega\int_0^\infty \Phi(f^n_{\text{init}})\,\mathrm{d}\xi\,\mathrm{d}x \le C<\infty.$$

Using characteristics, we show that the property extends to the solution f^n. Indeed, we have, with obvious notation,

$$f^n(t,x,\xi)=f^n_{\text{init}}\big(x,\Xi^n(0;t,x,\xi)\big)\ J^n(0;t,x,\xi).$$

Since $0\le J^n(0;t,x,\xi)\le 1$ and $\Phi(0)=0$, the convexity of Φ yields

$$\Phi\big(f^n(t,x,\xi)\big)\le \Phi\big(f^n_{\text{init}}\big(x,\Xi^n(0;t,x,\xi)\big)\big)\ J^n(0;t,x,\xi).$$

Integrating leads to the following uniform estimate

$$\int_\Omega\int_0^\infty \Phi\big(f^n(t,x,\xi)\big)\,\mathrm{d}\xi\,\mathrm{d}x \le \int_{\Xi^n(0;t,x,0)}^\infty \Phi\big(f^n_{\text{init}}(x,\xi)\big)\,\mathrm{d}\xi\le \int_0^\infty \Phi\big(f^n_{\text{init}}(x,\xi)\big)\,\mathrm{d}\xi\le C<\infty.$$

Since moreover the first moment with respect to ξ of f^n is controlled, the Dunford-Pettis theorem, see e. g. [10, Theorem 4.21.2], implies that f^n is relatively compact in $L^1((0,T)\times\Omega\times(0,\infty))$ for the weak topology. We can thus assume that

$$f^n \rightharpoonup f \text{ weakly in } L^1((0,T)\times\Omega\times(0,\infty)).$$

Furthermore, we can apply the De La Vallée Poussin Lemma again to exhibit a non negative function Ψ such that

$$\Psi(0)=0,\qquad \lim_{\tau\to+\infty}\frac{\Psi(\tau)}{\tau}=+\infty,\qquad \Psi \text{ is convex},$$

and

$$\sup_{n\in\mathbb{N}}\int_\Omega\int_0^\infty \Psi(\xi)\ f^n_{\text{init}}\,\mathrm{d}\xi\,\mathrm{d}x\le \int_\Omega\int_0^\infty \Psi(\xi)\ f_{\text{init}}\,\mathrm{d}\xi\,\mathrm{d}x\le C<\infty.$$

This is the De La Vallée Lemma applied to the function $(\xi\mapsto\xi)\in L^1(\Omega\times(0,\infty), f_{\text{init}}\,\mathrm{d}\xi\,\mathrm{d}x)$. As remarked in [3, Proposition I.1.1], we can suppose moreover that $\Psi'(\tau)\ge 0$ and Ψ' is concave. Therefore we have (see [12, Lemma A.1])

$$\Psi(\xi)\le \xi\Psi'(\xi)\le 2\Psi(\xi).$$

Integrating the equation satisfied by f^n we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \Psi(\xi)\ f^n(t,x,\xi)\,\mathrm{d}\xi &= \int_0^\infty \Psi'(\xi)\big(a(\xi)c^n(t,x)-1\big)\ f^n(t,x,\xi)\,\mathrm{d}\xi\\ &\le K_T\int_0^\infty \Psi'(\xi)a(\xi)\ f^n(t,x,\xi)\,\mathrm{d}\xi.\end{aligned}$$

We evaluate the right hand side by separating small and large sizes: let $\xi_0>0$ and write

$$\begin{aligned}\int_0^\infty \Psi'(\xi)a(\xi)\ f^n(t,x,\xi)\,\mathrm{d}\xi &= \int_0^{\xi_0} \Psi'(\xi)a(\xi)\ f^n(t,x,\xi)\,\mathrm{d}\xi+\int_{\xi_0}^\infty \Psi'(\xi)a(\xi)\ f^n(t,x,\xi)\,\mathrm{d}\xi\\ &\le \sup_{0\le z\le\xi_0}\big(\Psi'(z)a(z)\big)\int_0^\infty f^n(t,x,\xi)\,\mathrm{d}\xi+L_{a,0}\int_0^\infty \Psi'(\xi)\xi f^n(t,x,\xi)\,\mathrm{d}\xi\\ &\le \sup_{0\le z\le\xi_0}\big(\Psi'(z)a(z)\big)C_0+2L_{a,0}\int_0^\infty \Psi(\xi)f^n(t,x,\xi)\,\mathrm{d}\xi\end{aligned}$$

where C_0 is defined in Equation (14). Hence applying the Grönwall lemma yields the uniform estimate

$$\int_\Omega \int_0^\infty \Psi(\xi)\ f^n(t,x,\xi)\,\mathrm{d}\xi \le C_T$$

on $0 \le t \le T < \infty$ with $C_T > 0$ depending on C_0, ξ_0, Ω and T.

Therefore, for any function φ such that $|\varphi(\xi)| \le C(1+\xi)$, we can show that

$$\int_0^\infty \varphi(\xi) f^n(t,x,\xi)\,\mathrm{d}\xi \rightharpoonup \int_0^\infty \varphi(\xi) f(t,x,\xi)\,\mathrm{d}\xi \text{ weakly in } L^1((0,T)\times\Omega).$$

