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► **To cite this version:**

Ida Kruk, Francesco Russo. Malliavin-Skorohod calculus and Paley-Wiener integral for covariance singular processes. 2010. <inria-00540914>

HAL Id: inria-00540914

<https://hal.inria.fr/inria-00540914>

Submitted on 29 Nov 2010

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Malliavin-Skorohod calculus and Paley-Wiener integral for covariance singular processes

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November 29th 2010

Abstract

We develop a stochastic analysis for a Gaussian process X with singular covariance by an intrinsic procedure focusing on several examples such as covariance measure structure processes, bifractional Brownian motion, processes with stationary increments. We introduce some new spaces associated with the self-reproducing kernel space and we define the Paley-Wiener integral of first and second order even when X is only a square integrable process continuous in L^2 . If X has stationary increments, we provide necessary and sufficient conditions so that its paths belong to the self-reproducing kernel space. We develop Skorohod calculus and its relation with symmetric-Stratonovich type integrals and two types of Itô's formula. One of Skorohod type, which works under very general (even very singular) conditions for the covariance; the second one of symmetric-Stratonovich type, which works, when the covariance is at least as regular as the one of a fractional Brownian motion of Hurst index equal to $H = \frac{1}{4}$.

Key words and phrases: Malliavin-Skorohod calculus, singular covariance, symmetric integral, Paley-Wiener integral, Itô formulae.

2000 Mathematics Subject Classification: 60G12, 60G15, 60H05, 60H07.

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1 Introduction

Classical stochastic calculus is based on Itô's integral. It operates when the integrator X is a semimartingale. The present paper concerns some specific aspects of calculus for Gaussian non-semimartingales, with some considerations about Paley-Wiener integrals for non Gaussian processes. Physical modeling, hydrology, telecommunications, economics and finance has generated the necessity to make stochastic calculus with respect to more general processes than semimartingales. In the family of Gaussian processes, the most adopted and celebrated example is of course Brownian motion. In this paper we will consider Brownian motion B (and the processes having the same type of path regularity) as the frontier of two large classes of processes: those whose path regularity is more regular than B and those whose path regularity is more singular than B . It is well-known that B is a finite quadratic variation process in the sense of [13] or [40, 41]. Its quadratic variation process is given by $[B]_t = t$. If we have a superficial look to path regularity, we can *macroscopically* distinguish three classes.

1. Processes X which are more regular than Brownian motion B are such that $[X] = 0$.
2. Processes X which are as regular as B are finite non-zero quadratic variation processes.
3. Processes X which are more singular than B are processes which do not admit a quadratic variation.

Examples of processes X summarizing previous classification are given by the so called fractional Brownian motions B^H with Hurst index $0 < H < 1$. If $H > \frac{1}{2}$ (resp. $H = \frac{1}{2}$, $H < \frac{1}{2}$), B^H belongs to class 1. (resp. 2., 3.).

Calculus with respect to integrators which are not semimartingales is more than thirty-five years old, but a real activity acceleration was produced since the mid-eighties. A significant starting lecture note in this framework is [24]: well-written lecture articles from H. Kuo, D. Nualart, D. Ocone constitute excellent pedagogical articles on Wiener analysis (Malliavin calculus, white noise calculus) and its application to anticipative calculus via Skorohod integrals.

Since then, a huge amount of papers have been produced, and it is impossible to list them here, so we will essentially quote monographs.

There are two mainly techniques for studying non-semimartingales integrators.

- Wiener analysis, as we mentioned before. It is based on the so-called Skorohod integral (or divergence operator); it allows to “integrate” anticipative integrands with respect to various Gaussian processes. We quote for instance [33], [7], [47, 31], for Malliavin calculus and [17, 18] for white noise calculus. In principle, Malliavin calculus can be abstractly implemented on any Wiener space generated by any Gaussian process X , see for instance [49], but in the general abstract framework, integrands may live in some abstract spaces. In most of the present literature about Malliavin-Skorohod calculus, integrands are supposed to live in a space L which is isomorphic to the self-reproducing kernel space. One condition for an integrand Y to belong to the classical domain of the divergence operator is that its paths belong a.s. to L . More recent papers as [8, 32] allow integrands to live outside the classical domain of the divergence operator. Some activity about Skorohod integration was also performed in the framework of Poisson measures integrators, see e.g. [11].
- Pathwise and quasi-pathwise related techniques, as rough paths techniques [29], regularization (see for instance [39, 42]) or discretization techniques [13], but also fractional integrals techniques [50, 51] and also [20] for connections between rough paths and fractional calculus.

This paper is the continuation of [25], which focused on the processes belonging to categories 1. and 2. Here we are mainly interested in processes of category 3. We develop some intrinsic stochastic analysis with respect to those processes. The qualification *intrinsic* is related to the fact that in opposition to [1, 2, 8, 32], where there is no underlying Wiener process. We formulate a class of general assumptions Assumptions (A), (B), (C(ν)), (D) under which the calculus runs. Many properties hold only under the two three first hypotheses. Sometimes, however, we also make use of a supplementary assumption that we believe to be technical, i.e.

$$X_t = X_T, \quad t > T. \tag{1.1}$$

In some more specific situations, we also introduce Hypothesis (6.35) which intervenes for instance to guarantee, that X itself belongs to the *classical* domain of the divergence operator. At Section 6.1 we also define a suitable Hilbert space L_R for integrand processes, which is related to self-reproducing kernel space. We describe the content of L_R in many situations. We implement the analysis and we verify the assumptions in the following examples: the case when the process X is defined through a kernel integration with respect to a Wiener process, the case of processes with a covariance measure structure, the processes

with stationary increments, the case of bifractional Brownian motion. We provide a stochastic analysis framework starting from Paley-Wiener integral for second order processes. The Wiener integral with respect to the subclass of processes with stationary increments was studied with different techniques in [22]. In our paper, we also define the notion to a multiple Paley-Wiener integral, involving independent processes X^1, \dots, X^n . If $n = 2$ those integrals have a natural relation with the notion of Lévy's area in rough path theory.

Starting from Section 7, we concentrate on Malliavin-Skorohod calculus, with Itô's type formulae and connections with symmetric-Stratonovich integrals via regularization. Calculus via regularizations was started by F. Russo and P. Vallois [38] developing a regularization procedure, whose philosophy is similar to the discretization. They introduced a forward (generalizing Itô) integral, and the symmetric (generalizing Stratonovich) integral.

As we said, in the first six sections, we redefine a Paley-Wiener type integral with respect to an L^2 -continuous square integrable process. We aim at showing some interesting features and difficulties, which are encountered if one wants to define the integral in a natural function space avoiding distributions.

This allows in particular, but not only, to settle the basis of Malliavin-Skorohod calculus for Gaussian processes with singular covariance.

As we said, Malliavin calculus, according to [49], can be developed abstractly for any Gaussian process $X = (X_t)_{t \in [0, T]}$. The Malliavin derivation can be naturally defined on a general Gaussian abstract Wiener space. A Skorohod integral (or divergence) can also be defined as the adjoint of the Malliavin derivative.

The crucial ingredient is the canonical Hilbert space \mathcal{H} (called also, improperly, by some authors reproducing kernel Hilbert space) of the Gaussian process X which is defined as the closure of the linear space generated by the indicator functions $\{1_{[0, t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0, t]}, 1_{[0, s]} \rangle_{\mathcal{H}} = R(t, s), \quad (1.2)$$

where R denotes the covariance of X . Nevertheless, this calculus remains more or less abstract if the structure of the elements of the Hilbert space is not known. When we say abstract, we refer to the fact that, for example, it is difficult to characterize the processes which are integrable with respect to X , or to establish Itô formulae.

In this paper, as we have anticipated, we formulate some natural assumptions (A), (B), (C(ν)), (D), that the underlying process has to fulfill, which let us efficiently define a Skorohod intrinsic calculus and Itô formulae, when integrators belong to categories 2. and

3. In particular, Assumption (D) truly translates the singular character of the covariance.

We link Skorohod integral with integrals via regularization (so of almost pathwise type) similarly to [25], where the connection was established with forward integrals. We recall that the process X is forward integrable (in symbols $\int_0^\cdot X d^-X$ exists) if and only if X has a finite quadratic variation, see for instance [15] or [16]. Therefore if X is a fractional Brownian motion with Hurst index H , the forward integral $\int_0^\cdot X d^-X$ exists if and only if $H \geq \frac{1}{2}$; on the other hand the symmetric integral $\int_0^\cdot X d^\circ X$ always exists. Since we are mainly interested in singular covariance processes (category 3.), which are not of finite quadratic variation, Skorohod type integrals will be linked with the symmetric integrals.

As we have mentioned before, a particular case was deeply analyzed in the literature. We refer here to the situation when the covariance R can be explicitly written as

$$R(t, s) = \int_0^{t \wedge s} K(t, u)K(s, u)du,$$

where $K(t, s)$, $0 < s < t < T$, is a deterministic kernel satisfying some regularity conditions. Enlarging, if needed, our probability space, we can express the process X as

$$X_t = \int_0^t K(t, s)dW_s, \tag{1.3}$$

where $(W_t)_{t \in [0, T]}$ is a standard Wiener process and the above integral is understood in the Wiener sense. In this case, more concrete results can be proved, see [2, 9, 32]. In this framework the underlying Wiener process (W_t) is strongly used for developing anticipating calculus.

For illustration, we come back to the case, when X is a fractional Brownian motion B^H and H is the Hurst index. The process B^H admits the Wiener integral representation (1.3) and the kernel K together with the space \mathcal{H} can be characterized by the mean of fractional integrals and derivatives, see [2, 3, 10, 36, 8, 4] among others. As a consequence, one can prove for any $H > \frac{1}{4}$ (to guarantee that B^H is in the domain of the divergence), the following Itô's formula:

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H)\delta B_s^H + H \int_0^t f''(B_s^H)s^{2H-1}ds.$$

[32] puts emphasis on the case $K(t, s) = g(t - s)$, when the variance scale of the process is as general as possible, including logarithmic scales.

In section 5, we establish some connections between the "kernel approach" discussed in the literature and the "covariance intrinsic approach" studied here.

As we mentioned, if the deterministic kernel K in the representation (1.3) is not explicitly known, then the Malliavin calculus with respect to the Gaussian process X remains in an abstract form and there are of course many situations when this kernel is not explicitly known. As a typical example, we have in mind the case of the *bifractional Brownian motion (BFBM)* $B^{H,K}$, where $H \in]0, 1[$, $K \in]0, 1[$; a kernel representation is known in a particular case, but with respect to a space-time white noise: in fact the solution $F(t) = u(t, x)$ of a classical stochastic heat equation driven by a white noise with zero initial condition, is distributed as $B^{H,K}$, for any fixed x , see [46]. Another interesting representation is provided by [27], which shows the existence of real constants c_1, c_2 and an absolutely continuous process $X(H, K)$ independent of $B^{H,K}$, such that $c_1 B^{H,K} + c_2 X(H, K)$ is distributed as a fractional Brownian motion with parameter HK . In spite of those considerations, finding a kernel K , such that $B_t^{H,K} = \int_0^t K(t, s) dW_s$, is still an open problem. Bifractional Brownian motion was introduced in [19] and a *quasi-pathwise type* of regularization ([42]) type approach to stochastic calculus was provided in [37]. It is possible for instance to obtain an Itô formula of the Stratonovich type (see [37]), i.e.

$$f(B_t^{H,K}) = f(0) + \int_0^t f'(B_s^{H,K}) d^\circ B_s^{H,K} \quad (1.4)$$

for any parameters $H \in (0, 1)$ and $K \in (0, 1]$ such that $HK > \frac{1}{6}$. An interesting property of $B^{H,K}$ consists in the expression of its quadratic variation, defined as usual, as a limit of Riemann sums, or in the sense of regularization. The following properties hold true.

- If $2HK > 1$, then the quadratic variation of $B^{H,K}$ is zero and $B^{H,K}$ belongs to category 1.
- If $2HK < 1$ then the quadratic variation of $B^{H,K}$ does not exist and $B^{H,K}$ belongs to category 3.
- If $2HK = 1$ then the quadratic variation of $B^{H,K}$ at time t is equal to $2^{1-K}t$ and $B^{H,K}$ belongs to category 2.

The last property is remarkable; indeed, for $HK = \frac{1}{2}$ we have a Gaussian process which has the same quadratic variation as the Brownian motion. Moreover, the process is not a semimartingale (except for the case $K = 1$ and $H = \frac{1}{2}$), it is self-similar, has no stationary increments.

Motivated by the consideration above, one developed in [25] a Malliavin-Skorohod calculus with respect to Gaussian processes X having a *covariance measure structure* in

sense that the covariance is the distribution function of a (possibly signed) measure μ_R on $\mathcal{B}([0, T]^2)$. We denote by D_t the diagonal set

$$\{(s, s) | s \in [0, t]\}$$

The processes having a *covariance measure structure* belong to the category 1. (resp. 2.), i.e. they are more regular than Brownian motion (resp. as regular as Brownian motion) if μ_R restricted to the diagonal D_T vanishes (resp. does not vanish). In particular, it was shown that in this case X is a finite quadratic variation process and $[X]_t = \mu(D_t)$. This paper continues the spirit of [25], but it concentrates on the case when X is less regular or equal than Brownian motion.

A significant paper, is [35], which establishes a Itô-Stratonovich (of quasi-pathwise type, in the discretization spirit) for processes belonging to class 3., i.e. less regular than Brownian motion. In particular, the paper rediscovers Itô's formula of [37] for $B^{H,K}$, if $HK > \frac{1}{6}$; for this purpose the authors implement innovating Malliavin calculus techniques. Their main objective was however not to obtain a Skorohod type calculus, but more to use some Malliavin calculus ideas to recover pathwise type techniques.

In this paper, for simplicity of notations and without restriction of generality, we consider processes indexed by the whole first quarter of the completed plane $\bar{\mathbb{R}}_+^2 = [0, \infty]^2$. In particular we suppose that X is a continuous process in L^2 , such that $\lim_{s \rightarrow \infty} X_s$ exists, and it is denoted X_∞ and under some circumstances we suppose even (1.1). Let $R(s_1, s_2), s_1, s_2 \in [0, \infty]$ be the covariance function of X . As we said, we introduce a class of natural assumptions which have to be fulfilled in most of the results in order to get an efficient Skorohod calculus.

The processes of class 2. and 3., so essentially less regular than Brownian motion, will fulfill the following:

- $R(ds, \infty)$ is a non-negative real measure.
- If D is the first diagonal of \mathbb{R}_+^2 , the Schwartz distribution $\partial_{s_1, s_2}^2 R$ restricted to $\mathbb{R}_+^2 \setminus D$ is a non-positive σ -finite measure.

This will constitute the convenient Assumption (D).

The basic space of integrands for which Paley-Wiener integral is defined is L_R . This space, under Assumption (D), plays the role of self-reproducing kernel space. A necessary

condition for the process X itself to be in the natural domain ($Dom\delta$) of the divergence operator is that it belongs a.s. to L_R . Following the ideas of [8, 32] one defines for our general class of processes an extended domain called $Dom^*\delta$ which allows to proceed when X does not always belong a.s. to L_R .

Other products of this paper are the following.

- We define a corresponding appropriate Paley-Wiener integral with respect to second order processes in Section 6. In particular, see Section 6.17, we extend some significant considerations of [36] made in the context of fractional Brownian motion; [36] illustrates that the natural space where Wiener integral is defined, is complete if the Hurst index is smaller or equal to $\frac{1}{2}$.
- The link between symmetric and Skorohod integrals, see Theorem 13.5, is given by suitable trace of Malliavin derivative of the process.
- If the process X is continuous, Gaussian and has stationary increments, we provide necessary and sufficient conditions such that the paths of X belong to L_R , see Corollary 6.29.
- We establish an Itô type formula for Skorohod integrals for very singular covariation when the underlying process X is quite general, continuing the work of [8] and [32]. This is done in Proposition 11.7: if $f \in C^\infty$ with bounded derivatives

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\delta X_s + \frac{1}{2} \int_0^t f''(X_s)d\gamma(s), \quad (1.5)$$

where $\gamma(t)$ is the variance of X_t .

We recall that if X is a bifractional Brownian motion with indexes H, K such that $HK = \frac{1}{2}$, then $\gamma(t) = t$ and so equation (1.5) looks very similar to the one related to classical Wiener process.

Formula (1.5) implies the corresponding formula with respect to the symmetric integral, see Corollary 13.7, i.e.

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)d^\circ X_s.$$

We organize our paper as follows. After some preliminaries stated at Section 2, we introduce the basic assumptions (A), (B), (C) (or only its restricted version (C(ν))), (D)

in Section 3 followed by the motivating examples in Section 4, including the case when the process has a covariance measure structure treated in [25]. Section 5 discusses the link with the case that the process is of the type $X_t = \int_0^t K(t, s)dW_s$ for a suitable kernel K . At Section 6 we define the Wiener integral for second order processes together with a multiorder version; in the same section we discuss some path properties of the underlying process and the relation with the integrals via regularization. Starting from Section 7 until 9, we introduce and discuss the basic notions of Malliavin calculus. At Section 10 we introduce Skorohod integrals, at Section 11 we discuss Itô formula in the very singular case. Section 12 shows that Skorohod integral is truly an extension of Wiener integral. Finally Section 13 provides the link with integrals via regularization and Itô's formula with respect to symmetric integrals.

2 Preliminaries

Let J be a closed set of the type \mathbb{R}_+ , \mathbb{R}^m or $\mathbb{R}_+^2 = [0, +\infty[\times [0, +\infty[$, and $k \geq 1$. In this paper $C_0^\infty(J)$ (resp. $C_b^\infty(J)$, $C_0^k(J)$, $C_{pol}^k(J)$, $C_b(J)$) stands for the set of functions $f : J \rightarrow \mathbb{R}$ which are infinitely differentiable with compact support (resp. smooth with all bounded partial derivatives, of class C^k with compact support, of class C^k such that the partial derivatives of order smaller or equal to k have polynomial growth, bounded functions).

If $g_1, g_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$, we denote $g = g_1 \otimes g_2$, the function $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ defined by $g(s_1, s_2) = g_1(s_1)g_2(s_2)$.

Let I be a subset of \mathbb{R}_+^2 of the form

$$I =]a_1, b_1[\times]a_2, b_2]$$

Given $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ we will denote

$$\Delta_I g = g(b_1, b_2) + g(a_1, a_2) - g(a_1, b_2) - g(b_1, a_2).$$

It constitutes the **planar increment** of g .

Definition 2.1. $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ will be said to have a **bounded planar variation** if

$$\sup_{\tau} \sum_{i,j=0}^{n-1} \left| \Delta_{]t_i, t_{i+1}[\times]t_j, t_{j+1}] } g \right| < \infty.$$

where $\tau = \{0 = t_0 < \dots < t_n < \infty\}$, $n \geq 1$, i.e. τ is a subdivision of \mathbb{R}_+ . Previous quantity will be denoted by $\|g\|_{pv}$.

If g has bounded planar variation and is vanishing on the axes, then there exists a signed measure χ (difference of two positive measures) such that

$$g(t_1, t_2) = \chi([0, t_1] \times [0, t_2]). \quad (2.1)$$

For references, see a slight adaptation of Lemma 2.2 in [25] and Theorem 12.5 in [5].

Some elementary calculations allow to show the following.

Proposition 2.1. *Let $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ of class C^2 , g has a bounded planar variation if and only if*

$$\|g\|_{pv} := \int_{\mathbb{R}_+^2} \left| \frac{\partial^2 g}{\partial t_1 \partial t_2} \right| dt_1 dt_2 < \infty.$$

In particular, if g has compact support, then g has a bounded planar variation.

Let $X = (X_t)_{t \geq 0}$, be a zero-mean continuous process in $L^2(\Omega)$ such that $X_0 = 0$ a.s. For technical reasons we will suppose that

$$\lim_{t \rightarrow \infty} X_t = X_\infty \text{ in } L^2(\Omega). \quad (2.2)$$

(2.2) is verified if for instance

$$X_t = X_T, \quad t \geq T \quad (2.3)$$

for some $T > 0$.

We denote by R the covariance function, i.e. such that:

$$R(s_1, s_2) = Cov(X_{s_1}, X_{s_2}) = E(X_{s_1} X_{s_2}), \quad s_1, s_2 \in \overline{\mathbb{R}_+^2} = [0, \infty]^2.$$

In particular R is continuous and vanishes on the axes.

We convene that all the continuous functions on \mathbb{R}_+ are extended by continuity to \mathbb{R}_- . A continuous function $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ such that $f(s) = 0$, if s belongs to the axes, will also be extended by continuity to the whole plane.

In this paper D will denote the diagonal $\{(t, t) | t \geq 0\}$ of the first plane quarter \mathbb{R}_+^2 .

Definition 2.2. *X is said to have a **covariance measure structure** if $\frac{\partial^2 R}{\partial s_1 \partial s_2}$ is a finite Radon measure μ on \mathbb{R}_+^2 with compact support. We also say that X has a covariance measure μ .*

A priori $\frac{\partial R}{\partial s_1}, \frac{\partial R}{\partial s_2}, \frac{\partial^2 R}{\partial s_1 \partial s_2}$ are Schwartz distributions. In particular for $s = (s_1, s_2)$ we have

$$R(s_1, s_2) = \mu([0, s_1] \times [0, s_2])$$

Remark 2.2. *The class of processes defined in Definition 2.2 was introduced in [25], where the parameter set was $[0, T]$, for some $T > 0$, instead of \mathbb{R}_+ . Such processes can be easily extended by continuity to \mathbb{R}_+ setting $X_t = X_T$, if $t \geq T$. In that case the support of the measure is $[0, T]^2$.*

The present paper constitutes a natural continuation of [25] trying to extend Wiener integral and Malliavin-Skorohod calculus to a large class of more singular processes.

The covariance approach is an intrinsic way of characterizing square integrable processes. These processes include Gaussian processes defined through a kernel, as for instance [2, 32].

We will see later that a process $X_t = \int_0^t K(t, s) dW_s$, where $(W_t)_{t \geq 0}$ is a classical Wiener process and K is a deterministic kernel with some regularity, provide examples of processes with covariance measure structure. Other examples were given in [25].

One relevant object of [25] was Wiener integral with respect to X . Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ has locally bounded variation with compact support. We set

$$\int_0^\infty \varphi dX = - \int_0^\infty X d\varphi.$$

If φ is a Borel function such that

$$\int_{\mathbb{R}_+^2} |\varphi \otimes \varphi| d|\mu| < \infty, \tag{2.4}$$

then similarly to Section 5 of [25], the Wiener integral $\int_0^\infty \varphi dX$ can be defined through the isometry property

$$E \left(\int_0^\infty \varphi dX \right)^2 = \int_{\mathbb{R}_+^2} \varphi \otimes \varphi d\mu.$$

Remark 2.3. *If φ fulfills (2.4), then the process $Z_t = \int_0^t \varphi dX$ has again a covariance measure structure with measure ν defined by*

$$d\nu = \varphi \otimes \varphi d\mu.$$

3 Basic assumptions

In this section we formulate a class of fundamental hypotheses, which will be in force for the present paper.

Assumption (A)

- i) $\forall s \in \overline{\mathbb{R}}_+ : R(s, dx)$ is a signed measure,
- ii) $s \mapsto \int_0^\infty |R|(s, dx)$ is a bounded function.

Remark 3.1. *Since R is symmetric, Assumption (A) implies the following.*

- i)' $\forall s \in \overline{\mathbb{R}}_+ : R(dx, s)$ is a signed measure,
- ii)' $s \mapsto \int_0^\infty |R|(dx, s)$ is a bounded function.

Remark 3.2. *Suppose that X has a covariance measure μ . For $s \geq 0$ we have*

$$x \mapsto R(s, x) = \int_{[0, s] \times [0, x]} d\mu.$$

which is a bounded variation function whose total variation is clearly given by

$$\int_0^\infty |R|(s, dx) = \int_{[0, s] \times \mathbb{R}_+} d|\mu|(s_1, s_2) \leq \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2).$$

Hence Assumption (A) is fulfilled.

Assumption (B) We suppose that

$$\bar{\mu}(ds_1, ds_2) := \frac{\partial^2 R}{\partial s_1 \partial s_2}(s_1, s_2)(s_1 - s_2) \tag{3.1}$$

is a Radon measure. In this paper by a Radon measure we mean the difference of two (positive Radon) measures.

Remark 3.3. i) *The right-hand side of (3.1) is well defined being the product of a C^∞ function and a Schwartz distribution.*

- ii) *If D is the diagonal introduced before in Definition 2.2, Assumption (B) implies that $\frac{\partial^2 R}{\partial s_1 \partial s_2}$ restricted to $\mathbb{R}_+^2 \setminus D$ is a σ -finite measure. Indeed, given $\varphi \in C_0^\infty(\mathbb{R}_+^2 \setminus D)$, we symbolize by d the distance between $\text{supp } \varphi$ and D . Setting $g(s_1, s_2) = s_1 - s_2$, since $\frac{\varphi}{g} \in C_0^\infty(\mathbb{R}_+^2 \setminus D)$, we have*

$$\left| \left\langle \frac{\partial^2 R}{\partial s_1 \partial s_2}, \varphi \right\rangle \right| = \left| \left\langle \frac{\partial^2 R}{\partial s_1 \partial s_2} \cdot g, \frac{\varphi}{g} \right\rangle \right| = \left| \int_{\mathbb{R}_+^2} d\bar{\mu} \frac{\varphi}{g} \right| \leq \frac{1}{\inf_{|x| \geq d} |g|(x)} \|\varphi\|_\infty |\bar{\mu}|(\mathbb{R}_+^2).$$

iii) On each compact subset of $\mathbb{R}_+^2 \setminus D$, the total variation measure $|\mu|$ is absolutely continuous with density $\frac{1}{|g|}$ with respect to $|\bar{\mu}|$.

Assumption (C(ν)) We suppose the existence of a positive Borel measure ν on \mathbb{R}_+ such that:

- i) $R(ds, \infty) \ll \nu$,
- ii) The marginal measure of the symmetric measure $|\bar{\mu}|$ is absolutely continuous with respect to ν .

If Assumption (C(ν)) is realized with $\nu(ds) = |R|(ds, \infty)$ then we will simply say that **Assumption (C)** is fulfilled.

Proposition 3.4. *Suppose that X has a covariance measure structure and $\mu = \frac{\partial^2 R}{\partial s_1 \partial s_2}$ has compact support, then Assumption (C(ν)) is fulfilled with ν being the marginal measure of $|\mu|$.*

Proof: Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded non-negative Borel function.

- i) $\left| \int_{\mathbb{R}_+} f(s) |R|(ds, \infty) \right| \leq \int_{\mathbb{R}_+ \times \mathbb{R}_+} f(s_1) d|\mu|(s_1, s_2) = \int_{\mathbb{R}_+} f(s) d\nu(s)$. Take f being the indicator of a null set related to ν .
- ii) $\left| \int_{\mathbb{R}_+^2} f(s_1) d|\bar{\mu}|(s_1, s_2) \right| \leq k \left| \int_{\mathbb{R}_+^2} f(s_1) d|\mu|(s_1, s_2) \right| = k \int_0^\infty f d\nu$, where k is the diameter of the compact support of μ .

Corollary 3.5. *If X has a covariance measure μ , which is non-negative and with compact support, then Assumption (C) is verified.*

Proof: This follows because $|R|(ds, \infty) = R(ds, \infty)$ is the marginal measure of μ . ■

Next proposition is technical but useful.

Proposition 3.6. *We suppose Assumptions (A), (B). Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded variation and $\frac{1}{2}$ -Hölder continuous function with compact support. Then*

$$\int_{\mathbb{R}_+^2} R(s_1, s_2) df(s_1) df(s_2) = \int_{\mathbb{R}_+} f^2(s) R(ds, \infty) - \frac{1}{2} \int_{\mathbb{R}_+^2 \setminus D} (f(s_1) - f(s_2))^2 d\mu(s_1, s_2). \quad (3.2)$$

Remark 3.7. *The statement holds of course if f is Lipschitz with compact support.*

Proof: a) We extend R to the whole plane by continuity. We suppose first $f \in C_0^\infty(\mathbb{R}_+)$. Let ρ be a smooth, real function with compact support and $\int \rho(x)dx = 1$. We set $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$, for any $\varepsilon > 0$. The left-hand side of (3.2) can be approximated by

$$\int_{\mathbb{R}_+^2} R_\varepsilon(s_1, s_2)df(s_1)df(s_2), \quad (3.3)$$

where

$$R_\varepsilon = (\rho_\varepsilon \otimes \rho_\varepsilon) * R.$$

We remark that R_ε is smooth and

$$\frac{\partial^2 R_\varepsilon}{\partial s_1 \partial s_2} = (\rho_\varepsilon \otimes \rho_\varepsilon) * \frac{\partial^2 R}{\partial s_1 \partial s_2}, \quad (3.4)$$

where we recall that $\frac{\partial^2 R}{\partial s_1 \partial s_2}$ is a distribution. By Fubini's theorem on the plane, (3.3) gives

$$\int_{\mathbb{R}^2} \frac{\partial^2 R_\varepsilon}{\partial s_1 \partial s_2}(s_1, s_2)f(s_1)f(s_2)ds_1ds_2.$$

Let $\chi_\varepsilon \in C^\infty(\mathbb{R})$ such that $\chi_\varepsilon = 1$ for $|x| \leq \frac{1}{\varepsilon}$ and $\chi_\varepsilon(x) = 0$ for $|x| \geq \frac{1}{\varepsilon} + 1$. Moreover we choose $\varepsilon > 0$ large enough such that $[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$ includes the support of f . We have

$$\int_{\mathbb{R}^2} \frac{\partial^2 R_\varepsilon}{\partial s_1 \partial s_2}(s_1, s_2)f(s_1)f(s_2)\chi_\varepsilon(s_1)\chi_\varepsilon(s_2)ds_1ds_2 = \frac{1}{2}(-I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)),$$

where

$$\begin{aligned} I_1(\varepsilon) &= \int_{\mathbb{R}^2} \frac{\partial^2 R_\varepsilon}{\partial s_1 \partial s_2}(s_1, s_2)(f(s_1) - f(s_2))^2\chi_\varepsilon(s_1)\chi_\varepsilon(s_2)ds_1ds_2, \\ I_2(\varepsilon) &= \int_{\mathbb{R}^2} \frac{\partial^2 R_\varepsilon}{\partial s_1 \partial s_2}(s_1, s_2)f(s_1)^2\chi_\varepsilon(s_1)\chi_\varepsilon(s_2)ds_1ds_2, \\ I_3(\varepsilon) &= \int_{\mathbb{R}^2} \frac{\partial^2 R_\varepsilon}{\partial s_1 \partial s_2}(s_1, s_2)f(s_2)^2\chi_\varepsilon(s_1)\chi_\varepsilon(s_2)ds_1ds_2. \end{aligned}$$

Since $\chi_\varepsilon = 1$ on $\text{supp} f$, $I_2(\varepsilon)$ gives

$$\begin{aligned} & - \int_0^\infty ds_1 (f^2)'(s_1) \int_0^\infty ds_2 \frac{\partial R_\varepsilon}{\partial s_2}(s_1, s_2)\chi_\varepsilon(s_2) \\ &= - \int_0^\infty ds_1 (f^2\chi_\varepsilon)'(s_1) \int_0^\infty R_\varepsilon(s_1, ds_2)\chi_\varepsilon(s_2) = -(I_{2,1} + I_{2,2})(\varepsilon), \end{aligned}$$

where

$$\begin{aligned} I_{2,1}(\varepsilon) &= \int_0^\infty ds_1 (f^2)'(s_1) \int_0^\infty R_\varepsilon(s_1, ds_2)(\chi_\varepsilon - 1)(s_2), \\ I_{2,2}(\varepsilon) &= \int_0^\infty ds_1 (f^2)'(s_1) \int_0^\infty R_\varepsilon(s_1, ds_2). \end{aligned}$$

$I_{2,1}(\varepsilon)$ is bounded by

$$\int_0^\infty ds_1 \left| (f^2)' \right| (s_1) \int_{\frac{1}{\varepsilon}}^\infty |R_\varepsilon|(s_1, ds_2). \quad (3.5)$$

For each $s_1 \geq 0$, we have

$$\int_{\frac{1}{\varepsilon}}^\infty |R_\varepsilon|(s_1, ds_2) = \int_{\frac{1}{\varepsilon}}^\infty \left| \frac{\partial R_\varepsilon}{\partial s_2}(s_1, s_2) \right| ds_2. \quad (3.6)$$

Now

$$\begin{aligned} \frac{\partial R_\varepsilon}{\partial s_2}(s_1, s_2) &= \int_{\mathbb{R}^2} dy_1 dy_2 R(y_1, y_2) \rho_\varepsilon(s_1 - y_1) \rho'_\varepsilon(s_2 - y_2) \\ &= \int_0^\infty dy_1 \rho_\varepsilon(s_1 - y_1) \int_0^\infty R(y_1, dy_2) \rho_\varepsilon(s_2 - y_2). \end{aligned}$$

Using Fubini's theorem, (3.6) gives

$$\begin{aligned} \int_{\frac{1}{\varepsilon}}^\infty ds_2 \int_0^\infty dy_1 \rho_\varepsilon(s_1 - y_1) \int_0^\infty R(y_1, dy_2) \rho_\varepsilon(s_2 - y_2) \\ = \int_0^\infty dy_1 \rho_\varepsilon(s_1 - y_1) \int_0^\infty R(y_1, dy_2) \int_{\frac{1}{\varepsilon}}^\infty ds_2 \rho_\varepsilon(s_2 - y_2). \end{aligned}$$

But

$$\int_{\frac{1}{\varepsilon}}^\infty ds_2 \rho_\varepsilon(s_2 - y_2) = \int_{\frac{1}{\varepsilon} - y_2}^\infty ds_2 \rho_\varepsilon(s_2).$$

Let $M > 0$ such that $\text{supp} f \subset [-M, M]$. Hence (3.5) is bounded by

$$\begin{aligned} \sup \left| (f^2)' \right| \int_0^M dy_1 \int_0^\infty ds_1 \rho_\varepsilon(s_1 - y_1) \int_0^\infty |R|(y_1, dy_2) \int_{\frac{1}{\varepsilon}(\frac{1}{\varepsilon} - y_2)}^\infty ds_2 \rho(s_2) \\ \leq \sup \left| (f^2)' \right| \int_0^M dy_1 \int_0^\infty |R|(y_1, dy_2) \int_{\frac{1}{\varepsilon}(\frac{1}{\varepsilon} - y_2)}^\infty ds_2 \rho(s_2) \end{aligned}$$

because $\int_{-\infty}^\infty \rho_\varepsilon(y) dy = 1$. This is bounded by $(I_{2,1,1}(\varepsilon) + I_{2,1,2}(\varepsilon)) \sup \left| (f^2)' \right|$ with

$$\begin{aligned} I_{2,1,1}(\varepsilon) &= \int_0^M dy_1 \int_0^{\frac{1}{\varepsilon}} |R|(y_1, dy_2) \int_{(\frac{1}{\varepsilon} - y_2)\frac{1}{\varepsilon}}^\infty ds_2 \rho(s_2), \\ I_{2,1,2}(\varepsilon) &= \int_0^M dy_1 \int_{\frac{1}{\varepsilon}}^\infty |R|(y_1, dy_2). \end{aligned}$$

Both expressions above converge to zero because of Assumption (A) ii) and Lebesgue dominated convergence theorem. Hence $I_{2,1}(\varepsilon) \rightarrow 0$.

As far as $I_{2,2}(\varepsilon)$ is concerned, when $\varepsilon \rightarrow 0$ we get

$$\int_0^\infty df^2(s_1)R_\varepsilon(s_1, \infty) \rightarrow \int_0^\infty df^2(s_1)R(s_1, \infty) = - \int_0^\infty f^2(s_1)R(ds_1, \infty)$$

according to Assumption (A) i). Consequently $\lim_{\varepsilon \rightarrow 0} I_2(\varepsilon) = \int_0^\infty f^2(s)R(ds, \infty)$. Since $I_2(\varepsilon) = I_3(\varepsilon)$, we also have $\lim_{\varepsilon \rightarrow 0} I_3(\varepsilon) = \int_0^\infty f^2(s)R(ds, \infty)$.

It remains to prove that $\lim_{\varepsilon \rightarrow 0} I_1(\varepsilon) = \int_{\mathbb{R}_+^2 \setminus D} (f(s_1) - f(s_2))^2 d\mu(s_1, s_2)$. By (3.4), transferring the convolution against $\rho_\varepsilon \otimes \rho_\varepsilon$ to the test function, $I_1(\varepsilon)$ becomes the expression

$$\left\langle \frac{\partial^2 R}{\partial s_1 \partial s_2}, (f^\varepsilon(s_1) - f^\varepsilon(s_2))^2 \chi_\varepsilon^\varepsilon(s_1) \chi_\varepsilon^\varepsilon(s_2) \right\rangle,$$

where

$$\begin{aligned} f^\varepsilon &= f * \rho^\varepsilon, \\ \chi_\varepsilon^\varepsilon &= \chi_\varepsilon * \rho^\varepsilon. \end{aligned}$$

This gives

$$\int_{\mathbb{R}_+^2} d\bar{\mu}(s_1, s_2) \frac{(f^\varepsilon(s_1) - f^\varepsilon(s_2))^2}{(s_1 - s_2)} \chi_\varepsilon^\varepsilon(s_1) \chi_\varepsilon^\varepsilon(s_2). \quad (3.7)$$

We observe that the functions

$$g_\varepsilon(s_1, s_2) = \begin{cases} \frac{(f^\varepsilon(s_1) - f^\varepsilon(s_2))^2}{s_1 - s_2} & , s_1 \neq s_2 \\ 0 & , s_1 = s_2. \end{cases} \quad (3.8)$$

and

$$g(s_1, s_2) = \begin{cases} \frac{(f(s_1) - f(s_2))^2}{s_1 - s_2} & , s_1 \neq s_2 \\ 0 & , s_1 = s_2. \end{cases} \quad (3.9)$$

are bounded by the square of the $\frac{1}{2}$ -Hölder norm of f .