As a consequence A^n and B^n converge weakly to $A(t,x) = \int_0^\infty a(\xi) f(t,x,\xi)\,\mathrm{d}\xi$ and $B(t,x) = \int_0^\infty f(t,x,\xi)\,\mathrm{d}\xi$ in $L^1((0,T)\times\Omega)$, respectively. Since c^n is uniformly bounded and converges a.e. to c, a classical application of the Dunford-Pettis and Egoroff theorems proves that $c^n f^n$ converges weakly to cf in $L^1((0,T)\times\Omega\times(0,\infty))$. Similarly $A^n c^n$ converges weakly to Ac in $L^1((0,T)\times\Omega)$. Note also that $\partial_t f^n$ is bounded in $L^\infty((0,T)\times\Omega; W^{-1,1}(0,\infty))$[1], so that f^n is compact in $C^0([0,T]; L^1(\Omega\times(0,\infty)) - \text{weak})$. Finally, we can let n go to ∞ in (19); it shows that the pair (c,f) satisfies

$$\begin{cases} \partial_t f(t,x,\xi) + \partial_\xi((a(\xi)c(t,x) - 1) f(t,x,\xi)) = 0 & t \ge 0,\ x \in \Omega,\ \xi \ge 0, \\[1em] \partial_t c(t,x) - \Delta_x c(t,x) + A(t,x) c(t,x) = B(t,x) & t \ge 0,\ x \in \Omega, \\ \partial_\nu c = 0 & \text{on } \partial\Omega, \\ A(t,x) = \displaystyle\int_0^\infty a(\xi) f(t,x,\xi)\,\mathrm{d}\xi, \quad B(t,x) = \int_0^\infty f(t,x,\xi)\,\mathrm{d}\xi, & \\ f_{|t=0} = f_{\mathrm{init}}, \quad c_{|t=0} = c_{\mathrm{init}}. & \end{cases} \tag{20}$$

Note that we also get the mass conservation relation

$$\int_\Omega c(t,x)\,\mathrm{d}x + \int_\Omega \int_0^\infty \xi f(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \int_\Omega c_{\mathrm{init}}(x)\,\mathrm{d}x + \int_\Omega \int_0^\infty \xi f_{\mathrm{init}}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x.$$

Remark 4.1 *We remark that in the arguments developed here, the estimates do not involve the Lipschitz constant L_a that appears in Hypothesis 3.1. Therefore, we can adapt straightforwardly the proof to deal with non smooth coefficients $a(\xi)$, as stated in Hypothesis 1.1, including the physical case $a(\xi) = \xi^{1/3}$. It suffices to consider a sequence of smooth coefficients which converges pointwise to $a(\xi)$. A subsequence extracted from the associated solutions converges to (c,f), solution with the coefficient a. We refer to [12] for such an extension in the context of the homogeneous Lifschitz-Slyozov equation.*

4.2 Uniqueness

Let us consider $(c^{(1)}, f^{(1)})$ and $(c^{(2)}, f^{(2)})$ solution of (3) as obtained in the previous Section and let $0 < T < \infty$ be fixed once for all. We wish to prove that $c^{(1)} = c^{(2)}$ and $f^{(1)} = f^{(2)}$ for a.e $(t,x) \in (0,T)\times\Omega$ and $\xi \ge 0$ when the initial data coincide. We start by deriving an L^1 estimate for the monomers concentration instead of the usual L^2 energy estimate. To this end, let $\eta > 0$ and introduce the function $S_\eta(z) = z/\sqrt{\eta + z^2}$ which approaches the sign function. Observe that $S_\eta \in C^1(\mathbb{R})$ with $S'_\eta(s) = \frac{\eta}{(s^2+\eta)^{3/2}} \ge 0$ so that by Stampacchia's theorem for $w \in H^1(\Omega)$, $S_\eta(w)$

[1]Here, for $1 \le q \le \infty$, we denote by $W^{-1,q}(\Omega)$ the space of distributions which write as finite sums of zeroth and first order derivatives of functions belonging to $L^q(\Omega)$. Given $1 \le p < \infty$, for $1/p + 1/q = 1$, $W^{-1,q}(\Omega)$ identifies with the dual space of $W_0^{1,p}(\Omega)$, the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$, see [26, Definition 31.3 & Proposition 31.3].

belongs to $H^1(\Omega)$ too. Note also that $Z_\eta(z) = \int_0^z S_\eta(\tau)\, d\tau$ approaches $|s|$ as η goes to 0, with $0 \le Z_\eta(z) \le |z|$. We have

$$\big(\partial_t - \Delta_x + A^{(1)}\big)(c^{(1)} - c^{(2)}) = B^{(1)} - B^{(2)} + (A^{(2)} - A^{(1)})c^{(2)}.$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega Z_\eta(c^{(1)} - c^{(2)})\,\mathrm{d}x &+ \int_\Omega |\nabla_x(c^{(1)} - c^{(2)})|^{(2)}\ S'_\eta(c^{(1)} - c^{(2)})\,\mathrm{d}x\\ &+ \int_\Omega A^{(1)}(c^{(1)} - c^{(2)})S_\eta(c^{(1)} - c^{(2)})\,\mathrm{d}x\\ &= \int_\Omega \big(B^{(1)} - B^{(2)} + (A^{(2)} - A^{(1)})c^{(2)}\big)S_\eta(c^{(1)} - c^{(2)})\,\mathrm{d}x.\end{aligned}$$

Since $|S_\eta(z)| \le 1$, $S'_\eta(z) \ge 0$ and $zS_\eta(z) \ge 0$, we arrive at the following estimate

$$\begin{aligned}\int_\Omega Z_\eta(c^{(1)} - c^{(2)})(t,x)\,\mathrm{d}x \quad \le \quad & \int_\Omega Z_\eta(c^{(1)}_{\text{init}} - c^{(2)}_{\text{init}})(t,x)\,\mathrm{d}x\\ &+\int_0^t\int_\Omega |B^{(1)} - B^{(2)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ &+K_T\int_0^t\int_\Omega |A^{(2)} - A^{(1)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Letting $\eta \to 0$ yields

$$\begin{aligned}\int_\Omega |c^{(1)} - c^{(2)}|(t,x)\,\mathrm{d}x \quad \le \quad & \int_\Omega |c^{(1)}_{\text{init}} - c^{(2)}_{\text{init}}|(t,x)\,\mathrm{d}x\\ &+\int_0^t\int_\Omega |B^{(1)} - B^{(2)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s + K_T\int_0^t\int_\Omega |A^{(2)} - A^{(1)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}\tag{21}$$

The next step of the proof of uniqueness relies on an adaptation of the reasoning and estimates in [12] for the homogeneous case. We associate to $f^{(k)}$ $(k = 1, 2)$ the repartition function

$$F^{(k)}(t,x,\xi) = \int_\xi^\infty f^{(k)}(t,x,\zeta)\,\mathrm{d}\zeta.$$

As a matter of fact, we have

$$\partial_\xi F^{(k)} = -f^{(k)},$$

and

$$F^{(k)}(t,x,0) = \int_0^\infty f^{(k)}(t,x,\xi)\,\mathrm{d}\xi = B^{(k)}(t,x), \qquad \int_0^\infty F^{(k)}(t,x,\xi)\,\mathrm{d}\xi = \int_0^\infty \xi\ f^{(k)}(t,x,\xi)\,\mathrm{d}\xi.$$

We need to introduce $\xi_T > 0$ such that for any $0 \le \xi \le \xi_T$, we have $a(\xi)K_T - 1 \le a(\xi_T)K_T - 1 < 0$, which makes sense owing to Hypothesis 1.1. Furthermore, we can pick $r > 1$ large enough such that

$$K_T a(\xi) - 1 \le -2\frac{K_T a(\xi_T) + 1}{r} < 0 \qquad \text{holds for any } 0 \le \xi \le \xi_T.$$

In what follows, $L_{a,T}$ will stand for the Lipschitz constant of a on $[\xi_T, \infty)$. We will use weighted L^1 estimate, which relies of defining the auxiliary function

$$W_T(\xi) = \begin{cases} \dfrac{1}{a(\xi_T) + 1 - a(\xi)} & \text{for } 0 \le \xi \le \xi_T,\\ 1 & \text{for } \xi \ge \xi_T.\end{cases}$$

Note that

$$0 < \frac{1}{a(\xi_T) + 1} \leq W_T(\xi) \leq 1.$$

We have

$$\partial_t (f^{(1)} - f^{(2)}) + \partial_\xi \big((ac^{(1)} - 1)(f^{(1)} - f^{(2)}) \big) = \partial_\xi \big(a(c^{(2)} - c^{(1)}) f^{(2)} \big),$$

and thus

$$\partial_t (F^{(1)} - F^{(2)}) + (ac^{(1)} - 1) \partial_\xi (F^{(1)} - F^{(2)}) = -a(c^{(2)} - c^{(1)}) f^{(2)}.$$

Up to a regularization argument we deduce the following inequality (obtained formally by multiplying the previous relation by $|W_T(\xi)|^r \mathrm{sgn}(F^{(1)} - F^{(2)})$ and integrating over $\xi \in (0, \infty)$).

$$\begin{aligned}
&\int_0^\infty |W_T(\xi)|^r \ |F^{(1)} - F^{(2)}|(t, x, \xi) \,\mathrm{d}\xi + \int_0^t |W_T(0)|^r \ |F^{(1)} - F^{(2)}|(s, x, 0) \,\mathrm{d}s \\
&\quad \leq \int_0^\infty |W_T(\xi)|^r \ |F^{(1)}_{\mathrm{init}} - F^{(2)}_{\mathrm{init}}|(t, x, \xi) \,\mathrm{d}\xi + \int_0^t \int_0^\infty a(\xi) \ |c^{(2)} - c^{(1)}|(s, x) \ f^{(2)}(s, x, \xi) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \int_0^t \int_0^\infty \partial_\xi \big((a(\xi) c^{(1)}(s, x) - 1) |W_T(\xi)|^r \big) |F^{(1)} - F^{(2)}|(s, x, \xi) \,\mathrm{d}\xi \,\mathrm{d}s.
\end{aligned}$$

The last integral in the right hand side can be recast as

$$\begin{aligned}
&\int_0^t \int_0^{\xi_T} r a'(\xi) |W_T(\xi)|^{r+1} \Big(\frac{a(\xi_T) + 1 - a(\xi)}{r} c^{(1)}(s, x) + a(\xi) c^{(1)}(s, x) - 1 \Big) |F^{(1)} - F^{(2)}|(s, x, \xi) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \int_0^t \int_{\xi_T}^\infty a'(\xi) c^{(1)}(s, x) \ |F^{(1)} - F^{(2)}|(s, x, \xi) \,\mathrm{d}\xi \,\mathrm{d}s.
\end{aligned}$$

When $0 \leq \xi \leq \xi_T$, the integrand is dominated by

$$\begin{aligned}
&r a'(\xi) |W_T(\xi)|^{r+1} \Big[K_T \Big(\frac{a(\xi_T) + 1}{r} + a(\xi) \Big) - 1 \Big] |F^{(1)} - F^{(2)}|(s, x, \xi) \\
&\leq -(K_T a(\xi_T) + 1) a'(\xi) |W_T(\xi)|^{r+1} |F^{(1)} - F^{(2)}|(s, x, \xi) \leq 0
\end{aligned}$$

according to the definition of ξ_T and the choice of r. When $\xi \geq \xi_T$ we can simply use the fact that $a'(\xi)$ is bounded far away from $\xi = 0$. Note that we can also dominate, for some $\xi_0 > 0$,