Using Lebesgue's dominated convergence theorem, (3.7) goes to

$$\int_{\mathbb{R}_+^2} d\bar{\mu}(s_1, s_2) \frac{(f(s_1) - f(s_2))^2}{(s_1 - s_2)} = \int_{\mathbb{R}_+^2 \setminus D} d\mu(s_1, s_2) (f(s_1) - f(s_2))^2.$$

This justifies the case $f \in C_0^\infty(\mathbb{R})$. We consider now the general case. Let ρ_n be a sequence of mollifiers converging to the Dirac delta function and we set $f_n = \rho_n * f$. Taking into account previous arguments, identity (3.2) holds for f replaced with f_n . Therefore we have

$$\begin{aligned} \int_{\mathbb{R}_+^2} R(s_1, s_2) df_n(s_1) df_n(s_2) &= \int_{\mathbb{R}_+} f_n^2(s) R(ds, \infty) \\ &- \frac{1}{2} \int_{\mathbb{R}_+^2 \setminus D} (f_n(s_1) - f_n(s_2))^2 d\mu(s_1, s_2). \end{aligned} \quad (3.10)$$

The total variation of f_n is bounded by a constant times the total variation of f and $f_n \rightarrow f$ pointwise. So $df_n \rightarrow df$ weakly and also $df_n \otimes df_n \rightarrow df \otimes df$ by use of monotone class theorem. Since R is continuous and bounded,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}_+^2} R(s_1, s_2) df_n(s_1) df_n(s_2) = \int_{\mathbb{R}_+^2} R(s_1, s_2) df(s_1) df(s_2).$$

Therefore the left-hand side of (3.10) converges to the left-hand side of (3.2). On the other hand $f_n(s_1) - f_n(s_2) \rightarrow_{n \rightarrow \infty} f(s_1) - f(s_2)$ for every $(s_1, s_2) \in \mathbb{R}_+^2$. Since $\bar{\mu}$ and $|R|(ds, \infty)$ are finite non-atomic measures and because of considerations around (3.8) and (3.9), the sequence of right-hand sides of (3.10) converges to the right-hand side of (3.2). \blacksquare

Assumption (D)

- i) $R(ds_1, \infty)$ is a non-negative, σ -finite measure,
- ii) $\frac{\partial^2 R}{\partial s_1 \partial s_2} \Big|_{\mathbb{R}_+^2 \setminus D}$ is a non-positive measure.

In the next section, we will expand some examples of processes for which Assumptions (A), (B), (C) and (D) are fulfilled.

4 Examples

4.1 Processes with covariance measure structure

The first immediate example arises if X has a covariance measure μ with compact support, see Definition 2.2. In this case $\frac{\partial^2 R}{\partial s_1 \partial s_2}$ is a measure μ . In Remark 3.2 we proved that Assumption (A) is satisfied; obviously Assumption (B) is fulfilled and $\bar{\mu}$ is absolutely continuous with respect to μ . We recall that D is the diagonal of the first quarter of the plane.

Remark 4.1. *i) If μ restricted to D vanishes, Assumption (D) cannot be satisfied except if process X is deterministic. Indeed, suppose that for some $a > 0$, X_a is non-deterministic. Then, taking $f = 1_{[0, a]}$, we have*

$$\int_{\mathbb{R}_+^2} f(x_1) f(x_2) d\mu(x_1, x_2) = \text{Var}(X_a),$$

which is strictly positive. On the other hand previous integral equals

$$\int_{\mathbb{R}_+^2 \setminus D} f(x_1) f(x_2) d\mu(x_1, x_2) = \mu([0, a]^2) \leq 0.$$

ii) However, Assumption (D) is satisfied if $\text{supp } \mu \subset D$ and $R(ds_1, \infty)$ is positive. In this case $\mu|_{\mathbb{R}_+^2 \setminus D}$ is even zero. Consider as an example the case of classical Brownian motion or a martingale.

4.2 Fractional Brownian motion

Let $X = B^H$, $0 < H < 1$, $H \neq \frac{1}{2}$ be a fractional Brownian motion with Hurst parameter H stopped at some fixed time $T > 0$. Therefore we have $X_t = X_T$, $t \geq T$. Its covariance is

$$R(s_1, s_2) = \frac{1}{2}(\tilde{s}_1^{2H} + \tilde{s}_2^{2H} - |\tilde{s}_2 - \tilde{s}_1|^{2H})$$

with $\tilde{s}_i = s_i \wedge T$. Now

$$s_1 \mapsto R(s_1, \infty) = R(s_1, T)$$

has bounded variation and is even absolutely continuous since

$$\frac{\partial R}{\partial s_1}(s_1, \infty) = \begin{cases} H[s_1^{2H-1} + (T - s_1)^{2H-1}] & \text{if } s_1 < T \\ 0 & \text{if } s_1 > T. \end{cases}$$

So Assumption (A) is verified. Assumptions (B) and (C) are fulfilled because

$$\begin{aligned} \bar{\mu}(ds_1, ds_2) &= (s_1 - s_2) \frac{\partial^2 R(s_1, s_2)}{\partial s_1 \partial s_2} \\ &= H(2H - 1)|s_1 - s_2|^{2H-1} 1_{[0, T]^2}(s_1, s_2) \text{sign}(s_1 - s_2) ds_1 ds_2, \\ \left. \frac{\partial^2 R}{\partial s_1 \partial s_2} \right|_{\mathbb{R}^2 \setminus D} &= H(2H - 1)|s_1 - s_2|^{2H-2} 1_{[0, T]^2}(s_1, s_2). \end{aligned}$$

Remark 4.2. In this example $R(ds, \infty)$ is non-negative; μ is non-positive if and only if $H \leq \frac{1}{2}$. In that case Assumption (D) is fulfilled.

4.3 Bifractional Brownian motion

Suppose that $X = B^{H,K}$ is a bifractional Brownian motion with parameters $H \in]0, 1[$, $K \in]0, 1[$ stopped at some fixed time $T > 0$. We recall that $B^{H,1}$ is a fractional Brownian motion with Hurst index H . Moreover its covariance function, see [37] and [19], is given by

$$R(s_1, s_2) = 2^{-K} [(\tilde{s}_1^{2H} + \tilde{s}_2^{2H})^K - |\tilde{s}_1 - \tilde{s}_2|^{2HK}]$$

again with $\tilde{s}_i = s_i \wedge T$.

We have,

$$\frac{\partial R}{\partial s_1}(s_1, s_2) = 2HK2^{-K} \left[[s_1^{2H} + s_2^{2H}]^{K-1} s_1^{2H-1} - |s_1 - s_2|^{2HK-1} \text{sign}(s_1 - s_2) \right] \quad (4.1)$$

for $s_1, s_2 \in]0, T[$. Hence

$$\begin{aligned} & \left. \frac{\partial^2 R}{\partial s_1 \partial s_2} \right|_{[0, T]^2 \setminus D} \\ &= 2^{-K} \left[(4H^2 K(K-1)(s_1^{2H} + s_2^{2H})^{K-2} (s_1 s_2)^{2H-1} + 2HK(2HK-1)|s_1 - s_2|^{2HK-2} \right]. \end{aligned}$$

Consequently

$$\frac{\partial R}{\partial s_1}(s_1, \infty) = \begin{cases} 2HK2^{-K} \left[(s_1^{2H} + T^{2H})^{K-1} s_1^{2H-1} + (T - s_1)^{2HK-1} \right] & \text{if } s_1 \in]0, T[\\ 0 & \text{if } s_1 > T. \end{cases}$$

Moreover

$$\begin{aligned} & \bar{\mu}(ds_1, ds_2) \\ &= 1_{[0, T]^2}(s_1, s_2) 2^{-K} \left[4H^2 K(K-1)(s_1^{2H} + s_2^{2H})^{K-2} (s_1 s_2)^{2H-1} (s_1 - s_2)^2 \right. \\ & \quad \left. + 2HK(2HK-1)|s_1 - s_2|^{2HK} ds_1 ds_2 \right]. \end{aligned}$$

Conclusions

i) Assumptions (A) and (B) are verified. Assumption (C) is verified because $R(ds, \infty)$ and the marginal measure of $|\bar{\mu}|$ are equivalent to Lebesgue measure on $]0, T[$ and they vanish on $]T, \infty[$. If $HK \geq \frac{1}{2}$, X has even a covariance measure μ , see [25], Section 4.4.

ii) Assumption (D) is verified only if $HK \leq \frac{1}{2}$. Indeed, $R(ds, \infty)$ is non-negative and $\left. \frac{\partial^2 R}{\partial s_1 \partial s_2} \right|_{\mathbb{R}_+^2 \setminus D}$ is non-positive.

Remark 4.3. If $HK = \frac{1}{2}$, then Assumption (D) is verified even if $K \neq 1$. In that case $B^{H,K}$ is not a semimartingale, see [37], Proposition 3. This shows existence of a finite quadratic variation process which verifies Assumption (D) and it is not a local martingale.

4.4 Processes with weak stationary increments

Definition 4.1. A square $(\tilde{X}_t)_{t \geq 0}$, such that $\tilde{X}_0 = 0$, is said **with weak stationary increments** if for every $s, t, \tau \geq 0$

$$\text{Cov}(\tilde{X}_{s+\tau} - \tilde{X}_\tau, \tilde{X}_{t+\tau} - \tilde{X}_\tau) = \text{Cov}(\tilde{X}_s, \tilde{X}_t).$$

In particular setting $Q(t) = \text{Var}(\tilde{X}_t)$ we have

$$\text{Var}(\tilde{X}_{t+\tau} - \tilde{X}_\tau) = Q(t), \quad \forall t \geq 0.$$

In general \tilde{X} does not fulfill the technical assumption (2.2) and therefore we will work with X , where $X_t = \tilde{X}_{t \wedge T}$. This is no longer a process with weak stationary increments and its covariance is the following:

$$R(s_1, s_2) = \begin{cases} \frac{1}{2}(Q(s_1) + Q(s_2) - Q(s_1 - s_2)) & , s_1, s_2 \leq T, \\ \frac{1}{2}(Q(s_1) + Q(T) - Q(T - s_1)) & , s_2 > T, s_1 \leq T, \\ \frac{1}{2}(Q(s_2) + Q(T) - Q(T - s_2)) & , s_2 \leq T, s_1 > T, \\ Q(T) & , s_1, s_2 > T. \end{cases} \quad (4.2)$$

Remark 4.4. Without restriction of generality we will suppose

$$Q(t) = Q(T), \quad t \geq T \quad (4.3)$$

so that Q is bounded and continuous and can be extended to the whole line.

Proposition 4.5. Assumption (A) is verified if Q has bounded variation.

Proof: We have

$$R(\infty, s_2) = \begin{cases} \frac{Q(s_2) + Q(T) - Q(T - s_2)}{2} & , s_2 \leq T, \\ Q(T) & , s_2 > T, \end{cases} \quad (4.4)$$

so that

$$R(\infty, ds_2) = \begin{cases} \frac{1}{2} (Q(ds_2) - Q(T - ds_2)) & , s_2 \leq T, \\ 0 & , s_2 > T; \end{cases} \quad (4.5)$$

(4.2), (4.4) and (4.5) imply the validity of Assumption (A). ■

Proposition 4.6. We suppose the following.

- i) Q is absolutely continuous with derivative Q' .
- ii) $F_Q(s) := sQ'(s)$, $s > 0$ prolongates to zero by continuity to a bounded variation function, which is therefore bounded.

Then $\bar{\mu}$ is the finite Radon measure

$$\mathbf{1}_{[0, T]^2}(s_1, s_2) (-Q'(s_1 - s_2) ds_1 ds_2 + F_Q(s_1) ds_1 \delta_0(ds_2) - F_Q(s_1 - ds_2) ds_1).$$

Moreover Assumption (B) is verified as well as Assumption (C(ν)) with $\nu(ds_2) = \mathbf{1}_{]0, T[}(s_2) ds_2$.

Remark 4.7. i) F_Q can be prolonged to \mathbb{R} by setting $F_Q(-s) = F_Q(s)$, $s \geq 0$.

ii) In the sense of distributions we have

$$(Q'(s)s)' = Q''(s)s + Q'(s).$$

Under i), ii) is equivalent to saying that $Q''(ds) \cdot s$ is a finite measure.

iii) Consequently for any $\rho \neq 0$, $Q''|_{]-\infty, -\rho] \cup [\rho, +\infty[}$ is a finite, signed measure.

Proof (of Proposition 4.6): We will evaluate

$$\left\langle \frac{\partial R^2}{\partial s_1 \partial s_2}(s_1, s_2)(s_1 - s_2), \varphi \right\rangle \quad (4.6)$$

for $\varphi \in C_0^\infty(\mathbb{R}_+^2)$. This gives

$$\int_{\mathbb{R}_+^2} R(s_1, s_2) \frac{\partial^2}{\partial s_1 \partial s_2} (\varphi(s_1, s_2)(s_1 - s_2)) ds_1 ds_2 = \frac{1}{2}(I_1 + I_2 - I_3),$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}_+^2} Q(s_1) \frac{\partial^2}{\partial s_1 \partial s_2} (\varphi(s_1, s_2)(s_1 - s_2)) ds_1 ds_2, \\ I_2 &= \int_{\mathbb{R}_+^2} Q(s_2) \frac{\partial^2}{\partial s_1 \partial s_2} (\varphi(s_1, s_2)(s_1 - s_2)) ds_1 ds_2, \\ I_3 &= \int_{\mathbb{R}_+^2} \tilde{Q}(s_1, s_2) \frac{\partial^2}{\partial s_1 \partial s_2} (\varphi(s_1, s_2)(s_1 - s_2)) ds_1 ds_2, \end{aligned}$$

where

$$\tilde{Q}(s_1, s_2) = \begin{cases} Q(s_1 - s_2) & , s_1, s_2 \leq T, \\ Q(T - s_1) & , s_2 > T, s_1 \leq T, \\ Q(T - s_2) & , s_1 > T, s_2 \leq T, \\ 0 & , s_1, s_2 > T. \end{cases}$$

First we evaluate I_3 . Using assumption i) it is clear that for every $s_2 \geq 0$, $s_1 \mapsto \tilde{Q}(s_1, s_2)$ is absolutely continuous. Similarly, for every $s_1 \geq 0$, $s_2 \mapsto \tilde{Q}(s_1, s_2)$ has the same property.

Therefore integrating by parts we obtain

$$\begin{aligned} I_3 &= - \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\partial \tilde{Q}}{\partial s_2}(s_1, s_2) \frac{\partial}{\partial s_1} (\varphi(s_1, s_2)(s_1 - s_2)) \\ &= \int_0^T ds_1 \left(\int_0^T ds_2 Q'(s_1 - s_2) \frac{\partial}{\partial s_1} (\varphi(s_1, s_2)(s_1 - s_2)) \right) \\ &+ \int_T^\infty ds_1 \left(\int_0^T ds_2 Q'(T - s_2) \frac{\partial}{\partial s_1} (\varphi(s_1, s_2)(s_1 - s_2)) \right). \end{aligned}$$

Using Fubini's theorem we get

$$\begin{aligned} & \int_0^T ds_2 \int_0^T ds_1 Q'(s_1 - s_2) \left(\frac{\partial}{\partial s_1} \varphi(s_1, s_2)(s_1 - s_2) + \varphi(s_1, s_2) \right) \\ & - \int_0^T ds_2 Q'(T - s_2) \varphi(T, s_2)(T - s_2). \end{aligned}$$

Therefore

$$\begin{aligned} I_3 = & - \int_0^T ds_2 \int_0^T ds_1 F_Q(ds_1 - s_2) \varphi(s_1, s_2) + \int_0^T ds_2 \{F_Q(T - s_2) \varphi(T, s_2) - F_Q(-s_2) \varphi(0, s_2)\} \\ & + \int_0^T ds_2 \int_0^T ds_1 Q'(s_1 - s_2) \varphi(s_1, s_2) - \int_0^T ds_2 Q'(T - s_2) \varphi(T, s_2)(T - s_2). \end{aligned}$$

Consequently

$$\begin{aligned} I_3 = & - \int_0^T ds_2 \int_0^T ds_1 F_Q(ds_1 - s_2) \varphi(s_1, s_2) \\ & + \int_0^T ds_2 \int_0^T ds_1 Q'(s_1 - s_2) \varphi(s_1, s_2) - \int_0^T ds_2 F_Q(s_2) \varphi(0, s_2). \end{aligned} \tag{4.7}$$

Concerning I_1 we obtain

$$\begin{aligned} & \int_0^\infty ds_1 Q(s_1) \frac{\partial}{\partial s_1} (\varphi(s_1, s_2)(s_1, s_2))|_{s_2=0}^\infty = - \int_0^\infty ds_1 Q(s_1) \left(\frac{\partial \varphi}{\partial s_1} \varphi(s_1, 0) s_1 + \varphi(s_1, 0) \right) \\ & = - \int_0^\infty ds_1 Q(s_1) \frac{d}{ds_1} (\varphi(s_1, 0) s_1) = \int_0^T Q'(s_1) s_1 \varphi(s_1, 0) ds_1 = \int_0^T F_Q(s_1) \varphi(s_1, 0) ds_1. \end{aligned} \tag{4.8}$$

Concerning I_2 we obtain

$$I_2 = - \int_0^T F_Q(s_2) \varphi(0, s_2) ds_2. \tag{4.9}$$

Using (4.8), (4.9) and (4.7) it follows

$$\begin{aligned} I_1 + I_2 - I_3 = & \int_{\mathbb{R}_+} F_Q(s_1) \varphi(s_1, 0) ds_1 \\ & + \int_0^T ds_2 \int_0^T F_Q(ds_1 - s_2) \varphi(s_1, s_2) - \int_{\mathbb{R}_+^2} ds_1 ds_2 \varphi(s_1, s_2) Q'(s_1 - s_2). \end{aligned}$$

This allows to conclude taking into account the fact, by symmetry,

$$ds_1 F_Q(ds_1 - s_2) = ds_2 F_Q(s_1 - ds_2).$$

At this point Assumptions (B) and (C(ν)) follow directly. ■

Corollary 4.8. *Under the assumptions of Proposition 4.6, Assumption (D) is verified if Q is increasing and Q'' restricted to $]0, T[$ is a non-positive measure.*

Remark 4.9. *Previous Assumption (D) is equivalent to Q increasing and concave.*

Proof (of Corollary 4.8): By (4.5), $R(ds_1, \infty)$ is a non-negative measure if Q is increasing. Q'' restricted to $]0, \infty[$ is a Radon measure, hence Q' restricted to $]0, \infty[$ is of locally bounded variation and

$$\frac{\partial^2 R}{\partial s_1 \partial s_2} \Big|_{\mathbb{R}_+^2 \setminus D} = Q''(s_1 - ds_2) 1_{]0, T[}(s_1, s_2) ds_1$$

and the result follows. ■

Corollary 4.10. *Under the same assumptions as Proposition 4.6, if $Q'1_{]0, T[} > 0$ then Assumption (C) is verified.*

Proof: The result follows because in this case $\nu = |R|(\infty, ds) = R(\infty, ds) = \frac{1}{2}(Q'(s) + Q'(T - s))1_{]0, T[} ds$. ■

Example 4.11. Processes with weak stationary increments (particular cases).

Let $(\tilde{X}_t)_{t \geq 0}$ be a zero-mean second order process with weakly stationary increments. We set

$$Q(t) = \text{Var}(\tilde{X}_t)$$

We refer again to $X_t = \tilde{X}_{t \wedge T}$.

1. Suppose

$$Q(t) = \begin{cases} t^{2H} & , t < T \\ T^{2H} & , t \geq T. \end{cases}$$

Then Assumptions (A), (B) and (C) are verified. If $H \leq \frac{1}{2}$, Assumption (D) is fulfilled.

2. We consider a more singular kernel than every fractional scale. We set $T = 1$.

$$Q(t) = \begin{cases} \frac{1}{\log(\frac{1}{t})} & , 0 < t < e^{-2} \\ 0 & , t = 0, \\ \frac{1}{2} & , t \geq e^{-2}. \end{cases}$$

Then Assumption (A) is verified since Q is increasing. Moreover Q is absolutely continuous with

$$Q'(t) = \begin{cases} (\log \frac{1}{t})^{-2} \frac{1}{t} & , 0 < t < e^{-2}, \\ 0 & , t > e^{-2}. \end{cases}$$

We observe that

$$F_Q(t) = tQ'(t) = \begin{cases} (\log \frac{1}{t})^{-2} & , 0 < t < e^{-2}, \\ 0 & , t > e^{-2}. \end{cases}$$

and $\lim_{t \rightarrow 0} F_Q(t) = 0$. It is not difficult to show that F_Q has bounded variation, therefore Assumption (B) is fulfilled by Proposition 4.6. Since $Q' > 0$, on $]0, T[$ a.e., Assumption (C) is verified because of Corollary 4.10. Finally

$$Q''(t) = \begin{cases} -(\log \frac{1}{t})^{-2} t^{-2} [1 - 2 \log \frac{1}{t}] & , 0 < t < e^{-2}, \\ 0 & , t > e^{-2}. \end{cases}$$

Since Q'' is negative, Assumption (D) is fulfilled.

5 Comparison with the kernel approach

We consider a process X continuous in $L^2(\Omega)$ of the type

$$X_t = \int_0^t K(t, s) dW_s, t \in [0, T], \quad (5.10)$$

where $K : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is a measurable function, such that for every $t \geq 0$, $\int_0^t K^2(t, s) ds < \infty$.

Remark 5.1. $(X_t)_{t \in [0, T]}$ is a Gaussian process with covariance

$$R(t_1, t_2) = \int_0^{t_1 \wedge t_2} K(t_1, s) K(t_2, s) ds.$$

We extend (X_t) to the whole line, setting $X_t = X_T$, $t \geq T$, $X_t = 0$, $t < 0$.

We want to investigate here natural, sufficient conditions on K so that X has a covariance measure structure. We take inspiration from a paper of Alos-Mazet-Nualart [2], which discusses Malliavin calculus with respect to general processes of type (5.10). That paper distinguishes between the **regular** and **singular** case.

The aim of this section is precisely to provide some general considerations related to the approach presented in [2] in relation to ours. In their regular context, we will show

that the process has covariance measure structure. Concerning their singular case, we will restrict to the case that $K(t, s) = \kappa(t - s)$, $t \geq s \geq 0$, where $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}$. We will provide natural conditions so that Assumptions (A) and (B) are verified. We formulate first two general assumptions on K .

Assumption (K1) For each $s \geq 0$, $\bar{K}(dt, s) = K(dt, s)(t - s)$ is a finite measure. This implies in particular, for $\varepsilon > 0$,

$$K(dt, s)1_{(s+\varepsilon, \infty)}(t) \text{ is a finite measure.} \quad (5.11)$$

Assumption (K2)

$$\varepsilon \sup_s K(s + \varepsilon, s) \rightarrow 0.$$

Let $T > 0$. We extend K to $\tilde{K} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, so that

$$\tilde{K}(t, s) = \begin{cases} K(t, s) & , 0 < s < t < T, \\ K(T, s) & , 0 < s < T < t, \\ 0 & , \text{otherwise.} \end{cases} \quad (5.12)$$

Let $(W_t)_{t \geq 0}$ be a standard Brownian motion. Indeed

$$\tilde{X}_t = \int_0^t \tilde{K}(t, s) dW_s, \quad t \in \mathbb{R}_+ \quad (5.13)$$

extends X by continuity in $L^2(\Omega)$ from $[0, T]$ to \mathbb{R}_+ . In the sequel \tilde{K} and \tilde{X} will often be denoted again by K and X . For processes X defined for $t \in [0, T]$, [2] introduces two maps G and G^* . Similarly to [2], we define

$$G : L^2[0, T] \rightarrow L^2[0, T]$$

by

$$G\varphi(t) = \int_0^t K(t, s)\varphi(s)ds, \quad t \in [0, T].$$

Let $W^{1, \infty}([0, T])$ be the space of $\varphi \in L^2([0, T])$ absolutely continuous such that $\varphi' \in L^\infty([0, T])$. We set

$$G^* : W^{1, \infty}([0, T]) \rightarrow L^2[0, T],$$

by

$$G^*\varphi(s) = \varphi(s)K(T, s) + \int_{[s, T]} (\varphi(t) - \varphi(s))K(dt, s).$$

We remark that G^* is well defined because (K1) is verified.

Remark 5.2. In order to better understand the definition of G^* , we consider the following "regular" case: for $s \geq 0$, $t \mapsto K(t, s)$, $0 \leq s \leq t \leq T$ has bounded variation and

$$\sup_{s \in [0, T]} |K|(dt, s) < \infty.$$

Then integration by parts shows that

$$G^* \varphi(s) = \int_s^T \varphi(t) K(dt, s),$$

since $K(s-, s) = 0$.

Lemma 5.3. Under Assumptions (K1) and (K2), for $\varphi \in C_0^1(\mathbb{R}_+)$ we have

$$\int_0^T G^* \varphi dW = \varphi(T) X_T - \int_0^T X_s d\varphi_s. \quad (5.14)$$

Proof: Let $\varepsilon > 0$. Since

$$\begin{aligned} \int_s^T |\varphi(t) - \varphi(s)| |K|(dt, s) &= \int_s^t \frac{|\varphi(t) - \varphi(s)|}{|t - s|} |\bar{K}|(dt, s) \\ &\leq \sup |\varphi'| |\bar{K}|([s, T], s) < \infty, \end{aligned}$$

Lebesgue's dominated convergence theorem gives

$$(G^* \varphi)(s) = \varphi(s) K(T, s) + \lim_{\varepsilon \rightarrow 0} \int_{s+\varepsilon}^T (\varphi(t) - \varphi(s)) K(dt, s).$$

Integration by parts gives

$$\begin{aligned} &\varphi(s) K(T, s) + \lim_{\varepsilon \rightarrow 0} \{(\varphi(T) - \varphi(s)) K(T, s) + (\varphi(s + \varepsilon) - \varphi(s)) K(s + \varepsilon, s)\} \\ &\quad - \int_{s+\varepsilon}^T \varphi'(t) K(t, s) dt. \end{aligned}$$

Again Lebesgue's dominated convergence theorem implies

$$\varphi(T) K(T, s) - \int_s^T \varphi'(t) K(t, s) dt - \lim_{\varepsilon \rightarrow 0} (\varphi(s + \varepsilon) - \varphi(s)) K(s + \varepsilon, s).$$

Since $\varphi \in C_0^1$, Assumption (K2) says that the limit above is zero. Through stochastic Fubini's, the left member of (5.14) gives

$$\begin{aligned} &\varphi(T) \int_0^T K(T, s) dW_s - \int_0^T dt \varphi'(t) \int_0^T dW(s) K(t, s) \\ &= \varphi(T) X_T - \int_0^T X_s d\varphi(s). \end{aligned}$$

So the result is proven. ■

We leave now the general case and consider one assumption stated in [2].

Remark 5.4. [2] considers the following assumption

$$\int_0^T |K(\cdot|s, T], s)^2 ds < \infty, \quad (5.15)$$

which characterizes their "regular" context. Proposition 5.6 below shows that, (5.15) implies that X has a covariance measure structure.

Remark 5.5. Under (5.15), assumptions (K1) and (K2) are in particular fulfilled.

Proposition 5.6. Let $(X_t)_{t \in [0, T]}$ be a process defined by

$$X_t = \int_0^t K(t, s) dW_s, t \in [0, T],$$

where $(W_t)_{t \geq 0}$ is a classical Wiener process. Then X has a covariance measure structure if (5.15) is verified.

Proof: We recall that here K (resp. X) is prolongedated to \mathbb{R}_+^2 (resp. \mathbb{R}) in conformity with (5.12) and (5.13). It is enough to show that there is a constant C , such that

$$\left| \left\langle \frac{\partial^2 R}{\partial t_1 \partial t_2}, \varphi \right\rangle \right| \leq C \|\varphi\|_\infty, \forall \varphi \in C_0^\infty(\mathbb{R}_+^2).$$

Let $\varphi \in C_0^\infty(\mathbb{R}_+^2)$. We have

$$X_t = \int_0^\infty K(t, s) dW_s, t \geq 0,$$

with

$$R(t_1, t_2) = \int_0^\infty K(t_1, s) K(t_2, s) ds.$$

Indeed, using Fubini's, we have

$$\begin{aligned} \left\langle \frac{\partial^2 R}{\partial t_1 \partial t_2}, \varphi \right\rangle &= \int_{\mathbb{R}_+^2} R(t_1, t_2) \frac{\partial^2 \varphi}{\partial t_1 \partial t_2}(t_1, t_2) dt_1 dt_2 \\ &= \int_0^\infty ds \int_{\mathbb{R}_+^2} \frac{\partial^2 \varphi}{\partial t_1 \partial t_2}(t_1, t_2) K(t_1, s) K(t_2, s) dt_1 dt_2. \end{aligned} \quad (5.16)$$

Now $K(dt_1, s)K(dt_2, s)$ is a Radon measure on \mathbb{R}_+^2 because of Remark 5.2. According to [5], Theorem 12.5, its total variation is the supremum over $s = t_0 < t_1 < \dots < t_N = T$, of

$$\begin{aligned} & \sum_{i,j=0}^{N-1} |K(t_{i+1}, s)K(t_{j+1}, s) + K(t_i, s)K(t_j, s) - K(t_i, s)K(t_{j+1}, s) - K(t_{i+1}, s)K(t_j, s)| \\ &= \sum_{i,j=0}^{N-1} \left| \int_{]t_i, t_{i+1}] \times]t_j, t_{j+1}] } K(dt_1, s)K(dt_2, s) \right| \leq \sum_{i,j=0}^{N-1} \int_{]t_i, t_{i+1}] } |K|(dt_1, s) \int_{]t_j, t_{j+1}] } |K|(dt_2, s) \\ &= \left(\int_{]s, T]} |K|(dt, s) \right)^2 = |K|(\cdot]s, T], s)^2. \end{aligned}$$

Hence (5.16) equals

$$\int_0^\infty \int_{\mathbb{R}_+^2} \varphi(t_1, t_2) K(dt_1, s) K(dt_2, s)$$

and

$$\left| \left\langle \frac{\partial^2 R}{\partial t_1 \partial t_2}, \varphi \right\rangle \right| \leq \|\varphi\|_\infty \int_0^\infty ds \|K(dt_1, s)K(dt_2, s)\|_{var} \leq C \|\varphi\|_\infty.$$

with $C = \int_0^\infty |K|^2(\cdot]s, T], s) ds$ and $\|\cdot\|_{var}$ denotes the total variation norm. \blacksquare

In order to prepare the sequel, we specify $\frac{\partial^2 R}{\partial s_1 \partial s_2}$ if $X_t = \int_0^t \kappa(t-s) dW_s$, where $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}$ has bounded variation, supposed cadlag by convention. So we remain for the moment in the regular case.

Remark 5.7. a) We prolongate κ to $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ setting $\kappa(t) = 0$ if $t < 0$.

b) $R(t_1, t_2) = \int_0^{t_1 \wedge t_2} \kappa(t_1 - s) \kappa(t_2 - s) ds = \int_0^\infty \kappa(t_1 - s) \kappa(t_2 - s) ds$.

c) If κ has bounded variation then $\kappa 1_{] \varepsilon, \infty[}$ has bounded variation for any $\varepsilon > 0$, which will constitute Assumption (K1') below. It is equivalent to Assumption (K1), when the kernel K is not necessarily homogeneous.

Lemma 5.8. For $\phi \in C_0^\infty(\mathbb{R}_+^2)$

$$\left\langle \frac{\partial^2 R}{\partial t_1 \partial t_2}, \phi \right\rangle = \langle I_1 + I_2 + I_3 + I_4, \phi \rangle,$$

where I_1, I_2, I_3, I_4 are the following Radon measures:

$$\begin{aligned} I_1 &= \kappa^2(0)dt_1\delta(dt_2 - t_1), \\ I_2 &= \kappa(0)1_{[t_2, \infty[}(t_1)\kappa(dt_1 - t_2), \\ I_3 &= \kappa(0)1_{[t_1, \infty[}(t_2)\kappa(dt_2 - t_1), \\ \langle I_4, \phi \rangle &= \int_{\mathbb{R}_+^2} \kappa(dt_1)\kappa(dt_2) \int_0^\infty \phi(t_1 + s, t_2 + s)ds. \end{aligned}$$

Remark 5.9. If κ has bounded variation, Lemma 5.8 shows that X has a covariance measure structure.

In view of the verification of Assumption (B) we have the following result.

Corollary 5.10. Suppose that $\kappa(0) = 0$, κ with bounded variation. Let $\phi \in C_0^\infty(\mathbb{R}_+^2)$. We have

$$\left\langle \frac{\partial^2 R}{\partial t_1 \partial t_2}(t_1, t_2)(t_1 - t_2), \phi \right\rangle = \int_{\mathbb{R}_+^2} \kappa(dt_1)\kappa(dt_2)(t_1 - t_2) \int_0^\infty ds \phi(t_1 + s, t_2 + s).$$

Proof (of Lemma 5.8): By density arguments we will reduce to the case, where $\phi = \varphi \otimes \psi$, $\varphi, \psi \in C_0^\infty(\mathbb{R}_+)$. The left-hand side equals

$$\int_{\mathbb{R}_+^2} R(t_1, t_2) \frac{\partial^2 \varphi}{\partial t_1 \partial t_2}(t_1, t_2) dt_1 dt_2 = \int_0^\infty dt_1 \int_0^\infty dt_2 \varphi'(t_1) \psi'(t_2) \int_0^{t_1 \wedge t_2} \kappa(t_1 - s) \kappa(t_2 - s) ds.$$

We recall that by convention we extend κ to \mathbb{R} by setting zero on $] - \infty, 0[$. Hence, by Fubini's theorem it equals

$$\begin{aligned} & \int_0^\infty ds \int_s^\infty dt_1 \varphi'(t_1) \kappa(t_1 - s) \int_s^\infty \psi'(t_2) \kappa(t_2 - s) dt_2 \\ &= \int_0^\infty ds \left\{ -\varphi(s) \kappa(0) - \int_s^\infty \varphi(t_1) \kappa(dt_1 - s) \right\} \left\{ -\psi(s) \kappa(0) - \int_s^\infty \psi(t_2) \kappa(dt_2 - s) \right\} \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \kappa^2(0) \int_0^\infty \varphi(s) \psi(s) ds = \kappa(0)^2 \int_{\mathbb{R}_+^2} ds_1 \delta(ds_2 - s_1) \varphi(s_1) \psi(s_2), \\ I_2 &= \int_0^\infty \psi(s) \kappa(0) \int_s^\infty \varphi(t_1) \kappa(dt_1 - s) = \int_{\mathbb{R}_+^2} \varphi(t_1) \psi(s) 1_{[s, \infty[}(t_1) \kappa(dt_1 - s) ds \kappa(0), \\ I_3 &= \int_{\mathbb{R}_+^2} \varphi(s) \psi(t_2) 1_{[s, \infty[}(t_2) \kappa(dt_2 - s) ds \kappa(0), \\ I_4 &= \int_{\mathbb{R}_+^2} \varphi(t_1) \psi(t_2) \int_0^{t_1 \wedge t_2} \kappa(dt_1 - s) \kappa(dt_2 - s). \end{aligned}$$

By Fubini's theorem

$$\begin{aligned} I_4 &= \int_0^\infty ds \int_s^\infty \varphi(t_1)\kappa(dt_1 - s) \int_s^\infty \psi(t_2)\kappa(dt_2 - s) \\ &= \int_0^\infty ds \int_0^\infty \varphi(t_1 + s)\kappa(dt_1) \int_0^\infty \psi(t_2 + s)\kappa(dt_2). \end{aligned}$$

This concludes the proof. ■

We examine now some aspects related to the singular case. It is of course possible to give sufficient conditions on the kernel K , so that $X_t = \int_0^t K(t, s)dW_s$ fulfills Assumptions (A) and (B), however these conditions are too technical and not readable.

So we decided to consider the homogeneous case in the sense that $K(t, s) = \kappa(t - s)$, $\kappa : \mathbb{R} \rightarrow \mathbb{R}$, where $\kappa|_{\mathbb{R}_-} = 0$. Clearly the minimal assumption, so that X is defined, is $\kappa \in L^2([0, t])$, $\forall t \geq 0$. This is equivalent to $\kappa \in L^2(\mathbb{R}_+)$.

We formulate first an assumption on κ .

Assumption (K1') $\kappa|_{] \varepsilon, \infty[}$ is with bounded variation for any $\varepsilon > 0$.

We recall that this is equivalent to (K1), when K is homogeneous.