$$\begin{aligned}
&\int_0^t \int_0^\infty a(\xi) \ |c^{(2)} - c^{(1)}|(s, x) \ f^{(2)}(s, x, \xi) \,\mathrm{d}\xi \,\mathrm{d}s = \int_0^t \left(\int_0^{\xi_0} ... \,\mathrm{d}\xi + \int_{\xi_0}^\infty ... \,\mathrm{d}\xi \right) \mathrm{d}s \\
&\quad \leq \sup_{0 \leq \xi \leq \xi_0} \big(a(\xi) \big) \int_0^t \int_0^\infty |c^{(2)} - c^{(1)}|(s, x) \ f^{(2)}(s, x, \xi) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad\qquad + L_{a,0} \int_0^t \int_0^\infty |c^{(2)} - c^{(1)}|(s, x) \ \xi f^{(2)}(s, x, \xi) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\quad \leq \left(\sup_{0 \leq \xi \leq \xi_0} \big(a(\xi) \big) \ C_0 + L_{a,0} \ C_T \right) \int_0^t |c^{(2)} - c^{(1)}|(s, x) \,\mathrm{d}s.
\end{aligned}$$

Finally, we are led to the following estimate

$$
\begin{aligned}
&\int_0^\infty |W_T(\xi)|^r \ |F^{(1)} - F^{(2)}|(t,x,\xi)\,\mathrm{d}\xi \\
&\quad + \int_0^t |W_T(0)|^r \ |F^{(1)} - F^{(2)}|(s,x,0)\,\mathrm{d}s \\
&\quad + (K_T a(\xi_T)+1) \int_0^t \int_0^{\xi_T} |W_T(\xi)|^{r+1} \ a'(\xi) \ |F^{(1)} - F^{(2)}|(s,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\qquad \le \int_0^\infty |W_T(\xi)|^r \ |F^{(1)}_{\text{init}} - F^{(2)}_{\text{init}}|(t,x,\xi)\,\mathrm{d}\xi \\
&\qquad\quad + \Big(\sup_{0\le\xi\le\xi_0} \big(a(\xi)\big) \ C_0 + L_{a,0} \ C_T \Big) \ \int_0^t |c^{(2)} - c^{(1)}|(s,x)\,\mathrm{d}s \\
&\qquad\quad + L_{a,T} \int_0^t \int_0^\infty |F^{(1)} - F^{(2)}|(s,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}s,
\end{aligned}
\tag{22}
$$

where C_T is the bound on $\int_0^\infty \xi f^{(k)}(t,x,\xi)\,\mathrm{d}\xi$.

We combine the obtained relations, bearing in mind that W_T is bounded from below and above and that $B^{(k)}(t,x) = F^{(k)}(t,x,0)$. Let $\lambda > 0$ to be precised. By using (21) and (22), we are led to

$$
\begin{aligned}
&\frac{1}{(a(\xi_T)+1)^r} \int_\Omega \int_0^\infty |F^{(1)} - F^{(2)}|(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x + \lambda \int_\Omega |c^{(1)} - c^{(2)}|(t,x)\,\mathrm{d}x \\
&\quad + \Big(\frac{1}{(a(\xi_T)+1)^r} - \lambda \Big) \int_0^t \int_\Omega |B^{(1)} - B^{(2)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{K_T a(\xi_T)+1}{(a(\xi_T)+1)^{r+1}} \int_0^t \int_\Omega \int_0^{\xi_T} a'(\xi) \ |F^{(1)} - F^{(2)}|(s,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_\Omega \int_0^\infty |F^{(1)}_{\text{init}} - F^{(2)}_{\text{init}}|(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x + \lambda \int_\Omega |c^{(1)}_{\text{init}} - c^{(2)}_{\text{init}}|(t,x)\,\mathrm{d}x \\
&\quad + \Big(\sup_{0\le\xi\le\xi_0} \big(a(\xi)\big) \ C_0 + L_{a,0} \ C_T \Big) \int_0^t \int_\Omega |c^{(1)} - c^{(2)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + L_{a,T} \int_0^t \int_\Omega \int_0^\infty |F^{(1)} - F^{(2)}|(s,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\quad + \lambda K_T \int_0^t \int_\Omega |A^{(2)} - A^{(1)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

on $0 \le t \le T < \infty$. It remains to discuss the last integral of the right hand side. We split as follows

$$
\begin{aligned}
|A^{(2)} - A^{(1)}|(s,x) &= \Big| \int_0^\infty a(\xi)(f^{(2)} - f^{(1)})(s,x,\xi)\,\mathrm{d}\xi \Big| = \Big| \int_0^\infty a(\xi)\partial_\xi (F^{(1)} - F^{(2)})(s,x,\xi)\,\mathrm{d}\xi \Big| \\
&= \Big| \int_0^\infty a'(\xi)(F^{(1)} - F^{(2)})(s,x,\xi)\,\mathrm{d}\xi \Big| = \Big| \int_0^{\xi_T} \dots + \int_{\xi_T}^\infty \dots \Big| \\
&\le \int_0^{\xi_T} a'(\xi) \ |F^{(1)} - F^{(2)}|(s,x,\xi)\,\mathrm{d}\xi + L_{a,T} \int_{\xi_T}^\infty |F^{(1)} - F^{(2)}|(s,x,\xi)\,\mathrm{d}\xi.
\end{aligned}
$$

We now rearrange terms to obtain

$$\begin{aligned}
&\frac{1}{(a(\xi_T)+1)^r}\int_\Omega\int_0^\infty |F^{(1)}-F^{(2)}|(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x+\lambda\int_\Omega |c^{(1)}-c^{(2)}|(t,x)\,\mathrm{d}x\\
&\quad+\Big(\frac{1}{(a(\xi_T)+1)^r}-\lambda\Big)\int_0^t\int_\Omega |B^{(1)}-B^{(2)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\Big(\frac{K_T a(\xi_T)+1}{(a(\xi_T)+1)^{r+1}}-\lambda K_T\Big)\int_0^t\int_\Omega\int_0^{\xi_T} a'(\xi)\ |F^{(1)}-F^{(2)}|(s,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}s\\
&\le \int_\Omega\int_0^\infty |F^{(1)}_{\mathrm{init}}-F^{(2)}_{\mathrm{init}}|(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x+\lambda\int_\Omega |c^{(1)}_{\mathrm{init}}-c^{(2)}_{\mathrm{init}}|(t,x)\,\mathrm{d}x\\
&\quad+\Big(\sup_{0\le\xi\le\xi_0}\big(a(\xi)\big)\ C_0+L_{a,0}\ C_T\Big)\int_0^t\int_\Omega |c^{(1)}-c^{(2)}|(s,x)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+L_{a,T}(1+\lambda K_T)\int_0^t\int_\Omega\int_0^\infty |F^{(1)}-F^{(2)}|(s,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Thus, we pick $\lambda>0$ so that

$$\frac{1}{(a(\xi_T)+1)^r}>\lambda>0 \qquad\text{and}\qquad \frac{a(\xi_T)+1/K_T}{(a(\xi_T)+1)^{r+1}}>\lambda>0.$$

It suffices to apply the Grönwall lemma to conclude with a continuity estimate where

$$\int_\Omega\int_0^\infty |F^{(1)}-F^{(2)}|(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \qquad\text{and}\qquad \int_\Omega |c^{(1)}-c^{(2)}|(t,x)\,\mathrm{d}x$$

are dominated on $0\le t\le T$ by

$$\Gamma_T\left(\int_\Omega\int_0^\infty |F^{(1)}_{\mathrm{init}}-F^{(2)}_{\mathrm{init}}|(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x+\int_\Omega |c^{(1)}_{\mathrm{init}}-c^{(2)}_{\mathrm{init}}|(x)\,\mathrm{d}x\right)$$

with a suitable constant $\Gamma_T>0$.

5 Numerical simulations

In this Section we present a numerical scheme to simulate the behavior of the density of particles and monomers concentration, when monomers are subject to space diffusion, namely we design a scheme for (3). We consider the problem set on the one-dimensional slab $x\in(0,L)$.

5.1 Presentation of the algorithm

We consider time, space and size steps $\Delta t>0$, $\Delta x>0$, and $\Delta\xi>0$, respectively. We define discrete time $t^n=n\Delta t$, discrete size $\xi_j=j\Delta\xi$, and position $x_i=i\Delta x$ for $n,i,j\in\mathbb{N}$. We consider the discrete cells $C_j=(\xi_{j-1/2},\xi_{j+1/2})$ centered on ξ_j. The discrete unknowns c_i^n and $f_{i,j}^n$ are intended to be approximations of $c(t^n,x_i)$ and $\frac{1}{\Delta\xi}\int_{C_j} f(t^n,x_i,\zeta)\,\mathrm{d}\zeta$, respectively. The scheme is based on the following time–splitting:

- The updating of the particles distribution follows by integrating the advection equation over the finite volume cells C_j; for any fixed i, we set

$$f_{i,j}^{n+1}=f_{i,j}^n-\frac{\Delta t}{\Delta\xi}\Big((V\,f)^n_{i,j+1/2}-(V\,f)^n_{i,j-1/2}\Big)\quad\text{with}\ \ V(t,x,\xi)=a(\xi)c(t,x)-1,$$

  which requires a suitable definition of the numerical fluxes at the interfaces $\xi_{j\pm1/2}$. In our simulation we use the Rusanov scheme where

$$(V\,f)^n_{i,j+1/2}=\frac{1}{2}\big[(V\,f)^n_{i,j}+(V\,f)^n_{i,j+1}\big]-\frac{L_i^n}{2}\big(f^n_{i,j+1}-f^n_{i,j}\big),\qquad L_i^n=\max_{j\in\mathbb{N}}|V^n_{i,j}|$$

for all fixed space indices i. Then we have the following approximation

$$f_{i,j}^{n+1} = \big(1 - L_i^n \frac{\Delta t}{\Delta \xi}\big) f_{i,j}^n - \frac{\Delta t}{2\Delta \xi}\bigg(f_{i,j+1}^n \big(V_{i,j+1}^n - L_i^n\big) - f_{i,j-1}^n \big(V_{i,j-1}^n + L_i^n\big)\bigg).$$

In practice, the index j spans a finite set $\{0, ..., j_{max}\}$ and we need fictitious mesh points, where data and unknowns are defined as follows:

$$V_{i,j_{max+1}}^n = V_{i,j_{max}}^n, \qquad V_{i,-1}^n = V_{i,0}^n, \qquad f_{i,j_{max+1}}^n = f_{i,j_{max}}^n, \qquad f_{i,-1}^n = f_{i,0}^n.$$

The stability of the scheme is guaranteed by the CFL condition $\Delta t \leq \dfrac{\Delta x}{L_i^n}$. We point out that we tried other classical finite volume schemes like WENO (Weighted Essentially Non-Oscillatory method) or the ADM (Anti Dissipative Method) method described in [25] but we did not observe any substantial changes in the results (for short times).