Proposition 5.11. *Let $(X_t)_{t \geq 0}$ be a process defined by*

$$X_t = \int_0^t \kappa(t - s)dW_s, \quad t \geq 0.$$

We suppose (K1'), (K2) and moreover

a) κ has compact support,

b)

$$\sup_{s \geq 0} \int_0^\infty du \left| \int_0^u (\kappa(dx)\kappa(s + x - u) - \kappa(s - u)) \right| < \infty. \quad (5.17)$$

Then Assumption (A) is fulfilled.

Remark 5.12. 1. *If we assume (K1'), then $\kappa(dx)$ is a finite measure on $] \varepsilon, \infty[$, so the left-hand side of (5.17) is a priori not always finite. Indeed $|\kappa|(dx)$ on $[0, \infty[$ is only a σ -finite measure which may be infinite. $\int_0^u \kappa(dx)(\kappa(s + x - u) - \kappa(s - u))$ is evaluated as*

$$\lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty \kappa(dx)(\kappa(s + x - u) - \kappa(s - u))$$

2. Assumption (K2) implies here that $\kappa(\varepsilon)\varepsilon \xrightarrow{\varepsilon \rightarrow 0^+} 0$.

Proof (of Proposition 5.11): Let $\alpha \in C_0^\infty(\mathbb{R}_+)$. We want to show the existence of a constant \mathcal{C} such that, for any $s \geq 0$

$$\left| \int_0^\infty dx \alpha'(x) R(s, x) \right| \leq \mathcal{C} \|\alpha\|_\infty. \quad (5.18)$$

This would establish the validity of Assumption (A). The left-hand side of (5.18) is given by

$$\begin{aligned} & \int_0^\infty dx \alpha'(x) \int_0^\infty du \kappa(s-u) \kappa(x-u) = \int_0^\infty du \kappa(s-u) \int_u^\infty dx \alpha'(x) \kappa(x-u) \\ &= \int_0^s du \kappa(s-u) \int_0^\infty dx \alpha'(x+u) \kappa(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^s du \kappa(s-u) \int_\varepsilon^\infty dx \frac{d}{dx} (\alpha(x+u) - \alpha(u)) \kappa(x). \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^s du \kappa(s-u) \left\{ (\alpha(u+\varepsilon) - \alpha(u)) \kappa(\varepsilon) - \int_\varepsilon^\infty \kappa(dx) (\alpha(x+u) - \alpha(u)) \right\} \\ &= \lim_{\varepsilon \rightarrow 0} - \int_0^s du \kappa(s-u) \int_\varepsilon^\infty \kappa(dx) (\alpha(x+u) - \alpha(u)) \end{aligned} \quad (5.19)$$

since $\lim_{\varepsilon \rightarrow 0} \kappa(\varepsilon)\varepsilon = 0$ and $\kappa \in L^2([0, s])$.

Now (5.19) gives

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty \kappa(dx) \int_0^\infty du \kappa(s-u) (\alpha(x+u) - \alpha(u)) \\ &= \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty \kappa(dx) \left\{ \int_x^\infty d\tilde{u} \kappa(s+x-\tilde{u}) \alpha(\tilde{u}) - \int_0^\infty du \kappa(s-u) \alpha(u) \right\} \\ &= \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty du \alpha(u) \int_\varepsilon^u \kappa(dx) \kappa(s+x-u) - \int_\varepsilon^\infty du \alpha(u) \int_\varepsilon^\infty \kappa(dx) \kappa(s-u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^\infty du \alpha(u) \int_\varepsilon^u \kappa(dx) (\kappa(s+x-u) - \kappa(s-u)) - \int_\varepsilon^\infty du \alpha(u) \int_u^\infty \kappa(dx) \kappa(s-u) \\ & \xrightarrow{\varepsilon \rightarrow 0} I_1(\alpha) - I_2(\alpha), \end{aligned}$$

where

$$\begin{aligned} I_1(\alpha) &= \int_0^\infty du \alpha(u) \int_0^u \kappa(dx) (\kappa(s+x-u) - \kappa(s-u)), \\ I_2(\alpha) &= \int_0^\infty du \alpha(u) \int_u^\infty \kappa(dx) \kappa(s-u). \end{aligned}$$

We remark that

$$|I_2(\alpha)| \leq \|\alpha\|_\infty \int_0^\infty du \kappa^2(u)$$

because of Cauchy-Schwarz. Moreover

$$|I_1(\alpha)| \leq \|\alpha\|_\infty \int_0^\infty du \left| \int_0^u \kappa(dx) (\kappa(s+x-u) - \kappa(s-u)) \right|.$$

The right-hand side is bounded because of (5.17). ■

Remark 5.13. *We remark that (5.17) is a quite general assumption. It is for instance verified if*

$$\int_0^\infty |\kappa|(dx) \int_0^\infty |\kappa(x+u) - \kappa(u)| du < \infty \quad (5.20)$$

In particular, taking $\kappa(x) = x^{H-\frac{1}{2}}$, $H > 0$, (5.20) is always verified.

We go on establishing sufficient conditions so that Assumption (B) is verified.

Proposition 5.14. *We suppose again (K1'). In particular $|\kappa|_{var}(x) := -\int_x^\infty d|\kappa|(y)$, $x > 0$ exists. Suppose there is $\delta > 0$ with*

$$\int_0^\delta |\kappa|_{var}^2(y) dy < \infty \quad (5.21)$$

Then Assumption (B) is fulfilled.

Remark 5.15. *If κ is monotonous and $\kappa(+\infty) = 0$, then (5.21) is always fulfilled since $|\kappa|_{var}(x) = -\kappa(x)$, which is square integrable.*

Proof(of Proposition 5.14): Let $\varphi \in C_0^\infty(\mathbb{R}_+^2)$. We need to show that

$$\left| \int_{\mathbb{R}_+^2} R(t_1, t_2) \frac{\partial^2}{\partial t_1 \partial t_2} (\varphi(t_1, t_2)(t_1 - t_2)) \right| \leq \text{const.} \|\varphi\|_\infty, \quad (5.22)$$

where

$$R(t_1, t_2) = \int_0^{t_1 \wedge t_2} \kappa(t_1 - s) \kappa(t_2 - s) ds.$$

The left-hand side of (5.22) is the limit when $\varepsilon \rightarrow 0$ of

$$\int_{\mathbb{R}_+^2} R_\varepsilon(t_1, t_2) \frac{\partial^2}{\partial t_1 \partial t_2} (\varphi(t_1, t_2)(t_1 - t_2)) dt_1 dt_2, \quad (5.23)$$

where

$$R_\varepsilon(t_1, t_2) = \int_0^{t_1 \wedge t_2} \kappa_\varepsilon(t_1 - s) \kappa_\varepsilon(t_2 - s) ds,$$

$$\kappa_\varepsilon(u) = 1_{] \varepsilon, \infty[} \kappa(u),$$

κ_ε being of bounded variation. Applying Lemma 5.8 and the fact that $\kappa_\varepsilon(0) = 0$, expression (5.23) gives

$$\int_{\mathbb{R}_+^2} \kappa_\varepsilon(dt_1)\kappa_\varepsilon(dt_2)(t_1 - t_2) \int_0^\infty ds\varphi(t_1 + s, t_2 + s) \quad (5.24)$$

We set $|\kappa_\varepsilon|_{var}(x) := -\int_{x \vee \varepsilon}^\infty d|\kappa|(y)$. Let $M > 0$ such that $supp\varphi \subset [0, M]^2$. Previous quantity is bounded by

$$\|\varphi\|_\infty M^2 \int_{\mathbb{R}_+^2} |\kappa_\varepsilon|_{var}(dt_1)|\kappa_\varepsilon|_{var}(dt_2)|t_1 - t_2|.$$

We have

$$\int_{\mathbb{R}_+^2} |\kappa_\varepsilon|_{var}(dt_1)|\kappa_\varepsilon|_{var}(dt_2)(t_1 - t_2) = 2 \int_0^\infty |\kappa_\varepsilon|_{var}(dt_1) \int_0^{t_1} |\kappa_\varepsilon|_{var}(dt_2)(t_1 - t_2).$$

Integrating by parts, previous expression equals

$$\begin{aligned} & 2 \int_0^\infty |\kappa_\varepsilon|_{var}(dt_1) \int_0^\infty |\kappa_\varepsilon|_{var}(t_2)dt_2 = 2 \int_0^\infty dt_2 |\kappa_\varepsilon|_{var}(t_2) \int_{t_2}^\infty |\kappa_\varepsilon|_{var}(dt_1) \\ & = 2 \int_\varepsilon^\infty dt_2 |\kappa|_{var}^2(t_2) \xrightarrow{\varepsilon \rightarrow 0} 2 \int_0^\infty dt_2 |\kappa|_{var}^2(t_2), \end{aligned} \quad (5.25)$$

which is finite because of Assumption (5.21). ■

6 Definition of the Paley-Wiener integral

6.1 Functional spaces and related properties

The aim of this section is to define a natural class of integrands for the so called Paley-Wiener integral (or simply Wiener integral). Let $X = (X_t)$ be a second order process, i.e. a square integrable process which is continuous in $L^2(\Omega)$. We suppose moreover $X_0 = 0$ and $\lim_{t \rightarrow \infty} X_t = X_\infty$ in $L^2(\Omega)$, as in (2.2). As observed for instance in [22], the natural strategy is to extend the linear map

$$I : C_0^1(\mathbb{R}_+) \longrightarrow L^2(\Omega) \quad (6.1)$$

defined by $I(f)$:

$$I(f) := \int_0^\infty f dX = f(\infty)X_\infty - \int_0^\infty X df. \quad (6.2)$$

In this section, we introduce a natural Banach space of integrands for which the Wiener integral is defined through prolongation of operator I . In the whole section X will be supposed to verify Assumptions (A), (B) and (C(ν)) for some ν by default.

We recall that $\mu = \frac{\partial^2 R}{\partial s_1 \partial s_2}$ restricted to $\mathbb{R}_+^2 \setminus D$ is a σ -finite measure but $\frac{\partial^2 R}{\partial s_1 \partial s_2}$ is only a distribution.

Definition 6.1. We denote by \tilde{L}_R the linear space of Borel functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, such that

- i) $\int_0^\infty f^2(s)|R|(ds, \infty) < \infty$,
- ii) $\int_{\mathbb{R}_+^2 \setminus D} (f(s_1) - f(s_2))^2 d|\mu|(s_1, s_2) < \infty$,

where $|\mu|$ is the total variation measure of the σ -finite measure μ .

Remark 6.1. The integral in ii) equals

$$\int_{\mathbb{R}_+^2} |f(s_1) - f(s_2)|^2 d|\mu|(s_1, s_2).$$

For $f \in \tilde{L}_R$ we define

$$\|f\|_{\mathcal{H}}^2 = \int_0^\infty f^2(s)R(ds, \infty) - \frac{1}{2} \int_{\mathbb{R}_+^2} (f(s_1) - f(s_2))^2 d\mu(s_1, s_2) \quad (6.3)$$

$$\|f\|_R^2 = \int_0^\infty f^2(s)|R|(ds, \infty) + \frac{1}{2} \int_{\mathbb{R}_+^2} (f(s_1) - f(s_2))^2 d|\mu|(s_1, s_2) \quad (6.4)$$

Let H_X be the Hilbert subspace of $L^2(\Omega)$ constituted by the closure of $I(f)$, $f \in C_0^1(\mathbb{R}_+)$.

Remark 6.2. If $f \in C_0^1(\mathbb{R}_+)$, then (6.2) and Proposition 3.6 give

$$E(I(f)^2) = E\left(\int_0^\infty X_u df(u)\right)^2 = \int_{\mathbb{R}_+^2} R(s_1, s_2) df(s_1) df(s_2) = \|f\|_{\mathcal{H}}^2.$$

So $C_0^1(\mathbb{R}_+)$ equipped with $\|\cdot\|_{\mathcal{H}}$ is isometrically embedded into $L^2(\Omega)$.

Let \mathcal{H} be an abstract completion of $C_0^1(\mathbb{R}_+)$ with respect to $\|\cdot\|_{\mathcal{H}}$. \mathcal{H} will be called "self-reproducing kernel space". The application $I : C_0^1(\mathbb{R}_+) \rightarrow H_X$ uniquely prolongates to \mathcal{H} . If \mathcal{H} were a space of functions, the prolongation I would be candidate to be called Paley-Wiener integral.

Remark 6.3. i) For $f \in \tilde{L}_R$, we have

$$\|f\|_{\mathcal{H}} \leq \|f\|_R,$$

ii) $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_R$ are seminorms on \tilde{L}_R because they derive from the semi-scalar products

$$\begin{aligned}\langle f, g \rangle_R &= \int_0^\infty (fg)(s)|R|(ds, \infty) + \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2)(f(s_1) - f(s_2))(g(s_1) - g(s_2)) \\ \langle f, g \rangle_{\mathcal{H}} &= \int_0^\infty (fg)(s)R(ds, \infty) - \int_{\mathbb{R}_+^2} d\mu(s_1, s_2)(f(s_1) - f(s_2))(g(s_1) - g(s_2))\end{aligned}$$

iii) $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is a semi-scalar product because it is a bilinear symmetric form; moreover it is positive definite on C_0^1 because $\langle \varphi, \varphi \rangle_{\mathcal{H}} = E(I(\varphi)^2)$.

iv) We remark however that $\|f\|_R = 0$ implies in any case $f = 0$ $|R|(ds, \infty)$ a.e.

v) \tilde{L}_R / \sim is naturally equipped with a scalar product inherited by $\langle \cdot, \cdot \rangle_R$, where $f \sim g$ if $f = g$ $|R|(ds, \infty)$ a.e. and

$$f(s_1) - f(s_2) - (g(s_1) - g(s_2)) = 0 \quad |\mu| \quad \text{a.e. .}$$

In particular

$$f(s_1) - f(s_2) - (g(s_1) - g(s_2)) = 0 \quad |\bar{\mu}| \quad \text{a.e. .}$$

However it may not be complete.

vi) The linear space of $\frac{1}{2}$ -Hölder continuous functions with compact support S included in \tilde{L}_R . Indeed, if f belongs to such space then

$$\int_0^\infty f^2(s)|R|(ds, \infty) \leq \|f\|_\infty^2 \int_0^\infty |R|(ds, \infty)$$

Moreover, expression ii) in Definition 6.1 is bounded by

$$k^2 \int_S |s_1 - s_2| d|\mu|(s_1, s_2) = k^2 |\bar{\mu}|(S \times S),$$

where k is a Hölder constant for f .

We denote by L_R the closure of C_0^1 onto \tilde{L}_R with respect to $\|\cdot\|_R$.

Remark 6.4. L_R is a normed linear space (as \tilde{L}_R) which is not necessarily complete.

We repeat that we will not consider the (abstract) completion of $C_0^1(\mathbb{R}_+)$ with respect to norm $\|\cdot\|_{\mathcal{H}}$ excepted if it is identifiable with a concrete space of functions.

Next proposition shows that in many situations L_R is a rich subspace of \tilde{L}_R .

Proposition 6.5. Suppose the existence of an even function $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$, such that $d|\mu|$ is equivalent to $\phi(x_1 - x_2)dx_1dx_2$. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded Borel function with compact support with at most countable jumps. Then $f \in \tilde{L}_R \Rightarrow f \in L_R$.

Proof (of Proposition 6.5): Let $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ smooth with $\int_0^\infty \rho(y)dy = 1$. We set $\rho_\varepsilon = \frac{1}{\varepsilon}\rho(\frac{\cdot}{\varepsilon})$, for $\varepsilon > 0$. We show that f is a limit with respect to $\|\cdot\|_R$ of a sequence of smooth bounded functions with compact support (which belong to L_R). We consider the sequence $f_n : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$f_n(x) = \int_0^\infty f(x-y)\rho_{\frac{1}{n}}(y)dy = \int_0^\infty f\left(x - \frac{y}{n}\right)\rho(y)dy.$$

Clearly $\|f_n\|_\infty \leq \|f\|_\infty$, $f_n(x) \rightarrow f(x)$, for every continuity point x of f . Therefore

$$\int_0^\infty |f_n(s) - f(s)||R|(ds, \infty) \xrightarrow{n \rightarrow \infty} 0,$$

since $|R|(ds, \infty)$ is a non-atomic finite measure. It remains to prove

$$\int_{\mathbb{R}_+^2} (f_n(x_1) - f(x_1) - f_n(x_2) + f(x_2))^2 d|\mu|(x_1, x_2) \xrightarrow{n \rightarrow \infty} 0. \quad (6.5)$$

The left-hand side of (6.5) equals

$$\begin{aligned} & \int_{\mathbb{R}_+^2} d|\mu|(x_1, x_2) \left[\int_{\mathbb{R}_+} dy \rho(y) f\left(x_1 - \frac{y}{n}\right) - f(x_1) - f\left(x_2 - \frac{y}{n}\right) + f(x_2) \right]^2 \\ & \leq \int_{\mathbb{R}_+^2} d|\mu|(x_1, x_2) \int_{\mathbb{R}_+} dy \rho(y) \left[f\left(x_1 - \frac{y}{n}\right) - f(x_1) - f\left(x_2 - \frac{y}{n}\right) + f(x_2) \right]^2. \end{aligned}$$

Last inequality comes from Jensen's. By Fubini's the right-hand side of previous expression equals

$$\begin{aligned} & \int_0^\infty dy \rho(y) \int_{\mathbb{R}_+^2} d|\mu|(x_1, x_2) \left[f\left(x_1 - \frac{y}{n}\right) - f(x_1) - f\left(x_2 - \frac{y}{n}\right) + f(x_2) \right]^2 \\ & = \int_0^\infty dy \rho(y) I_n(y), \end{aligned} \quad (6.6)$$

where

$$I_n(y) = \int_{\mathbb{R}_+^2} d|\mu|(x_1, x_2) \left[f\left(x_1 - \frac{y}{n}\right) - f(x_1) - f\left(x_2 - \frac{y}{n}\right) + f(x_2) \right]^2.$$

$I_n(y)$ is bounded by

$$\begin{aligned} & 2 \left[\int_{\mathbb{R}_+^2} dx_1 dx_2 \phi(x_1 - x_2) \left(f\left(x_1 - \frac{y}{n}\right) - f\left(x_2 - \frac{y}{n}\right) \right)^2 \right. \\ & \quad \left. + \int_{\mathbb{R}_+^2} dx_1 dx_2 \phi(x_1 - x_2) (f(x_1) - f(x_2))^2 \right] \end{aligned} \quad (6.7)$$

Setting $\tilde{x}_i = x_i - \frac{y}{n}$, $i = 1, 2$, in the first integral, (6.7) is upper bounded by

$$2 \int_{\mathbb{R}_+^2} \phi(x_1 - x_2)(f(x_1) - f(x_2))^2 dx_1 dx_2 = 2 \int_{\mathbb{R}_+^2} d|\mu|(x_1, x_2)(f(x_1) - f(x_2))^2.$$

On the other hand I_n converges pointwise to zero. Lebesgue's dominated convergence theorem applied to (6.6) allows to conclude (6.5). \blacksquare

Remark 6.6. 1) We recall that for instance a cadlag function with compact support is a bounded function with at most countable jumps.

2) The assumption related to μ in the statement of Proposition 6.5 is for instance verified if X is a fractional (or bifractional) Brownian motion.

3) If $|R|(ds, \infty)$ is absolutely continuous with respect to Lebesgue, it is easy to show that the statement of Proposition 6.5 holds for every bounded Borel function with compact support $f : \mathbb{R}_+ \rightarrow \mathbb{R}$. In particular no jump condition is required.

An easy consequence of the definition of the L_R -norm is the following.

Proposition 6.7. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function, $f \in L_R$. Then $g \circ f \in L_R$.

Proof: $\nu(ds) = |R|(ds, \infty)$. Calling k the Lipschitz constant, since $|g|(s) \leq k(1+|s|)$ it follows

$$\begin{aligned} & \int_0^\infty g^2(f(s))\nu(ds) + \frac{1}{2} \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2)(g(f(s_1)) - g(f(s_2)))^2 \\ & \leq k^2\nu(\mathbb{R}_+) + k^2 \int_0^\infty f^2(s)\nu(ds) + \frac{k^2}{2} \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2)(f(s_1) - f(s_2))^2 \\ & = k^2\nu(\mathbb{R}_+) + k^2\|f\|_R^2, \end{aligned}$$

This shows that $g \circ f \in \tilde{L}_R$. $g \circ f$ is indeed in L_R since, if $f_n \in C_0^1(\mathbb{R}_+)$ converges to $f \in L_R$, then performing similar calculations as before, we have

$$\|g \circ f_n - g \circ f\|_R^2 \rightarrow 0.$$

By analogous arguments, we obtain the following result.

Proposition 6.8. Let $f, g \in L_R$ and bounded. Then $fg \in L_R$.

Proposition 6.9. *Let $V : \mathbb{R} \rightarrow \mathbb{R}_+$ increasing on \mathbb{R}_+ such that*

$$\int_{\mathbb{R}_+^2} V^2(x_1 - x_2) d|\mu|(x_1, x_2) < \infty \quad (6.8)$$

Then every Borel function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ with compact support such that

$$|f(x_1) - f(x_2)| \leq V(x_1 - x_2) \quad (6.9)$$

belongs to L_R .

Corollary 6.10. *Every bounded $\frac{1}{2}$ -Hölder continuous function f with compact support belongs to L_R .*

Proof of Corollary 6.10: We apply Proposition 6.9 with $V(x) = |x|^{\frac{1}{2}}$. ■

Remark 6.11. i) *If V is continuous, condition (6.9) is equivalent to saying that V is the continuity modulus of f .*

ii) *If V is continuous, then f fulfilling (6.9) is continuous.*

Proof(of Proposition 6.9): It follows the same scheme as the proof of Proposition 6.5. Let f_n be as in that proof.

i) $f_n \rightarrow f$ pointwise and $|f_n| \leq \|f\|_\infty$,

ii) $\int_0^\infty |f_n - f|(s) |R|(ds, \infty) \rightarrow 0$ because of Lebesgue's dominated convergence theorem.

iii) (6.5) is again a consequence of Lebesgue's; (6.6) still holds with

$$|I_n(y)| \leq 2 \int_{\mathbb{R}_+^2} d|\mu|(x_1, x_2) V^2(x_1 - x_2).$$

■

Remark 6.12. *If X has a covariance measure with compact support, then every bounded function with compact support belongs to L_R . This follows by Proposition 6.9 taking $V(x) \equiv 2 \sup |f|$. However the property of compact support will be raised in Proposition 6.13.*

Proposition 6.13. *If X has a covariance measure with compact support, any bounded function still belongs to L_R .*

Proposition 6.14. \tilde{L}_R *is a Hilbert space if Assumption (C) is fulfilled.*

Remark 6.15. Suppose that X has a covariance measure with compact support, with corresponding signed measure μ . We link below the present approach with the one in [25].

1) One alternative approach would be to consider the measure ν which was introduced in [25], Section 5, i.e. the marginal measure of $|\mu|$. The space $L^2(d\nu) := L^2(\mathbb{R}_+, d\nu)$ was a natural space, where the Wiener integral could be defined.

2) In this paper $\|\cdot\|_R$ is the norm which allows to prolongate the Wiener integral operator I . Point 1) suggests that $\|\cdot\|_R$ should be somehow related to $\|\cdot\|_{L^2(\nu)}$. By Cauchy-Schwarz, we have

$$\int_{\mathbb{R}_+^2} |f(s_1)f(s_2)|d|\mu|(s_1, s_2) \leq \int_0^\infty f^2(s)d\nu(s).$$

This shows that the seminorm of $L^2(d\nu)$ is equivalent to $\|f\|_{R,\nu}$ where

$$\|f\|_{R,\nu}^2 = \int_0^\infty f^2(s)\nu(ds) + \frac{1}{2} \int_{\mathbb{R}_+^2} (f(s_1) - f(s_2))^2 d|\mu|(s_1, s_2).$$

This looks similarly to $\|\cdot\|_R$ norm but they could be different. Indeed, we only have

$$\|f\|_{L^2(d\nu)} \sim \|f\|_{R,\nu} \geq \|f\|_R,$$

which implies that

$$L^2(d\nu) \subset L_R. \tag{6.10}$$

This implies that $\|\cdot\|_R$ will provide a larger space, where the Wiener integral is defined.

3) In particular, if μ is non-negative (as for the case X being a fractional Brownian motion with Hurst index $H \geq \frac{1}{2}$ stopped at some time T), we have

$$\|\cdot\|_R = \|\cdot\|_{R,\nu} \sim \|\cdot\|_{L^2(d\nu)}.$$

Consequently by item 2) it follows that $L_R = L^2(d\nu)$.

4) If μ is non-negative, then $L_R = \tilde{L}_R$ since $C_0^1(\mathbb{R})$ is dense in $L^2(d\nu)$, see Lemma 3.8 of [25].

Proof (of Proposition 6.13): This follows by Remark 6.15, point 2). Indeed, any bounded function belongs to $L^2(d\nu)$ because ν is finite. ■

Proof (of Proposition 6.14): It is enough to show that \tilde{L}_R is complete. We set $\chi(ds) = |R|(ds, \infty)$. Let (f_n) be a Cauchy sequence in \tilde{L}_R . We recall that

$$\|f_n - f_m\|_R^2 = \int_0^\infty (f_n - f_m)^2(s)\chi(ds) + \frac{1}{2} \int_{\mathbb{R}_+^2} |\mu|(ds_s, ds_2)(g_n - g_m)^2(s_1, s_2),$$

where $g_n(s_1, s_2) = f_n(s_1) - f_n(s_2)$. Since both integrals are non-negative, the Cauchy sequence (f_n) is Cauchy in $L^2(d\chi)$ and (g_n) is Cauchy in $L^2(d|\mu|)$. Since $L^2(d\chi)$ is complete, there is $f \in L^2(d\chi)$ being the limit of f_n when $n \rightarrow \infty$. On the other hand, since $L^2(\mathbb{R}_+^2, |\mu|)$ is complete, g_n converges to some $g \in L^2(\mathbb{R}_+^2; d|\mu|)$.

It remains to show that

$$g(s_1, s_2) = f(s_1) - f(s_2) \text{ } \mu \text{ a.e., } s_1, s_2 > 0.$$

Since $f_n \rightarrow f$ in $L^2(d\chi)$, there is a subsequence (n_k) such that $f_{n_k}(s) \rightarrow f(s)$, for $s \notin N$, where $\chi(N) = 0$. Consequently for $(s_1, s_2) \in N^c \times N^c$

$$f_n(s_1) - f_n(s_2) \rightarrow f(s_1) - f(s_2). \quad (6.11)$$

Moreover, obviously if $s_1 = s_2$, (6.11) holds. Hence for $(s_1, s_2) \in (N^c \times N^c) \cup D$,

$$g(s_1, s_2) = f(s_1) - f(s_2),$$

where we recall that $D = \{(s, s) | s \in \mathbb{R}_+\}$ is the diagonal. This concludes the proof if we show that $((N^c \times N^c) \cup D)^c$ is $|\mu|$ -null.

This set equals

$$(N^c \times N^c)^c \cap D^c$$

and it is included in $((N \times \mathbb{R}_+) \cap D^c) \cup ((\mathbb{R}_+ \times N) \cap D^c)$. Since $|\mu|$ and $|\bar{\mu}|$ are equivalent outside D^c , it is enough to show that

$$|\bar{\mu}|((N \times \mathbb{R}_+) \cap D^c) = 0 \text{ and } |\bar{\mu}|((\mathbb{R}_+ \times N) \cap D^c) = 0.$$

Previous quantities are bounded by

$$|\bar{\mu}|(N \times \mathbb{R}_+), \text{ and } |\bar{\mu}|(\mathbb{R}_+ \times N),$$

which coincide with the marginal measures of $|\bar{\mu}|$ evaluated on N . Assumption (C) allows to conclude. ■

Corollary 6.16. *If Assumptions (C), (D) are in force, then L_R is the closure of $C_0^1(\mathbb{R}_+)$ under $\|\cdot\|_{\mathcal{H}}$; in particular L_R is a "self-reproducing kernel space".*

Proof: According to Assumption (D), we have $\|\cdot\|_R = \|\cdot\|_{\mathcal{H}}$ for $\psi \in C_0^1(\mathbb{R}_+)$. According to Proposition 6.14 L_R is a Hilbert space equipped with $\|\cdot\|_{\mathcal{H}}$, which is by definition the closure of $C_0^1(\mathbb{R}_+)$. ■

Remark 6.17. 1) We will see that the Paley-Wiener integral can be naturally defined on space L_R .

2) Corollary 6.16 is interesting because it shows that a natural space where the Wiener integral will be defined is complete under Assumptions (A), (B), (C), (D).

3) This result is of the same nature as the one of [36], which shows that the space, where its Wiener integral is defined, is also complete with respect to the norm $\|\cdot\|_{\mathcal{H}}$ when X is a fractional Brownian motion, with parameter $H \leq \frac{1}{2}$. In Section 4.2 we have proved that Assumptions (A), (B), (C), (D) are indeed fulfilled in that case.

Proposition 6.18. We suppose that Assumption (D) is fulfilled. Then any bounded variation function with compact support belongs to L_R and

$$\|\varphi\|_R^2 = E \left(- \int_0^\infty X d\varphi \right)^2. \quad (6.12)$$

That property does not seem easy to prove in the general case.

Corollary 6.19. We suppose Assumption (D) to be fulfilled. Then every step function belongs to L_R . In particular, if $t > 0$, $1_{[0,t]} \in L_R$

and

$$\|1_{[0,t]}\|_R^2 = E(X_t^2).$$

Corollary 6.20. Under Assumption (D), if $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a bounded variation function with compact support, then

$$\int_{\mathbb{R}_+^2} (f(t_1) - f(t_2))^2 d|\mu|(t_1, t_2) < \infty.$$

Proof (of Corollary 6.20): This follows from Proposition 6.18 and the fact that $L_R \subset \tilde{L}_R$. ■

Proof: (of Proposition 6.18): Let φ be a bounded variation function with compact support, defined on \mathbb{R}_+ . Let (ρ_n) be a sequence of mollifiers converging to the Dirac delta function. We set

$$\varphi_n = \rho_n * \varphi.$$

Since φ_n is smooth with compact support, it belongs to L_R . Now $d\varphi_n \rightarrow d\varphi$ and

$$d\varphi_n \otimes d\varphi_m \xrightarrow{n,m \rightarrow \infty} d\varphi \otimes d\varphi \text{ weakly.}$$

Since R is continuous it follows that

$$\int_{\mathbb{R}_+^2} R(s_1, s_2) d\varphi_n(s_1) d\varphi_m(s_2) \longrightarrow \int_{\mathbb{R}_+^2} R(s_1, s_2) d\varphi(s_1) d\varphi(s_2). \quad (6.13)$$

By Remark 6.2, we have

$$\begin{aligned} \|\varphi_n\|_R^2 &= E \left(- \int_0^\infty X d\varphi_n \right)^2 \\ &= \int_{\mathbb{R}_+^2} R(s_1, s_2) d\varphi_n(s_1) d\varphi_n(s_2). \end{aligned} \quad (6.14)$$

Using (6.13), the limit of the right-hand side of (6.14) gives

$$E \left(- \int_0^\infty X d\varphi \right)^2.$$

Again using (6.13), it follows that

$$\lim_{n, m \rightarrow \infty} \|\varphi_n - \varphi_m\|_R^2 = 0,$$

so (φ_n) is Cauchy in L_R . Since L_R is complete, there is $\psi \in L_R$ such that

$$\|\varphi_n - \psi\|_R \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand $\varphi_n \longrightarrow \varphi R(ds, \infty)$ a.e. since $R(ds, \infty)$ is a non-atomic measure. By Lebesgue's dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\|_{R(ds, \infty)} = 0.$$

But we also have

$$\|\varphi_n - \psi\|_{R(ds, \infty)} \leq \|\varphi_n - \psi\|_R \xrightarrow{n \rightarrow \infty} 0.$$

By uniqueness of the limit, $\varphi = \psi R(ds, \infty)$ a.e. and so $\psi = \varphi d\nu$ a.e. this shows that $\varphi \in L_R$. The limit of the left-hand side in (6.14) gives $\|\varphi\|_R^2$, which finally shows (6.12). ■

A natural question concerns whether the constant function 1 belongs to L_R . The answer is of course well-known if X has a covariance measure structure with compact support because of Proposition 6.13. Again it will be also the case if Assumptions (A), (B), (C) and (D) are fulfilled.

Proposition 6.21. *If Assumptions (C) and (D) are fulfilled, then $1 \in L_R$ and $\|1\|_{\mathcal{H}}^2 = \|1\|_R^2 = R(\infty, \infty) = E(X_\infty^2)$.*

Proof: For $n \in \mathbb{N}^*$, we consider a smooth, decreasing function $\chi_{[0,n]} : \mathbb{R}_+ \rightarrow [0, 1]$ which equals 1 on $[0, n]$ and zero on $[n+1, \infty[$ and it is bounded by 1. Clearly $\chi_{[0,n]} \in L_R$ for any n and

$$\begin{aligned} \|\chi_{[0,n]} - \chi_{[0,m]}\|_{\mathcal{H}}^2 &= E \left(\int_0^\infty (\chi_{[0,n]} - \chi_{[0,m]}) dX \right)^2 \\ &= E \left(- \int_0^\infty X d(\chi_{[0,n]} - \chi_{[0,m]}) \right)^2 = I(n, n) + I(m, m) - 2I(n, m), \end{aligned}$$

where

$$\begin{aligned} I(n, m) &= \int_{\mathbb{R}_+^2} R(s_1, s_2) d\chi_{[0,n]}(s_1) d\chi_{[0,m]}(s_2), \\ &= \int_{[n, n+1] \times [m, m+1]} R(s_1, s_2) d(1 - \chi_{[0,n]})(s_1) d(1 - \chi_{[0,m]})(s_2). \end{aligned}$$

We have

$$\inf_{\xi \in [n, n+1] \times [m, m+1]} R(\xi) \leq I(n, m) \leq \sup_{\xi \in [n, n+1] \times [m, m+1]} R(\xi).$$

Since $\lim_{s_1, s_2 \rightarrow \infty} R(s_1, s_2) = R(\infty, \infty)$, it follows that

$$I(n, m) \xrightarrow{n, m \rightarrow \infty} R(\infty, \infty)$$

and $\chi_{[0,n]}$ is a Cauchy sequence in $\|\cdot\|_{\mathcal{H}}$. On the other hand $\chi_{[0,n]} \rightarrow 1$ pointwise when $n \rightarrow \infty$ and in particular a.e. with respect to the measure $|R|(dt, \infty)$. \blacksquare

An important question concerns the separability of the Hilbert space L_R .

Proposition 6.22. *Suppose the validity of Assumptions (C) and (D). Then the Hilbert space L_R is separable. Moreover there is an orthonormal basis (e_n) in C_0^1 of L_R .*

Proof: We denote by \mathcal{S} the closed linear span $\{1_{[0,t]}, t \geq 0\}$ into \tilde{L}_R . We first prove

$$\mathcal{S} = L_R \tag{6.15}$$

a) $1_{[0,t]}, \forall t \geq 0$ belongs to L_R because of Corollary 6.19, so it follows that $\mathcal{S} \subset L_R$.

b) We prove the converse inclusion. It is enough to show that $C_0^1(\mathbb{R}_+) \subset \mathcal{S}$. Let $\varphi \in C_0^1(\mathbb{R}_+)$ and consider a sequence of step functions of the type

$$\varphi_n(t) = \sum_l 1_{[t_l, t_{l+1}[}(t) \varphi(t_l),$$

which converges pointwise to φ . Since the total variation of φ_n is bounded by the total variation of φ , then $\varphi_n \rightarrow \varphi$ weakly. Let (Y_t) be a Gaussian process with the same covariance as X . A consequence of previous observations shows that

$$\int_0^\infty \varphi_n dY := - \int_0^\infty Y d\varphi_n \xrightarrow{n \rightarrow \infty} - \int_0^\infty Y d\varphi \text{ a.s.} \quad (6.16)$$

Since Y is a Gaussian process, the sequence in the left-hand side of (6.16) is Cauchy in $L^2(\Omega)$, so (φ_n) is Cauchy in L_R by Proposition 6.18; the result follows because $\varphi_n \in \mathcal{S}$ for every n . This concludes b) and (6.15).

Since $\|\cdot\|_R = \|\cdot\|_{\mathcal{H}}$, taking into account the consideration preceding Remark 6.3 and (6.15), H_X is the closure in $L^2(\Omega)$ of $-\int_0^\infty X d\varphi$, φ of the type $1_{[0,t]}$, $t \geq 0$. Since X is continuous, H_X (and therefore L_R) is separable. The existence of an orthonormal basis in $C_0^1(\mathbb{R}_+)$ follows by Gram-Schmidt orthogonalization procedure. \blacksquare

6.2 Path properties of some processes with stationary increments

In this subsection we are interested in expressing necessary and sufficient conditions under which the paths of Gaussian continuous processes with stationary increments restricted to any compact intervals, belong to L_R . We have some relatively complete elements of answer.

We reconsider the example treated in Section 4.4. Let \tilde{X} be a process with weak, stationary increments, continuous in L^2 such that $\tilde{X}_0 = 0$. We denote by $Q(t) = \text{Var}(\tilde{X}_t)$, and we consider again X defined by $X_t = \tilde{X}_{t \wedge T}$. We recall that without restrictions to generality, we can suppose $Q(t) = Q(T)$, $t \geq T$.

We recall that in Proposition 4.6 we provided conditions so that Assumptions (A) and (B) are verified, i.e.

Hypothesis 6.23. i) Q is absolutely continuous with derivative Q' ,

ii) $F_Q(s) := sQ'(s)$, $s > 0$ prolongates to zero by continuity to a bounded variation function.

In Corollary 4.8 we provided conditions so that Assumption (D) is verified. This gave the following

Hypothesis 6.24. i) Q is non-decreasing,

ii) Q'' non-positive σ -finite measure.

Proposition 6.25. *We suppose the validity of Hypothesis 6.23. If*

$$\int_{0+} Q(y)|Q''|(dy) < \infty, \quad (6.17)$$

then almost all paths of X belong to \tilde{L}_R .

Proof: We recall that Hypothesis 6.23 implies that Q'' is a finite Radon measure on $]\delta, \infty[$ for every $\delta > 0$. Hence (6.17) implies that

$$\int_0^\infty Q(y)|Q''|(dy) < \infty. \quad (6.18)$$

Since $\int_0^\infty |R|(ds, \infty)X_s^2 < \infty$ a.s. being $|R|(ds, \infty)$ a finite measure, it remains to prove that

$$\int_{[0, T]^2} (X_{s_1} - X_{s_2})^2 |Q''|(ds_2 - s_1) ds_1 < \infty. \quad \text{a.s.} \quad (6.19)$$

To prove (6.19), it is enough to evaluate the expectation of its left-hand side. We get

$$\int_{[0, T]^2} |Q''|(ds_2 - s_1) ds_1 Q(s_1 - s_2) = 2 \int_0^T ds_1 \int_0^{s_1} Q(s_2) |Q''|(ds_2).$$

This concludes the proof. ■

Remark 6.26. *If X has a covariance measure structure, then Assumption (6.17) is trivially verified.*

Remark 6.27. *1) Assumption (6.17) implies (6.18) which ensures (6.43) in Proposition 6.48. Suppose (6.18), if Assumptions (C), (D) are fulfilled, then Proposition 6.48 says that a.s. $X \in L_R$.*

2) In the sequel we will express necessary conditions.

Proposition 6.28. *We suppose X Gaussian and continuous. We suppose again the validity of Hypotheses 6.23, 6.24 and the following technical conditions. There are $c_1, c_2 > 0$, $\alpha_1 < 1$, $\alpha_2 > 0$ such that*

$$c_1 t^{\alpha_1} \leq Q(t) \leq c_2 t^{\alpha_2}, \quad (6.20)$$

$$- \int_{0+} Q(y)Q''(dy) = \infty \quad (6.21)$$

then $X \notin \tilde{L}_R$ a.s.

Corollary 6.29. *Let \tilde{X} be a continuous mean-zero Gaussian process with stationary increments such that $\tilde{X}_0 = 0$ a.s. $Q(t) = \text{Var}\tilde{X}_t$. Set $X_t = \tilde{X}_{t \wedge T}$, $t \geq 0$. We suppose Hypotheses 6.23 and 6.24 together with (6.20).*

Then $X \in L_R$ a.s. if and only if

$$\int_{0+} Q(y)|Q''|(dy) < \infty.$$

Proof: *It follows from Remark 6.27 and Proposition 6.28, and the fact that $L_R \subset \tilde{L}_R$.*

Remark 6.30. *1) The importance of Corollary 6.29 is related to the problem of finding sufficient and necessary conditions on the paths of a continuous Gaussian process X to belong to its "self-reproducing kernel space".*

When it is the case, X belongs to the natural domain of the divergence operator in Malliavin calculus (Skorohod integral); in the other cases X will be shown to belong the extended domain $\text{Dom}\delta^$, see Definition 10.2 introduced in the spirit of [8, 32].*

2) We conjecture that assumption (6.20) and Hypothesis 6.24 can be omitted, but this would have considerably complicated the proof.

Proof (of Proposition 6.28): Since X is continuous, therefore locally bounded, we observe that

$$\int_0^T X_s^2 |R|(ds, \infty) < \infty \text{ a.s.}$$

To prove that $X \notin \tilde{L}_R$ a.s., it will be enough to prove that

$$\int_{\mathbb{R}_+^2} (X_{s_1} - X_{s_2})^2 |Q''|(ds_2 - s_1) = \infty \text{ a.s.} \quad (6.22)$$

The left-hand side of (6.22) gives

$$\begin{aligned} & 2 \int_0^T ds_1 \int_0^{s_1} (X_{s_1} - X_{s_2})^2 (-Q'')(ds_2 - s_1) = 2 \int_0^T ds_1 \int_0^{s_1} (X_{s_1} - X_{s_1-s_2})^2 (-Q'')(ds_2) \\ & = 2 \int_0^T (-Q'')(ds_2) Q(s_2) \Phi(s_2), \end{aligned} \quad (6.23)$$

where

$$\Phi(s_2) = \int_{s_2}^T ds_1 \frac{(X_{s_1} - X_{s_1-s_2})^2}{Q(s_2)}.$$

In Lemma 6.31 below we will show that

$$\Phi(s_2) \xrightarrow{s_2 \rightarrow 0+} T \text{ a.s.}$$

so a.s. $s_2 \mapsto \Phi(s_2)$ can be extended by continuity to $[0, T]$. If (6.21) holds, then (6.23) is also infinite and so (6.22) is established. It remains to establish the following lemma.

Lemma 6.31. *Under the hypotheses of Proposition 6.28, we have*

$$Z_\varepsilon := \frac{1}{Q(\varepsilon)} \int_\varepsilon^T ds (X_s - X_{s-\varepsilon})^2 \xrightarrow{\varepsilon \rightarrow 0} T \text{ a.s.} \quad (6.24)$$

Proof: 1) We have $E(Z_\varepsilon) = T - \varepsilon$ and this obviously converges to T when $\varepsilon \rightarrow 0$. In order to prove that the convergence in (6.24) holds in $L^2(\Omega)$, it would be enough to show that

$$\text{Var}(Z_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

2) In order to prove the a.s. convergence we will implement the program of [14], see in particular Lemma 3.1. This will only be possible because of technical assumption (6.20). We will show that

$$\text{Var}(Z_\varepsilon) = O\left(\frac{\varepsilon}{Q(\varepsilon)}\right). \quad (6.25)$$

Consequently

$$\text{Var}(Z_\varepsilon) = O(t^\alpha),$$

$\alpha = 1 - \alpha_1$ and (3.1) in [14] is verified. The upper bound of (6.20) allows to show that X is Hölder continuous, by use of Kolmogorov lemma.

3) We prove finally (6.25). We remark that $Q(\varepsilon) \neq 0$ for ε in a neighbourhood of zero, otherwise (6.22) cannot be true. We have

$$\text{Var}(Z_\varepsilon) = \frac{1}{Q(\varepsilon)^2} \int_\varepsilon^T ds_1 \int_\varepsilon^T ds_2 \text{Cov}((X_{s_1} - X_{s_1-\varepsilon})^2, (X_{s_2} - X_{s_2-\varepsilon})^2).$$

It is well-known that given two mean-zero Gaussian random variables ξ and η

$$\text{Cov}(\xi^2, \eta^2) = 3\text{Cov}(\xi, \eta)^2.$$

This, together with the stationary increments property, implies that

$$\text{Var}(Z_\varepsilon) = \frac{6}{Q^2(\varepsilon)} \int_\varepsilon^T ds_1 \int_0^{s_1-\varepsilon} ds_2 [\text{Cov}(X_{s_2+\varepsilon} - X_{s_2}, X_\varepsilon)]^2.$$

Since, by Cauchy-Schwarz

$$\text{Cov}(X_{s_2+\varepsilon} - X_{s_2}, X_\varepsilon) \leq Q(\varepsilon)$$

then

$$\text{Var}(Z_\varepsilon) = \frac{3}{Q^2(\varepsilon)}I(\varepsilon) + O(\varepsilon),$$

where

$$I(\varepsilon) = \int_0^T ds_1 \int_\varepsilon^{s_1} ds_2 (-\text{Cov}(X_{s_2+\varepsilon} - X_{s_2}, X_\varepsilon))^2.$$

Since Hypothesis 6.24 holds, Assumption (D) is verified and

$$-\text{Cov}(X_{s_2+\varepsilon} - X_{s_2}, X_\varepsilon) \geq 0.$$

Hence

$$\begin{aligned} I(\varepsilon) &\leq Q(\varepsilon) \int_0^T ds_1 \int_\varepsilon^{s_1} ds_2 (2Q(s_2) - Q(s_2 + \varepsilon) - Q(s_2 - \varepsilon)) \\ &\leq Q(\varepsilon) \int_0^T ds_1 \int_\varepsilon^{s_1} ds_2 \left(\int_{s_2-\varepsilon}^{s_2} Q'(y) dy - \int_{s_2}^{s_2+\varepsilon} Q'(y) dy \right). \end{aligned}$$

Using Fubini's theorem, we obtain

$$\begin{aligned} \frac{I(\varepsilon)}{Q^2(\varepsilon)} &= \frac{1}{Q(\varepsilon)} \int_0^T ds_1 \left(\int_0^{s_1} dy Q'(y) \int_{\varepsilon \vee y}^{(y+\varepsilon) \wedge s_1} ds_2 - \int_\varepsilon^{s_1+\varepsilon} dy Q'(y) \int_{(y-\varepsilon) \vee \varepsilon}^{y \wedge s_1} ds_2 \right) \\ &= \frac{1}{Q(\varepsilon)} \left(\int_0^T ds_1 \left\{ \int_0^\varepsilon dy Q'(y) y + \varepsilon \int_\varepsilon^{s_1-\varepsilon} dy Q'(y) + \int_{s_1-\varepsilon}^{s_1} dy Q'(y) (s_1 - y) \right\} \right. \\ &\quad \left. - \frac{1}{Q(\varepsilon)} \left(\int_0^T ds_1 \left\{ \int_\varepsilon^{2\varepsilon} dy Q'(y) (y - \varepsilon) + \varepsilon \int_{2\varepsilon}^{s_1} dy Q'(y) + \int_{s_1}^{s_1+\varepsilon} dy Q'(y) (s_1 - y + \varepsilon) \right\} \right) \right). \end{aligned}$$

Performing carefully the calculations, in particular commuting ds_1 and dy through Fubini's, it is possible to show that

$$\frac{I(\varepsilon)}{Q^2(\varepsilon)} \leq O(\varepsilon) + O\left(\frac{\varepsilon}{Q(\varepsilon)}\right).$$

Assumption (6.20) allows to conclude. ■

6.3 Paley-Wiener integral and integrals via regularization

We start introducing the definition of Paley-Wiener integral.

Proposition 6.32. *Let $g \in L_R$, then*

$$E \left(\int_0^\infty g dX \right)^2 = \|g\|_{\mathcal{H}}^2. \tag{6.26}$$

Therefore the map $g \rightarrow \int_0^\infty g dX$ is continuous with respect to $\|\cdot\|_{\mathcal{H}}$.

Proof: (6.26) follows from Remark 6.2. The second part of the statement follows because

$$\|g\|_{\mathcal{H}} \leq \|g\|_R.$$

■

At this point the map $I : C_0^1(\mathbb{R}_+) \subset L_R \longrightarrow L^2(\Omega)$ defined as $g \longrightarrow I(g)$ admits a linear continuous extension to L_R . It will still be denoted by I .

Definition 6.2. Let $g \in L_R$. We define **the Paley-Wiener integral** of g with respect to X denoted by $\int_0^\infty g dX$ the random variable $I(g)$.

Proposition 6.33. Under Assumptions (C), (D), if φ has bounded variation with compact support, then

$$\int_0^\infty \varphi dX = - \int_0^\infty X d\varphi. \quad (6.27)$$

In particular

$$\int_0^\infty 1_{[0,t]} dX = X_t.$$

Proof: By definition, (6.27) holds for $g \in C_0^1$. We introduce the same sequence (φ_n) as in the proof of Proposition 6.18. By (6.26)

$$E \left(\int_0^\infty (\varphi_n - \varphi) dX \right)^2 = \|\varphi_n - \varphi\|_R^2.$$

This converges to zero when $n \longrightarrow \infty$ as it was shown in the proof of Proposition 6.18. In the same proof it was established that $\int_0^\infty X d\varphi_n \longrightarrow \int_0^\infty X d\varphi$ in $L^2(\Omega)$. ■

We recall briefly the notion of integrals via regularization in the spirit of [39] or [42]. We propose here a definite type integral.

Definition 6.3. Let Y be a process with paths in $L_{loc}^1(\mathbb{R})$. We say that the **forward** (resp. **backward, symmetric**) **integral** of Y with respect to X exists, if the following conditions hold.

a) For $\varepsilon > 0$ small enough the following Lebesgue integral

$$\begin{aligned} I(\varepsilon, Y, dX) &= \int_0^\infty Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon} ds \\ (\text{resp. } &\int_0^\infty Y_s \frac{X_s - X_{s-\varepsilon}}{\varepsilon} ds \\ &\int_0^\infty Y_s \frac{X_{s+\varepsilon} - X_{s-\varepsilon}}{\varepsilon} ds) \end{aligned}$$

with the usual condition $X_s = 0, s \leq 0$ exists.

b) $\lim_{\varepsilon \rightarrow 0} I(\varepsilon, Y, dX)$ exists in probability.

The limit above will be denoted by

$$\int_0^\infty Y d^- X \text{ (resp. } \int_0^\infty Y d^+ X, \int_0^\infty Y d^0 X).$$

Proposition 6.34. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ cadlag bounded. Suppose the existence of $V_f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

i)

$$|f(s_2) - f(s_1)| \leq V_f(s_2 - s_1), \quad s_1, s_2 \geq 0,$$

ii)

$$\int_{\mathbb{R}_+^2} V_f^2(s_2 - s_1) d|\mu|(s_1, s_2) < \infty \quad (6.28)$$

Then

$$\int_0^\infty f d^* X = \int_0^\infty f dX, \quad * \in \{-, +, 0\}.$$

In particular $\int_0^\infty f d^* X$ exists.

Proof: We consider the case $* = -$, the other cases being similar. The quantity

$$\int_0^\infty f(s) \frac{X_{s+\varepsilon} - X_s}{\varepsilon} ds$$

equals

$$\int_0^\infty f_\varepsilon(u) dX_u,$$

where

$$f_\varepsilon(u) = \frac{1}{\varepsilon} \int_{u-\varepsilon}^u f(s) ds = \frac{1}{\varepsilon} \int_{-\varepsilon}^0 f(s+u) ds,$$

with the convention that f is prolonged by zero on \mathbb{R}_- .

It remains to show that $f_\varepsilon \rightarrow f$ in L_R . By Lebesgue's dominated convergence theorem and the fact that f is bounded, cadlag and $|R|(ds, \infty)$ is non-atomic, we have

$$\int_0^\infty (f_\varepsilon - f)^2(s) d|R|(ds, \infty) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

It remains to show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2) ((f_\varepsilon - f)(s_1) - (f_\varepsilon - f)(s_2))^2 = 0.$$

Indeed

$$\begin{aligned}
|(f_\varepsilon - f)(s_1) - (f_\varepsilon - f)(s_2)| &= \frac{1}{\varepsilon} \left| \int_{-\varepsilon}^0 [(f(y + s_1) - f(s_1)) - (f(y + s_2) - f(s_2))] dy \right| \\
&= \frac{1}{\varepsilon} \left| \int_{-\varepsilon}^0 [(f(y + s_1) - f(y + s_2)) - (f(s_1) - f(s_2))] dy \right| \\
&\leq \frac{1}{\varepsilon} \int_{-\varepsilon}^0 |f(y + s_1) - f(y + s_2)| dy + |f(s_1) - f(s_2)| \leq 2V(s_2 - s_1).
\end{aligned}$$

Since $f_\varepsilon \rightarrow f$, (6.28) and Lebesgue's dominated convergence theorem allow to conclude. \blacksquare

6.4 About some second order Paley-Wiener integral

We introduce now a second order Wiener integral of the type:

$$I_2(g) := \int_{\mathbb{R}_+^2} g(s_1, s_2) dX_{s_1}^1 dX_{s_2}^2,$$

where $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is a suitable function and X^1, X^2 are two independent copies of X .

In fact all the considerations can be extended to Wiener integrals with respect to n copies X^1, \dots, X^n of X , but in order not to introduce technical complications we only consider the case $n = 2$. This case will be helpful in section 9 in order to topologize the tensor product $L_R \otimes L_R$.

We will make use of tensor product spaces in the Hilbert framework. For a complete information about tensor product spaces and topologies the reader can consult [43]. We suppose here the validity of Assumption (C). We denote $\nu = |R|(dt, \infty)$ as before. If $g = g_1 \otimes g_2$, $g_1, g_2 \in L_R$ then we set

$$I_2(g) = \int_0^\infty g_1 dX_1 \int_0^\infty g_2 dX_2. \quad (6.29)$$

We remark that $g(s_1, s_2) = g_1(s_1)g_2(s_2)$. We denote by $L_R \otimes L_R$ the algebraic tensor product space of linear combinations of functions of the type $g_1 \otimes g_2$, $g_1, g_2 \in L_R$.

We define $\tilde{L}_{2,R}$ as the space of Borel functions $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ such that

$$\|g\|_{2,R}^2 = \int_0^\infty \nu(dt) \|g(t, \cdot)\|_R^2 + \frac{1}{2} \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2) \|g(s_1, \cdot) - g(s_2, \cdot)\|_R^2 < \infty. \quad (6.30)$$

An easy property which can be established by inspection is given below.

Lemma 6.35. For $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ we have

$$\begin{aligned}
\|g\|_{2,R}^2 &= \int_{\mathbb{R}_+^2} \nu(ds_1)\nu(ds_2)g^2(s_1, s_2) \\
&\quad + \frac{1}{4} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} (g(s_1, t_1) - g(s_2, t_1) - g(s_1, t_2) + g(s_2, t_2))^2 d|\mu|(t_1, t_2) d|\mu|(s_1, s_2) \\
&\quad + \frac{1}{2} \int_0^\infty d\nu(s) \left\{ \int_{\mathbb{R}_+^2} (g(s, t_1) - g(s, t_2))^2 d|\mu|(t_1, t_2) + \int_{\mathbb{R}_+^2} (g(s_1, s) - g(s_2, s))^2 d|\mu|(s_1, s_2) \right\} \\
&= \int_0^\infty \nu(dt) \|g(\cdot, t)\|_R^2 + \frac{1}{2} \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \|g(\cdot, t_1) - g(\cdot, t_2)\|_{\mathbb{R}_+^2}^2.
\end{aligned}$$

Remark 6.36. We observe that second term of the right-hand side equals

$$\frac{1}{4} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} (\Delta_{]s_1, s_2] \times]t_1, t_2]} g)^2 d|\mu|(t_1, t_2) d|\mu|(s_1, s_2),$$

where $\Delta_{]s_1, s_2] \times]t_1, t_2]} g$ is the planar increment introduced in Section 2.

Remark 6.37. 1. The (semi)-norm $\|\cdot\|_{2,R}$ derives from an inner product. We have

$$\begin{aligned}
\langle f, g \rangle_{2,R} &= \int_0^\infty \nu(ds) \langle f(s, \cdot), g(s, \cdot) \rangle_R \\
&\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2) \langle f(s_1, \cdot) - f(s_2, \cdot), g(s_1, \cdot) - g(s_2, \cdot) \rangle_R
\end{aligned}$$

2. An analogous expression to Lemma's 6.35 statement, can be written for $\langle \cdot, \cdot \rangle_{2,R}$ instead of $\|\cdot\|_{2,R}$.

3. If $f = f_1 \otimes f_2$, $f_1, f_2 \in L_R$ then $f \in \tilde{L}_{2,R}$. If $g = g_1 \otimes g_2$, $g_1, g_2 \in L_R$

$$\langle f, g \rangle_{2,R} = \langle f_1, g_1 \rangle_R \langle f_2, g_2 \rangle_R.$$

4. $L_R \otimes L_R$ is included in $\tilde{L}_{2,R}$. In particular any linear combination of the type $\phi \otimes \phi$ belong to $L_R \otimes L_R$.

5. Similarly to the proof of Proposition 6.14, taking into account Assumption (C), it is possible to show that $\tilde{L}_{2,R}$ is complete and it is therefore a Hilbert space.

6. Since $\langle \cdot, \cdot \rangle_{2,R}$ is a scalar product and because of 3., it follows that $\|\cdot\|_{2,R}$ is the Hilbert tensor norm of $L_R \otimes L_R$. For more information about tensor topologies, see e.g. [43]. The closure of $L_R \otimes L_R$ with respect to $\|\cdot\|_{2,R}$ can be identified with the Hilbert tensor product space $L_R \otimes^h L_R$; it will be denoted by $L_{2,R}$.

7. $\tilde{L}_{2,R}$ can also be equipped with the scalar product $\langle \cdot, \cdot \rangle_{2,\mathcal{H}}$

$$\begin{aligned} \langle f, g \rangle_{2,\mathcal{H}} &= \int_0^\infty R(ds, \infty) \langle f(s, \cdot), g(s, \cdot) \rangle_{\mathcal{H}} \\ &\quad - \frac{1}{2} \int_{\mathbb{R}_+^2} \mu(ds_1, ds_2) \langle f(s_1, \cdot) - f(s_2, \cdot), g(s_1, \cdot) - g(s_2, \cdot) \rangle_{\mathcal{H}}. \end{aligned}$$

Remark 6.38. Similar considerations as in Remark 6.37 can be made for the inner product $\langle \cdot, \cdot \rangle_{2,\mathcal{H}}$.

1. If f, g are as in 3. of Remark 6.37, then

$$\langle f, g \rangle_{2,\mathcal{H}} = \langle f_1, g_1 \rangle_{\mathcal{H}} \langle f_2, g_2 \rangle_{\mathcal{H}}.$$

2. We denote by $\|\cdot\|_{2,\mathcal{H}}$ the associated norm. Analogous expressions as for Lemma 6.35 can be found for $\|\cdot\|_{2,\mathcal{H}}$.

3. If Assumption (D) is fulfilled then L_R can be identified with \mathcal{H} and $\langle \cdot, \cdot \rangle_{2,\mathcal{H}}$ coincides with $\langle \cdot, \cdot \rangle_{2,R}$. The Hilbert tensor product $\mathcal{H} \otimes^h \mathcal{H}$ can be identified with $L_{2,R}$.

4. If $f \in \tilde{L}_{2,R}$ then

$$\|f\|_{2,\mathcal{H}} \leq \|f\|_{2,R}.$$

The double integral application $g \mapsto I_2(g)$ extends by linearity through (6.29) to the algebraic tensor product $L_R \otimes L_R$.

Proposition 6.39. $I_2 : L_R \otimes L_R \longrightarrow L^2(\Omega, \mathcal{F}, P)$ extends continuously to $L_{2,R}$. In particular for every $g \in L_{2,R}$ we have

$$E(I_2(g)^2) = \|g\|_{2,R}^2.$$

Proof: Let $g \in L_{2,R}$, so $g = \sum_{i=1}^n g_{i1} \otimes g_{i2}$, $g_{i1}, g_{i2} \in L_R$, then

$$\begin{aligned} E(I_2(g)^2) &= \sum_{i,j=1}^n E(I_2(g_{i1} \otimes g_{i2}) I_2(g_{j1} \otimes g_{j2})) \\ &= \sum_{i,j=1}^n E\left(\int_0^\infty g_{i1} dX^1 \int_0^\infty g_{i2} dX^2 \int_0^\infty g_{j1} dX^1 \int_0^\infty g_{j2} dX^2\right) \\ &= \sum_{i,j=1}^n E\left(\int_0^\infty g_{i1} dX^1 \int_0^\infty g_{j1} dX^1\right) E\left(\int_0^\infty g_{i2} dX^2 \int_0^\infty g_{j2} dX^2\right) \end{aligned}$$

using the independence of X^1 and X^2 . Therefore by Remark 6.38 1. and bilinearity of the inner product, it follows

$$\begin{aligned} E(I_2(g))^2 &= \sum_{i,j=1}^n \langle g_{i1}, g_{j1} \rangle_{\mathcal{H}} \langle g_{i2}, g_{j2} \rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^n \langle g_{i1} \otimes g_{i2}, g_{j1} \otimes g_{j2} \rangle_{2,\mathcal{H}} = \|g\|_{2,\mathcal{H}}^2 \leq \|g\|_{2,R}^2. \end{aligned}$$

This allows to conclude the proof of the proposition. ■

Remark 6.40. *The proof of Proposition 6.39 allows to establish (as a by product) that for $g \in L_{2,R}$ the double integral is unambiguously defined.*

Given $g \in L_{2,R}$, we denote

$$I_2(g) = \int_{\mathbb{R}_+^2} g(s_1, s_2) dX_{s_1}^1 dX_{s_2}^2.$$

This quantity is called **double (Paley-)Wiener integral** of g with respect to X^1 and X^2 .

We will characterize now some significant functions which belongs to $L_{2,R}$.

Lemma 6.41. *Suppose that $1_{[0,t]} \in L_R$ for every $t > 0$ and Assumption (C). Then for any $t_1, t_2 > 0$, $y_1, y_2 > 0$, $h = 1_{]t_1, t_2] \times]y_1, y_2]}$ belongs to $L_{2,R}$.*

Proof: Since $1_{]t_1, t_2]}, 1_{]y_1, y_2]} \in L_R$, clearly $h \in L_R \otimes L_R \subset L_{2,R}$. ■

Remark 6.42. 1) *If g is a sum of functions of the type $g^1 \otimes g^2$, where $g^1, g^2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ are bounded variation functions with compact support, then g has bounded planar variation.*

2) *If g is as in item 1) and X^1, X^2 are independent copies of X , then*

$$\int_{\mathbb{R}_+^2} g(t_1, t_2) dX_{t_1}^1 dX_{t_2}^2 = \int_{]0, \infty[^2} X_{t_1}^1 X_{t_2}^2 dg(t_1, t_2),$$

where the right-hand side is a Lebesgue integral with respect to the signed measure χ such that

$$g(t_1, t_2) = \chi(]0, t_1] \times]0, t_2]).$$

This follows because of the following reasons.

If $g = g_1 \otimes g_2$, g_1, g_2 have bounded variation with compact support then

i) $I_2(g) = \int_0^\infty g_1 dX^1 \int_0^\infty g_2 dX^2,$

ii) $\int_{]0,\infty[^2} \varphi dg = \int_{]0,\infty[^2} \varphi(s_1, s_2) dg_1(s_1) dg_2(s_2),$

iii) $\int_0^\infty g_1 dX^1 = - \int_0^\infty X_s^1 dg_1(s),$ because of Proposition 6.33.

A significant proposition characterizing elements of $L_{2,R}$ under Assumption (D) is the following.

Proposition 6.43. *We suppose, that Assumptions (C) and (D) are verified. Moreover we suppose that $h : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ has bounded planar variation. Then $h \in L_{2,R}$.*

Remark 6.44. *If $X_t = X_T, t \geq T$ then the statement of Proposition 6.43 holds if $h|_{[0,T]^2}$ has bounded planar variation.*

Indeed, by definition of $L_{2,R}$, if h is prolonged by zero outside $[0,T]^2$ denoted by \bar{h} , then $\|h - \bar{h}\|_{2,R} = 0$. In particular $h = \bar{h} \nu_\infty \otimes \nu_\infty$ a.e. since

$$\|h\|_{L^2(\nu_\infty) \otimes L^2(\nu_\infty)} \leq \|h - \bar{h}\|_{2,R},$$

where $\nu_\infty = R(ds, \infty)$.

Proof of Proposition 6.43: Let $N > 0$ and $t_i := t_i^N := \frac{i}{N}, 0 \leq i \leq N^2$. According to Corollary 6.19 $1_{]t_i, t_{i+1}[}, 1_{]t_j, t_{j+1}[}$ belongs to L_R for any $0 \leq i, j \leq N^2$. We denote

$$h^N(s_1, s_2) = \sum_{i,j=0}^N h(t_i, t_j) 1_{]t_i, t_{i+1}[}(s_1) 1_{]t_j, t_{j+1}[}(s_2).$$

Of course h^N belongs to $L_R \otimes L_R \subset L_{2,R}$.

On the other hand $h^N \rightarrow h$ for every continuity point. The total variation of dh^N is bounded by

$$\sum_{i,j=0}^{N^2} 1_{]t_i, t_{j+1}[} \otimes 1_{]t_j, t_{j+1}[} \left| \Delta h_{]t_i, t_{i+1}[\times]t_j, t_{j+1}[} \right|$$

Previous quantity is bounded by $\|h\|_{pv}$. Finally h^N converges weakly to h , by the theory of two-parameter distribution functions of measures. Therefore, if X^1 and X^2 are two independent copies of X , then

$$\int_{]0,\infty[^2} X_{t_1} X_{t_2} dh^N(t_1, t_2) \xrightarrow{N \rightarrow \infty} \int_{]0,\infty[^2} X_{t_1} X_{t_2} dh(t_1, t_2) \text{ a.s.} \quad (6.31)$$

By Remark 6.42 2), we have

$$\int_{\mathbb{R}_+^2} (h^N - h^M)(t_1, t_2) dX_{t_1}^1 dX_{t_2}^2 = \int_{]0,\infty[^2} X_{t_1}^1 X_{t_2}^2 d(h^N - h^M)(t_1, t_2). \quad (6.32)$$

By Fubini's, the fact that X^1 and X^2 are independent and (6.32), it follows

$$\begin{aligned}\|h^N - h^M\|_{2,R}^2 &= E \left(\int_{]0,\infty[^2} (h^N - h^M)(t_1, t_2) dX_{t_1}^1 dX_{t_2}^2 \right)^2 \\ &= \int_{]0,\infty[^4} R(t_1, s_1) R(t_2, s_2) d(h^N - h^M)(t_1, t_2) d(h^N - h^M)(s_1, s_2).\end{aligned}$$

This converges to zero because $dh^N \otimes dh^M$ weakly converges when $N, M \rightarrow \infty$ and $(t_1, s_1, t_2, s_2) \mapsto R(t_1, s_1) R(t_2, s_2)$ is a continuous function. Consequently the sequence (h^N) is Cauchy in $L_{2,R}$.

Since $L_{2,R}$ is complete, there is $\psi : \mathbb{R}_+^2 \rightarrow \mathbb{R} \in L_{2,R}$ such that

$$\|h^N - \psi\|_{2,R} \xrightarrow{N \rightarrow \infty} 0.$$

By definition of $\|\cdot\|_{2,R}$, we have

$$\|h^N - \psi\|_{L^2(d\nu_\infty)^{\otimes 2}} \leq \|h^N - \psi\|_{2,R} \rightarrow 0, \quad (6.33)$$

where again

$$\nu_\infty = R(ds, \infty).$$

So there is a subsequence (N_k) such that $\|h^{N_k} - \psi\| \rightarrow 0$ $\nu_\infty \otimes \nu_\infty$ a.e.

Since $h^N \rightarrow h$ excepted on a countable quantity of points and $\nu_\infty \otimes \nu_\infty$ is nonatomic, then $h^N \rightarrow h$ $\nu_\infty \otimes \nu_\infty$ a.e. Finally $h = \psi$ $\nu_\infty \otimes \nu_\infty$ a.e. and therefore

$$\|h - h^N\|_{2,R} = 0$$

and so $h \in L_{2,R}$. ■

A side-effect of the proof of Proposition 6.43 is the following.

Proposition 6.45. *If $h : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ has bounded planar variation, then*

$$\int_{\mathbb{R}_+^2} h(s_1, s_2) dX_{s_1}^1 dX_{s_2}^2 = \int_{]0,\infty[^2} X_{s_1}^1 X_{s_2}^2 d\chi(s_1, s_2),$$

where as usual $\chi(]0, s_1] \times]0, s_2]) = \Delta_{]0, s_1] \times]0, s_2]} h$.

Remark 6.46. *From Proposition 6.39 and Proposition 6.45 we obtain*

$$\begin{aligned}\|h\|_{2,R}^2 &= E \left(\int_{\mathbb{R}_+^2} h(s_1, s_2) dX_{s_1}^1 dX_{s_2}^2 \right)^2 \\ &= \int_{]0,\infty[^4} R(t_1, s_1) R(t_2, s_2) dh(t_1, t_2) dh(s_1, s_2).\end{aligned} \quad (6.34)$$

Another consequence of Proposition 6.43 is the following.

Proposition 6.47. *We suppose the following.*

- a) *Assumptions (C) and (D);*
- b) *there is $r_0 > 0$ such that*

$$\sup_{r \in]0, r_0]} \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \text{Var}(X_{t_1+r} - X_{t_2+r}) < \infty, \quad (6.35)$$

- c) $X_t = X_T, t \geq T.$

Then, for $a \in \mathbb{R}$ small enough, $h(s, t) = 1_{[0, (t+a)_+]}(s) \in L_{2,R}$. Moreover

$$\|h\|_{2,R}^2 = \int_{\mathbb{R}_+^2} d(-\mu)(t_1, t_2) \text{Var}(X_{(t_2+a)_+} - X_{(t_1+a)_+}) + \int_0^\infty R(ds, \infty) \text{Var}(X_{(s+a)_+}).$$

Proof: We regularize the function h in t_1 . Let ρ be a smooth function with compact support on \mathbb{R}_+ , $\rho_\varepsilon(x) = \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}$. We set

$$\begin{aligned} F^\varepsilon(s, t) &= \int_{\mathbb{R}} \rho_\varepsilon(s - s_1) 1_{[0, (t+a)_+]}(s_1) ds_1 \\ &= \int_{\frac{s-(t+a)_+}{\varepsilon}}^{\frac{s}{\varepsilon}} \rho(\tilde{s}_1) d\tilde{s}_1 + \int_0^\infty R(dt, \infty) \text{Var}(X_t). \end{aligned} \quad (6.36)$$

Of course we have

$$\frac{\partial F^\varepsilon}{\partial s}(s, t) = \rho\left(\frac{s}{\varepsilon}\right) - \rho\left(\frac{s - (t+a)_+}{\varepsilon}\right). \quad (6.37)$$

Since F^ε is smooth, by Remark 6.44 and Proposition 6.43, it follows that $F^\varepsilon \in L_{2,R}$.