- For updating the monomers concentration, we use the following numerical finite difference approximation

$$(E) \qquad \frac{c_i^{n+1} - c_i^n}{\Delta t} = \frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{\Delta x^2} - \frac{\Delta \xi}{\Delta t} \sum_{j=0}^{j_{max}} \xi_j \big(f_{i,j}^{n+1} - f_{i,j}^n\big) \quad \forall i \in \mathbb{N},$$

or the implicit version

$$(I) \qquad \frac{c_i^{n+1} - c_i^n}{\Delta t} = \frac{c_{i+1}^{n+1} - 2c_i^{n+1} + c_{i-1}^{n+1}}{\Delta x^2} - \frac{\Delta \xi}{\Delta t} \sum_{j=0}^{j_{max}} \xi_j \big(f_{i,j}^{n+1} - f_{i,j}^n\big) \quad \forall i \in \mathbb{N}.$$

It can be written in matrix form

$$A_1 C^{n+1} = A_2 C^n - r^{n+1/2} \tag{23}$$

with $C^n = (c_i^n)_{i \in \{0,...,i_{max}\}}$, $r^{n+1/2} = \big(\Delta \xi \sum_{j=0}^{j_{max}} \xi_j (f_{i,j}^{n+1} - f_{i,j}^n)\big)_{i \in \{0,...,i_{max}\}}$,

$$\mathbb{A} = \begin{pmatrix} -2 & 1 & 0 & ... & \\ 1 & -2 & 1 & 0 & ... \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & ... & 1 & -2 & 1 \\ 0 & ... & 0 & 1 & -2 \end{pmatrix}$$

and either $A_1 = \mathbb{I}$, the identity matrix, $A_2 = \frac{\Delta t}{\Delta x^2}\mathbb{A}$ for scheme (E) or $A_1 = \mathbb{I} - \frac{\Delta t}{\Delta x^2}\mathbb{A}$, $A_2 = \mathbb{I}$ for scheme (I). The stability of the explicit scheme (E) requires the CFL condition $\Delta t \leq \Delta x^2/2$. Since this condition is usually more restrictive than the one obtained at the previous step, it can be efficient to use a subcycling method where we perform one time step Δt_{adv} for f while several time steps $\Delta t_{diff} \ll \Delta t_{adv}$ for c. Anyway, the parabolic CFL condition leads to a prohibitive computational cost for multi-dimension simulations where the implicit scheme (I) will be preferred. It requires the inversion of the sparse positive definite matrix $\mathbb{I} - \frac{\Delta t}{\Delta x^2}\mathbb{A}$, that can be done by using performing algorithms like the conjugate gradient method. In numerical simulations we do not observe significant discrepancies between results obtained by either the explicit or the implicit scheme. The numerical results in the next section are provided by the explicit one. Owing to the Neumann boundary conditions, the discrete mass conservation relation

$$\Delta x \sum_i c_i^{n+1} + \Delta x \Delta \xi \sum_i \sum_j \xi_j f_{i,j}^{n+1} = \Delta x \sum_i c_i^n + \Delta x \Delta \xi \sum_i \sum_j \xi_j f_{i,j}^n \tag{24}$$

holds. We check numerically that this quantity is indeed exactly conserved.

5.2 Numerical results

The numerical simulations are performed in the slab $x \in [0, 100]$ with 10 points by length unit. The size variable is truncated to $\xi \in [0, 100]$ meshed with 20 points by size unit. The initial data are defined by

$$\begin{cases} c_{\text{init}}(x) = 0.5\ \mathbf{1}_{x\in[20,35]}, \\ \\ f_{\text{init}}(x,\xi) = 0.01\ \mathbf{I}_{x\in[20,35]} \times \mathbf{1}_{\xi\in[30,35]}. \end{cases} \tag{25}$$

Figure 1 shows the initial data $f_{\text{init}}(x, \xi)$. On Figure 2, the solution $f_{\text{init}}(T, x, \xi)$ at the final time $T = 20$ can be compared to the solution obtained by getting rid of the diffusion term in the monomers equation. We clearly observe the influence of the diffusion of monomers on the space repartition of the macro-particles.

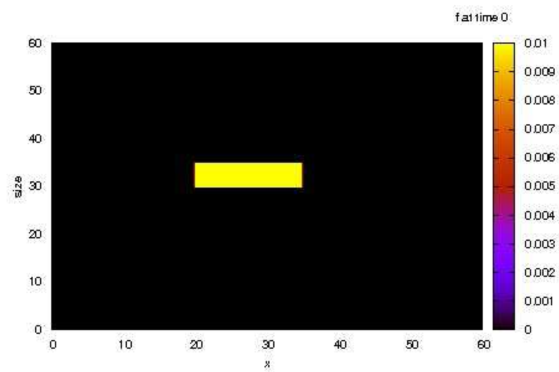

Figure 1: initial density.

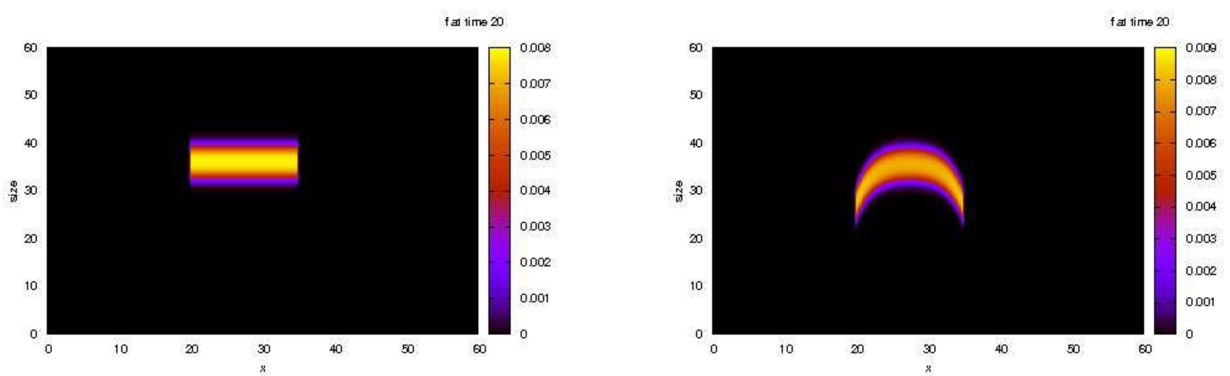

Figure 2: left: density at time 20 without diffusion term; right: density at time 20 with diffusion term.