It remains to show that $F^\varepsilon \rightarrow h$ in $L_{2,R}$. First of all, we observe that $F^\varepsilon \rightarrow h$ pointwise. We need to show that $\|F^\varepsilon - h\|_{2,R} \rightarrow 0$ when $\varepsilon \rightarrow 0$. We have

$$\|F^\varepsilon - h\|_{2,R} = I_1(\varepsilon) + I_2(\varepsilon),$$

where

$$\begin{aligned} I_1(\varepsilon) &= \int_0^\infty R(dt, \infty) \|F^\varepsilon(\cdot, t) - 1_{[0, (t+a)_+]} \|_R^2 \\ I_2(\varepsilon) &= \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \|F^\varepsilon(\cdot, t_1) - F^\varepsilon(\cdot, t_2) - (1_{[0, (t_1+a)_+]} - 1_{[0, (t_2+a)_+]}) \|_R^2. \end{aligned}$$

We will first evaluate

$$\|F^\varepsilon(\cdot, t)\|_R^2, \quad (6.38)$$

$$\|F^\varepsilon(\cdot, t_1) - F^\varepsilon(\cdot, t_2)\|_R^2. \quad (6.39)$$

Now

$$\begin{aligned} \|F^\varepsilon(\cdot, t)\|_R^2 &= E \left(\int_0^\infty F^\varepsilon(s, t) dX_s \right)^2 = E \left(\int_0^\infty X_s \frac{\partial F^\varepsilon}{\partial s}(s, t) ds \right)^2 \\ &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}_+^2} ds_1 ds_2 R(s_1, s_2) \left[\rho\left(\frac{s_1}{\varepsilon}\right) - \rho\left(\frac{s_1 - (t+a)_+}{\varepsilon}\right) \right] \\ &\quad \left[\rho\left(\frac{s_2}{\varepsilon}\right) - \rho\left(\frac{s_2 - (t+a)_+}{\varepsilon}\right) \right] \\ &= I_{++}(\varepsilon) - I_{+-}(\varepsilon, t) - I_{-+}(\varepsilon, t) + I_{--}(\varepsilon, t), \end{aligned}$$

where after an easy change of variable, one can easily see that

$$\sup_{t \geq 0} |I_{++}(\varepsilon) + I_{+-}(\varepsilon, t) + I_{-+}(\varepsilon, t)| \xrightarrow{\varepsilon \rightarrow 0} 0$$

because $R(0, s_2) = R(s_1, 0) \equiv 0$, $\forall s_1, s_2 > 0$. On the other hand

$$\begin{aligned} I_{--}(\varepsilon, t) &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}_+^2} ds_1 ds_2 R(s_1, s_2) \rho\left(\frac{s_1 - (t+a)_+}{\varepsilon}\right) \rho\left(\frac{s_2 - (t+a)_+}{\varepsilon}\right) \\ &= \int_{\mathbb{R}_+^2} d\tilde{s}_1 d\tilde{s}_2 R((t+a)_+ + \varepsilon\tilde{s}_1, (t+a)_+ + \varepsilon\tilde{s}_2) \rho(\tilde{s}_1) \rho(\tilde{s}_2). \end{aligned}$$

By Lebesgue's dominated convergence theorem,

$$\begin{aligned} I_1(\varepsilon) &= \int_{\mathbb{R}_+^2} d\rho(s_1) d\rho(s_2) [R((t+a)_+ + \varepsilon s_1, (t+a)_+ + \varepsilon s_2) - R((t+a)_+, (t+a)_+)] \frac{R(dt, \infty)}{2} \\ &\quad + J(\varepsilon), \end{aligned}$$

where $\lim_{\varepsilon \rightarrow 0} J(\varepsilon) = 0$, so

$$\int_0^\infty \|F^\varepsilon(\cdot, t)\|_R^2 R(dt, \infty) \xrightarrow{\varepsilon \rightarrow 0} \int_0^\infty \|1_{[0, (t+a)_+]}\|_R^2 R(dt, \infty) = \int_0^\infty \text{Var}(X_{(t+a)_+}) R(dt, \infty).$$

Similarly, we can show that

$$\int_0^\infty \langle F^\varepsilon(\cdot, t), 1_{[0, (t+a)_+]} \rangle_R R(dt, \infty) \xrightarrow{\varepsilon \rightarrow 0} \int_0^\infty \|1_{[0, (t+a)_+]}\|_R^2 R(dt, \infty).$$

This implies that $\lim_{\varepsilon \rightarrow 0} I_1(\varepsilon) = 0$. Concerning $I_2(\varepsilon)$, we need to evaluate (6.39). We observe that

$$\begin{aligned} \|F^\varepsilon(\cdot, t_1) - F^\varepsilon(\cdot, t_2)\|_R^2 &= E \left(\int_0^\infty X_s \left(\frac{\partial F^\varepsilon}{\partial s}(s, t_1) - \frac{\partial F^\varepsilon}{\partial s}(s, t_2) \right) ds \right)^2 \\ &= E \left(\int_0^\infty X_s \left[\rho \left(\frac{s - (t_2 + a)_+}{\varepsilon} \right) - \rho \left(\frac{s - (t_1 + a)_+}{\varepsilon} \right) \right] ds \right)^2 \\ &= K_{++}(\varepsilon, t_1, t_2) - K_{-+}(\varepsilon, t_1, t_2) - K_{+-}(\varepsilon, t_1, t_2) + K_{--}(\varepsilon, t_1, t_2), \end{aligned}$$

where

$$\begin{aligned} K_{++}(\varepsilon, t_1, t_2) &= \int_{\mathbb{R}_+^2} ds_1 ds_2 R(s_1, s_2) \rho \left(\frac{s_1 - (t_2 + a)_+}{\varepsilon} \right) \rho \left(\frac{s_2 - (t_2 + a)_+}{\varepsilon} \right), \\ K_{+-}(\varepsilon, t_1, t_2) &= \int_{\mathbb{R}_+^2} ds_1 ds_2 R(s_1, s_2) \rho \left(\frac{s_1 - (t_2 + a)_+}{\varepsilon} \right) \rho \left(\frac{s_2 - (t_1 + a)_+}{\varepsilon} \right), \\ K_{-+}(\varepsilon, t_1, t_2) &= \int_{\mathbb{R}_+^2} ds_1 ds_2 R(s_1, s_2) \rho \left(\frac{s_1 - (t_1 + a)_+}{\varepsilon} \right) \rho \left(\frac{s_2 - (t_2 + a)_+}{\varepsilon} \right), \\ K_{--}(\varepsilon, t_1, t_2) &= \int_{\mathbb{R}_+^2} ds_1 ds_2 R(s_1, s_2) \rho \left(\frac{s_1 - (t_1 + a)_+}{\varepsilon} \right) \rho \left(\frac{s_2 - (t_1 + a)_+}{\varepsilon} \right). \end{aligned}$$

Consequently

$$\begin{aligned} K_{++}(\varepsilon, t_1, t_2) &= \int_{\mathbb{R}_+^2} ds_1 ds_2 \rho(s_1) \rho(s_2) R((t_2 + a)_+ + \varepsilon s_1, (t_2 + a)_+ + \varepsilon s_2), \\ K_{+-}(\varepsilon, t_1, t_2) &= \int_{\mathbb{R}_+^2} ds_1 ds_2 \rho(s_1) \rho(s_2) R((t_2 + a)_+ + \varepsilon s_1, (t_1 + a)_+ + \varepsilon s_2), \\ K_{-+}(\varepsilon, t_1, t_2) &= \int_{\mathbb{R}_+^2} ds_1 ds_2 \rho(s_1) \rho(s_2) R((t_1 + a)_+ + \varepsilon s_1, (t_2 + a)_+ + \varepsilon s_2), \\ K_{--}(\varepsilon, t_1, t_2) &= \int_{\mathbb{R}_+^2} ds_1 ds_2 \rho(s_1) \rho(s_2) R((t_1 + a)_+ + \varepsilon s_1, (t_1 + a)_+ + \varepsilon s_2). \end{aligned}$$

Hence

$$\int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \|F^\varepsilon(\cdot, t_1) - F^\varepsilon(\cdot, t_2)\|_R^2 \quad (6.40)$$

$$\begin{aligned} &= \int_{\mathbb{R}_+^2} d\rho(s_1) d\rho(s_2) \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \\ &\quad \text{Cov} \left(X_{(t_2+a)_+ + \varepsilon s_1} - X_{(t_1+a)_+ + \varepsilon s_1}, X_{(t_2+a)_+ + \varepsilon s_2} - X_{(t_1+a)_+ + \varepsilon s_2} \right). \quad (6.41) \end{aligned}$$

By Fubini's, Cauchy-Schwarz, choosing the support of ρ small enough, and taking into

account hypothesis b) of the statement, previous expression equals

$$\begin{aligned} & \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \int_0^\infty d\rho(s) \text{Var} (X_{(t_2+a)_+ + \varepsilon s} - X_{(t_1+a)_+ + \varepsilon s}) \\ &= \int_0^\infty d\rho(s) \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \text{Var} (X_{(t_2+a)_+ + \varepsilon s} - X_{(t_1+a)_+ + \varepsilon s}). \end{aligned}$$

Lebesgue's dominated convergence theorem says that previous expression goes to

$$\begin{aligned} & \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \text{Var} (X_{(t_2+a)_+} - X_{(t_1+a)_+}) \\ &= \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \|1_{[0, (t_2+a)_+]} - 1_{[(t_1+a)_+]}\|_R^2. \end{aligned} \tag{6.42}$$

This shows that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \|F^\varepsilon(\cdot, t_1) - F^\varepsilon(\cdot, t_2)\|_R^2$$

equals the right-hand side of (6.42). By similar arguments we can show that

$$\int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \langle F^\varepsilon(\cdot, t_1) - F^\varepsilon(\cdot, t_2), 1_{[0, (t_1+a)_+]} - 1_{[0, (t_2+a)_+]} \rangle_R$$

converges again to the right-hand side of (6.42). This finally shows $\lim_{\varepsilon \rightarrow 0} I_2(\varepsilon) = 0$ and the final result. \blacksquare

An interesting consequence is the following.

Proposition 6.48. *We suppose Assumptions (C), (D). Let $g \in L_{2,R}$. Suppose that X fulfills the following assumption*

$$\int_{\mathbb{R}_+^2} \text{Var}(X_{t_1} - X_{t_2}) d|\mu|(t_1, t_2) < \infty. \tag{6.43}$$

Then $g(s, \cdot) \in L_R$, $R(ds, \infty)$ a.e., $s \mapsto \int_0^\infty g(s, t) dX_t \in L_R$ a.s. and it belongs to $L^2(\Omega; L_R)$. Moreover

$$\int_{\mathbb{R}_+^2} g(s, t) dX_s^1 dX_t = \int_0^\infty \left(\int_0^\infty g(s, t) dX_t \right) dX_s^1. \tag{6.44}$$

if X^1 is an independent copy distributed as X .

Corollary 6.49. *We suppose Assumptions (C), (D), (6.35) and $X_t = X_T$ for $t \geq T$.*

1) We have $s \mapsto X_s \in L_R$ a.s. and it belongs to $L^2(\Omega; L_R)$.

2) Let X^1 be an independent copy of X . If $h(s_1, s_2) = 1_{[0, s_1 \wedge T]}(s_2)$, then

i)

$$\int_{\mathbb{R}_+^2} h(s_1, s_2) dX_{s_1}^1 dX_{s_2} = \int_0^\infty X_s dX_s^1.$$

ii)

$$E \left(\int_{\mathbb{R}_+^2} h(s_1, s_2) dX_{s_1}^1 dX_{s_2} \right)^2 = E (\|X\|_R^2). \quad (6.45)$$

Proof(of Corollary 6.49): It is a consequence of Proposition 6.47 setting $g = h$, with $a = 0$ and Proposition 6.48. \blacksquare

Proof(of Proposition 6.48): The fact that $g(s, \cdot) \in L_R$ for $R(ds, \infty)$ a.e. comes from the definition of $L_{2,R}$. Let $g^N \in L_R \otimes L_R$ of the type $g^N(s, t) = \sum_{i=1}^N f_i(s) h_i(t)$, $f_i, h_i \in L_R$ and

$$\|g^N - g\|_{2,R}^2 \xrightarrow{N \rightarrow \infty} 0. \quad (6.46)$$

We denote by $Z(s) = \int_0^\infty g(s, t) dX_t$, $Z^N(s) = \int_0^\infty g^N(s, t) dX_t$. We observe that $Z^N(s) = \sum_{i=1}^N (\int_0^\infty h_i(t) dX_t) f_i(s)$. Clearly $Z^N \in L_R$ a.s. The result would follow if we show the existence of a subsequence (N_k) such that $Z^{N_k} \rightarrow Z$ a.s. in L_R . For this it will be enough to show that

$$E (\|Z^N - Z\|_R^2) \xrightarrow{N \rightarrow \infty} 0.$$

We have indeed

$$\begin{aligned} E (\|Z^N - Z\|_R^2) &= E \left(\int_0^\infty (Z^N - Z)^2(s) R(ds, \infty) \right) \\ &\quad - \frac{1}{2} E \left(\int_{\mathbb{R}_+^2} ((Z^N - Z)(s_1) - (Z^N - Z)(s_2))^2 d\mu(s_1, s_2) \right) \\ &= \int_0^\infty R(ds, \infty) E \left(\int_0^\infty (g^N - g)(s, t) dX_t \right)^2 \\ &\quad - \frac{1}{2} \int_{\mathbb{R}_+^2} d\mu(s_1, s_2) E \left(\int_0^\infty [(g^N - g)(s_1, t) - (g^N - g)(s_2, t)] dX_t \right)^2. \end{aligned}$$

By Assumption (D) and Corollary 6.32, previous expression equals

$$\begin{aligned} &\int_0^\infty R(ds, \infty) \|(g^N - g)(s, \cdot)\|_R^2 \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} d\mu(s_1, s_2) \|(g^N - g)(s_1, \cdot) - (g^N - g)(s_2, \cdot)\|_R^2 \\ &= \|g^N - g\|_{2,R}^2. \end{aligned} \quad (6.47)$$

The result follows by (6.46). \blacksquare

7 Basic considerations on Malliavin calculus

The aim of this paper is to implement Wiener analysis in the case when our basic process X fulfills Assumptions (A), (B) and $(C(\nu))$. In the sequel we will also often suppose the validity of Assumptions (C), (D). The spirit is still the one of [25] in which the process X was supposed to have a covariance measure structure but in a much more singular context. The target of this is the study of a suitable framework of Skorohod calculus with Itô formulae including the case when the covariance is singular. We also explore the connection with calculus via regularization.

Let $X = (X_t)_{t \in [0, \infty]}$ be an L^2 -continuous process with continuous paths. For simplicity we suppose $X_0 = 0$. We denote by $C^{0,0}(\overline{\mathbb{R}}_+)$ the set of continuous functions defined on \mathbb{R}_+ vanishing at zero with a limit at infinity. As in [25], we will also suppose that the law Ξ of process X on $C^{0,0}(\overline{\mathbb{R}}_+)$ has full support, i.e. the probability that X belongs to any non-empty, open subset of $C^{0,0}(\overline{\mathbb{R}}_+)$ is strictly positive. This allows to state the following result.

Proposition 7.1. *We set $\Omega_0 = C^{0,0}(\overline{\mathbb{R}}_+)$, equipped with its Borel σ -algebra and probability Ξ . We denote by \mathcal{FC}_b^∞ the linear space of $f(l_1, \dots, l_m)$, $m \in \mathbb{N}^*$, $f \in C_b^\infty(\mathbb{R}^m)$, $l_1, \dots, l_m \in \Omega_0^*$. Then \mathcal{FC}_b^∞ is dense into $L^2(\Omega_0, \Xi)$.*

Remark 7.2. i) *A reference for this result is [30], Section II.3.*

ii) *We apply Proposition 7.1 on the canonical probability space related to a continuous square integrable process X .*

We introduce a technical assumption, which will be verified in the most examples.

$$1_{[0,t]} \in L_R, \quad \forall t \geq 0, 1 \in L_R. \quad (7.1)$$

For instance it is fulfilled if Assumptions (C) and (D) hold or if X has a covariance measure structure, see Corollary 6.19 and Proposition 6.13.

Remark 7.3. *Taking into account (7.1), we denote by \bar{L}_R the linear space of functions $\bar{f} : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that there is $f \in L_R$ with*

$$\bar{f}(t) = \langle f, 1_{[0,t]} \rangle_{\mathcal{H}}. \quad (7.2)$$

\bar{L}_R is the classical self-reproducing kernel space appearing in the literature. We equip \bar{L}_R with the Hilbert norm inherited from L_R i.e. $\|\bar{f}\|_{\bar{L}_R} = \|f\|_R$. Therefore $\bar{f}_n \rightarrow \bar{f}$ in \bar{L}_R if and only if $f_n \rightarrow f$ in L_R . We set $\gamma_\infty = \sup_{t \geq 0} \sqrt{\text{Var}(X_t)}$. Since for $0 \leq s < t$,

$$|\bar{f}(t) - \bar{f}(s)| = \left| E \left((X_t - X_s) \int_0^\infty f dX \right) \right| \leq \{E(X_t - X_s)^2\}^{\frac{1}{2}} \|f\|_{\mathcal{H}}$$

and X is continuous in $L^2(\Omega)$. We have the following.

1. If $f \in L_R$, then $\sup_{t \geq 0} |\bar{f}(t)| \leq \gamma_\infty \|f\|_{\mathcal{H}} \leq \gamma_\infty \|f\|_R = \gamma_\infty \|\bar{f}\|_{\bar{L}_R}$.
2. $\bar{L}_R \subset C_b(\mathbb{R}_+)$.

We denote by Cyl the set of smooth and cylindrical random variables of the form

$$F = f \left(\int_0^\infty \phi_1 dX, \dots, \int_0^\infty \phi_m dX \right), \quad (7.3)$$

where $f \in C_b^\infty(\mathbb{R}^m)$, $\phi_1, \dots, \phi_m \in C_0^1(\mathbb{R})$ and $\int_0^\infty \phi_i dX, 1 \leq i \leq m$ still denotes the Paley-Wiener integral developed in Section 6.

An important basic consequence of Proposition 7.1 for developing Malliavin calculus is the following.

Theorem 7.4. *Cyl is dense into $L^2(\Omega)$.*

Before entering the proof we make some preliminary considerations. We first suppose that Ω coincides with the canonical space Ω_0 and $X_t(\omega) = \omega(t)$, $t \geq 0$, so $P = \Xi$. In this case, if $f \in C_0^1(\mathbb{R}_+)$ (which is an element of L_R), the following Wiener integral

$$\int_0^\infty f dX(\omega) = - \int_0^\infty X_s(\omega) df(s) = - \int_0^\infty \omega(s) df(s)$$

is pathwise defined.

Lemma 7.5. *Let $l : \Omega_0 \rightarrow \mathbb{R}$ be linear and continuous. There is a sequence (g_n) in $C_0^1(\mathbb{R})$, $a_n \in \mathbb{R}$ with $(\int_0^\infty g_n dX)(\omega) + a_n X_\infty \rightarrow l(\omega)$, $\forall \omega \in \Omega_0$ and so in particular Ξ a.s.*

Proof: Since $l : \Omega_0 \rightarrow \mathbb{R}$ is linear and continuous, there is a finite signed, Borel measure ℓ on $\bar{\mathbb{R}}_+$ such that for every $\tilde{h} \in \Omega_0$

$$l(\tilde{h}) = - \int_{]0, \infty[} \tilde{h} d\ell + \tilde{h}(+\infty) \ell(\{+\infty\})$$

Now

$$l(X) = - \int_0^\infty X dl + X_\infty \ell(\{+\infty\})$$

We set

$$g_n(x) = \int_0^n \rho_n(x-y)g(y)dy,$$

$$g(x) = \ell([x, \infty[).$$

where (ρ_n) is the usual sequence of mollifiers with compact support approaching Dirac delta function. In particular $dg_n \rightarrow \ell|_{\mathbb{R}_+}$ weakly. We set

$$l_n(\tilde{h}) = - \int_0^\infty \tilde{h} dg_n, \quad a_n = \ell(\{+\infty\})$$

so that

$$l_n(X) = \int_0^\infty g_n dX.$$

Since $l_n(\tilde{h}) \rightarrow - \int_0^\infty \tilde{h} dl$ pointwise, the result follows. ■

Lemma 7.6. *The statement of Theorem 7.4 holds whenever $\Omega = \Omega_0$, $P = \Xi$.*

Proof: By Proposition 7.1 it is enough to show that any element of \mathcal{FC}_b^∞ can be approached by a sequence of random variables in *Cyl*. Let $F \in \mathcal{FC}_b^\infty$ given by $f(l_1, \dots, l_m)$ as in Proposition 7.1. By truncation it is clear that we can reduce to the case, when f is bounded. Lemma 7.5 implies that it can be pointwise approximated (so a.s.) by a sequence of random variables of the type $f(\int_0^\infty \phi_0 dX, \dots, \int_0^\infty \phi_m dX)$ for $\phi_0 = 1$, and $\phi_1, \dots, \phi_n \in C_0^1(\mathbb{R}_+)$. Since f is bounded, the convergence also holds in $L^2(\Omega; \Xi)$. ■

Proof (of Theorem 7.4): Any r.v. $h \in L^2(\Omega)$ can be represented through $F(X)$, where $F \in L^2(\Omega_0, \Xi)$. According to Lemma 7.6 there is a sequence of elements of the type $f(\int_0^\infty \phi_0 di, \dots, \int_0^\infty \phi_m di)$, $i(s) = s$, $\phi_0 = 1, \phi_1, \dots, \phi_n \in C_0^1(\mathbb{R}_+)$, $f \in C_b^\infty(\mathbb{R})$ converging in $L^2(\Omega_0, \Xi)$ to F . Since Wiener integrals $\int_0^\infty \phi_j dX$, $1 \leq j \leq n$, can be pathwise represented, then

$$f\left(\int_0^\infty \phi_0 dX, \dots, \int_0^\infty \phi_m dX\right) = f\left(\int_0^\infty \phi_0 di, \dots, \int_0^\infty \phi_m di\right)\Big|_{i=X}$$

and the general result follows. ■

8 Malliavin derivative and related properties

In this section we suppose again Assumptions (A), (B) and (C(ν)). We will suppose from now on that X is Gaussian. We start with a technical lemma.

Lemma 8.1. *Let $\phi_1, \dots, \phi_n \in C_0^1(\mathbb{R}_+)$ orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, not vanishing. Then then the law of the vector*

$$V = \left(\int_0^\infty \phi_1 dX, \dots, \int_0^\infty \phi_m dX \right)$$

has full support in the sense that for any non empty open set I of \mathbb{R}^n , $P\{V \in I\} > 0$.

Proof: Clearly we can reduce the question to the case $I = \prod_{j=1}^m]a_j, b_j[$, $a_j < b_j$. Since the random variables $\int_0^\infty \phi_1 dX, \dots, \int_0^\infty \phi_m dX$ are independent, it is enough to write the proof in the case $m = 1$, $\phi = \phi_1 \in C_0^1(\mathbb{R})$, $\phi \neq 0$.

Let Ξ be the law of X on Ω_0 . Then

$$P \left\{ \int_0^\infty \phi dX \in]a_1, b_1[\right\} = \Xi \{ \omega \in \Omega_0 | l(\omega) \in]a_1, b_1[\}, \quad (8.1)$$

where $l(\omega) = - \int \omega d\phi$, which is clearly an element of the topological dual Ω_0^* . Since l is continuous,

$$\{ \omega \in \Omega_0 | l(\omega) \in]a_1, b_1[\} \quad (8.2)$$

is an open subset of Ω_0 . Since Ξ has full support, it remains to show that the set (8.2) is non empty.

It is always possible to find $\omega_0 \in \Omega_0$ such that $l(\omega_0) \neq 0$. Otherwise the derivative $\dot{\phi}$ would be orthogonal with respect to the $L^2(\mathbb{R}_+)$ norm to the linear space

$$\{ \omega \in L^2(\mathbb{R}_+) \cap C(\mathbb{R}_+) | \omega(0) = 0 \}.$$

This would not be possible since that space is dense in $L^2(\mathbb{R}_+)$. Consequently, there exists $\lambda \in \mathbb{R}$ such that

$$l(\lambda\omega_0) = \lambda l(\omega_0) \in]a_1, b_1[.$$

It is enough to choose λ between $\frac{a_1}{l(\omega_0)}$ and $\frac{b_1}{l(\omega_0)}$. Finally $\lambda\omega_0$ belongs to the set defined in (8.2). ■

For $F \in Cyl$ of the form (7.3), we define

$$D_t F = \sum_{i=1}^n \partial_i f \left(\int_0^\infty \phi_1 dX, \dots, \int_0^\infty \phi_m dX \right) \phi_i(t). \quad (8.3)$$

Remark 8.2. Let $F \in Cyl$. Since $\phi_i \in L_R$ and $\partial_i f, 1 \leq i \leq n$ are bounded, then $t \rightarrow D_t F \in L_R$ a.s. Moreover

$$E(\|DF\|_R^2) < \infty.$$

Consequently $DF \in L^2(\Omega; L_R)$.

Proposition 8.3. Expression (8.3) does not depend on the explicit form (7.3).

Proof: We can of course reduce the problem as follows. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $f \in C_b^\infty(\mathbb{R}^n)$, $\phi_1, \dots, \phi_n \in C_0^1$ such that

$$f(Z_1, \dots, Z_n) = 0,$$

where $Z_i = \int_0^\infty \phi_i dX$. We need to prove that

$$\sum_{k=1}^n \partial_k f(Z_1, \dots, Z_n) \phi_k = 0 \quad a.s. \quad (8.4)$$

By a classical orthogonalization procedure with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, there is $m \leq n$, $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ such that $\phi_i = \sum_{j=1}^m a_{ij} \psi_j$, ψ_1, \dots, ψ_m being orthogonal. Writing $Y_j = \int_0^\infty \psi_j dX$, we also have

$$\tilde{f}(Y_1, \dots, Y_m) = 0, \quad (8.5)$$

with

$$\tilde{f}(y_1, \dots, y_m) = f\left(\sum_{j=1}^m a_{1j} y_j, \dots, \sum_{j=1}^m a_{nj} y_j\right).$$

By usual rules of calculus, (8.4) implies that

$$\sum_{l=1}^m \partial_l \tilde{f}(Y_1, \dots, Y_m) \psi_l = \sum_{k=1}^n \partial_k f(Z_1, \dots, Z_n) \phi_k. \quad (8.6)$$

(8.5) implies that

$$\int_{\mathbb{R}^m} \tilde{f}^2(y_1, \dots, y_m) d\mu_V(y_1, \dots, y_m) = 0, \quad (8.7)$$

where μ_V is the law of (Y_1, \dots, Y_m) . Since \tilde{f} is continuous and because of Lemma 8.1, (8.7) implies that $\tilde{f} \equiv 0$. This finally allows to conclude (8.4). \blacksquare

Before going on, we need to show that $D : Cyl \rightarrow L^2(\Omega)$ is closable. We observe first the following property.

Proposition 8.4. *Let $F \in Cyl$, $h \in L_R$. Then*

$$E(\langle DF, h \rangle_{\mathcal{H}}) = E\left(F \int_0^\infty h dX\right).$$

Proof: It is very similar to Lemma 6.7 of [25] or Lemma 1.1 of [33]. ■

Remark 8.5. *Cyl is a vector algebra. Moreover, if $F, G \in Cyl$, then*

$$D(FG) = GDF + FDG. \tag{8.8}$$

A consequence of Proposition 8.4 and Remark 8.5 is the following.

Corollary 8.6. *Let $F, G \in Cyl$, $h \in L_R$. Then*

$$\begin{aligned} E(G \langle DF, h \rangle_{\mathcal{H}}) \\ = E(-F \langle DG, h \rangle_{\mathcal{H}}) + E(FG \int_0^\infty h dX). \end{aligned}$$

Finally we can state the following result.

Proposition 8.7. *The map $D : Cyl \longrightarrow L^2(\Omega; L_R)$ is closable.*

Proof: Let F_n be a sequence in Cyl such that $\lim_{n \rightarrow \infty} E(F_n^2) = 0$ and there is $Z \in L^2(\Omega; L_R)$ such that $\lim_{n \rightarrow \infty} E(\|DF_n - Z\|_R^2) = 0$. We need to prove that $Z = 0$ a.s. It is enough to show that $\|Z\|_{\mathcal{H}}^2 = 0$ a.s. Since \mathcal{H} is separable and C_0^1 is dense in \mathcal{H} , it is enough to show that $\langle Z, h \rangle_{\mathcal{H}} = 0$ a.s. Since Cyl is dense in $L^2(\Omega)$, we only have to prove that

$$E(\langle Z, h \rangle_{\mathcal{H}} G) = 0 \quad \forall G \in Cyl.$$

By Corollary 8.6, previous expectation equals

$$\begin{aligned} \lim_{n \rightarrow \infty} E(\langle DF_n, h \rangle_{\mathcal{H}} G) \\ = \lim_{n \rightarrow \infty} \left(E(-F_n \langle DG, h \rangle_{\mathcal{H}}) + E\left(F_n G \int_0^\infty h dX\right) \right) = 0. \end{aligned} \tag{8.9}$$

This concludes the proof of the proposition. ■

We denote by $|\mathbb{D}^{1,2}|$ the space constituted by $F \in L^2(\Omega)$ such that there is a sequence (F_n) of the form (7.3) verifying the following conditions.

i) $F_n \longrightarrow F$ in $L^2(\Omega)$,

$$\text{ii) } E \left(\|DF_n - Z\|_R^2 \right) \xrightarrow{n \rightarrow \infty} 0,$$

for some $Z \in L^2(\Omega; L_R)$. In agreement with Proposition 8.7, we denote $DF = Z$.

The set $\mathbb{D}^{1,2}$ will stand for the vector subspace of $L^2(\Omega)$ constituted by functions F such that there is a sequence (F_n) of the form (7.3) with

$$\text{i) } F_n \longrightarrow F \text{ in } L^2(\Omega),$$

$$\text{ii) } E \left(\|DF_n - DF_m\|_{\mathcal{H}}^2 \right) \xrightarrow{n, m \rightarrow \infty} 0.$$

Note that $|\mathbb{D}^{1,2}| \subset \mathbb{D}^{1,2}$. $|\mathbb{D}^{1,2}|$, equipped with the scalar product

$$\langle F, G \rangle_{1,2} = E(FG) + E(\langle DF, DG \rangle_R)$$

is a Hilbert space.

From previous definitions we can easily prove the following.

Proposition 8.8. *Let (F_n) be a sequence in $|\mathbb{D}^{1,2}|$ (resp. $\mathbb{D}^{1,2}$), $F \in L^2(\Omega)$, $\mathcal{Y} \in L^2(\Omega; L_R)$ such that*

$$E \left((F_n - F)^2 + \|DF_n - \mathcal{Y}\|_R^2 \right) \xrightarrow{n \rightarrow \infty} 0.$$

(resp.

$$E \left((F_n - F)^2 + \|D(F_n - F_m)\|_{\mathcal{H}}^2 \right) \xrightarrow{m, n \rightarrow \infty} 0.)$$

Then $F \in |\mathbb{D}^{1,2}|$ and $\mathcal{Y} = DF$ (resp. $F \in \mathbb{D}^{1,2}$).

Remark 8.9. *If Assumption (D) is fulfilled, then $|\mathbb{D}^{1,2}| = \mathbb{D}^{1,2}$ and*

$$\langle F, G \rangle_{1,2} = E(FG) + E(\langle DF, DG \rangle_{\mathcal{H}}).$$

Remark 8.10. *The notation $|\mathbb{D}^{1,2}|$ does not have the same meaning as in [25]. Indeed $\|\cdot\|_{|\mathcal{H}|}$ introduced there is not exactly a norm.*

Remark 8.11. *By definition of $\mathbb{D}^{1,2}$ the statement of Corollary 8.6 extends to $F, G \in \mathbb{D}^{1,2}$. We have therefore the following*

$$E(G \langle DF, h \rangle_{\mathcal{H}}) = E(-F \langle DG, h \rangle_{\mathcal{H}}) + E \left(FG \int_0^\infty h dX \right)$$

for every $F, G \in \mathbb{D}^{1,2}$, $h \in L_R$.

Proposition 8.12. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$, be absolutely continuous. Let $\phi \in L_R$. We suppose that f' is subexponential.*

Then $f\left(\int_0^\infty \phi dX\right) \in \mathbb{D}^{1,2}$ and

$$D_r f\left(\int_0^\infty \phi dX\right) = f'\left(\int_0^\infty \phi dX\right) \phi(r).$$

Remark 8.13. 1. *A function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be **subexponential** if there is $\gamma > 0, c > 0$ with $|f(x)| \leq ce^{\gamma|x|}, \forall x \in \mathbb{R}^n$.*

2. *In particular if f is a polynomial, previous result holds.*

Proof (of Proposition 8.12) i) We first suppose $f \in C_b^\infty(\mathbb{R})$. There is a sequence ϕ_n in C_0^1 such that $\|\phi - \phi_n\| \xrightarrow{n \rightarrow \infty} 0$. Clearly

$$E\left(f\left(\int_0^\infty \phi dX\right) - f\left(\int_0^\infty \phi_n dX\right)\right)^2 \xrightarrow{n \rightarrow \infty} 0$$

since

$$\int_0^\infty \phi_n dX \rightarrow \int_0^\infty \phi dX$$

in $L^2(\Omega)$ and by Lebesgue dominated convergence theorem. On the other hand

$$D_t f\left(\int_0^\infty \phi_n dX\right) = f'\left(\int_0^\infty \phi_n dX\right) \phi_n(t), \quad t \geq 0,$$

so

$$\begin{aligned} & E\left(\left\|Df\left(\int_0^\infty \phi_n dX\right) - f'\left(\int_0^\infty \phi dX\right) \phi\right\|_R^2\right) \\ & \leq \|\phi - \phi_n\|_R^2 E\left(f'\left(\int_0^\infty \phi dX\right)^2\right) + E\left(f'\left(\int_0^\infty \phi_n dX\right) - f'\left(\int_0^\infty \phi dX\right)\right)^2 \|\phi_n\|_R^2. \end{aligned}$$

This converges to zero by usual integration theory arguments. The result for $f \in C_b^\infty(\mathbb{R})$ follows by Proposition 8.8.

ii) We suppose now that f' is subexponential nad let $\phi \in L_R$. $\int_0^\infty \phi dX$ is Gaussian zero-mean variable with covariance $\sigma^2 = \|\phi\|_R^2$. In fact it is the limit in $L^2(\Omega)$ of r.v. of the type $\int_0^\infty \phi_n dX$, $\phi_n \in C_0^1$. We proceed setting $\tilde{f}'_M = (f' \wedge M) \vee (-M)$ for $M > 0$ and

$$\tilde{f}_M := f(0) + \int_0^x \tilde{f}'_M(y) dy.$$

We also set

$$f_M(x) = \int_{\mathbb{R}} \rho_{\frac{1}{M}}(x-y) \tilde{f}_M(y) dy,$$

where ρ_ε is a sequence of Gaussian mollifiers converging to the Dirac delta function. It is easy to show that

$$\int_{\mathbb{R}} (f_M - f)^2(x) p_\sigma(x) dx \xrightarrow{M \rightarrow \infty} 0, \quad (8.10)$$

$$\int_{\mathbb{R}} (f'_M - f')^2(x) p_\sigma(x) dx \xrightarrow{M \rightarrow \infty} 0, \quad (8.11)$$

where p_σ is the density related to the Gaussian law $N(0, \sigma^2)$. (8.10) implies that

$$(f_M - f) \left(\int_0^\infty \phi dX \right) \rightarrow 0$$

in $L^2(\Omega)$. By point i) of the running proof we have

$$D_r f^M \left(\int_0^\infty \phi dX \right) = (f^M)' \left(\int_0^\infty \phi dX \right) \phi(r).$$

(8.11) implies

$$E \left(\left\| D_r f_M \left(\int_0^\infty \phi dX \right) - f' \left(\int_0^\infty \phi dX \right) \phi \right\|^2 \right) \xrightarrow{M \rightarrow \infty} 0.$$

which together with Proposition 8.8 clearly gives the result. ■

Proposition 8.12 extends to the case, where f depends on more than one variable. The proof is a bit more complicated, but it follows the same idea. Therefore we omit it.

Proposition 8.14. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^1 , with subexponential partial derivatives. Let $\phi_1, \dots, \phi_n \in L_R$. Then*

$$f \left(\int_0^\infty \phi_1 dX, \dots, \int_0^\infty \phi_n dX \right) \in \mathbb{D}^{1,2}$$

and

$$\begin{aligned} & D_r f \left(\int_0^\infty \phi_1 dX, \dots, \int_0^\infty \phi_n dX \right) \\ &= \sum_{j=1}^n \partial_j f \left(\int_0^\infty \phi_1 dX, \dots, \int_0^\infty \phi_n dX \right) \phi_j(r). \end{aligned} \quad (8.12)$$

We establish some immediate properties of the Malliavin derivative.

Lemma 8.15. *Let $F \in Cyl$, $G \in |\mathbb{D}^{1,2}|$. Then $F \cdot G \in |\mathbb{D}^{1,2}|$ and (8.8) still holds.*

Proof: According to the definition of $|\mathbb{D}^{1,2}|$, let (G_n) be a sequence in Cyl with the following properties.

- i) $E(G_n - G)^2 \xrightarrow[n \rightarrow \infty]{} 0$,
- ii) $E \left(\int_0^\infty (D_r(G_n - G))^2 |R|(dr, \infty) \right) \xrightarrow[n \rightarrow \infty]{} 0$,
- iii) $E \left(\int_{\mathbb{R}_+^2} d|\mu|(r_1, r_2) (D_{r_1}(G_n - G) - D_{r_2}(G_n - G))^2 \right) \xrightarrow[n \rightarrow \infty]{} 0$.

Since $F \in L^\infty(\Omega)$ then $FG_n \rightarrow FG$ in $L^2(\Omega)$. Remark 8.5 implies that

$$D(FG_n) = G_n DF + FDG_n.$$

It remains to show ii) and iii) for G_n (resp. G) replaced with FG_n . We only check ii), because iii) follows similarly. If F is of the type (7.3) then

$$DF = \sum_{i=1}^m Z_i \phi_i,$$

where $\phi_i \in L_R$, $Z_i \in L^\infty(\Omega)$. This implies, by subadditivity, that

$$\int_0^\infty |R|(dr, \infty) (D_r F)^2 \leq 2^m \left(\sum_{i=1}^m \|Z_i\|_\infty^2 \int_0^\infty \phi_i^2(r) |R|(dr, \infty) \right).$$

Hence

$$\begin{aligned} & E \left(\int_0^\infty |R|(dr, \infty) (G_n - G)^2 (D_r F)^2 \right) \\ & \leq E(G_n - G)^2 \max_{i \in \{1, \dots, m\}} \|Z_i\|_\infty^2 \left(2^m \sum_{i=1}^m \int_0^\infty \phi_i^2(r) |R|(dr, \infty) \right) \xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

Moreover, since $F \in L^\infty$ and taking into account ii)

$$E \left(\int_0^\infty |R|(dr, \infty) (FD_r(G_n - G))^2 \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

Hence ii) is proven for $F(G_n - G)$ instead of $G_n - G$. ■

A natural question is the following. Does X_t belong to $\mathbb{D}^{1,2}$ for fixed t ? The proposition and corollary below partially answers the question.

Proposition 8.16. *If $\psi \in L_R$, then $\int_0^\infty \psi dX \in |\mathbb{D}^{1,2}|$ and $D_t(\int_0^\infty \psi dX) = \psi(t)$.*

Proof: We consider a sequence (ψ_n) in $C_0^1(\mathbb{R})$ such that $\|\psi - \psi_n\|_R \xrightarrow{n \rightarrow \infty} 0$. We know that $\int_0^\infty \psi_n dX \in Cyl$. Obviously

$$E \left(\int_0^\infty (\psi_n - \psi) dX \right)^2 = \|\psi - \psi_n\|_{\mathcal{H}}^2 \leq \|\psi - \psi_n\|_R^2 \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand

$$D_r \int_0^\infty \psi_n dX = \psi_n(r), \quad r > 0,$$

so

$$E \left(\left\| D \int_0^\infty (\psi_n - \psi_m) dX \right\|_R^2 \right) = \|\psi_n - \psi_m\|_R^2 \xrightarrow{n, m \rightarrow \infty} 0.$$

■

Corollary 8.17. *If $1_{[0,t]} \in L_R$, then $X_t \in |\mathbb{D}^{1,2}|$ and $DX_t = 1_{[0,t]}$.*

Remark 8.18. *The conclusion of Corollary 8.17 holds if Assumptions (C) and (D) hold, see Corollary 6.19.*

9 About vector valued Malliavin-Sobolev spaces

We suppose here the validity of Assumption (C) and we use the notations introduced in Section 6.4. We denote again $\nu(dt) = |R|(dt, \infty)$.

We will first define $Cyl(L_R)$ as the set of smooth cylindrical random elements of the form

$$u(t) = \sum_{l=1}^n \psi_l(t) G_l, \quad t \in \mathbb{R}_+,$$

$G_l \in Cyl$, $\psi_l \in C_0^1(\mathbb{R}_+)$. If $u \in Cyl(L_R)$, we define

$$\tilde{D}_s u(t) = \sum_{l=1}^n \psi_l(t) D_s G_l, \quad s, t \geq 0.$$

Clearly $\tilde{D}u = (\tilde{D}_s u(t))$ belongs to $L_{2,R}$ for each underlying $\omega \in \Omega$.

Remark 9.1. 1. *If $u \in Cyl(L_R)$, it is easy to see that a.s. the paths of $\tilde{D}u$ belong to $L_R \otimes L_R$.*

2. *Taking into account Assumption (C), if $u \in Cyl(L_R)$, then $u(t) \in |\mathbb{D}^{1,2}|$.*

3. By analogous arguments as in Proposition 8.7, it is possible to show that $\tilde{D} : \text{Cyl}(L_R) \longrightarrow L^2(L_{2,R})$ is well-defined and closable. This allows to set $Z = \tilde{D}u$, called the **Malliavin derivative** of process u .

Similarly to $|\mathbb{D}^{1,2}|$ we will define $|\mathbb{D}^{1,2}(L_R)|$. We denote $|\mathbb{D}^{1,2}(L_R)|$ the vector space of random elements $u : \Omega \longrightarrow L_R$ such that there is a sequence (u_n) in $\text{Cyl}(L_R)$ and

i) $\|u - u_n\|_R^2 \xrightarrow[n \rightarrow \infty]{} 0$ in $L^2(\Omega)$.

ii) There is $Z : \Omega \longrightarrow L_{2,R}$ with $\|\tilde{D}u_n - Z\|_{2,R} \longrightarrow 0$ in $L^2(\Omega)$.

We denote Z again by $\tilde{D}u$.

Remark 9.2. i) Let (u_n) be a sequence in $|\mathbb{D}^{1,2}(L_R)|$, $u \in L^2(\Omega; L_R)$, $Z \in L^2(\Omega; L_{2,R})$. If

$$\lim_{n \rightarrow \infty} E \left(\|u - u_n\|_R^2 + \|\tilde{D}u_n - Z\|_{2,R}^2 \right) = 0,$$

it is not difficult to show that $u \in |\mathbb{D}^{1,2}(L_R)|$ and

$$\tilde{D}u = Z.$$

- ii) Let $u_t = \psi(t)G$, $t \geq 0$; $G \in \mathbb{D}^{1,2}$, $\psi \in L_R$. Then $u \in |\mathbb{D}^{1,2}(L_R)|$. Moreover $\tilde{D}_r u(t) = \psi(t)D_r G$, $r \geq 0$. This follows by point i) and the fact that u can be approximated by $u_t^n = \psi_n(t)G_n$, where $\psi_n \in C_0^1$ and $G_n \in \text{Cyl}$.

Remark 9.3. 1. $|\mathbb{D}^{1,2}(L_R)|$ is a Hilbert space if equipped with the norm $\|\cdot\|$ associated with the inner product

$$\langle u, v \rangle = E \left(\langle u, v \rangle_R + \langle \tilde{D}u, \tilde{D}v \rangle_{2,R} \right).$$

Moreover $\text{Cyl}(L_R)$ is dense in $|\mathbb{D}^{1,2}(L_R)|$.

2. We convene here that

$$\tilde{D}u : (s, t) \longmapsto \tilde{D}_s u(t).$$

3. If Assumption (D) is fulfilled, it is possible to show that $|\mathbb{D}^{1,2}(L_R)| = \mathbb{D}^{1,2}(L_R)$, where $\mathbb{D}^{1,2}(L_R)$ is constituted by the vector space of random elements $u : \Omega \longrightarrow L_R$ such that there is a sequence (u_n) in $\text{Cyl}(L_R)$ with the following properties

i) $\|u - u_n\|_{\mathcal{H}}^2 \xrightarrow[n \rightarrow \infty]{} 0$ in $L^2(\Omega)$.

ii) There is $Z : \Omega \longrightarrow L_R \otimes^h L_R = L_{2,R}$ with

$$\|\tilde{D}u_n - Z\|_{2,R} \xrightarrow[n \rightarrow \infty]{} 0$$

in $L^2(\Omega)$.

4. If there is a sequence u_n verifying points i), ii), then it is not difficult to show that $u \in \mathbb{D}^{1,2}(L_R)$. Of course $Z = \tilde{D}u$.

We focus the attention on some technical point. The derivative \tilde{D} of process $(u(t))$ may theoretically not be compatible with the family of derivatives of random variables $u(t)$.

Proposition 9.4. *Let $u \in |\mathbb{D}^{1,2}(L_R)|$. Then $\nu(dt)$ a.e. $u(t) \in |\mathbb{D}^{1,2}|$ and*

$$D_r u(t) = \tilde{D}_r u(t), \quad \nu \otimes \nu \otimes P. \text{ a.e.}$$

Proof: Since $u \in |\mathbb{D}^{1,2}(L_R)|$, there is a sequence $u_n \in Cyl(L_R)$ such that $u_n \longrightarrow u$ in $|\mathbb{D}^{1,2}(L_R)|$. According to (6.30) and Point ii), it follows that

$$E \left(\int_0^\infty \nu(dt) \|\tilde{D}.u_n(t) - \tilde{D}.u(t)\|_R^2 \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

Consequently $\nu(dt)$ a.e. we have

$$E \left(\|\tilde{D}.u_n(t) - \tilde{D}.u(t)\|_R^2 \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

By a similar argument, it follows that

$$\lim_{n \rightarrow \infty} \int_0^\infty \nu(dt) E(u_n(t) - u(t))^2 = 0.$$

We observe that $u_n(t) \in Cyl$ for every $t \geq 0$. By definition of \tilde{D} on $Cyl(L_R)$, we have

$$\tilde{D}u_n(t) = Du_n(t).$$

Finally the result follows. ■

From now on we will not distinguish between D and \tilde{D} .

A delicate point consists in proving that the process $X \in \mathbb{D}^{1,2}(L_R)$. First we state a lemma.

Lemma 9.5. *We suppose Assumption (D). Let $g \in C^1$ such that there is $T > 0$ with $g(t) = g(T)$, $t \geq T$. Then $g \in L_R$ and for every $f \in C_0^1$*

$$\langle f, g \rangle_R = \int_{\mathbb{R}_+^2} f'(s_1) g'(s_2) R(s_1, s_2) ds_1 ds_2. \quad (9.1)$$

Proof: We consider a family of functions χ^n in $C_b^\infty(\mathbb{R}_+)$ such that $\chi^n = 1$ on $[0, n]$ and $\chi^n = 0$ on $[n + 1, \infty]$. We define $g_n = g\chi^n$. For $n, m, n > m$, we have

$$\begin{aligned} \|g_n - g_m\|_R^2 &= E \left(\int_0^\infty (g_n - g_m) dX \right)^2 \\ &= E \left(\int_{]0, \infty[^2} X_{s_1} X_{s_2} d(g_n - g_m)(s_1) d(g_n - g_m)(s_2) \right) \xrightarrow{n, m \rightarrow \infty} 0. \end{aligned}$$

This shows that g_n is Cauchy; g_n is also Cauchy in $L^2(\nu)$. Consequently, there is a subsequence (n_k) such that $g_{n_k} \rightarrow g$ in $L^2(d\nu)$. Since $g_n \rightarrow g$ pointwise, then $g \in L_R$. By Remark 6.2, we recall that (9.1) holds for every $f, g \in C_0^1$. Therefore it holds for f and g_n . Letting $n \rightarrow \infty$ on both sides, the result follows. \blacksquare

We operate now a restriction on X , supposing the existence of $T > 0$ with $X_t = X_T$ if $t \geq T$.

Proposition 9.6. *We suppose Assumption (D), (6.35) and $X_t = X_T, t \geq T$. Let $f \in L_R$; there is $\varphi = \varphi_f \in L_R$ such that*

$$\langle f, X \rangle_R = \int_0^\infty \varphi dX \text{ a.s.} \quad (9.2)$$

Proof: By Lemma 9.5, we observe for every $f \in C_0^1(\mathbb{R}), g \in C^1$, constant after some $T > 0$, we observe

$$\langle f, g \rangle_R = - \int_0^\infty d\varphi_f(s) g(s), \quad (9.3)$$

where

$$\varphi_f(s) = \int_0^\infty R(s_1, s) f'(s_1) ds_1.$$

Taking into account Assumption (A), φ_f has bounded variation.

The next step will be to prove that

$$\langle f, X \rangle_R = - \int_0^\infty d\varphi_f(s) X_s, \forall f \in C_0^1(\mathbb{R}). \quad (9.4)$$

We will set $g = X$.

1) We denote $h(s_1, s_2) = 1_{[0, s_1 \wedge T]}(s_2)$ and we consider again the approximating sequence (F^ε) as in the proof of Proposition 6.47. We recall that each F^ε verify has bounded planar variation and therefore, belongs to $L_{2,R}$. We also had

$$F^\varepsilon \rightarrow h$$

in $L_{2,R}$. By construction it also converges pointwise.

2) Let X^1 be an independent copy of X . By isometry of the double Wiener integral, it follows that

$$E \left(\int_{\mathbb{R}_+^2} (F^\varepsilon(t_1, t_2) - h(t_1, t_2)) dX_{t_1}^1 dX_{t_2} \right)^2 \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (9.5)$$

3) Taking into account Remark 6.42 and Proposition 6.33, we can easily show that

$$\int_{\mathbb{R}_+^2} F^\varepsilon(t_1, t_2) dX_{t_1}^1 dX_{t_2} = - \int_0^\infty dX_{t_1}^1 \int_0^\infty X_{t_2} \frac{\partial F^\varepsilon}{\partial t_2}(t_1, t_2) dt_2,$$

where F^ε is given in (6.36).

4) By Proposition 6.48 and item 3), we have

$$\int_{\mathbb{R}_+^2} (F^\varepsilon(t_1, t_2) - h(t_1, t_2)) dX_{t_1}^1 dX_{t_2} = - \int_0^\infty dX_{t_1}^1 \Phi^\varepsilon(t_1, X),$$

where

$$\begin{aligned} \Phi^\varepsilon(t_1, x) &= \int_0^\infty dt_1 x(t_2) \frac{\partial F^\varepsilon}{\partial t_1}(t_1, t_2) - x(t_1) \\ &= \frac{1}{\varepsilon} \int_0^\infty dt_2 x(t_2) \rho\left(\frac{t_1 - t_2}{\varepsilon}\right) - x(t_1). \end{aligned} \quad (9.6)$$

(9.5) gives

$$E(R^\varepsilon(X)^2) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (9.7)$$

where

$$R^\varepsilon(x) = \int_0^\infty dX_{t_1}^1 \Phi^\varepsilon(t_1, x).$$

Taking the conditional expectation with respect to X , we get

$$\begin{aligned} E(R^\varepsilon(X)^2) &= E(\tilde{R}^\varepsilon(X)), \\ \tilde{R}^\varepsilon(x) &= E(R^\varepsilon(x))^2 = \|\Phi^\varepsilon(\cdot, x)\|_R^2. \end{aligned}$$

Therefore there is a sequence (ε_n) such that

$$\|\Phi^{\varepsilon_n}(\cdot, X)\|_R^2 \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

Setting

$$X_t^\varepsilon = \Phi^\varepsilon(\cdot, X),$$

we have shown that $\|X^\varepsilon - X\|_R \xrightarrow{\varepsilon \rightarrow 0} 0$.

5) By (9.6), obviously $X^\varepsilon \rightarrow X$ pointwise a.s.

6) By (9.3), we have

$$\langle f, X^\varepsilon \rangle_R = - \int_0^\infty X_s^\varepsilon d\varphi_f(s), \forall f \in C_0^1(\mathbb{R}). \quad (9.8)$$

Since $X^\varepsilon \rightarrow X$ a.s. in L_R , $X^\varepsilon \rightarrow X$ pointwise. Lebesgue's dominated convergence theorem allows to take the limit, when $\varepsilon \rightarrow 0$ in (9.8). This establishes (9.4).

In order to conclude the validity of (9.2), taking into the isometry property of stochastic integral, we need to show that the linear operator $f \mapsto \int_0^\infty \varphi_f dX$ from C_0^1 to $L^2(\Omega)$ is continuous with respect to $\|\cdot\|_R$.

Let (f_n) be a sequence in C_0^1 converging to 0 according to the L_R -norm. Corollary 6.49 implies that $X \in L^2(\Omega; L_R)$. Cauchy-Schwarz implies that

$$\begin{aligned} E \left(\int_0^\infty \varphi_f dX \right)^2 &= E \left(\langle f_n, X \rangle_R^2 \right) \\ &\leq \|f_n\|_R^2 E(\|X\|_R^2) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

This concludes the proof of (9.4). ■

Proposition 9.7. *Under Assumptions (C) and (D) and again (6.35) together with $X_t = X_T$, $t \geq T$, we have*

$$X \in \mathbb{D}^{1,2}(L_R)$$

and

$$D_{t_2} X_{t_1} = 1_{[0, t_1 \wedge T]}(t_2).$$

Proof: Let (e_n) be an orthonormal basis of $L_R = \mathcal{H}$ which is separable by Proposition 6.22. By Corollary 6.49, $X \in L_R$, so

$$X = \sum_{i=1}^{\infty} F_i e_i \text{ in } \mathcal{H} \text{ a.s.},$$

where

$$F_i = \langle X, e_i \rangle_{\mathcal{H}}.$$

We recall that

$$\begin{aligned} E(\|X\|_{\mathcal{H}}^2) &= E \left(\int_0^\infty R(ds, \infty) X_s^2 - \frac{1}{2} \int_{\mathbb{R}_+^2} d\mu(s_1, s_2) (X_{s_1} - X_{s_2})^2 \right) \\ &= \int_0^\infty R(ds, \infty) E(X_s^2) + \frac{1}{2} \int_{\mathbb{R}^2} d(-\mu)(s_1, s_2) \text{Var}(X_{s_1} - X_{s_2}) \end{aligned}$$

which is finite by assumption. Since

$$\|X\|_{\mathcal{H}}^2 = \sum_{i=0}^{\infty} |F_i|^2,$$

taking the expectation we get

$$\sum_{i=0}^{\infty} E(F_i^2) < \infty.$$

This shows

$$\lim_{n \rightarrow \infty} \|X - X^n\|_R^2 = 0. \quad (9.9)$$

It remains to show that the sequence $(D(\sum_{i=0}^n F_i e_i))_{n \geq 0}$ is Cauchy in $L_{2,R}$. It is enough to show

$$E \left(\left\| \sum_{i=n}^{\infty} DF_i \otimes e_i \right\|_{\mathcal{H} \otimes \mathcal{H}}^2 \right) \xrightarrow{n \rightarrow \infty} 0. \quad (9.10)$$

According to Proposition 9.6 there is $\phi_i \in L_R$ such that $F_i = \int_0^\infty \phi_i dX$. Proposition 8.16 says that $DF_i = \phi_i$. Consequently the left-hand side of (9.10) equals

$$\begin{aligned} E \left(\sum_{i=n}^{\infty} \|DF_i\|_{\mathcal{H}}^2 \right) &= E \left(\sum_{i=n}^{\infty} \|\phi_i\|_{\mathcal{H}}^2 \right) \\ &= \sum_{i=1}^n E(F_i^2) \xrightarrow{n \rightarrow \infty} 0, \end{aligned} \quad (9.11)$$

where the last equality is explained by Proposition 6.32.

It remains to show that

$$D_{t_1} F_{t_2} = h(t_1, t_2),$$

with $h(t_1, t_2) = 1_{[0, t_1 \wedge T]}(t_2)$. We observe that

$$DX^n = \sum_{i=1}^n e_i \otimes DF_i = \sum_{i=1}^n e_i \otimes \phi_i,$$

so that

$$\begin{aligned} \|DX^n - h\|_R^2 &= \int_0^\infty R(dt, \infty) \left\| \sum_{i=1}^n e_i(t) \phi_i - 1_{[0, t \wedge T]} \right\|_R^2 \\ &\quad + \int_{\mathbb{R}_+^2} (-d\mu)(t_1, t_2) \left\| \sum_{i=1}^n e_i(t_1) \phi_i - 1_{[0, t_1 \wedge T]} - e_i(t_2) \phi_i + 1_{[0, t_2 \wedge T]} \right\|_R^2. \end{aligned} \quad (9.12)$$

We have

$$\begin{aligned}
& \left\| \sum_{i=1}^n e_i(t) \phi_i - 1_{[0, t \wedge T]} \right\|_R^2 = E \left\{ \sum_{i=1}^n \int_0^\infty (e_i(t) \phi_i - 1_{[0, t \wedge T]}) dX \right\}^2 \\
& = \sum_{i,j=1}^n E \left(e_i(t) e_j(t) \int_0^\infty \phi_i dX \int_0^\infty \phi_j dX \right) \\
& \quad - 2 \sum_{i=1}^n e_i(t) E \left(\int_0^\infty \phi_i dX X_t \right) + E(X_t^2) \\
& = E \left(\sum_{i=1}^n e_i(t) \int_0^\infty \phi_i dX - X_t \right)^2 = E(X_t^n - X_t)^2.
\end{aligned} \tag{9.13}$$

By a similar reasoning, it follows that

$$\begin{aligned}
& \left\| \sum_{i=1}^n (e_i(t_1) - e_i(t_2)) \phi_i - 1_{[0, t_1 \wedge T]} + 1_{[0, t_2 \wedge T]} \right\|_R^2 \\
& = E((X_{t_1}^n - X_{t_1}) - (X_{t_2}^n - X_{t_2}))^2.
\end{aligned} \tag{9.14}$$

Therefore coming back to (9.12) and taking into account (9.13) and (9.14), we have

$$\|DX^n - h\|_R^2 = \|X - X^n\|_R^2 \xrightarrow{n \rightarrow \infty} 0$$

because of (9.9). ■

Remark 9.8. *Adapting slightly the proof of Proposition 9.7, under the same assumptions, we have $X_{\cdot+r} \in |\mathbb{D}^{1,2}(L_R)|$, for $r \in \mathbb{R}$ small enough.*

Proposition 9.9. *Let $f \in C_b^2$ and $Y \in \mathbb{D}^{1,2}(L_R)$ such that*

$$\sup_{t \leq T} \|DY_t\| \in L^\infty. \tag{9.15}$$

Then $f(Y) \in \mathbb{D}^{1,2}(L_R)$ and

$$Df(Y) = f'(Y)DY \tag{9.16}$$

in the sense that

$$D_{t_2} f(Y_{t_1}) \equiv f'(Y_{t_1}) D_{t_2} Y_{t_1}.$$

Corollary 9.10. Under Assumptions (C), (D), (6.35), $X_t = X_T$ if $t \geq T$, we have

$$f(X) \in \mathbb{D}^{1,2}(L_R)$$

and

$$D_r f(X_t) = f'(X_t)1_{[0,t]}(r).$$

Proof: This is a consequence of Proposition 9.9 and Proposition 9.7.

Remark 9.11. If $Y \in Cyl(L_R)$ of the form $\sum_{i=1}^m F^i \psi_i$, $\psi_i \in C_0^1$, $F^i \in Cyl$, $f \in C_b^2$ we have

$$D_{t_2} f(Y_{t_1}) = \sum_{i=1}^n f'(Y_{t_1}) \psi_i(t_1) D_{t_2} F^i.$$

It is obviously a.s. an element of $L_{2,R}$ since $DF^i \in L_R$ a.s. and $f'(Y)\psi_i \in L_R$ by Propositions 6.7 and 6.8.

Proof (of Proposition 9.9): We proceed in five steps.

a) We suppose that $Y \in Cyl(L_R)$, $f \in C_b^\infty(\mathbb{R})$. Complications come from the fact that $f(Y)$ does not necessarily belong to $Cyl(L_R)$. Let $\psi \in L_R$. We show that

$$\langle f(Y), \cdot \rangle \in Cyl$$

and

$$D(\langle f(Y), \psi \rangle) = \langle f'(Y)DY, \psi \rangle. \quad (9.17)$$

b) We make some general considerations about approximations.

c) We suppose that $f \in C_b^2(\mathbb{R})$, $Y \in Cyl$. For $\psi \in L_R$, we show that $\langle f(Y), \psi \rangle \in \mathbb{D}^{1,2}$ and (9.17) holds.

d) We suppose that $Y \in \mathbb{D}^{1,2}(L_R)$, $f \in C_b^2(\mathbb{R})$. For $\psi \in L_R$ we show that

$$\langle f(Y), \psi \rangle \in \mathbb{D}^{1,2}$$

and (9.17) holds.

e) We conclude the proof.

We will proceed now in details step by step.

a) Let $F^1, \dots, F^m \in Cyl$, $\psi_1, \dots, \psi_m \in C_0^1$ such that $Y = \sum_{i=1}^m F^i \psi_i$. Since $f \in C_b^\infty$, using the definition of inner product on L_R and the definition of Malliavin derivative on Cyl , it follows that $\langle f(Y), \psi \rangle_R \in Cyl$ and

$$D(\langle f(Y), \psi \rangle) = \sum_{i=1}^m DF^i \langle f'(Y)\psi_i, \psi \rangle_R.$$

This coincides with

$$\langle f'(Y)DY, \psi \rangle_R$$

taking into account Remark 9.11.

b) Consider the case $f \in C_b^2(\mathbb{R})$. We regularize setting

$$f_\varepsilon(y) = \int_{\mathbb{R}} dz f(y + \varepsilon z) \rho(z),$$

where $\rho(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$. We denote $C_f = \|f'\|_\infty$, so

$$\sup_y |f_\varepsilon(y) - f(y)| \leq \varepsilon C_f \int_{\mathbb{R}} |z| \rho(z) dz = \varepsilon C_f \sqrt{\frac{2}{\pi}}.$$

We observe that for every $\varepsilon > 0$

$$|f_\varepsilon(y_1) - f_\varepsilon(y_2)| \leq C_f |y_1 - y_2|. \quad (9.18)$$

Let $Y \in \mathbb{D}^{1,2}(L_R)$ fulfilling (9.15). In this framework, we prove the following results

$$E(\|f(Y)\|_R^2) < \infty, \quad (9.19)$$

$$E(\|f'(Y)DY\|_{2,R}^2) < \infty. \quad (9.20)$$

$$E(\|(f - f_\varepsilon)(Y)\|_R^2) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (9.21)$$

$$E(\|[f'_\varepsilon(Y) - f'(Y)]DY\|_{2,R}^2) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (9.22)$$

Indeed (9.19) and (9.20) follow by similar arguments as for (9.21) and (9.22). We only prove the two latter formulae.

$$\begin{aligned} E(\|(f_\varepsilon - f)(Y)\|_R^2) &= E\left(\int_0^\infty |R|(dt, \infty)(f_\varepsilon - f)^2(Y_t)\right) \\ &\quad + E\left(\int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2) [(f_\varepsilon - f)(Y_{s_1}) - (f_\varepsilon - f)(Y_{s_2})]^2\right). \end{aligned}$$

(9.18) implies that

$$|(f_\varepsilon - f)(Y_{s_1}) - (f_\varepsilon - f)(Y_{s_2})| \leq 2C_f |Y_{s_1} - Y_{s_2}|. \quad (9.23)$$

Since $f_\varepsilon \rightarrow f$ pointwise when $\varepsilon \rightarrow 0$ and using Lebesgue's dominated convergence theorem, (9.21) follows.

Concerning (9.22), using similar arguments and the fact that f'' is bounded, we obtain

$$E \left(\|f'_\varepsilon(Y) - f'(Y)\|_R^2 \right) \xrightarrow{n \rightarrow \infty} 0.$$

So

$$\begin{aligned} E \left(\|(f'_\varepsilon(Y) - f'(Y))DY\|_{2,R}^2 \right) &\leq E \left(\|(f_\varepsilon - f)'(Y)\|_{D.Y}\|_R\|_R^2 \right) \\ &= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon), \end{aligned}$$

where

$$\begin{aligned} I_1(\varepsilon) &= E \left(\int_0^\infty |R|(dt, \infty) [(f_\varepsilon - f)'(Y_t)]^2 \|DY_t\|^2 \right), \\ I_2(\varepsilon) &= E \left(\int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) [(f_\varepsilon - f)'(Y_{t_1}) - (f_\varepsilon - f)'(Y_{t_2})]^2 \|DY_{t_1}\|_R^2 \right), \\ I_3(\varepsilon) &= E \left(\int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) (f_\varepsilon - f)'(Y_{t_2}) (\|DY_{t_1}\|_R - \|DY_{t_2}\|_R)^2 \right). \end{aligned}$$

All the integrands converge a.s. and for any (t_1, t_2) when $\varepsilon \rightarrow 0$. We apply (9.23) replacing f_ε, f with f'_ε, f' . The fact that $\sup_{t \leq T} \|DY_t\|_R \in L^2$, Lebesgue's dominated convergence theorem and Cauchy-Schwarz show that $I_i(\varepsilon) \rightarrow 0$, $i = 1, 2, 3$.

c) We go on with the proof. If $Y \in Cyl$ clearly $Y \in \mathbb{D}^{1,2}(L_R)$ and (9.15) is verified. Using (9.17), it remains to show

$$E \left(\langle (f - f_\varepsilon)(Y), \psi \rangle_R^2 \right) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (9.24)$$

$$E \left(\langle (f'_\varepsilon(Y) - f'_\delta(Y))DY, \psi \rangle_{2,R}^2 \right) \xrightarrow{\varepsilon, \delta \rightarrow 0} 0. \quad (9.25)$$

The left-hand side of (9.24) is bounded by

$$\|\psi\|_R^2 E(\|(f - f_\varepsilon)(Y)\|_R^2).$$

because of Cauchy-Schwarz. This together with (9.21) implies (9.24). (9.25) holds again because of Cauchy-Schwarz and (9.22).

d) We first observe that $f(Y) \in L_R$ a.s. by Proposition 6.7. Let $Y \in \mathbb{D}^{1,2}(L_R)$ and a sequence (Y^n) in $Cyl(L_R)$ such that

$$\lim_{n \rightarrow \infty} E \left(\|Y - Y^n\|_{2,R}^2 \right) = 0.$$

We have

$$E(\|f(Y^n) - f(Y)\|_R^2) \leq \|f'\|_\infty^2 E(\|Y^n - Y\|_R^2) \xrightarrow{n \rightarrow \infty} 0 \quad (9.26)$$

and

$$E(\|f'(Y^n)DY^n - f'(Y)DY\|_{2,R}^2) \xrightarrow{n \rightarrow \infty} 0. \quad (9.27)$$

Then

$$\begin{aligned} \|f'(Y^n)DY^n - f'(Y)DY\|_{2,R}^2 &\leq \|f'(Y^n)(DY^n - DY)\|_{2,R}^2 + \|(f'(Y^n) - f'(Y))DY\|_{2,R}^2 \\ &\leq \|f'\|_\infty^2 \|DY^n - DY\|_{2,R}^2 + \|(f'(Y^n) - f'(Y))DY\|_{2,R}^2. \end{aligned}$$

The first term goes to zero since $Y^n \rightarrow Y$ in $\mathbb{D}^{1,2}(L_R)$. The second one converges because f'' is bounded, using Lebesgue's dominated convergence theorem. This shows the validity of (9.26) and (9.27).

The next difficulty consists in showing that $U := \langle f(Y), \psi \rangle \in \mathbb{D}^{1,2}$ if $\psi \in L_R$. This will be the case approximating it through U^n , where

$$U^n = \langle f(Y^n), \psi \rangle_R.$$

Indeed, by item c) we have $U^n \in \mathbb{D}^{1,2}$ and taking into account Proposition 8.8 it remains to show that

- i) $E((U^n - U)^2) \xrightarrow{n \rightarrow \infty} 0$,
- ii) $E(\|DU^n - DU^m\|_R^2) \xrightarrow{n, m \rightarrow \infty} 0$.

Concerning i) we can easily obtain

$$E(U^n - U)^2 \leq \|\psi\|_R^2 E(\|f(Y^n) - f(Y)\|_R^2).$$

This converges to zero because of (9.26). As far as ii) is concerned, we can prove that

$$\lim_{n \rightarrow \infty} E\|DU^n - \langle f'(Y)DY, \psi \rangle_R\|_R^2 = 0. \quad (9.28)$$

Indeed, by item c) and (9.17)

$$DU^n = \langle f'(Y^n)D.Y, \psi \rangle,$$

so the left-hand side of (9.28) gives

$$\begin{aligned} &E(\langle f'(Y^n)DY^n - f'(Y)DY, \psi \rangle_R)^2 \\ &\leq \|\psi\|_R^2 E(\|f'(Y^n)DY^n - f'(Y)DY\|_{2,R}^2) \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

because of (9.27). This concludes the proof of d).

e) Since L_R is a separable Hilbert space, we consider an orthonormal basis $(e_n)_{n=0}^\infty$. We can expand a.s.

$$f(Y) = \lim_{N \rightarrow \infty} I_N(f(Y)),$$

where

$$I_N(f(Y)) = \sum_{n=0}^N \langle f(Y), e_n \rangle e_n$$

and the convergence holds in L_R . According to d) $\langle f(Y), e_n \rangle \in \mathbb{D}^{1,2}$ and so $I_N(f(Y)) \in \mathbb{D}^{1,2}(L_R)$. It remains to show

$$E(\|f(Y) - I_N(f(Y))\|_R^2) \xrightarrow{N \rightarrow \infty} 0, \quad (9.29)$$

$$\|DI_N(f(Y)) - DI_M(f(Y))\|_{2,R}^2 \xrightarrow{N,M \rightarrow \infty} 0. \quad (9.30)$$

(9.29) follows using Parseval's and Lebesgue's dominated convergence. Indeed

$$\begin{aligned} \|f(Y) - I_N(f(Y))\|_R^2 &= \sum_{n=N+1}^{\infty} \langle f(Y), e_n \rangle^2 \\ &\leq \sum_{n=0}^{\infty} \langle f(Y), e_n \rangle^2 = \|f(Y)\|_R^2. \end{aligned}$$

$\|f(Y)\|_R^2$ is integrable because of (9.19). Concerning (9.30), taking $M > N$, we observe that

$$DI_N(f(Y)) - DI_M(f(Y)) = \sum_{n=N+1}^M \sum_{m=0}^{\infty} \langle f'(Y)DY, e_n \otimes e_m \rangle e_n \otimes e_m,$$

so by Parseval's in $L_{2,R}$ we have

$$\|DI_N(f(Y)) - DI_M(f(Y))\|_{2,R}^2 = \sum_{n=N+1}^M \sum_{m=0}^{\infty} \langle f'(Y)DY, e_n \otimes e_m \rangle^2. \quad (9.31)$$

Now previous quantity converges a.s. to zero when $N, M \rightarrow \infty$. Moreover (9.31) is bounded by

$$\|f'(Y)DY\|_{2,R}^2.$$

Lebesgue's dominated convergence theorem finally implies (9.30). ■

An easier but similar result to Proposition 9.9 is the following

Proposition 9.12. *Let Z be a random variable in $\mathbb{D}^{1,2}$, $f \in C_b^2$, $DZ \in L^\infty$. Then $f(Z) \in \mathbb{D}^{1,2}$ and*

$$Df(Z) = f'(Z)DZ.$$

Proof: It follows by similar, but simpler arguments than those of Proposition 9.9.

■

Let Y be a stochastic process such that $Y_t \in \mathbb{D}^{1,2} \forall t \in \mathbb{R}_+$. Let $a : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be Borel integrable function. We look for conditions on a so that the process

$$Z_t = \int_0^\infty a(s, t) Y_s ds$$

belongs to $\mathbb{D}^{1,2}(L_R)$. A partial answer is given below. We first proceed formally. If it exists, the Malliavin derivative is given by $D_r Z_t = Z_1(r, t)$ where

$$Z_1(r, t) = \int_0^\infty a(s, t) D_r Y_s ds.$$

We need now another technical lemma.

Lemma 9.13. *Let $(d\rho_t)$ be a σ -finite signed Borel measure on \mathbb{R}_+ . Let (Y_t) be a stochastic process fulfilling the following properties*

- i) *For every $t \geq 0$ $Y_t \in \mathbb{D}^{1,2}$.*
- ii) *$t \mapsto Y_t$ is continuous and bounded on $\text{supp } d\rho_t$ in $\mathbb{D}^{1,2}$. In particular $t \mapsto Y_t$ is continuous and bounded on $\text{supp } d\rho_t$ in L^2 and $t \mapsto D.Y_t$ is continuous and bounded on $\text{supp } d\rho_t$ in $L^2(\Omega; L_R)$.*

Let $g \in L^2(d\rho_t)$. Then

$$\int_0^\infty g(t) Y_t d\rho_t \in \mathbb{D}^{1,2} \tag{9.32}$$

and

$$D_r \left(\int_0^\infty g(t) Y_t d\rho_t \right) = \int_0^\infty g(t) D_r Y_t d\rho_t. \tag{9.33}$$

Proof: We denote $t_i^n = i2^{-n}$, $i = 0, \dots, n2^n$. We set

$$\zeta^n = \sum_{i=1}^{n2^n} \int_{t_{i-1}^n}^{t_i^n} Y_{t_i^n} g(s) d\rho_s = \int_0^\infty g(s) Y_s^n d\rho_s,$$

where

$$Y_s^n = \begin{cases} Y_{t_i^n} & \text{if } s \in]t_{i-1}^n, t_i^n], s \leq n, \\ 0 & \text{if } s > n. \end{cases}$$

It follows

$$E \left(\int_0^\infty (Y_s - Y_s^n)^2 d\rho_s \right) \xrightarrow[n \rightarrow \infty]{} 0, \quad (9.34)$$

since $t \mapsto Y_t$ is continuous in $L^2(\Omega)$. By Cauchy-Schwarz, it follows that $\zeta^n \rightarrow \int_0^\infty g(s)Y_s d\rho_s$ belongs to $L^2(\Omega)$. Since ζ^n is a linear combination of random variables issued from process Y , then $\zeta^n \in \mathbb{D}^{1,2}$ and

$$D_r \zeta^n = \int_0^\infty g(t) D_r Y_t^n d\rho_t, \quad r \geq 0. \quad (9.35)$$

Since $t \mapsto DY_t$ is continuous in $L^2(\Omega; L_R)$, it follows

$$E \left(\int_0^\infty \|D_r Y_t^n - D_r Y_t\|_R^2 d\rho_t \right) \xrightarrow[n \rightarrow \infty]{} 0. \quad (9.36)$$

Again by Cauchy-Schwarz inequality it follows that

$$D_r \zeta^n \rightarrow \int_0^\infty D_r Y_t g(t) d\rho_t, \quad r \geq 0$$

in $L^2(\Omega; L_R)$. By Proposition 8.8, the conclusions (9.32) and (9.33) hold. \blacksquare

Proposition 9.14. *Let $(d\rho_t)$ be a finite Borel measure on \mathbb{R}_+ , $a : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be a Borel function, (Y_t) be a stochastic process. We suppose the following.*

- i) $a(s, \cdot) \in L_R$ for $d\rho_s$ a.e.
- ii) $\int_0^\infty \|a(s, \cdot)\|_R^2 d\rho_s < \infty$.
- iii) Assumptions i), ii) on Y stated in Lemma 9.13 hold.