The monomers concentration in the same situations is displayed in Figure 3 (diffusion case on the right, diffusion–free case on the left). As said above, the simulations also show a numerical evidence of the conservation of the total mass. The time evolution of the monomers concentration can be found in Figure 4. As expected the support of the concentration spreads as time increases, by contrast to the diffusion free case. Note however that the maximum of c seems unchanged between the two cases. Of course, since the space repartition of monomers is modified, it influences the dynamics of the whole system. In Figure 5 we show the time evolution of the mean value of c and f over space, that is compared to the usual solutions of the Lifschitz-Slyozov system. It clearly shows that, even considering only mean values, space diffusion changes the behavior of the solutions, for both the monomers concentration and the particles distribution function.

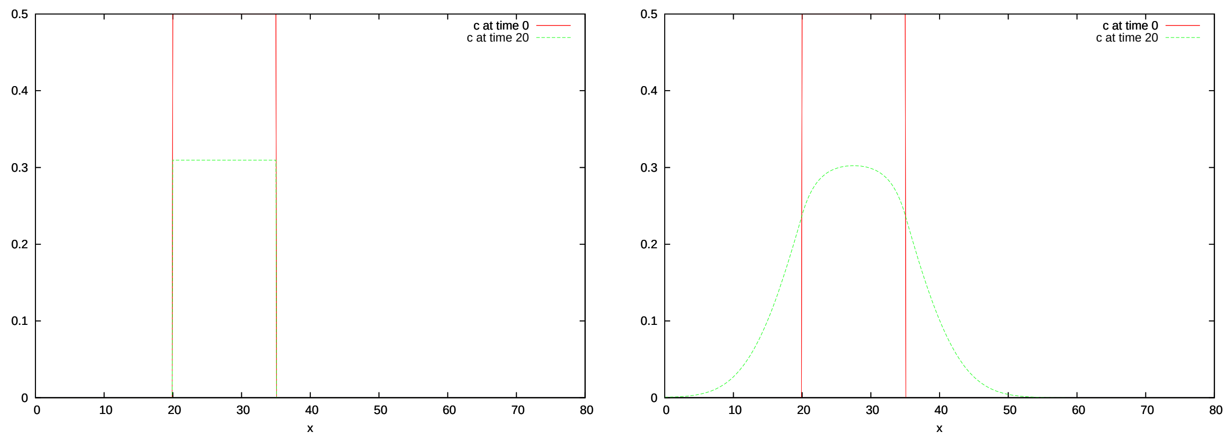

Figure 3: left: evolution of the monomers concentration without diffusion; right: evolution of the monomers concentration with diffusion.

As explained in the Introduction, many questions arise with the large time behavior of the solutions of the Lifschitz-Slyozov equations (1)-(2), and capturing the correct asymptotic profile is numerically challenging. Similar questions can be addressed for the modified model with space diffusion of monomers. Like for the standard model a numerical difficulty comes from the formation of particles with large sizes. As time goes, the support of $f(t, x, \xi)$ might reach the largest size of the numerical domain, which then induces a fictitious loss of mass. Increasing the size domain leads to a considerable increase of the computational cost because f now also depends on the space variable. Therefore, the present method is restricted to quite short times of simulations.

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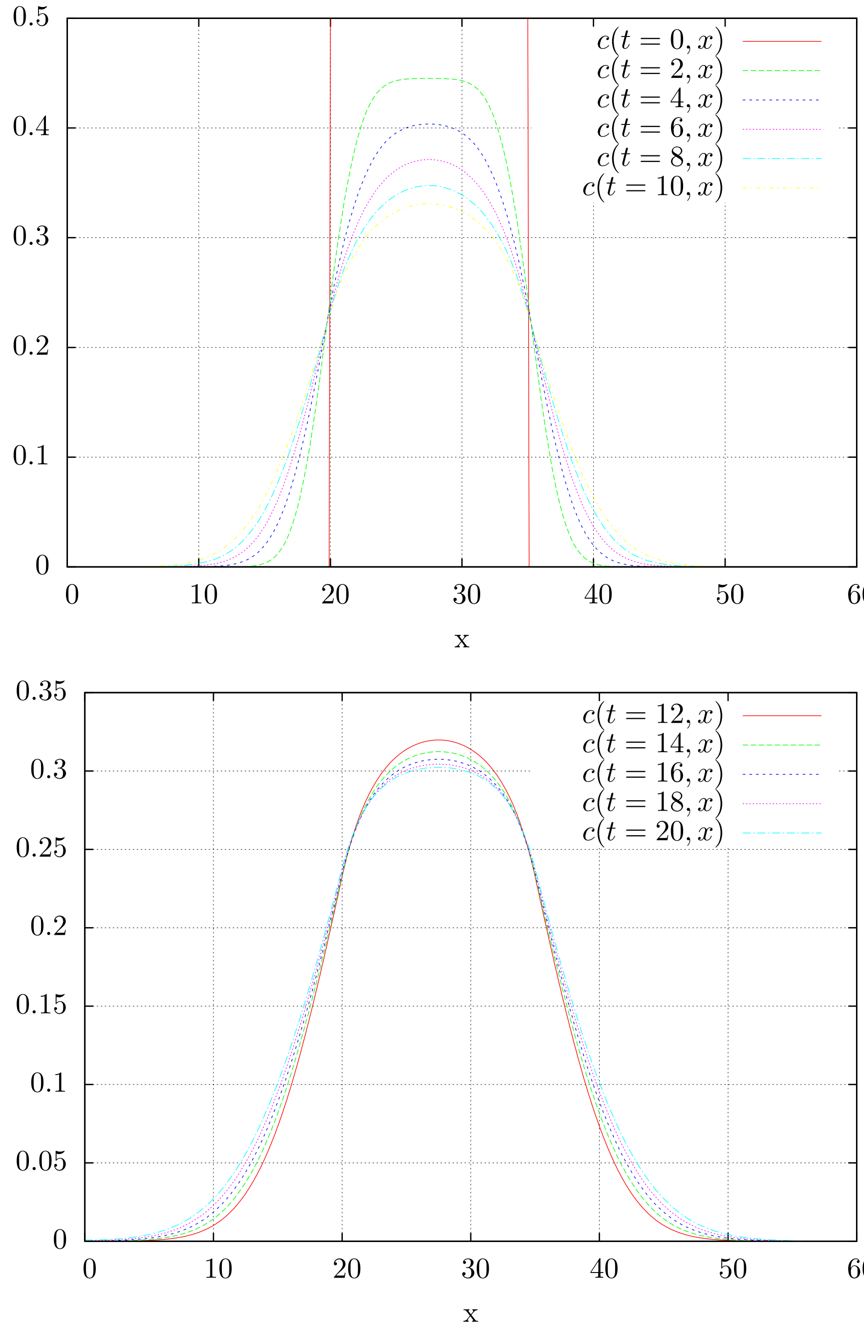