We suppose that Assumptions (C), (D) hold. Then the process

$$Z_t = \int_0^\infty a(s, t) Y_s d\rho_s, \quad t \geq 0,$$

belongs to $\mathbb{D}^{1,2}(L_R)$ and

$$D_r Z_t = \int_0^\infty a(s, t) D_r Y_s d\rho_s, \quad r \geq 0. \quad (9.37)$$

Proof: According to Proposition 6.22 there is an orthonormal basis (e_n) of L_R included in $C_0^1(\mathbb{R})$. For a.e. $d\rho_s$, we have

$$a(s, \cdot) = \lim_{n \rightarrow \infty} a^n(s, \cdot) \quad d\rho_s \text{ a.e.},$$

where for $n \geq 1$

$$a^n(s, \cdot) = \sum_{m=0}^n \langle a(s, \cdot), e_m \rangle_R e_m.$$

Moreover by Parseval's, for a.e. $d\rho_s$

$$\|a(s, \cdot)\|_R^2 = \sum_{m=0}^{\infty} \langle a(s, \cdot), e_m \rangle_R^2. \quad (9.38)$$

Let $m \geq 0$. According to hypothesis ii) and Cauchy-Schwarz it follows that $g_m(t) := \langle a(s, \cdot), e_m \rangle_R$ belongs to $L^2(d\rho_s)$. By Lemma 9.13, we obtain

$$\int_0^{\infty} \langle a(s, \cdot), e_m \rangle_R Y_s d\rho_s \in \mathbb{D}^{1,2}$$

and

$$D_r \left(\int_0^{\infty} \langle a(s, \cdot), e_m \rangle_R Y_s d\rho_s \right) = \int_0^{\infty} \langle a(s, \cdot), e_m \rangle_R D_r Y_s d\rho_s.$$

We denote

$$Z_t^m = \int_0^{\infty} a^m(s, t) Y_s d\rho_s.$$

By linearity and Remark 9.2 ii) $Z^m \in \mathbb{D}^{1,2}(L_R)$ and

$$DZ_t^m = \int_0^{\infty} a^m(s, t) DY_s d\rho_s, \quad t \geq 0.$$

Using Remark 9.2 i), it will be enough to show

a) $\lim_{m \rightarrow \infty} E(\|Z^m - Z\|_R^2) = 0,$

b) $\lim_{m \rightarrow \infty} E(\|DZ^m - \mathcal{Y}\|_{2,R}^2) = 0,$ where

$$\mathcal{Y}(r, t) = \int_0^{\infty} a(s, t) D_r Y_s d\rho_s.$$

a) First, we observe that $Z \in \tilde{L}_R$ a.s. because, avoiding some technical details, we have

$$\begin{aligned} & \left\| \int_0^{\infty} a(s, \cdot) Y_s d\rho_s \right\|_R \leq \text{const.} \int_0^{\infty} \|a(s, \cdot) Y_s\|_R d\rho_s \\ & = \text{const.} \int_0^{\infty} \|a(s, \cdot)\|_R |Y_s| d\rho_s \leq \text{const.} \sqrt{\int_0^{\infty} \|a(s, \cdot)\|_R^2 d\rho_s} \int_0^{\infty} |Y_s|^2 d\rho_s. \end{aligned}$$

Again Cauchy-Schwarz and condition ii), imply that

$$E \left(\left\| \int_0^\infty a(s, \cdot) Y_s d\rho_s \right\|_R^2 \right) < \infty.$$

Taking into account (9.38) and Lebesgue's dominated convergence theorem we can show that

$$\begin{aligned} E(\|Z - Z^m\|^2) \\ \leq \text{const.} \left\{ E \left(\int_0^\infty |Y_s|^2 d\rho_s \right) \int_0^\infty \|a - a^m(s, \cdot)\|_R^2 d\rho_s \right\} \xrightarrow{m \rightarrow \infty} 0. \end{aligned} \tag{9.39}$$

b) By similar arguments, we can show that

$$\mathcal{Y} \in \tilde{L}_{2,R} \text{ a.s.}$$

and

$$E(\|\mathcal{Y}\|_{2,R}^2) < \infty.$$

Moreover

$$DZ^m - \mathcal{Y} = \int_0^\infty (a^m - a)(s, t) D_r Y_s d\rho_s.$$

Consequently, by similar arguments as in (9.39) it follows that

$$E(\|DZ^m - \mathcal{Y}\|_{2,R}^2) \xrightarrow{m \rightarrow \infty} 0.$$

This concludes the proof of Proposition 9.14. ■

An application of previous proposition is the following. It holds under Assumptions (C), (D).

Proposition 9.15. *Let Y be a process, continuous in L^2 , such that $Y_t \in \mathbb{D}^{1,2}$ for every $t \geq 0$ and $t \mapsto D.Y_t$ is continuous in $L^2(\Omega; L_R)$. Let $\varepsilon > 0$ and denote*

$$Y_t^\varepsilon = \int_{(t-\varepsilon)^+}^{(t+\varepsilon) \wedge T} Y_s ds.$$

Then $Y^\varepsilon \in \mathbb{D}^{1,2}(L_R)$.

Proof: In view of applying Proposition 9.14, we set $\rho(t) = t \wedge T$,

$$a(s, t) = 1_{]t-\varepsilon, t+\varepsilon] \cap]0, T]}(s) 1_{[0, T]}(t),$$

which also gives

$$a(s, t) = 1_{[s-\varepsilon, s+\varepsilon] \cap [0, T+\varepsilon]}(t) 1_{[0, T]}(s).$$

We have $Y_t^\varepsilon = \int_0^\infty a(s, t) Y_s d\rho_s$. According to Assumption (D) and Corollary 6.19 $a(s, \cdot) \in L_R$, for every $s \geq 0$, Assumption i) of Proposition 9.14 is verified. Again by Corollary 6.19

$$\|a(s, \cdot)\|_R^2 = \text{Var}(X_{(T \wedge s)+\varepsilon} - X_{(s-\varepsilon)+}) \leq 2\text{Var}(X_{(T \wedge s)+\varepsilon}) + 2\text{Var}(X_{(s-\varepsilon)+}).$$

Since X is continuous in L^2 , $s \mapsto \|a(s, \cdot)\|_R$ is bounded and Assumption ii) of Proposition 9.14 is verified.

Point iii) of Proposition 9.14 follows by the continuity assumption on Y and DY and because ρ has compact support. \blacksquare

In the sequel, we will apply Proposition 9.15 to $Y = g(X)$ with g having polynomial growth.

The lemma below allows to improve slightly the statement of Proposition 9.14.

Lemma 9.16. *Let (Y_t) (resp. (Y_t^n)) be a process (resp. a sequence of processes) such that $Y_t, Y_t^n \in \mathbb{D}^{1,2}$, $\forall t \in \mathbb{R}_+$ and*

$$E \left(\int_0^\infty Y_t^2 d\rho_t + \int_0^\infty \|D.Y_t\|_R^2 d\rho_t \right) < \infty \quad (9.40)$$

$$E \left(\int_0^\infty (Y_t - Y_t^n)^2 d\rho_t + \int_0^\infty \|D.(Y_t - Y_t^n)\|_R^2 d\rho_t \right) \xrightarrow{n \rightarrow \infty} 0. \quad (9.41)$$

Let $a : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be a Borel function such that

$$\int_0^\infty d\rho_s \|a(s, \cdot)\|_R^2 < \infty. \quad (9.42)$$

We set

$$\begin{aligned} Z_t &= \int_0^\infty a(s, t) Y_s d\rho_s, \\ Z_1(r, t) &= \int_0^\infty a(s, t) D_r Y_s d\rho_s, \\ Z_t^n &= \int_0^\infty a(s, t) Y_s^n d\rho_s, \\ Z_1^n(r, t) &= \int_0^\infty a(s, t) D_r Y_s^n d\rho_s. \end{aligned}$$

We have the following properties:

i) $Z \in \tilde{L}_R$, $Z_1 \in \tilde{L}_{2,R}$ and

$$E(\|Z\|_R^2 + \|Z_1\|_{2,R}^2) < \infty.$$

ii)

$$\lim_{n \rightarrow \infty} E(\|Z^n - Z\|_R^2 + \|Z_1^n - Z_1\|_R^2) = 0.$$

Proof: We only prove point i) since the other follows similarly.

$$\begin{aligned} \|Z\|_R^2 &= I_1 + I_2, \\ \|Z_1\|_{2,R}^2 &= I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^\infty |R|(dt, \infty) \left(\int_0^\infty a(s, t) Y_s d\rho_s \right)^2, \\ I_2 &= \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \left(\int_0^\infty (a(s, t_1) - a(s, t_2)) Y_s d\rho_s \right)^2, \\ I_3 &= \int_0^\infty |R|(dt, \infty) \left\| \int_0^\infty a(s, t) D.Y_s d\rho_s \right\|_R^2, \\ I_4 &= \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \left\| \int_0^\infty (a(s, t_1) - a(s, t_2)) D.Y_s d\rho_s \right\|_R^2. \end{aligned}$$

Cauchy-Schwarz implies that

$$\begin{aligned} I_1 &\leq \int_0^\infty |R|(dt, \infty) \left(\int_0^\infty a^2(s, t) d\rho_s \right) \left(\int_0^\infty Y_u^2 d\rho_u \right), \\ I_2 &\leq \int_{\mathbb{R}_+^2} d|\mu|(t_1, t_2) \int_0^\infty (a(s, t_1) - a(s, t_2))^2 d\rho_s \int_0^\infty Y_s^2 d\rho_s. \end{aligned}$$

Consequently

$$E(I_1 + I_2) \leq E \left(\int_0^\infty Y_s^2 d\rho_s \right) \int_0^\infty \|a(s, \cdot)\|_R^2 d\rho_s < \infty.$$

On the other hand, Bochner integration theory implies

$$I_3 + I_4 \leq \int_0^\infty \|D.Y_s\|_R^2 d\rho_s \int_0^\infty d\rho_s \|a(s, \cdot)\|_R^2.$$

Taking the expectation it follows

$$E(I_3 + I_4) < \infty.$$

■

Next result allows to relax the boundedness property on $\|Y\|_R$ and $\|D.Y\|_R$ required in Proposition 9.14.

Corollary 9.17. *Let $(d\rho_t)$ be a finite measure on \mathbb{R}_+ , $a : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be a Borel function. Let (Y_t) be a stochastic process continuous in $\mathbb{D}^{1,2}$ such that*

$$E \left(\int_0^\infty Y_t^2 d\rho_t + \int_0^\infty \|D.Y_t\|_R^2 d\rho_t \right) < \infty.$$

We only suppose i), ii) as in Proposition 9.14. Then the same conclusion as therein holds.

Proof: We use Lemma 9.16 and we approximate Y by Y^m , where $Y^m = \phi^m(Y)$ for $\phi = \phi^m : \mathbb{R} \rightarrow \mathbb{R}$ smooth, with

$$\phi(y) = \begin{cases} y & , |y| \leq m \\ 0 & , |y| > m + 1. \end{cases}$$

Clearly

$$E \left(\int_0^\infty (Y_t - Y_t^m)^2 d\rho_t \right) \xrightarrow{m \rightarrow \infty} 0 \tag{9.43}$$

by Lebesgue's dominated convergence theorem. Moreover Proposition 9.9 implies

$$DY^m = \phi'(Y)DY,$$

and of course ϕ is smooth, bounded such that

$$\phi'(y) = \begin{cases} 1 & , |y| \leq m \\ 0 & , |y| > m + 1. \end{cases}$$

Therefore we have again

$$DY^m - DY = DY(1 - \phi'(Y))$$

and by similar arguments, we obtain

$$E \left(\int_0^\infty \|D.(Y_t^m - Y_t)\|_R^2 d\rho_t \right) \xrightarrow{m \rightarrow \infty} 0. \tag{9.44}$$

Finally (9.43), (9.44) and Lemma 9.16 allow to conclude. ■

10 Skorohod integrals

10.1 Generalities

We suppose again Assumptions (A), (B), (C) by default. Similarly as in [8] and [32], we will define two natural domains for the divergence operators, i.e. Skorohod integral, which will be in some sense the dual map of the related Malliavin derivative. We denote by $L^2(\Omega; L_R)$ the set of stochastic processes $(u_t)_{t \in [0, T]}$ verifying $E(\|u\|_R^2) < \infty$. We say that $u \in L^2(\Omega; L_R)$ belongs to $Dom(\delta)$ if there is a zero-mean square integrable random variable Z such that

$$E(FZ) = E(\langle DF, u \rangle_{\mathcal{H}}) \quad (10.1)$$

for every $F \in Cyl$. In other words we have

$$E(FZ) = E\left(\int_0^\infty R(ds, \infty) D_s F u_s\right) - E\left(\int_{\mathbb{R}_+^2} \mu(ds_1, ds_2) (D_{s_1} F - D_{s_2} F)(u_{s_1} - u_{s_2})\right) \quad (10.2)$$

for every $F \in Cyl$. Using the Riesz theorem, we can see that $u \in Dom(\delta)$ if and only if the map

$$F \longrightarrow E(\langle DF, u \rangle_{\mathcal{H}})$$

is continuous linear form with respect to $\|\cdot\|_{L^2(\Omega)}$. Since Cyl is dense in $L^2(\Omega)$, Z is uniquely characterized. We will call $Z = \int_0^\infty u \delta X$ the **Skorohod integral** of u towards X .

We continue investigating general properties of Skorohod integral.

Definition 10.1. *If $u1_{[0,t]} \in Dom(\delta)$, for any $t \geq 0$, then we define*

$$\int_0^t u_s \delta X_s := \int_0^\infty u_s 1_{[0,t]} \delta X_s.$$

A consequence of the duality formula defining Skorohod integral appears below.

Remark 10.1. *If (10.1) holds, then it will be valid by density for every $F \in |\mathbb{D}^{1,2}|$.*

Proposition 10.2. *Let $u \in Dom(\delta)$, $F \in |\mathbb{D}^{1,2}|$. Suppose $F \int_0^\infty u_s \delta X_s \in L^2(\Omega)$. Then $Fu \in Dom(\delta)$ and*

$$\int_0^\infty Fu_s \delta X_s = F \int_0^\infty u_s \delta X_s - \langle DF, u \rangle_{\mathcal{H}}.$$

Proof: The proof is very similar to the one of Proposition 6.4 in [25]. Let $F_0 \in Cyl$. We need to show

$$E \left(F_0 \left\{ F \int_0^\infty u_s \delta X_s - \langle DF, u \rangle_{\mathcal{H}} \right\} \right) = E \left(\langle DF_0, Fu \rangle_{\mathcal{H}} \right).$$

We proceed using Lemma 8.15, which says that $F_0 F \in |\mathbb{D}^{1,2}|$ and (8.8) holds (with $G = F_0$), together with Remark 10.1, which extends the duality relation. \blacksquare

We state now Fubini's theorem, which allows to interchange Skorohod and measure theory integrals. When X has a covariance measure structure, this was established in [25], Proposition 6.5. When X is a Brownian motion the result is stated in [33].

Proposition 10.3. *Let $(G, \mathcal{G}, \lambda)$ be a σ -finite measure space. Let $u : G \times \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a measurable random field with the following properties.*

i) *For every $x \in G, u(x, \cdot) \in Dom(\delta)$.*

ii)

$$E \left(\int_G d\lambda(x) \|u(x, \cdot)\|_{L^2}^2 \right) < \infty.$$

iii) *There is a measurable version in $\Omega \times G$ of the random field*

$$\left(\int_0^\infty u(x, t) \delta X_t \right)_{x \in G}.$$

iv) *It holds that*

$$\int_G d\lambda(x) E \left(\int_0^\infty u(x, t) \delta X_t \right)^2 < \infty.$$

Then $\int_G d\lambda(x) u(x, \cdot) \in Dom(\delta)$ and

$$\int_0^\infty \left(\int_G d\lambda(x) u(x, \cdot) \right) \delta X_t = \int_G d\lambda(x) \left(\int_0^\infty u(x, t) \delta X_t \right).$$

Proof: We will prove two following properties,

a) $\int_G d\lambda(x) u(x, \cdot) \in L^2(\Omega, L_R)$

b) For every $F \in Cyl$ we have

$$E \left(F \left(\int_G d\lambda(x) \int_0^\infty u(x, \cdot) \delta X_t \right) \right) = E \left(\left\langle DF, \int_G d\lambda(x) u(x, \cdot) \right\rangle_{\mathcal{H}} \right). \quad (10.3)$$

Without restriction to the generality we can suppose λ to be a finite measure. Concerning a), Jensen's inequality implies

$$E \left(\left\| \int_G d\lambda(x) u(x, \cdot) \right\|_R^2 \right) \leq \lambda(G) E \left(\int_G d\lambda(x) \|u(x, \cdot)\|_R^2 \right) = \lambda(G) \int_G d\lambda(x) \|u(x, \cdot)\|_R^2 < \infty$$

because of ii). For part b), by classical Fubini's theorem, the left-hand side of (10.3) gives

$$\begin{aligned} & \int_G d\lambda(x) E \left(F \int_0^\infty u(x, t) \delta X_t \right) = \int_G d\lambda(x) E (\langle DF, u(x, \cdot) \rangle_{\mathcal{H}}) \\ & = E \left(\int_G d\lambda(x) \langle DF, u(x, \cdot) \rangle_{\mathcal{H}} \right). \end{aligned} \tag{10.4}$$

This is possible because

$$|\langle DF, u(x, \cdot) \rangle_{\mathcal{H}}| \leq \|DF\|_{\mathcal{H}} \|u(x, \cdot)\|_R.$$

(10.4) equals the right-hand side of (10.3) because

$$\begin{aligned} & \int_G d\lambda(x) \langle DF, u(x, \cdot) \rangle_{\mathcal{H}} \\ & = \int_G d\lambda(x) \left(\int_0^\infty D_s F u(x, s) R(ds, \infty) \right. \\ & \quad \left. - \frac{1}{2} \int_{\mathbb{R}_+^2} (D_{s_1} F - D_{s_2} F)(u(x, s_1) - u(x, s_2)) d\mu(s_1, s_2) \right) \\ & = \int_0^\infty R(ds, \infty) D_s F \int_G u(x, s) d\lambda(x) \\ & \quad - \frac{1}{2} \int_{\mathbb{R}_+^2} (D_{s_1} F - D_{s_2} F) \int_G (u(x, s_1) - u(x, s_2)) d\lambda(x) d\mu(s_1, s_2). \end{aligned}$$

Last equality is possible by means of classical Fubini's theorem and assumption ii). This equals

$$\left\langle DF, \int_G d\lambda(x) u(x, \cdot) \right\rangle$$

and the proof is concluded. ■

10.2 Malliavin calculus and Hermite polynomials

We introduce here shortly Hermite polynomials. For more information, refer to [33], Section 1.1.1. Those polynomials have the following properties. For every integer $n \geq 1$

i) $nH_n(x) = xH_{n-1}(x) - H_{n-2}(x),$

ii) $H'_n(x) = H_{n-1}(x),$

iii) $H_0(x) \equiv 1, H_{-1}(x) = 0.$

From Proposition 8.12 the following result follows.

Proposition 10.4. *Let $h \in L_R.$*

i) *For any $n \geq 1, F := H_n \left(\int_0^\infty h dX \right) \in \mathbb{D}^{1,2}$ and*

$$D_t F = H_{n-1} \left(\int_0^\infty h dX \right) h(t).$$

ii) *$F = \exp \left(\int_0^\infty h dX \right).$ Then $F \in \mathbb{D}^{1,2}$ and*

$$D_t F = F h(t).$$

We recall that $\{H_k, k \geq n\}$ constitute a basis of the linear span generated by $\{1, \dots, x^n\}.$ We denote by \mathcal{E}_{Herm} the linear subspace of $L^2(\Omega)$ constituted by all finite linear combinations of elements of the type $H_n \left(\int_0^\infty \phi_n dX \right), \phi_n \in C_0^1(\mathbb{R}_+), n \in \mathbb{N}.$

Proposition 10.5. $\bar{\mathcal{E}}_{Herm} = L^2(\Omega).$

Proof: We first observe that

$$\left\{ \exp \left(\int_0^\infty h dX \right), h \in C_0^1(\mathbb{R}_+) \right\}$$

is total in $L^2(\Omega).$ The idea is to show that a random variable $F \in L^2(\Omega),$ such that $E \left(F \exp \left(\int_0^\infty h dX \right) \right) = 0$ for every $h \in C_0^1(\mathbb{R}_+),$ fulfills

$$E \left(F g \left(\int_0^\infty h_1 dX, \dots, \int_0^\infty h_n dX \right) \right) = 0,$$

for every $h_1, \dots, h_n \in C_0^1(\mathbb{R}_+)$ and $g \in C_b^\infty(\mathbb{R}^n).$ This can be done adapting the proof of Lemma 1.1.2 of [33].

Let us now consider $F \in L^2(\Omega), h \in C_0^1(\mathbb{R}_+)$ such that

$$E \left(F H_n \left(\int_0^\infty h dX \right) \right) = 0, \forall n \in \mathbb{N}. \tag{10.5}$$

It remains to show

$$E \left(F \exp \left(\int_0^\infty h dX \right) \right) = 0. \quad (10.6)$$

By (10.5) it follows obviously that

$$E \left(F \left(\int_0^\infty h dX \right)^n \right) = 0, \forall n \in \mathbb{N}$$

and consequently (10.6) holds. ■

We denote by \mathcal{E}_n the linear span of $H_n \left(\int_0^\infty \phi dX \right)$, $\phi \in C_0^1(\mathbb{R})$, $\|\phi\|_{\mathcal{H}} = 1$, and by \mathcal{H}_n the closure of \mathcal{E}_n in $L^2(\Omega)$. We recall that all the considered Wiener integrals are Gaussian random variables. Adapting Theorem 1.1.1 and Lemma 1.1.1 of [33], we obtain the following result.

Proposition 10.6. 1. *The spaces (\mathcal{H}_n) are orthogonal.*

2. $L^2(\Omega) = \bigoplus_{n=0}^\infty \mathcal{H}_n$.

We discuss here some technical points related to Malliavin derivative and chaos spaces.

Lemma 10.7. *Let $n \geq 1$. The map $D : \mathcal{E}_n \subset L^2(\Omega) \rightarrow L^2(\Omega; L_R)$ verifies the following. For any sequence (F_k) in \mathcal{E}_n converging to zero in $L^2(\Omega)$, (DF_k) is Cauchy in the sense that*

$$\lim_{k,l \rightarrow \infty} E \left(\|DF_k - DF_l\|_{\mathcal{H}}^2 \right) = 0.$$

Proof: The result will follow if, for every $F \in \mathcal{E}_n$ we prove

$$E \left(\|DF\|_{\mathcal{H}}^2 \right) = nE(F^2). \quad (10.7)$$

Let $F = \sum_{k=1}^m H_n \left(\int_0^\infty h_k dX \right)$, $h_k \in C_0^1$. Item i) of Proposition 10.4 and Lemma 1.1.1 of [33] imply that

$$\begin{aligned} E \left(\|DF\|_{\mathcal{H}}^2 \right) &= \sum_{k,l=1}^m E \left(H_{n-1} \left(\int_0^\infty h_k dx \right) H_{n-1} \left(\int_0^\infty h_l dX \right) \right) \langle h_l, h_k \rangle_{\mathcal{H}} \\ &= \sum_{k,l=1}^m \langle h_k, h_l \rangle_{\mathcal{H}} \frac{1}{(n-1)!} \left\{ E \left(\int_0^\infty h_k dX \int_0^\infty h_l dX \right) \right\}^{n-1}. \end{aligned}$$

In fact $\int_0^\infty h dX$ is a standard Gaussian random variable. This gives

$$\frac{1}{(n-1)!} \sum_{k,l=1}^m \langle h_k, h_l \rangle_{\mathcal{H}} \langle h_k, h_l \rangle_{\mathcal{H}}^{n-1} = n \frac{1}{n!} \sum_{k,l=1}^m \langle h_k, h_l \rangle_{\mathcal{H}}^n = nE(F^2)$$

again by Lemma 1.1.1 of [33].

Corollary 10.8. *Let $n \geq 1$.*

i) $\mathcal{H}_n \subset \mathbb{D}^{1,2}$

ii) *If Assumption (D) is verified, then $D : \mathcal{H}_n \subset L^2(\Omega) \rightarrow L^2(\Omega; L_R)$ is continuous.*

iii) *Suppose that Assumption (D) is verified. For every $F \in \mathcal{H}_n$, we have $\langle DF, h \rangle_{\mathcal{H}} \in \mathcal{H}_{n-1}$, $\forall h \in L_R$.*

Proof: i) Let $F \in \mathcal{H}_n$ and (F_k) a sequence in \mathcal{E}_n converging to F in $L^2(\Omega)$. By Lemma 10.7 and Proposition 8.8, $F \in \mathbb{D}^{1,2}$.

ii) It is an obvious consequence of Lemma 10.7.

iii) Let $h \in L_R$. By items i) and ii) $T_h : \mathcal{H}_n \rightarrow L^2(\Omega)$ defined by $T_h(F) = \langle DF, h \rangle_{\mathcal{H}}$ is continuous. By Proposition 10.4 i) the image of \mathcal{E}_n through T_h is included in \mathcal{H}_{n-1} . Since \mathcal{H}_{n-1} is a closed subspace of $L^2(\Omega)$, the result follows. \blacksquare

Proposition 10.9. *We suppose the validity of Assumptions (C) and (D). Let $F \in \mathbb{D}^{1,2}$, $h \in L_R$. Then there is a sequence (F_n) such that $F_n \in \mathcal{H}_n$, $h \in L_R$,*

$$F = \sum_{n=0}^{\infty} F_n,$$

$$\langle DF, h \rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \langle DF_n, h \rangle_{\mathcal{H}}, \quad \forall h \in L_R,$$

where the convergence holds in $L^2(\Omega)$.

Proof: Let $h \in L_R$ and $F = \sum_{n=0}^{\infty} F_n$ according to Proposition 10.6. By Corollary 10.8, iii), $\langle DF_{m+1}, h \rangle_{\mathcal{H}}$ belongs to \mathcal{H}_m ; since $\langle DF, h \rangle_{\mathcal{H}} \in L^2(\Omega)$, we need to show that for $m \geq 0$

$$E(\langle DF, h \rangle_{\mathcal{H}} \mathcal{G}_m) = E(\langle DF_{m+1}, h \rangle_{\mathcal{H}} \mathcal{G}_m) \quad (10.8)$$

for every $\mathcal{G}_m \in \mathcal{H}_m$. To prove (10.8), taking into account Corollary 10.8 i) and Remark 8.11, we write

$$\begin{aligned} E(\langle DF, h \rangle_{\mathcal{H}} \mathcal{G}_m) &= E\left(F \left\{ \mathcal{G}_m \int_0^{\infty} h dX - \langle D\mathcal{G}_m, h \rangle_{\mathcal{H}} \right\}\right) \\ &= \sum_{n=0}^{\infty} E\left(F_n \left\{ \mathcal{G}_m \int_0^{\infty} h dX - \langle D\mathcal{G}_m, h \rangle_{\mathcal{H}} \right\}\right) \\ &= \sum_{n=0}^{\infty} E(\langle DF_n, h \rangle_{\mathcal{H}} \mathcal{G}_m) = E(\langle DF_{m+1}, h \rangle_{\mathcal{H}} \mathcal{G}_m). \end{aligned}$$

\blacksquare

10.3 Generalized Skorohod integrals and Hermite polynomials

We define now, implementing the idea of [8] and [32], an extension of $Dom(\delta)$ denoted by $Dom(\delta)^*$. The idea is to use a similar duality relation to (10.2), but keeping in mind that an element u of $Dom(\delta)^*$ will not necessarily live in $L^2(\Omega; L_R)$. We denote by \mathcal{L}^2 the space of processes $(u_t)_{t \geq 0}$ with

$$E \left(\int_0^\infty u_s^2 |R|(ds, \infty) + \int_0^\infty |u_s|^2 \bar{m}(ds) \right) < \infty,$$

where \bar{m} is the marginal measure of $|\bar{\mu}|$.

We will denote by \mathcal{M} the linear space of Borel functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$\|f\|_{\mathcal{M}} := \int_0^\infty f^2(s) |R|(ds, \infty) + \int_{\mathbb{R}_+^2} (f(s_1) - f(s_2))^2 |\bar{\mu}|(ds_1, ds_2) < \infty. \quad (10.9)$$

Remark 10.10. 1. Obviously $\|\cdot\|_{\mathcal{M}} \leq \|\cdot\|_R$ and so $\tilde{L}_R \subset \mathcal{M}$.

2. \mathcal{M} is complete (therefore it is a Hilbert space) because of Assumption (C). The argument is similar to the one used in proof of Proposition 6.14.

3. Expanding the second integral of (10.9) and applying Cauchy-Schwarz, we can show that $\mathcal{L}^2 \subset \mathcal{M}$.

Definition 10.2. A process $u \in L^2(\Omega; \mathcal{M})$ is said to belong to $Dom(\delta)^*$ if there is a square integrable r.v. $Z \in L^2(\Omega)$ such that

$$\begin{aligned} E(FZ) = E \left(\int_0^\infty R(ds, \infty) D_s F u_s \right. \\ \left. - \int_{\mathbb{R}_+^2} \mu(ds_1, ds_2) ((D_{s_1} F - D_{s_2} F)(u_{s_1} - u_{s_2})) \right) \end{aligned} \quad (10.10)$$

for any $F = H_n \left(\int_0^\infty h dX \right)$, $h \in C_0^1$, $n \geq 0$.

Remark 10.11. 1. If $u \in Dom(\delta)^*$ then (10.10) holds by linearity for every $F \in \mathcal{E}_{Herm}$.

2. Since \mathcal{E}_{Herm} is dense in $L^2(\Omega)$, Z is uniquely determined. Z will be called **the *-Skorohod integral** of u with respect to X , it will be denoted $\int_0^\infty u_s \delta^* X_s$.

3. The right-hand side of (10.10) is well-defined for any $u \in L^2(\Omega, \mathcal{M})$ and $F = f(\int_0^\infty \varphi_1 dX, \dots, \int_0^\infty \varphi_n dX)$, $\varphi_1, \dots, \varphi_n \in C_0^1(\mathbb{R}_+)$, $f \in C_{pol}^1(\mathbb{R}^n)$, in particular for F as in Definition 10.2. We observe that in this case $F^i = \partial_i f(\int_0^\infty \varphi_1 dX, \dots, \int_0^\infty \varphi_n dX)$, $1 \leq i \leq n$, is a square integrable random variable.

Indeed, we only discuss the second addend of the right-hand side of (10.10), since the first one is obviously meaningful.

This is bounded by

$$\begin{aligned}
& E \left(\int_{\mathbb{R}_+^2} |\mu|(ds_1, ds_2) |(D_{s_1} F - D_{s_2} F)(u_{s_1} - u_{s_2})| \right) \\
&= \sum_{j=1}^n E \left(\int_{\mathbb{R}_+^2} |\mu|(ds_1, ds_2) |F^j(\varphi_j(s_1) - \varphi_j(s_2))(u_{s_1} - u_{s_2})| \right) \\
&\leq \sum_{j=1}^n \|\varphi_j'\|_\infty \left(\int_{\mathbb{R}_+^2} |\bar{\mu}|(ds_1, ds_2) |u_{s_1} - u_{s_2}| |F^j| \right) \\
&\leq \sum_{j=1}^n \|\varphi_j'\|_\infty \sqrt{E(F^j)^2 E \left(\int_{\mathbb{R}_+^2} |\bar{\mu}|(ds_1, ds_2) |u_{s_1} - u_{s_2}| \right)^2} \\
&\leq \sum_{j=1}^n \|\varphi_j'\|_\infty \|F^j\|_{L^2(\Omega)} \sqrt{|\bar{\mu}|(\mathbb{R}_+^2)} E \left(\int_{\mathbb{R}_+^2} |\bar{\mu}|(ds_1, ds_2) |u_{s_1} - u_{s_2}|^2 \right).
\end{aligned}$$

If $u \in L^2(\Omega; \mathcal{M})$, then previous quantity is finite.

Proposition 10.12. $Dom(\delta) \subset Dom(\delta)^*$.

Proof: Let $u \in Dom(\delta)$, $Z = \int_0^\infty u \delta X$. First of all $u \in L^2(\Omega; \mathcal{M})$ since $\|u\|_{\mathcal{M}} \leq \|u\|_R$. For any $F \in Cyl$ we have

$$E(FZ) = E(\langle DF, u \rangle_{\mathcal{H}}). \quad (10.11)$$

Relation (10.11) extends to the elements F of the type $f(\int_0^\infty h dX)$, $f \in C^1$ with subexponential derivative. For this, it is enough to provide the same type of approximation sequence as in the proof (item ii)) of Proposition 8.12. If (f^M) is such a sequence, setting $F^M = f^M(\int_0^\infty h dX)$ and taking into account Proposition 8.12, we clearly obtain

$$\begin{aligned}
\lim_{M \rightarrow \infty} E(\langle DF^M, u \rangle_{\mathcal{H}}) &= E(\langle DF, u \rangle_{\mathcal{H}}), \\
\lim_{M \rightarrow \infty} E(F^M Z) &= E(FZ).
\end{aligned} \quad (10.12)$$

This implies in particular that (10.11) holds for F of the type $H_n(\int_0^\infty h dX)$, where $n \geq 0$, $h \in C_0^1$. For such an element, the right-hand side of (10.11) coincides with the right-hand side of (10.10) and the result follows. \blacksquare

11 It formula in the very singular case

We suppose again Assumptions (A), (B), (C) by default. We start with a technical observation.

Lemma 11.1. *Let (G_1, G_2) be a Gaussian vector such that $\text{Var}G_2 = 1$. Let $f \in C^1(\mathbb{R})$ such that f' is subexponential. Then*

$$nE(f(G_1)H_n(G_2)) = E(f'(G_1)H_{n-1}(G_2))\text{Cov}(G_1, G_2).$$

Remark 11.2. *It follows in particular that $E(H_n(G_2)) = 0, \forall n \geq 1$.*

Proof: According to relation i) about Hermite polynomials, the left-hand side equals $I_1 - I_2$, where

$$\begin{aligned} I_1 &= E(f(G_1)G_2H_{n-1}(G_2)) \\ I_2 &= E(f(G_1)H_{n-2}(G_2)). \end{aligned}$$

According to Wick theorem, recalled briefly in Lemma 11.3 below, I_1 gives.

$$E(f'(G_1)H_{n-1}(G_2))\text{Cov}(G_1, G_2) + E(f(G_1)H'_{n-1}(G_2)).$$

Using relation ii) about Hermite polynomials, we have

$$E(f(G_1)H'_{n-1}(G_2)) = E(f(G_1)H_{n-2}(G_2)) = I_2$$

and the result follows. \blacksquare

The lemma below was recalled and for instance proved in [12].

Lemma 11.3. *(Wick) Let $\underline{Z} = (Z_1, \dots, Z_N)$ be a zero-mean Gaussian vector, $\phi \in C^1(\mathbb{R}^N, \mathbb{R})$ such that the derivatives are subexponential. Then for $1 \leq l \leq N$, we have*

$$E(Z_l \phi(\underline{Z})) = \sum_{j=1}^N \text{Cov}(Z_l, Z_j) E(\partial_j \phi(\underline{Z})).$$

Applying Lemma 11.1 iteratively, it is possible to show the following.

Proposition 11.4. *Let $f \in C^{n+2}(\mathbb{R})$, such that $f^{(n+2)}$ is subexponential. Let (G_1, G_2) be a Gaussian vector such that $\text{Var}(G_2) = 1$. We have the following.*

- a) $n!E(f(G_1)H_n(G_2)) = E(f^{(n)}(G_1))\text{Cov}(G_1, G_2)^n$,
- b) $(n-1)!E(f'(G_1)H_{n-1}(G_2)) = E(f^{(n)}(G_1))\text{Cov}(G_1, G_2)^{n-1}$,
- c) $n!E(f''(G_1)H_n(G_2)) = E(f^{(n+2)}(G_1))\text{Cov}(G_1, G_2)^n$.

Let (X_t) be a process such that $1_{[0,t]} \in L_R$ and $X_t = \int_0^\infty 1_{[0,t]} dX$ for every $t \geq 0$. We recall that under Assumption (D) this is always verified.

We denote $\gamma(t) = \text{Var}(X_t)$.

Remark 11.5. a) $R(t, \infty) = \text{Var}(X_t) - \text{Cov}(X_t, X_\infty - X_t)$

b) $t \rightarrow \text{Var}(X_t)$ has locally bounded variation if and only if $t \rightarrow \text{Cov}(X_t, X_\infty - X_t)$ has locally bounded variation.

Remark 11.6. *It is not easy to find general conditions so that $t \rightarrow \gamma(t)$ has locally bounded variation even though this condition is often realized. We give for the moment some examples.*

1. If X has a convergence measure structure γ has always bounded variation, see [25], Lemma 8.12.
2. If $X_t = \tilde{X}_{t \wedge T}$, and (\tilde{X}_t) is a process with weak stationary increments, then $\gamma(t) = Q(t)$ has always bounded variation, under for instance the assumptions of Proposition 4.6.
- 3.

$$X_t = \int_0^t G(t-s) dW_s, \quad G \in L_{loc}^2(\mathbb{R}).$$

In this case $\gamma(t) = \int_0^t G^2(u) du$, which is increasing and therefore locally of bounded variation.

4. In all explicit examples considered until now, e.g. fractional Brownian motion, bifractional Brownian motion, then γ has locally bounded variation.

We can now state the Itô's formula in the singular case. We recall that from the beginning we suppose $E(X_t) = 0, \forall t \geq 0$.

Proposition 11.7. *We denote $\gamma(t) = \text{Var}(X_t)$, which is supposed to have locally bounded variation. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ of class C_b^∞ . Then $f'(X)1_{[0,t]}$ belongs to $\text{Dom}(\delta)^*$ and*

$$\int_0^t f'(X_s)1_{[0,t]}(s)\delta^* X_s = f(X_t) - f(X_0) - \frac{1}{2} \int_0^t f''(X_s)d\gamma_s.$$

Proof (of Proposition 11.7): We proceed similarly to [8] and [32]. We first observe that

$$g(X_s)1_{[0,t]}(s)$$

belongs to $\mathcal{L}^2 \subset \mathcal{M}$, for every bounded function g . In particular this is true for $g = f'$. Moreover

$$Z_f := f(X_t) - f(X_0) - \frac{1}{2} \int_0^t f''(X_s)d\gamma_s$$

belongs to $L^2(\Omega)$, since f has linear growth, f'' is bounded and X is square integrable. In agreement with (10.10) for $u_s = f'(X_s)1_{[0,t]}(s)$ we have to prove that for any F of the type $H_n(\int_0^\infty \phi dX)$, $\phi \in C_0^1$, $n \geq 0$.

$$\begin{aligned} E(FZ_f) &= \int_0^t R(ds, \infty)E(D_s F f'(X_s)) \\ &\quad - \int_{\mathbb{R}_+^2} \mu(ds_1, ds_2)E((f'(X_{s_1})1_{[0,t]}(s_1) - f'(X_{s_2})1_{[0,t]}(s_2))(D_{s_1} F - D_{s_2} F)). \end{aligned} \tag{11.1}$$

Without restriction to generality we suppose $\|\phi\|_{\mathcal{H}} = 1$. In this case by Proposition 8.12, we have

$$D_s F = H_{n-1} \left(\int_0^\infty \phi dX \right) \phi(s).$$

The right-hand side of (11.1) becomes

$$\begin{aligned} &E \left(\int_0^t R(ds, \infty) H_{n-1} \left(\int_0^\infty \phi dX \right) \phi(s) f'(X_s) \right) \\ &\quad - \int_{\mathbb{R}_+^2} \mu(ds_1, ds_2) (\phi(s_1) - \phi(s_2)) E \left((f'(X_{s_1})1_{[0,t]}(s_1) - f'(X_{s_2})1_{[0,t]}(s_2)) H_{n-1} \left(\int_0^\infty \phi dX \right) \right) \\ &= E \left(H_n \left(\int_0^\infty \phi dX \right) Z_f \right). \end{aligned} \tag{11.2}$$

We recall that by convention H_{-1} is set to zero. We denote

$$p(\sigma, y) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{y^2}{2\sigma}\right), \sigma > 0, y \in \mathbb{R}$$

and we recall that

$$\frac{\partial p}{\partial \sigma} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2}.$$

Hence for all $n \in \mathbb{N}$ and $t > 0$, we evaluate

$$\frac{d}{dt} E \left(f^{(n)}(X_t) \right) = \frac{d}{dt} E \left(f^{(n)}(\sqrt{\gamma_t} N) \right),$$

where $N \sim N(0, 1)$. We show that

$$\frac{d}{dt} E \left(f^{(n)}(X_t) \right) = E \left(f^{(n+2)}(X_t) \right) \frac{d\gamma_t}{2} \quad (11.3)$$

in the sense of measures. Let $\alpha \in C_0^\infty(\mathbb{R})$ be a test function. With help of classical Lebesgue-Stieltjes integration theory, we have

$$\begin{aligned} \left\langle \frac{d}{dt} E(f^{(n)}(\sqrt{\gamma_t} N)), \alpha \right\rangle &= \int_0^\infty \alpha(t) \left(\frac{d}{dt} \int_{\mathbb{R}} f^{(n)}(y) p(\gamma_t, y) dy \right) dt \\ &= \int_0^\infty \alpha(t) \left(\int_{\mathbb{R}} dy f^{(n)}(y) \frac{\partial p}{\partial \sigma}(\gamma_t, y) \right) d\gamma_t = \int_0^\infty \alpha(t) \left(\int_{\mathbb{R}} dy f^{(n)}(y) \frac{1}{2} \frac{\partial^2 p}{\partial y^2}(\gamma_t, y) \right) d\gamma_t \\ &= \int_0^\infty \alpha(t) \left(\int_{\mathbb{R}} dy f^{(n+2)}(y) p(\gamma_t, y) \right) \frac{d\gamma_t}{2} = \int_0^\infty \frac{d\gamma_t}{2} \alpha(t) E(f^{(n+2)}(\sqrt{\gamma_t} N)) \\ &= \int_0^\infty \frac{d\gamma_t}{2} \alpha(t) E(f^{(n+2)}(X_t)). \end{aligned}$$

which proves (11.3). We prove now (11.2). The case of $n = 0$ holds if we prove that $E(Z_f) = 0$. This expectation gives

$$E \left(f(X_t) - f(0) - \frac{1}{2} \int_0^\infty f''(X_s) d\gamma_s \right),$$

which vanishes applying (11.3) for $n = 0$. It remains to prove (11.2) for $n \geq 1$. Proposition 11.4 implies

$$\mathbf{a)} \quad E \left(H_{n-1} \left(\int_0^\infty \phi dX \right) f'(X_s) \right) = \frac{1}{(n-1)!} E \left(f^{(n)}(X_s) \right) \langle 1_{[0,s]}, \phi \rangle_{\mathcal{H}}^{n-1}$$

$$\mathbf{b)} \quad E \left(H_n \left(\int_0^\infty \phi dX \right) f(X_s) \right) = \frac{1}{n!} E \left(f^{(n)}(X_s) \right) \langle 1_{[0,s]}, \phi \rangle_{\mathcal{H}}^n$$

$$\mathbf{c)} \quad E \left(H_n \left(\int_0^\infty \phi dX \right) f''(X_s) \right) = \frac{1}{n!} E \left(f^{(n+2)}(X_s) \right) \langle 1_{[0,s]}, \phi \rangle_{\mathcal{H}}^n$$

Similarly to [8], Lemma 4.3, we evaluate, as a measure,

$$\frac{d}{dt} E \left(f(X_t) H_n \left(\int_0^\infty \phi dX \right) \right). \quad (11.4)$$

In Lemma 11.8 below we will show that $t \mapsto \langle \phi, 1_{[0,t]} \rangle$ has bounded variation. By b), (11.4) gives

$$\begin{aligned} & \frac{1}{n!} \frac{d}{dt} \left(E(f^{(n)}(X_t) \langle 1_{[0,t]}, \phi \rangle_{\mathcal{H}}^n) \right) \\ &= I_{1,n}(dt) + I_{2,n}(dt). \end{aligned}$$

where

$$\begin{aligned} I_{1,n}(t) &= \frac{1}{n!} \frac{d}{dt} E \left(f^{(n)}(X_t) \right) \langle 1_{[0,t]}, \phi \rangle_{\mathcal{H}}^n \\ I_{2,n}(t) &= \frac{1}{(n-1)!} \int_0^t E(f^{(n)}(X_s)) \langle 1_{[0,s]}, \phi \rangle_{\mathcal{H}}^{n-1} d \langle 1_{[0,t]}, \phi \rangle_{\mathcal{H}} ds \end{aligned}$$

Using (11.3), we have

$$I_{1,n}(t) = \int_0^t \frac{1}{n!} E(f^{(n+2)}(X_s)) \langle 1_{[0,s]}, \phi \rangle_{\mathcal{H}}^n \frac{d\gamma_s}{2}$$

which gives

$$I_{1,n}(t) = \int_0^t E \left(f''(X_s) H_n \left(\int_0^\infty \phi dX \right) \right) \frac{d\gamma_s}{2} \quad (11.5)$$

using c). Concerning the second term, a) implies

$$I_{2,n}(t) = E \left(H_{n-1} \left(\int_0^\infty \phi dX \right) f'(X_s) d \langle 1_{[0,s]}, \phi \rangle_{\mathcal{H}}(s) \right). \quad (11.6)$$

By Remark 11.2, (11.5) and (11.6) we get

$$\begin{aligned} & E \left((f(X_t) - f(0)) H_n \left(\int_0^\infty \phi dX \right) \right) = E \left(f(X_t) H_n \left(\int_0^\infty \phi dX \right) \right) \\ &= I_{1,n}(t) + I_{2,n}(t) = \int_0^t E \left(f''(X_s) H_n \left(\int_0^\infty \phi dX \right) d\gamma_s \right) + I_{2,n}(t). \end{aligned}$$

This implies that

$$E \left(Z_f H_n \left(\int_0^\infty \phi dX \right) \right) = I_{2,n}(t).$$

It remains to prove that the left-hand side of (11.2) equals $I_{2,n}(t)$. Setting

$g(s) := E \left(f'(X_s) 1_{[0,s]} H_{n-1} \left(\int_0^\infty \phi dX \right) \right)$ we have to prove that

$$\begin{aligned} & \int_0^t R(ds, \infty) g(s) \phi(s) - \frac{1}{2} \int_{\mathbb{R}_+^2} \mu(ds_1, ds_2) (\phi(s_1) - \phi(s_2)) (g(s_1) - g(s_2)) \\ &= \int_0^t g(s) d \langle 1_{[0,\cdot]}, \phi \rangle_{\mathcal{H}}(s). \end{aligned} \quad (11.7)$$

This will be the object of the following lemma.

Lemma 11.8. *Let $\phi \in C_0^1(\mathbb{R}_+)$, $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ continuous and bounded.*

1. $t \mapsto \langle 1_{[0,t]}, \phi \rangle_{\mathcal{H}}$ has bounded variation,
2. (11.7) holds.

Proof: We denote by μ_ϕ the antisymmetric measure on \mathbb{R}_+^2 defined by

$$d\mu_\phi(s_1, s_2) = \int_{\mathbb{R}_+^2} d\bar{\mu}(s_1, s_2) \frac{\phi(s_1) - \phi(s_2)}{s_1 - s_2};$$

the right-hand side is well-defined since $\phi \in C_0^1(\mathbb{R}_+)$. It follows

$$\begin{aligned} & \langle 1_{[0,t]}, \phi \rangle_{\mathcal{H}} \\ &= \int_0^t R(ds, \infty) \phi(s) - \frac{1}{2} \int_{\mathbb{R}_+^2} (1_{[0,t]}(s_1) - 1_{[0,t]}(s_2)) (\phi(s_1) - \phi(s_2)) d\mu(s_1, s_2) \\ &= \int_0^t R(ds, \infty) \phi(s) - \frac{1}{2} \mu_\phi([0, t] \times \mathbb{R}_+) + \frac{1}{2} \mu_\phi(\mathbb{R}_+ \times [0, t]). \end{aligned} \quad (11.8)$$

Therefore

$$\langle 1_{[0,t]}, \phi \rangle_{\mathcal{H}} = \int_0^t R(ds, \infty) \phi(s) + m_\phi([0, t]), \quad (11.9)$$

where $m_\phi([0, t]) = \mu_\phi([0, t] \times \mathbb{R}_+)$. This concludes the proof of 1).

2) The left-hand side of (11.7) gives

$$\int_0^\infty g(s) \phi(s) R(ds, \infty) + \int_0^\infty g(s) dm_\phi(s) = \int_0^\infty g(s) d \langle 1_{[0,\cdot]}, \phi \rangle_{\mathcal{H}}(s)$$

■

At this point (11.1) is established for every $F = H_n(\int_0^\infty \phi dX)$, $\|\phi\|_{\mathcal{H}} = 1$. If $\|\phi\| = 0$ then (11.2) holds trivially. If $\|\phi\|_{\mathcal{H}} = \sigma > 0$ then (11.1) with $\|\phi\|_{\mathcal{H}} = 1$ can be extended to this case replacing X with σX .

12 Wiener and Skorohod integrals

If the integrand is deterministic, the Wiener integral equals Skorohod integral as Proposition 12.1 below shows. We list here some properties, whose proof is very close to the one of [25], where we supposed that X has a covariance measure structure. We suppose Assumptions (A), (B), (C) by default.

Proposition 12.1. *Let $h \in L_R$. Then $h \in \text{Dom}(\delta)$ and*

$$\int_0^\infty h \delta X = \int_0^\infty h dX.$$

Proof: It follows from Proposition 8.4 and the definition of Skorohod integral.

Proposition 12.2. *Let $u \in \text{Cyl}(L_R)$. Then $u \in \text{Dom}(\delta)$ and $\int_0^\infty u \delta X \in L^p(\Omega)$, $\forall p \geq 1$.*

Proof: Let $u = G\psi$, $\psi \in L_R$, $G \in \text{Cyl}$. Proposition 12.1 says that $\psi \in \text{Dom}(\delta)$. Applying Proposition 10.2 with $F = G$, $u = \psi$, it follows that $\psi G \in \text{Dom}(\delta)$ and

$$\int_0^\infty u \delta X = G \int_0^\infty \psi \delta X - \langle \psi, DG \rangle_{\mathcal{H}}.$$

Making explicit previous equality when $G = g(Y_1, \dots, Y_n)$, where $g \in C_0^\infty(\mathbb{R}^n)$, $Y_j = \int_0^\infty \varphi_j dX$, $1 \leq j \leq n$, then

$$\int_0^\infty u \delta X = g(Y_1, \dots, Y_n) \int_0^\infty \psi dX - \sum_{j=1}^n \langle \varphi_j, \psi \rangle_{\mathcal{H}} \partial_j g(Y_1, \dots, Y_n). \quad (12.1)$$

The right-hand side belongs obviously to each L^p since Y_j is a Gaussian random variable and g , $\partial_j g$ are bounded. The final result for $u \in \text{Cyl}(L_R)$ follows by linearity. \blacksquare

Remark 12.3. *(12.1) provides an explicit expression of $\int_0^\infty u \delta X$, if $u \in \text{Cyl}(L_R)$.*

Next result concerns the commutation property of the derivative and Skorohod integral. First we observe that $(D_t F) \in \text{Dom}(\delta)$ if $F \in \text{Cyl}$. Moreover, if $u \in \text{Cyl}(L_R)$, $(D_s u_t)$ belongs to $|\mathbb{D}^{1,2}(L_R)|$. Closely to Proposition 7.3 of [25] and to [33], Chapter 1, (1.46) we can prove the following result.

Proposition 12.4. *Let $u \in \text{Cyl}(L_R)$. Then*

$$\int_0^\infty u \delta X \in |\mathbb{D}^{1,2}|$$

and for every t

$$D_t \left(\int_0^\infty u \delta X \right) = u_t + \int_0^\infty (D_t u_s) \delta X.$$

We can now evaluate the $L^2(\Omega)$ -norm of the Skorohod integral.

Proposition 12.5. *Let $u \in |\mathbb{D}^{1,2}(L_R)|$. Then $u \in \text{Dom}(\delta)$, $\int_0^\infty u \delta X \in L^2(\Omega)$ and*

$$\begin{aligned} E \left(\int_0^\infty u \delta X \right)^2 &= E(\|u\|_{\mathcal{H}}^2) \\ &\quad - \frac{1}{2} E \left(\int_{\mathbb{R}_+} d\mu(t_1, t_2) \langle D \cdot (u_{t_1} - u_{t_2}), (D_{t_1} - D_{t_2})u \rangle_{\mathcal{H}} \right) \\ &\quad + E \left(\int_0^\infty R(dt, \infty) \|D_t u\|_{\mathcal{H}}^2 \right). \end{aligned} \quad (12.2)$$

Moreover

$$\begin{aligned} E \left(\int_0^\infty u \delta X \right)^2 \\ \leq E \left(\|u\|_R^2 + \int_{\mathbb{R}_+} d|R|(dt, \infty) \|D_t u\|_R^2 + \frac{1}{2} \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2) \|D \cdot u_{s_1} - D \cdot u_{s_2}\|_R^2 \right). \end{aligned} \quad (12.3)$$

Remark 12.6. *We denote $(D^1 u)(s, t) = D_t u_s$, $(Du)(s, t) = D_s u_t$, $s, t \geq 0$.*

i) *The right-hand side of (12.2) can be written as*

$$E \left(\|u\|_{\mathcal{H}}^2 + \langle Du, D^1 u \rangle_{\mathcal{H} \otimes \mathcal{H}} \right),$$

ii) *The right-hand side of (12.3) can be written as*

$$E \left(\|u\|_R^2 + \|Du\|_{2,R}^2 \right).$$

Proof(of Proposition 12.5): Let $u \in \text{Cyl}(L_R)$. By Proposition 12.4, $\int_0^\infty u \delta X \in |\mathbb{D}^{1,2}|$ and we get

$$\begin{aligned} E \left(\int_0^\infty u \delta X \right)^2 &= E \left(\left\langle u, D \int_0^\infty u \delta X \right\rangle_{\mathcal{H}} \right) \\ &= E \left(\int_0^\infty u_s D_s \left(\int_0^\infty u \delta X \right) R(ds, \infty) \right) \\ &\quad - \frac{1}{2} E \left(\int_{\mathbb{R}_+^2} (u_{s_1} - u_{s_2})(D_{s_1} - D_{s_2}) \left(\int_0^\infty u \delta X \right) \mu(ds_1, ds_2) \right) \\ &= E_1 - \frac{1}{2} E_2, \end{aligned}$$

where

$$E_1 = E \left(R(dt, \infty) u_t^2 + \int_0^\infty u_t \left(\int_0^\infty (D_t u_r) \delta X_r \right) R(dt, \infty) \right),$$

$$E_2 = E \left(\int_{\mathbb{R}_+^2} \mu(dt_1, dt_2) (u_{t_1} - u_{t_2})^2 + \int_{\mathbb{R}_+^2} \mu(dt_1, dt_2) (u_{t_1} - u_{t_2}) \int_0^\infty (D_{t_1} - D_{t_2}) u_r \delta X_r \right).$$

Consequently

$$E_1 - \frac{1}{2} E_2 = E (\|u\|_{\mathcal{H}}^2) + \int_0^\infty R(dt, \infty) E \left(u_t \int_0^\infty D_t u_r \delta X_r \right) - \frac{1}{2} \int_{\mathbb{R}_+^2} \mu(dt_1, dt_2) E \left((u_{t_1} - u_{t_2}) \int_0^\infty (D_{t_1} - D_{t_2}) u_r \delta X_r \right).$$

Using again the duality relation of Skorohod integral, we obtain

$$E(\|u\|_{\mathcal{H}}^2) + E \left(\int_0^\infty R(dt, \infty) \langle D \cdot u_t, D_t u \cdot \rangle_{\mathcal{H}} - \frac{1}{2} \int_{\mathbb{R}_+^2} d\mu(t_1, t_2) \langle D \cdot (u_{t_1} - u_{t_2}), (D_{t_1} - D_{t_2}) u \cdot \rangle_{\mathcal{H}} \right).$$

This proves (12.2) and the result for $u \in Cyl(L_R)$. (12.3) is then a consequence of Cauchy-Schwarz with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the fact that the norm $\|\cdot\|_R$ (resp. $\|\cdot\|_{2,R}$) dominates $\|\cdot\|_{\mathcal{H}}$ (resp. $\|\cdot\|_{\mathcal{H} \otimes \mathcal{H}}$). The general result for $u \in |\mathbb{D}^{1,2}(L_R)|$ follows because $Cyl(L_R)$ is dense in $|\mathbb{D}^{1,2}(L_R)|$. \blacksquare

13 Link between symmetric and Skorohod integral

We wish now to establish a relation between the symmetric integral via regularization and Skorohod integral. We recall that, in most of our examples, forward integrals do not exist. If $X = B$ is a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$, then X is not of finite quadratic variation, therefore $\int_0^\cdot X d^- X$ does not exist, see Introduction.

We suppose here again the validity of Assumptions (A), (B), (C) and $X_t = X_T$, $t \geq T$. We first need a technical lemma.

Lemma 13.1. *Let $Y \in |\mathbb{D}^{1,2}(L_R)|$ cadlag. For $\varepsilon > 0$, we set*

$$Y_t^\varepsilon = \frac{1}{2\varepsilon} \int_{(t-\varepsilon)^+}^{(t+\varepsilon) \wedge T} Y_s ds.$$

We suppose next the validity of next hypothesis.

Hypothesis TR *We set $\nu(dt) = |R|(dt, \infty)$*

1. For every $t \geq 0$, $Y_t \in \mathbb{D}^{1,2}$.
2. $t \mapsto Y_t$ is continuous and bounded in $\mathbb{D}^{1,2}$. In particular $t \mapsto Y_t$ is continuous and bounded in L^2 , $t \mapsto DY_t$ is continuous, bounded in $L^2(\Omega; L_R)$.
3. For $r \in \mathbb{R}$ sufficiently small, $t \mapsto Y_{t+r}$ belongs to $\mathbb{D}^{1,2}(L_R)$ and $r \mapsto Y_{\cdot+r}$ is continuous in $\mathbb{D}^{1,2}(L_R)$ at $r = 0$. In particular $r \mapsto Y_{\cdot+r}$ is continuous in $L^2(\Omega; L_R)$ and $r \mapsto D.Y_{\cdot+r}$ is continuous in $L^2(\Omega; L_{2,R})$ at $r = 0$.

Then Y^ε converges to Y in $\mathbb{D}^{1,2}(L_R)$.

Proof: We set $a(s, t) = 1_{[(t-\varepsilon)^+, (t+\varepsilon) \wedge T]}(s)$, $\rho_t = (t + \varepsilon) \wedge T$. By Hypothesis TR and Corollary 6.19, $a(s, \cdot) \in L_R$ for every $s \geq 0$. Hence Assumptions i), ii) of Proposition 9.14 are verified. By Hypothesis TR 2., Assumption iii) of the same Proposition 9.14 is also valid. Consequently $Y_t^\varepsilon = \frac{1}{2\varepsilon} \int_0^\infty a(s, t) Y_s d\rho_s$ defines a process in $\mathbb{D}^{1,2}(L_R)$. Moreover

$$D_r Y_t^\varepsilon = \frac{1}{2\varepsilon} \int_{(t-\varepsilon)^+}^{(t+\varepsilon) \wedge T} ds D_r Y_s. \quad (13.1)$$

We need to prove the following

$$E(\|Y - Y^\varepsilon\|_R^2) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (13.2)$$

$$E(\|D(Y - Y^\varepsilon)\|_{2,R}^2) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (13.3)$$

1) The left hand-side of (13.2), using Bochner integration properties and Jensen's inequality, is bounded by

$$\frac{1}{2\varepsilon} \int_{(-\varepsilon)^+}^\varepsilon dr E(\|Y_{s+r} - Y_s\|_R^2). \quad (13.4)$$

By Hypothesis TR 3. $\lim_{r \rightarrow 0} E(\|Y_{\cdot+r} - Y\|_R^2) = 0$ and (13.4) converges to zero.

Again by Bochner integration properties and Jensen's inequality the left-hand side of (13.3) is bounded by

$$\frac{1}{2\varepsilon} \int_{(-\varepsilon)^+}^\varepsilon dr E(\|D.Y_{\cdot+r} - DY\|_{2,R}^2).$$

Again this converges to zero since $r \mapsto D.Y_{\cdot+r}$ is continuous in $L_{2,R}$. ■

Hypothesis TR is quite technical. We provide a very important example, which is constituted by $Y = g(X)$, for suitable real functions g . Before treating this we need a preliminary lemma which looks similar, but is significantly different from Corollary 9.10.

Lemma 13.2. *Let $g \in C_b^2(\mathbb{R})$. Then $Y_t = g(X_t)$ belongs to $\mathbb{D}^{1,2}$ and*

$$D_r Y_t = g'(X_t)1_{[0,t]}(r), \quad t \geq 0. \quad (13.5)$$

Moreover the assumptions of Lemma 9.13 are verified with $\rho(t) = t$.

Proof: i) By Assumption (D) $1_{[0,t]} \in L_R$ because of Corollary 6.19. Hence the first part and (13.5) follow by Proposition 8.12.

ii) We continue verifying the assumption of Lemma 9.13. Y is continuous in L^2 because Y is pathwise continuous and $(Y_t)_{t \leq T}$ is uniformly integrable. Indeed g has linear growth and X is Gaussian, so there is constant *const.* with

$$\sup_{t \leq T} E(g(X_t))^4 \leq \text{const.} \left(1 + \left(\sup_{t \geq 0} \text{Var} X_t \right)^2 \right).$$

Y is bounded in L^2 since X by similar arguments as above ($g'(X_t)$) is continuous in L^2 . Let now $t_2 > t_1 > 0$. It follows that

$$\begin{aligned} \|D.Y_{t_2} - D.Y_{t_1}\|_R^2 &\leq (g'(X_{t_2})^2)\|1_{[t_1,t_2]}\|_R^2 + (g'(X_{t_2}) - g'(X_{t_1}))^2\|1_{[0,t_1]}\|_R^2 \\ &= 2g'(X_{t_2})^2 \text{Var}(X_{t_2} - X_{t_1}) + 2(g'(X_{t_2}) - g'(X_{t_1}))^2 \text{Var}(X_{t_1}). \end{aligned}$$

Since X and $g'(X)$ are continuous in L^2 and g' has linear growth, we obtain that $t \mapsto D.Y_t$ is continuous in L^2 . By similar arguments $t \mapsto \|DY_t\|_R^2$ is also bounded. This concludes the proof of the Lemma 13.2. ■

We go on with another step in the investigation between symmetric and Skorohod integral.

Proposition 13.3. *Together with the assumptions mentioned at the beginning of Section 13, we suppose*

$$\int_{\mathbb{R}_+^2} \sup_{r \in [-\varepsilon_0, \varepsilon_0]} \text{Var}(X_{s_1+r} - X_{s_2+r}) d|\mu|(s_1, s_2) < \infty \quad (13.6)$$

for some $\varepsilon_0 > 0$. Let $g \in C_b^2(\mathbb{R})$. Then $Y = g(X)$ verifies Hypothesis TR.

Proof :

- 1) Hypothesis TR 1. was the objective of Lemma 13.2.
- 2) We have

$$E(g(X_{t_1}) - g(X_{t_2}))^2 \leq \|g'\|_\infty^2 \text{Var}(X_{t_2} - X_{t_1}).$$

Since X is continuous in L^2 , then $g(X)$ is continuous in L^2 . On the other hand (13.5) holds and $t \mapsto 1_{[0,t]}$ is continuous from \mathbb{R}_+ to L_R because

$$\|1_{[0,t]} - 1_{[0,s]}\|_R^2 = \text{Var}(X_t - X_s).$$

taking into account Corollary 6.19. This implies that $t \mapsto DY_{t+r}$ is continuous (and bounded) from \mathbb{R}_+ to $\mathbb{D}^{1,2}$ for r small enough. Consequently Hypothesis TR 2. is valid.

3) i) By Proposition 9.7 and Remark 9.8 we know that $X_{\cdot+r} \in \mathbb{D}^{1,2}(L_R)$ for s small enough. Proposition 9.9 implies, that $Y = g(X_{\cdot+r})$ belongs to $\mathbb{D}^{1,2}(L_R)$ and $D_s Y_t = g'(X_{t+r})1_{[0,t+r]}(s)$.

ii) To conclude the validity of Hypothesis TR 3. we need to show that

a) $r \mapsto g(X_{\cdot+r})$ is continuous in $L^2(\Omega; L_R)$ in a neighbourhood of 0.

b) $r \mapsto (s, t) \mapsto g'(X_{t+r})1_{[0,t+r]}(s)$ is continuous in a neighbourhood of zero in $L^2(\Omega; L_{2,R})$.

Concerning a), by definition of $\|\cdot\|_R$,

$$\|g(X_{\cdot+r}) - g(X_{\cdot})\|_R^2 \leq \|g'\|_\infty \|X_{\cdot+r} - X_{\cdot}\|_R^2.$$

Taking the expectation and since

$$\begin{aligned} E(\|X_{\cdot+r} - X_{\cdot}\|_R^2) &= \int_0^\infty R(ds, \infty) E(X_{s+r} - X_s)^2 \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} d|\mu|(s_1, s_2) E(X_{s_1+r} - X_{s_1} - X_{s_2+r} + X_{s_2})^2. \end{aligned}$$

Since X is bounded and continuous in L^2 , (13.6) and Lebesgue's dominated convergence theorem imply that

$$\lim_{r \rightarrow 0} E(\|X_{\cdot+r} - X_{\cdot}\|_R^2) = 0. \quad (13.7)$$

Concerning b) we have to estimate,

$$E(\|DY_{\cdot+r} - DY_{\cdot}\|_{2,R}^2),$$

where $D_s Y_{t+r} = g'(X_{t+r})1_{[0,t+r]}(s)$. Previous expectation is bounded by

$$2(I_1(r) + I_2(r)),$$

where

$$\begin{aligned} I_1(r) &= E(g(X_{t+r}))^2 \|1_{[0,t+r]} - 1_{[0,t]}\|_R^2 \\ I_2(r) &= E(\|g(X_{\cdot+r}) - g(X_{\cdot})\|_R^2) \|1_{[0,t]}\|_R^2. \end{aligned}$$

We clearly have

$$I_1(r) \leq (g(0) + \|g'\|_\infty E(X_{t+r}^2)) \|Var(X_{\cdot+r}) - Var(X_\cdot)\|_R^2.$$

Since X is continuous and bounded in $L^2(\Omega)$ and taking into account the definition of $\|\cdot\|_R$, (13.6) and Lebesgue's dominated convergence theorem imply $\lim_{r \rightarrow 0} I_1(r) = 0$. On the other hand

$$\|1_{[0,t]}\|_R^2 = \int_0^\infty Var(X_t)R(dt, \infty) + \frac{1}{2} \int_{\mathbb{R}_+^2} Var(X_{t_2} - X_{t_1}) d|\mu|(t_1, t_2).$$

Since

$$E(\|g(X_{\cdot+r}) - g(X_\cdot)\|_R^2) \leq \|g'\|_\infty \|X_{\cdot+r} - X_\cdot\|_R^2,$$

(13.7) and (13.6) imply $\lim_{r \rightarrow 0} I_2(r) = 0$. This concludes the proof of Proposition 13.3. ■

Remark 13.4. If $X_t = \tilde{X}_{t \wedge T}$ and \tilde{X} has stationary increments, then (13.6) and (6.35) are equivalent to

$$\int_{0+} Q(r)|Q''|(dr) < \infty,$$

whenever $Q(t) = Var \tilde{X}_t$.

We are able now to state a theorem linking Skorohod integral and regularization integrals. We will introduce first a definition.

Definition 13.1. Let $Y \in \mathbb{D}^{1,2}(L_R)$. We say that DY admits a **symmetric trace** if

$$\lim_{\varepsilon \rightarrow 0} \int_0^\tau \langle DY_t, 1_{[t-\varepsilon, t+\varepsilon]} \rangle_{\mathcal{H}} \frac{dt}{\varepsilon}$$

for every $\tau > 0$ in probability. We denote by $(Tr^0 DY)(\tau)$ the mentioned quantity.

Theorem 13.5. Let Y be a process with the following assumptions

1. Assumption (D).
2. $Y \in \mathbb{D}^{1,2}(L_R)$.
3. Hypothesis TR holds.
4. Y admits a symmetric trace.

Then

$$\int_0^t Y d^0 X = \int_0^t Y \delta X + (Tr^0 DY)(t). \quad (13.8)$$

Proof: As in Lemma 13.1, we denote

$$Y_t^\varepsilon = \frac{1}{2\varepsilon} \int_{(t-\varepsilon)^+}^{(t+\varepsilon) \wedge T} ds Y_s.$$

The ε - approximation of the left-hand symmetric integral in (13.8) gives

$$\frac{1}{2\varepsilon} \int_0^t Y_s (X_{s+\varepsilon} - X_{(s-\varepsilon)^+}) ds.$$

Using Proposition 12.1 and Proposition 10.2 previous expression equals

$$\begin{aligned} & \frac{1}{2\varepsilon} \int_0^t ds Y_s \int_0^\infty \delta X_u 1_{[s-\varepsilon, s+\varepsilon]}(u) = \\ & = \int_0^t ds \int_0^\infty Y_s 1_{[s-\varepsilon, s+\varepsilon]}(u) - \frac{1}{2\varepsilon} \int_0^t ds \langle DY_s, 1_{[s-\varepsilon, s+\varepsilon]} \rangle_{\mathcal{H}}. \end{aligned} \quad (13.9)$$

Using Fubini's theorem Proposition 10.3, the last expression equals

$$(I_1 + I_2)(\varepsilon)$$

where

$$\begin{aligned} I_1(\varepsilon) &= \frac{1}{2\varepsilon} \int_0^\infty \delta X_u Y_u^\varepsilon, \\ I_2(\varepsilon) &= \int_0^t \frac{ds}{2\varepsilon} \langle DY_s, 1_{[s-\varepsilon, s+\varepsilon]} \rangle_{\mathcal{H}}, \end{aligned}$$

with

$$Y_u^\varepsilon = \frac{1}{2\varepsilon} \int_{(u-\varepsilon)^+}^{(u+\varepsilon) \wedge t} Y_r dr.$$

Lemma 13.1 implies that $I_1(\varepsilon) \rightarrow \int_0^t Y_u \delta X_u$. The definition of symmetric trace implies that $\lim_{\varepsilon \rightarrow 0} I_2(\varepsilon) = (Tr^0 DY)(t)$. ■

Next application will be an application to the case $Y = g(X)$, $g \in C_b^2(\mathbb{R})$.

Corollary 13.6. *We suppose the following*

1. Assumption (D), (13.6) are fulfilled.

2. $\forall T > 0$

$$\sup_{\varepsilon > 0} \int_{\varepsilon}^T |\text{Var}(X_{s+\varepsilon} - X_s) - \text{Var}(X_s - X_{s-\varepsilon})| ds < \infty. \quad (13.10)$$

3. $\gamma(t) = \text{Var}(X_t)$ has bounded variation.

Then, for $g \in C_b^3(\mathbb{R})$ we have

$$\int_0^t g(X) d^0 X = \int_0^t g(X) \delta X + \frac{1}{2} \int_0^t g'(X_s) d\gamma_s, \quad \forall t \in [0, T].$$

Finally we are able to state an Itô formula related to the symmetric (Stratonovich) integral via regularization.

Corollary 13.7. *Under the assumptions 1), 2), 3) of Corollary 13.6, if $f \in C_b^\infty(\mathbb{R})$, then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^0 X_s. \quad (13.11)$$

Remark 13.8. 1. *The mentioned Itô's formula has to be considered as a side effect of Theorem 13.5. We do not aim in providing minimal assumptions. A refinement of this formula could concern the case when the paths of X do not belong to L_R and X belongs only to $(\text{Dom}\delta)^*$. Various techniques developed by [16, 37, 8] for the specific case of fractional (or bifractional) Brownian motion will probably help.*

2. *If X has stationary increments, then (13.10) holds.*

Proof (of Corollary 13.6): The result follows as consequence of Theorem 13.5. Indeed $Y = g(X)$ belongs to $\mathbb{D}^{1,2}(L_R)$ because of Corollary 9.10. Hypothesis TR holds because of Proposition 13.3. So Hypotheses 1., 2., 3. of Theorem 13.5 are verified. It remains to check that Y admits a symmetric trace and

$$\text{Tr}^0 Dg(X)_\tau = \int_0^\tau g'(X_s) d\gamma_s, \quad \forall \tau > 0. \quad (13.12)$$

As we said, Corollary 9.10 implies that

$$D_r Y_t = g'(X_t) 1_{[0,t]}(r).$$

Hence for $\tau > 0$ the left-hand side of (13.12) is the limit when $\varepsilon \rightarrow 0$ of

$$\int_0^\tau \frac{1}{2\varepsilon} \langle DY_t, 1_{[t-\varepsilon, t+\varepsilon]} \rangle_{\mathcal{H}} dt = \frac{1}{2} \int_0^\tau g'(X_t) \langle 1_{[0,t]}, 1_{[t-\varepsilon, t+\varepsilon]} \rangle_{\mathcal{H}} dt. \quad (13.13)$$

We consider the bounded variation function

$$F_\varepsilon(\tau) = \int_0^\tau \langle 1_{[0,t]}, 1_{[t-\varepsilon, t+\varepsilon]} \rangle_{\mathcal{H}} \frac{dt}{2\varepsilon}.$$

Let $T > 0$. If we prove that

$$dF_\varepsilon(\tau) \Rightarrow \frac{d\gamma(\tau)}{2}, \quad \tau \in [0, T], \quad (13.14)$$

then (13.14) converges to $\frac{1}{2} \int_0^\tau g'(X_t) d\gamma_t$ a.s. for every $\tau \geq 0$ and the theorem would be established. To prove (13.14) we need to establish the following.

i) The total variation $d|F_\varepsilon|(T)$ a.e. is bounded in $\varepsilon > 0$.

ii) $F_\varepsilon(\tau) \longrightarrow \frac{\gamma(\tau)}{2}, \forall \tau > 0$.

Indeed we have

$$\begin{aligned} F_\varepsilon(\tau) &= \int_0^\tau \frac{dt}{2\varepsilon} (Cov(X_{t+\varepsilon}, X_t) - Cov(X_{t-\varepsilon}, X_t)) \\ &= \frac{1}{2\varepsilon} \int_0^\tau dt ((\gamma(t+\varepsilon) - \gamma(t)) - Var(X_{t+\varepsilon} - X_t)) \\ &\quad - \frac{1}{2\varepsilon} \int_0^\tau dt (\gamma(t) - \gamma(t-\varepsilon) - Var(X_t - X_{t-\varepsilon})). \end{aligned}$$

So

$$F_\varepsilon(\tau) = I_{1,\varepsilon}(\tau) + I_{2,\varepsilon}(\tau),$$

where

$$\begin{aligned} I_{1,\varepsilon}(\tau) &= \frac{1}{2\varepsilon} \int_{\tau-\varepsilon}^\tau dt \gamma(t), \\ I_{2,\varepsilon}(\tau) &= \frac{1}{2\varepsilon} \int_{\tau-\varepsilon}^\tau dt (Var(X_{t+\varepsilon} - X_t) - Var(X_t - X_{t-\varepsilon})) \end{aligned} \quad (13.15)$$

i) The total variations of $I_{1,\varepsilon}$ on $[0, T]$ are bounded by the total variation of γ . The total variation of $I_{2,\varepsilon}$ are bounded because of (13.10).

ii) Obviously

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_{1,\varepsilon}(\tau) &= \frac{\gamma(\tau)}{2} \\ \lim_{\varepsilon \rightarrow 0} I_{2,\varepsilon}(\tau) &= 0. \end{aligned} \quad (13.16)$$

Finally **i)** and **ii)** are proved. ■

ACKNOWLEDGEMENTS: Part of the work was done during the stay of the first and third named authors at the Bielefeld University (SFB 701 and BiBoS). They are grateful to Prof. Michael Röckner for the invitation.

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