Figure 4: Evolution of the monomers concentration all 2 time units with diffusion term.

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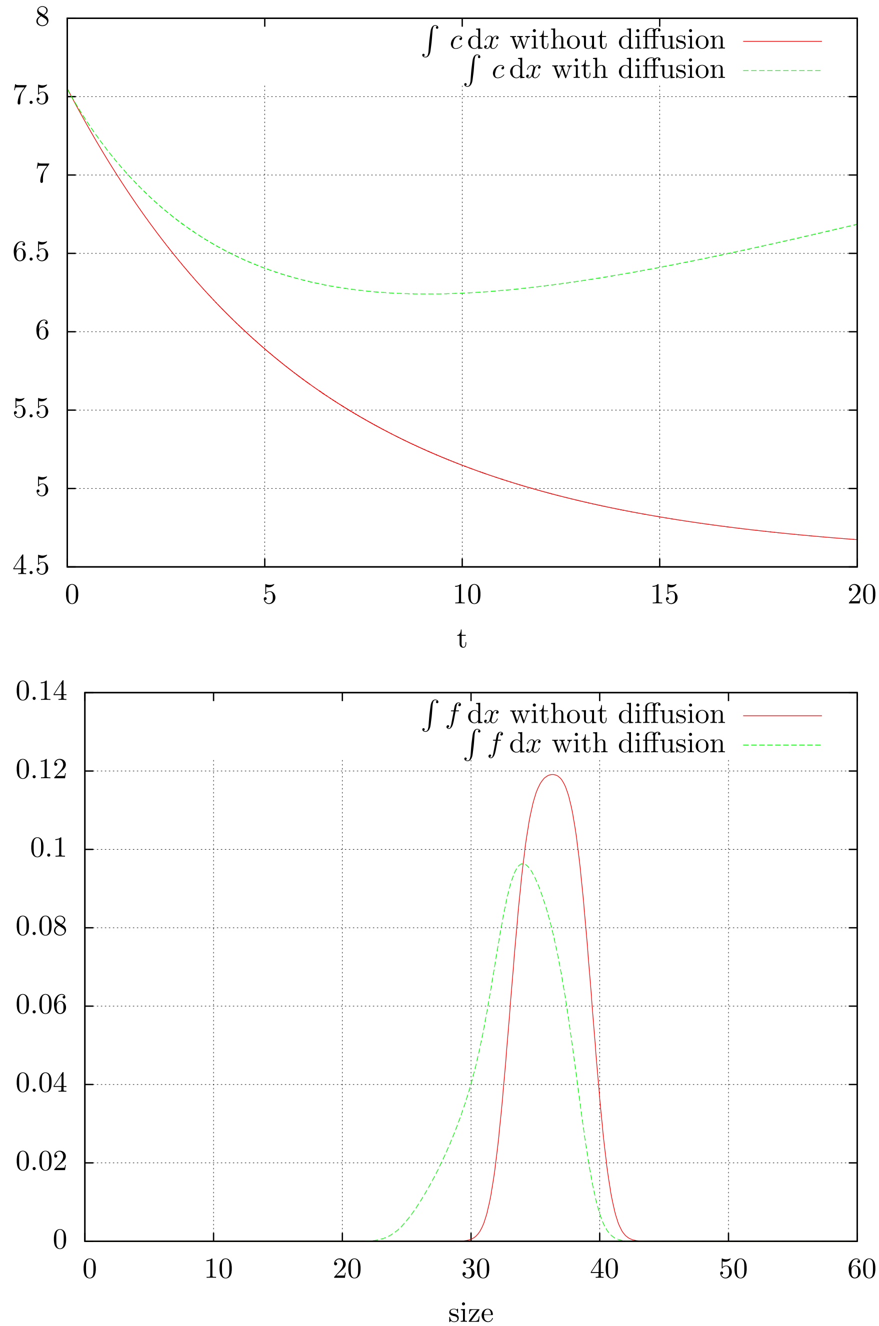

Figure 5: Comparison of mean values of the unknowns (dashed line=diffusion case). Top: time evolution of $\int c(t,x)\,\mathrm{d}x$; Bottom: size variation of $\int f(t=20,x,\xi)\,\mathrm{d}x$.

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